18.02 Multivariable Calculus
Fall 2007

D. Determinants

Given a square array A of numbers, we associate with it a number called the **determinant** of A, and written either $\det(A)$, or $|A|$. For 2 x 2 and **3** x **3** arrays, the number is defined by

$$(1)\qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc; \qquad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + dhc - ceg - bdi - fha\,.$$

Do not memorize these as formulas — learn instead the patterns which give the terms. The 2 x 2 case is easy: the product of the elements on one diagonal (the "main diagonal"), minus the product of the elements on the other (the "antidiagonal").

is this way

For the **3** x **3** case, three products get the + sign: those formed from the main diagonal, or having two factors parallel to the main diagonal. The other three get a negative sign: those from the antidiagonal, or having two factors parallel to it.[1] Try the following example on your own, then check your work against the solution.

Example **1.1** Evaluate $\begin{vmatrix} 1 & -2 & 1 \\ -1 & 3 & 2 \\ 2 & -1 & 4 \end{vmatrix}$ using (1).

Solution. Using the same order as in (1), we get $12 + (-8) + 1 - 6 - 8 - (-2) = -7$.

Important facts about $|A|$:

D-1. $|A|$ is multiplied by -1 if we interchange two rows or two columns. WHY?

D-2. $|A| = 0$ if one row or column is all zero, or if two rows or two columns are the same.

D-3. $|A|$ is multiplied by c, if every element of some row or column is multiplied by c.

D-4. The value of $|A|$ is unchanged if we add to one row (or column) a constant multiple of another row (resp. column).

All of these facts are easy to check for 2 x 2 determinants from the formula (1); from this, their truth also for **3** x **3** determinants will follow from the Laplace expansion.

Though the letters a, b, c, ... can be used for very small determinants, they can't for larger ones; it's important early on to get used to the standard notation for the entries of determinants. This is what the common software packages and the literature use. The determinants of order two and three would be written respectively

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

[1]There is another form for this rule which requires adding two extra columns to the determinant, but this wastes too much time in practice and leads to awkward write-ups; instead, learn to evaluate each of the six products mentally, writing it down with the correct sign, and then add the six numbers, as is done in Example **1.** Note that the word "determinant" is also used for the square array itself, enclosed between two vertical lines, as for example when one speaks of "the second row of the determinant".

In general, the **ij-entry**, written a_{ij}, is the number in the i-th row and j-th column.

Its **ij-minor**, written $|A_{ij}|$, is the determinant that's left after deleting from $|A|$ the row and column containing a_{ij}.

Its **ij-cofactor**, written here A_{ij}, is given as a formula by $A_{ij} = (-1)^{i+j}|A_{ij}|$. For a 3×3 determinant, it is easier to think of it this way: we put $+$ or $-$ in front of the ij-minor, according to whether $+$ or $-$ occurs in the ij-position in the checkerboard pattern

(2)
$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}.$$

Example 1.2 $|A| = \begin{vmatrix} 1 & 0 & 3 \\ 1 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix}$. Find $|A_{12}|$, A_{12}, $|A_{22}|$, A_{22}.

Solution. $|A_{12}| = \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = 1$, $A_{12} = -1$. $|A_{22}| = \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} = -7$, $A_{22} = -7$.

Laplace expansion by cofactors

This is another way to evaluate a determinant; we give the rule for a 3×3. It generalizes easily to an $n \times n$ determinant.

Select any row (or column) of the determinant. Multiply each entry a_{ij} in that row (or column) by its cofactor A_{ij}, and add the three resulting numbers; you get the value of the determinant.

As practice with notation, here is the formula for the Laplace expansion of a third order (i.e., a 3×3) determinant using the cofactors of the first row:

(3)
$$a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = |A|$$

recall what we did involving i, j, k in first row

and the formula using the cofactors of the j-th column:

(4)
$$a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j} = |A|$$

Example 1.3 Evaluate the determinant in Example 1.2 using the Laplace expansions by the first row and by the second column, and check by also using (1).

Solution. The Laplace expansion by the first row is

$$\begin{vmatrix} 1 & 0 & 3 \\ 1 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} + 3 \cdot \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1 \cdot (-1) - 0 \cdot 1 + 3 \cdot (-3) = -10.$$

The Laplace expansion by the second column would be

be careful with the sign patterns

$$\begin{vmatrix} 1 & 0 & 3 \\ 1 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix} = -0 \cdot \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} + 2 \cdot \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} = 0 + 2 \cdot (-7) - 1 \cdot (-4) = -10.$$

Checking by (1), we have $|A| = -2+0+3-12-0-(-1) = -10$.

Example 1.4 Show the Laplace expansion by the first row agrees with definition (1).

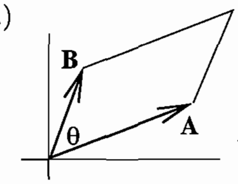
Solution. We have

think the a, b, c as vector i, j and k

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\cdot\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\cdot\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\cdot\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$
$$= a(ei-fh) - b(di-fg) + c(dh-eg),$$

whose six terms agree with the six terms on the right of definition (1).

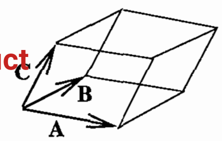
(A similar argument can be made for the Laplace expansion by any row or column.)



Area and volume interpretation of the determinant:

this is a repeating of the vector cross product

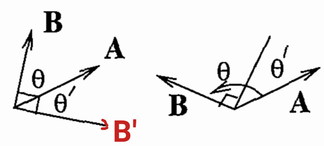
(5) $\pm\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ = area of parallelogram with edges $\mathbf{A} = (a_1, a_2)$, $\mathbf{B} = (b_1, b_2)$.



actually we were using Theorem 4: Scalar Triple Product
A = a*(bXc), {a, b, c vectors}

(6) $\pm\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ = volume of parallelepiped with edges row-vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$.

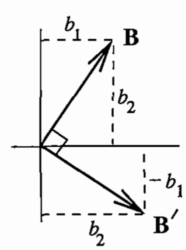
In each case, choose the sign which makes the left side non-negative.



Proof of (5). We begin with two preliminary observations.

Let θ be the positive angle from $\mathbf{A}$ to $\mathbf{B}$; we assume it is $< \pi$, so that $\mathbf{A}$ and $\mathbf{B}$ have the general positions illustrated.

Let $\theta' = \pi/2 - \theta$, as illustrated. Then $\cos\theta' = \sin\theta$.



Draw the vector $\mathbf{B}'$ obtained by rotating $\mathbf{B}$ to the right by $\pi/2$. The picture shows that $\mathbf{B}' = (b_2, -b_1)$, and $|\mathbf{B}'| = |\mathbf{B}|$.

To prove (5) now, we have a standard formula of Euclidean geometry,

$$\begin{aligned} \text{area of parallelogram} &= |\mathbf{A}||\mathbf{B}|\sin\theta \\ &= |\mathbf{A}||\mathbf{B}'|\cos\theta', && \text{by the above observations} \\ &= \mathbf{A}\cdot\mathbf{B}', && \text{by the geometric definition of dot product} \\ &= a_1b_2 - a_2b_1 && \text{by the formula for } \mathbf{B}' \end{aligned}$$

This proves the area interpretation (5) if $\mathbf{A}$ and $\mathbf{B}$ have the position shown. If their positions are reversed, then the area is the same, but the sign of the determinant is changed, so the formula has to read,

$$\text{area of parallelogram} = \pm\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}, \quad \text{whichever sign makes the right side } \geq 0.$$

The proof of the analogous volume formula (6) will be made when we study the scalar triple product $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$.

For $n \times n$ determinants, the analog of definition (1) is a bit complicated, and not used to compute them; that's done by the analog of the Laplace expansion, which we give in a moment, or by using Fact **D-4** in a systematic way to make the entries below the main diagonal all zero. Generalizing (5) and (6), $n \times n$ determinants can be interpreted as the hypervolume in n-space of a n-dimensional parallelotope.

For $n \times n$ determinants, the **minor** $|A_{ij}|$ of the entry a_{ij} is defined to be the determinant obtained by deleting the i-th row and j-th column; the **cofactor** A_{ij} is the minor, prefixed by a $+$ or $-$ sign according to the natural generalization of the checkerboard pattern (2). Then the Laplace expansion by the i-th row would be

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in} .$$

This is an inductive calculation — it expresses the determinant of order n in terms of determinants of order $n-1$. Thus, since we can calculate determinants of order 3, it allows us to calculate determinants of order 4; then determinants of order 5, and so on. If we take for definiteness $i = 1$, then the above Laplace expansion formula can be used as the basis of an inductive definition of the $n \times n$ determinant.

Example 1.5 Evaluate $\begin{vmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 1 & 4 \\ -1 & 4 & 1 & 0 \\ 0 & 4 & 2 & -1 \end{vmatrix}$ by its Laplace expansion by the first row.

Solution. $1 \cdot \begin{vmatrix} -1 & 1 & 4 \\ 4 & 1 & 0 \\ 4 & 2 & -1 \end{vmatrix} - 0 \cdot A_{12} + 2 \cdot \begin{vmatrix} 2 & -1 & 4 \\ -1 & 4 & 0 \\ 0 & 4 & -1 \end{vmatrix} - 3 \cdot \begin{vmatrix} 2 & -1 & 1 \\ -1 & 4 & 1 \\ 0 & 4 & 2 \end{vmatrix}$

$$= 1 \cdot 21 + 2 \cdot (-23) - 3 \cdot 2 = -31.$$

Exercises: Section 1C