18.02 Multivariable Calculus
Fall 2007

18.02 Lecture 6. – Tue, Sept 18, 2007

Handouts: Practice exams 1A and 1B.

Velocity and acceleration. Last time: position vector $\vec{r}(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}}$ $[+z(t)\hat{\boldsymbol{k}}]$.

E.g., cycloid: $\vec{r}(t) = \langle t - \sin t, 1 - \cos t\rangle$.

Velocity vector: $\vec{v}(t) = \frac{d\vec{r}}{dt} = \langle \frac{dx}{dt}, \frac{dy}{dt}\rangle$. E.g., cycloid: $\vec{v}(t) = \langle 1 - \cos t, \sin t\rangle$. (at $t = 0$, $\vec{v} = \vec{0}$: translation and rotation motions cancel out, while at $t = \pi$ they add up and $\vec{v} = \langle 2, 0\rangle$).

Speed (scalar): $|\vec{v}|$. E.g., cycloid: $|\vec{v}| = \sqrt{(1-\cos t)^2 + \sin^2 t} = \sqrt{2 - 2\cos t}$. (smallest at $t = 0, 2\pi, ...$, largest at $t = \pi$).

Acceleration: $\vec{a}(t) = \frac{d\vec{v}}{dt}$. E.g., cycloid: $\vec{a}(t) = \langle \sin t, \cos t\rangle$ (at $t = 0$ $\vec{a} = \langle 0, 1\rangle$ is vertical).

Remark: the speed is $|\frac{d\vec{r}}{dt}|$, which is NOT the same as $\frac{d|\vec{r}|}{dt}$!

Arclength, unit tangent vector. s = distance travelled along trajectory. $\boxed{\frac{ds}{dt} = \text{speed} = |\vec{v}|.}$ Can recover length of trajectory by integrating ds/dt, but this is not always easy... e.g. the length of an arch of cycloid is $\int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt$ (can't do).

Unit tangent vector to trajectory: $\hat{T} = \frac{\vec{v}}{|\vec{v}|}$. We have: $\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = \hat{T}\frac{ds}{dt} = \hat{T}|\vec{v}|$.

In interval Δt: $\Delta\vec{r} \approx \hat{T}\,\Delta s$, dividing both sides by Δt and taking the limit $\Delta t \to 0$ gives us the above identity.

Kepler's 2nd law. (illustration of efficiency of vector methods) Kepler 1609, laws of planetary motion: the motion of planets is in a plane, and area is swept out by the line from the sun to the planet at a constant rate. Newton (about 70 years later) explained this using laws of gravitational attraction.

Kepler's law in vector form: area swept out in Δt is area $\approx \frac{1}{2}|\vec{r} \times \Delta\vec{r}| \approx \frac{1}{2}|\vec{r} \times \vec{v}|\,\Delta t$
So $\frac{d}{dt}(\text{area}) = \frac{1}{2}|\vec{r} \times \vec{v}|$ is constant.

Also, $\vec{r} \times \vec{v}$ is perpendicular to plane of motion, so $\text{dir}(\vec{r} \times \vec{v})$ = constant. Hence, Kepler's 2nd law says: $\vec{r} \times \vec{v}$ = constant.

The usual product rule can be used to differentiate vector functions: $\frac{d}{dt}(\vec{a} \cdot \vec{b})$, $\frac{d}{dt}(\vec{a} \times \vec{b})$, being careful about non-commutativity of cross-product.

$$\frac{d}{dt}(\vec{r} \times \vec{v}) = \frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt} = \vec{v} \times \vec{v} + \vec{r} \times \vec{a} = \vec{r} \times \vec{a}.$$

So Kepler's law $\Leftrightarrow$ $\vec{r} \times \vec{v}$ = constant $\Leftrightarrow$ $\vec{r} \times \vec{a} = 0$ $\Leftrightarrow$ $\vec{a} // \vec{r}$ $\Leftrightarrow$ the force $\vec{F}$ is central.

(so Kepler's law really means the force is directed $//\vec{r}$; it also applies to other central forces – e.g. electric charges.)

18.02 Lecture 7. – Thu, Sept 20, 2007

Handouts: PS2 solutions, PS3.

Review. Material on the test = everything seen in lecture. The exam is similar to the practice exams, or very slightly harder. The main topics are (Problem numbers refer to Practice 1A):

1) vectors, dot product. $\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}|\cos\theta = \sum a_i b_i$. Finding angles. (e.g. Problem 1.)

2) cross-product, area of space triangles $\frac{1}{2}|\boldsymbol{A} \times \boldsymbol{B}|$; equations of planes (coefficients of equation = components of normal vector) (e.g. Problem 5.)

3) matrices, inverse matrix, linear systems (e.g. Problem 3.)

4) finding parametric equations by decomposing position vector as a sum; velocity, acceleration; differentiating vector identities (e.g. Problems 2,4,6).