18.02 Multivariable Calculus
Fall 2007

18.02 Lecture 3. – Tue, Sept 11, 2007

Remark: $\boldsymbol{A}\times\boldsymbol{B} = -\boldsymbol{B}\times\boldsymbol{A}$, $\boldsymbol{A}\times\boldsymbol{A} = 0$.

Application of cross product: equation of plane through P_1, P_2, P_3: $P = (x, y, z)$ is in the plane iff $\det(\overrightarrow{P_1P}, \overrightarrow{P_1P_2}, \overrightarrow{P_1P_3}) = 0$, or equivalently, $\overrightarrow{P_1P}\cdot\boldsymbol{N} = 0$, where $\boldsymbol{N}$ is the normal vector $\boldsymbol{N} = \overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}$. I explained this geometrically, and showed how we get the same equation both ways. the area, V_a * (V_b X V_c)

Matrices. Often quantities are related by linear transformations; e.g. changing coordinate systems, from $P = (x_1, x_2, x_3)$ to something more adapted to the problem, with new coordinates (u_1, u_2, u_3). For example

$$\begin{cases} u_1 = 2x_1 + 3x_2 + 3x_3 \\ u_2 = 2x_1 + 4x_2 + 5x_3 \\ u_3 = x_1 + x_2 + 2x_3 \end{cases}$$

Rewrite using matrix product: $\begin{bmatrix} 2 & 3 & 3 \\ 2 & 4 & 5 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, i.e. $AX = U$.

Entries in the matrix product = dot product between rows of A and columns of X. (here we multiply a 3x3 matrix by a column vector = 3x1 matrix).

More generally, matrix multiplication AB:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}\begin{bmatrix} \cdot & 0 \\ \cdot & 3 \\ \cdot & 0 \\ \cdot & 2 \end{bmatrix} = \begin{bmatrix} \cdot & 14 \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$$

(Also explained one can set up A to the left, B to the top, then each entry of AB = dot product between row to its left and column above it).

Note: for this to make sense, width of A must equal height of B.

What AB means: BX = apply transformation B to vector X, so $(AB)X = A(BX)$ = apply first B then A. (so matrix multiplication is like composing transformations, but from right to left!)

(Remark: matrix product is not commutative, AB is in general not the same as BA – one of the two need not even make sense if sizes not compatible).

Identity matrix: identity transformation $IX = X$. $\quad I_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Example: $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ = plane rotation by 90 degrees counterclockwise.

$R\hat{\imath} = \hat{\jmath}$, $R\hat{\jmath} = -\hat{\imath}$, $R^2 = -I$.

Inverse matrix. Inverse of a matrix A (necessarily square) is a matrix $M = A^{-1}$ such that $AM = MA = I_n$.

A^{-1} corresponds to the reciprocal linear relation.

E.g., solution to linear system $AX = U$: can solve for X as function of U by $X = A^{-1}U$.

Cofactor method to find A^{-1} (efficient for small matrices; for large matrices computer software uses other algorithms): $A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$ (adj(A) = "adjoint matrix").

Illustration on example: starting from $A = \begin{bmatrix} 2 & 3 & 3 \\ 2 & 4 & 5 \\ 1 & 1 & 2 \end{bmatrix}$

1) matrix of minors (= determinants formed by deleting one row and one column from A): $\begin{bmatrix} 3 & -1 & -2 \\ 3 & 1 & -1 \\ 3 & 4 & 2 \end{bmatrix}$ (e.g. top-left is $\begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} = 3$).

2) cofactors = flip signs according to checkerboard diagram $\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$: get $\begin{bmatrix} 3 & +1 & -2 \\ -3 & 1 & +1 \\ 3 & -4 & 2 \end{bmatrix}$

3) transpose = exchange rows / columns (read horizontally, write vertically) get the adjoint matrix $M^T = adj(A) = \begin{bmatrix} 3 & -3 & 3 \\ 1 & 1 & -4 \\ -2 & 1 & 2 \end{bmatrix}$

4) divide by $\det(A)$ (here = 3): get $A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & -3 & 3 \\ 1 & 1 & -4 \\ -2 & 1 & 2 \end{bmatrix}$.

18.02 Lecture 4. – Thu, Sept 13, 2007

Handouts: PS1 solutions; PS2.

Equations of planes. Recall an equation of the form $ax + by + cz = d$ defines a plane.

1) plane through origin with normal vector $\boldsymbol{N} = \langle 1, 5, 10\rangle$: $P = (x, y, z)$ is in the plane $\Leftrightarrow$ $\boldsymbol{N} \cdot \overrightarrow{OP} = 0 \Leftrightarrow \langle 1, 5, 10\rangle \cdot \langle x, y, z\rangle = x + 5y + 10z = 0$. Coefficients of the equation are the components of the normal vector.

2) plane through $P_0 = (2, 1, -1)$ with same normal vector $\boldsymbol{N} = \langle 1, 5, 10\rangle$: parallel to the previous one! P is in the plane $\Leftrightarrow \boldsymbol{N} \cdot \overrightarrow{P_0P} = 0 \Leftrightarrow (x-2) + 5(y-1) + 10(z+1) = 0$, or $x + 5y + 10z = -3$. Again coefficients of equation = components of normal vector.

(Note: the equation multiplied by a constant still defines the same plane).

So, to find the equation of a plane, we really need to look for the normal vector $\boldsymbol{N}$; we can e.g. find it by cross-product of 2 vectors that are in the plane.

Flashcard question: the vector $\boldsymbol{v} = \langle 1, 2, -1\rangle$ and the plane $x + y + 3z = 5$ are 1) parallel, 2) perpendicular, 3) neither?

(A perpendicular vector would be proportional to the coefficients, i.e. to $\langle 1, 1, 3\rangle$; let's test if it's in the plane: equivalent to being $\perp \boldsymbol{N}$. We have $\boldsymbol{v} \cdot \boldsymbol{N} = 1 + 2 - 3 = 0$ so $\boldsymbol{v}$ is parallel to the plane.)

because a*x + b*y + c*z = d is plane equation

Interpretation of 3x3 systems. A 3x3 system asks for the intersection of 3 planes. Two planes intersect in a line, and usually the third plane intersects it in a single point (picture shown). The unique solution to $AX = B$ is given by $X = A^{-1}B$. think about it, why saying so?

Exception: if the 3rd plane is parallel to the line of intersection of the first two? What can happen? (asked on flashcards for possibilities).

If the line $\mathcal{P}_1 \cap \mathcal{P}_2$ is contained in $\mathcal{P}_3$ there are infinitely many solutions (the line); if it is parallel to $\mathcal{P}_3$ there are no solutions. (could also get a plane of solutions if all three equations are the same)

These special cases correspond to systems with $\det(A) = 0$. Then we can't invert A to solve the system: recall $A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$. Theorem: A is invertible $\Leftrightarrow \det A \neq 0$.

imagine the picture in your head

Homogeneous systems: $AX = 0$. Then all 3 planes pass through the origin, so there is the because obvious ("trivial") solution $X = 0$. If $\det A \neq 0$ then this solution is unique: $X = A^{-1}0 = 0$. Otherwise, if $\det A = 0$ there are infinitely many solutions (forming a line or a plane).

Note: $\det A = 0$ means $\det(\boldsymbol{N}_1, \boldsymbol{N}_2, \boldsymbol{N}_3) = 0$, where $\boldsymbol{N}_i$ are the normals to the planes $\mathcal{P}_i$. This means the parallelepiped formed by the $\boldsymbol{N}_i$ has no area, i.e. they are coplanar (showed picture of 3 planes intersecting in a line, and their coplanar normals). The line of solutions is then perpendicular to the plane containing $\boldsymbol{N}_i$. For example we can get a vector along the line of intersection by taking a cross-product $\boldsymbol{N}_1 \times \boldsymbol{N}_2$. think what does this means

General systems: $AX = B$: compared to $AX = 0$, all the planes are shifted to parallel positions from their initial ones. If $\det A \neq 0$ then unique solution is $X = A^{-1}B$. If $\det A = 0$, either there are infinitely many solutions or there are no solutions.

(We don't have tools to decide whether it's infinitely many or none, although elimination will let us find out). looking for the intersection of 3 planes

18.02 Lecture 5. – Fri, Sept 14, 2007

Lines. We've seen a line as intersection of 2 planes. Other representation = parametric equation = as trajectory of a moving point.

E.g. line through $Q_0 = (-1, 2, 2)$, $Q_1 = (1, 3, -1)$: moving point $Q(t)$ starts at Q_0 at $t = 0$, moves at constant speed along line, reaches Q_1 at $t = 1$: its "velocity" is $\vec{v} = \overrightarrow{Q_0Q_1}$; $\overrightarrow{Q_0Q(t)} = t\overrightarrow{Q_0Q_1}$. On example: $\langle x+1, y-2, z-2\rangle = t\langle 2, 1, -3\rangle$, i.e. think Vector_s

$$\begin{cases} x(t) = -1 + 2t, \\ y(t) = 2 + t, \\ z(t) = 2 - 3t \end{cases}$$

Lines and planes. Understand where lines and planes intersect.

Flashcard question: relative positions of Q_0, Q_1 with respect to plane $x + 2y + 4z = 7$? (same side, opposite sides, one is in the plane, can't tell).

(A sizeable number of students erroneously answered that one is in the plane.)

Answer: plug coordinates into equation of plane: at Q_0, $x+2y+4z = 11 > 7$; at Q_1, $x+2y+4z = 3 < 7$; so opposite sides.

Intersection of line Q_0Q_1 with plane? When does the moving point $Q(t)$ lie in the plane? Check: at $Q(t)$, $x + 2y + 4z = (-1+2t) + 2(2+t) + 4(2-3t) = 11 - 8t$, so condition is $11 - 8t = 7$, or $t = 1/2$. Intersection point: $Q(t = \frac{1}{2}) = (0, 5/2, 1/2)$. it never crossed my stupid mind

(What would happen if the line was parallel to the plane, or inside it. Answer: when plugging the coordinates of $Q(t)$ into the plane equation we'd get a constant, equal to 7 if the line is contained in the plane – so all values of t are solutions – or to something else if the line is parallel to the plane – so there are no solutions.)

General parametric curves. that example should be sufficient enough for teaching us the way of studying motion with vector

Example: cycloid: wheel rolling on floor, motion of a point P on the rim. (Drew picture, then showed an applet illustrating the motion and plotting the cycloid).

Position of P? Choice of parameter: e.g., θ, the angle the wheel has turned since initial position. Distance wheel has travelled is equal to arclength on circumference of the circle $= a\theta$.

Setup: x-axis = floor, initial position of P = origin; introduce A = point of contact of wheel on floor, B = center of wheel. Decompose $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BP}$.

$\overrightarrow{OA} = \langle a\theta, 0\rangle$; $\overrightarrow{AB} = \langle 0, a\rangle$. Length of $\overrightarrow{BP}$ is a, and direction is θ from the $(-y)$-axis, so $\overrightarrow{BP} = \langle -a\sin\theta, -a\cos\theta\rangle$. Hence the *position vector* is $\overrightarrow{OP} = \langle a\theta - a\sin\theta, a - a\cos\theta\rangle$.

Q: What happens near bottom point? (flashcards: corner point with finite slopes on left and right; looped curve; smooth graph with horizontal tangent; vertical tangent (cusp)).

Answer: use Taylor approximation: for $t \to 0$, $f(t) \approx f(0) + tf'(0) + \frac{1}{2}t^2 f''(0) + \frac{1}{6}t^3 f'''(0) + \dots$. This gives $\sin\theta \approx \theta - \theta^3/6$ and $\cos\theta \approx 1 - \theta^2/2$. So $x(\theta) \simeq \theta^3/6$, $y(\theta) \simeq \theta^2/2$ Hence for $\theta \to 0$, $y/x \simeq (\frac{1}{2}\theta^2)/(\frac{1}{6}\theta^3) = 3/\theta \to \infty$: vertical tangent.