18.02 Multivariable Calculus
Fall 2007

18.02 Lecture 1. – Thu, Sept 6, 2007

Handouts: syllabus; PS1; flashcards.

Goal of multivariable calculus: tools to handle problems with several parameters – functions of several variables.

Vectors. A vector (notation: $\vec{A}$) has a direction, and a length ($|\vec{A}|$). It is represented by a directed line segment. In a coordinate system it's expressed by components: in space, $\vec{A} = \langle a_1, a_2, a_3 \rangle = a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}$. (Recall in space x-axis points to the lower-left, y to the right, z up).

Scalar multiplication

Formula for length? Showed picture of $\langle 3, 2, 1 \rangle$ and used flashcards to ask for its length. Most students got the right answer ($\sqrt{14}$).

You can explain why $|\vec{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ by reducing to the Pythagorean theorem in the plane (Draw a picture, showing $\vec{A}$ and its projection to the xy-plane, then derived $|\vec{A}|$ from length of projection + Pythagorean theorem).

Vector addition: $\vec{A}+\vec{B}$ by head-to-tail addition: Draw a picture in a parallelogram (showed how the diagonals are $\vec{A}+\vec{B}$ and $\vec{B}-\vec{A}$); addition works componentwise, and it is true that $\vec{A} = 3\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$ on the displayed example.

Dot product.

Definition: $\vec{A}\cdot\vec{B} = a_1b_1 + a_2b_2 + a_3b_3$ (a scalar, not a vector).

Theorem: geometrically, $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$.

Explained the theorem as follows: first, $\vec{A}\cdot\vec{A} = |\vec{A}|^2\cos 0 = |\vec{A}|^2$ is consistent with the definition. Next, consider a triangle with sides $\vec{A}$, $\vec{B}$, $\vec{C} = \vec{A}-\vec{B}$. Then the law of cosines gives $|\vec{C}|^2 = |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}|\cos\theta$, while we get

$$|\vec{C}|^2 = \vec{C}\cdot\vec{C} = (\vec{A}-\vec{B})\cdot(\vec{A}-\vec{B}) = |\vec{A}|^2 + |\vec{B}|^2 - 2\vec{A}\cdot\vec{B}.$$

Hence the theorem is a vector formulation of the law of cosines.

Applications. 1) computing lengths and angles: $\cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}$.

Example: triangle in space with vertices $P = (1,0,0)$, $Q = (0,1,0)$, $R = (0,0,2)$, find angle at P:

$$\cos\theta = \frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{|\overrightarrow{PQ}||\overrightarrow{PR}|} = \frac{\langle -1,1,0\rangle\cdot\langle -1,0,2\rangle}{\sqrt{2}\sqrt{5}} = \frac{1}{\sqrt{10}}, \qquad \theta \approx 71.5^\circ.$$

Note the sign of dot product: positive if angle less than 90°, negative if angle more than 90°, zero if perpendicular.

2) detecting orthogonality.

dot product

Example: what is the set of points where $x + 2y + 3z = 0$? (possible answers: empty set, a point, a line, a plane, a sphere, none of the above, I don't know).

Answer: plane; can see "by hand", but more geometrically use dot product: call $\vec{A} = \langle 1,2,3\rangle$, $P = (x,y,z)$, then $\vec{A}\cdot\overrightarrow{OP} = x + 2y + 3z = 0 \Leftrightarrow |\vec{A}||\overrightarrow{OP}|\cos\theta = 0 \Leftrightarrow \theta = \pi/2 \Leftrightarrow \vec{A}\perp\overrightarrow{OP}$. So we get the plane through O with normal vector $\vec{A}$.

18.02 Lecture 2. – Fri, Sept 7, 2007

We've seen two applications of dot product: finding lengths/angles, and detecting orthogonality. A third one: finding components of a vector. If $\hat{\boldsymbol{u}}$ is a unit vector, $\vec{A}\cdot\hat{\boldsymbol{u}} = |\vec{A}|\cos\theta$ is the component of $\vec{A}$ along the direction of $\hat{\boldsymbol{u}}$. E.g., $\vec{A}\cdot\hat{\boldsymbol{\imath}}$ = component of $\vec{A}$ along x-axis.

Example: pendulum making an angle with vertical, force = weight of pendulum $\vec{F}$ pointing downwards: then the physically important quantities are the components of $\vec{F}$ along tangential direction (causes pendulum's motion), and along normal direction (causes string tension).

Area. E.g. of a polygon in plane: break into triangles. Area of triangle $= \frac{1}{2}$ base $\times$ height $= \frac{1}{2}|\vec{A}||\vec{B}|\sin\theta$ (= 1/2 area of parallelogram). Could get $\sin\theta$ using dot product to compute $\cos\theta$ and $\sin^2+\cos^2=1$, but it gives an ugly formula. Instead, reduce to complementary angle $\theta' = \pi/2-\theta$ by considering $\vec{A}' = \vec{A}$ rotated 90° counterclockwise (drew a picture). Then area of parallelogram $= |\vec{A}||\vec{B}|\sin\theta = |\vec{A}'||\vec{B}|\cos\theta' = \vec{A}'\cdot\vec{B}$.

negative theta plus Pi/2

Q: if $\vec{A} = \langle a_1, a_2\rangle$, then what is $\vec{A}'$? (showed picture, used flashcards). Answer: $\vec{A}' = \langle -a_2, a_1\rangle$. (explained on picture). So area of parallelogram is $\langle b_1, b_2\rangle\cdot\langle -a_2, a_1\rangle = a_1b_2 - a_2b_1$.

Determinant. Definition: $\det(\vec{A},\vec{B}) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$.

Geometrically: $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \pm$ area of parallelogram.

The sign of 2D determinant has to do with whether $\vec{B}$ is counterclockwise or clockwise from $\vec{A}$, without details.

Determinant in space: $\det(\vec{A},\vec{B},\vec{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$.

Geometrically: $\det(\vec{A},\vec{B},\vec{C}) = \pm$ volume of parallelepiped. Referred to the notes for more about determinants.

Cross-product. (only for 2 vectors in space); gives a vector, not a scalar (unlike dot-product).

Definition: $\vec{A}\times\vec{B} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{\boldsymbol{\imath}}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{\boldsymbol{\jmath}}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{\boldsymbol{k}}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$.

(the 3x3 determinant is a *symbolic* notation, the actual formula is the expansion).

Geometrically: $|\vec{A}\times\vec{B}|$ = area of space parallelogram with sides $\vec{A}$, $\vec{B}$; direction = normal to the plane containing $\vec{A}$ and $\vec{B}$.

How to decide between the two perpendicular directions = right-hand rule. 1) extend right hand in direction of $\vec{A}$; 2) curl fingers towards direction of $\vec{B}$; 3) thumb points in same direction as $\vec{A}\times\vec{B}$.

Flashcard Question: $\hat{\boldsymbol{\imath}}\times\hat{\boldsymbol{\jmath}} = ?$ (answer: $\hat{\boldsymbol{k}}$, checked both by geometric description and by calculation).

Triple product: volume of parallelepiped = area(base) $\cdot$ height $= |\vec{B}\times\vec{C}|\,(\vec{A}\cdot\hat{n})$, where $\hat{n} = \vec{B}\times\vec{C}/|\vec{B}\times\vec{C}|$. So volume $= \vec{A}\cdot(\vec{B}\times\vec{C}) = \det(\vec{A},\vec{B},\vec{C})$. The latter identity can also be checked directly using components.