18.03 Practice Hour Exam I (2010)

1. A certain computer chip sheds heat at a rate proportional to the difference between its temperature and that of its environment.

(a) Write down a differential equation controlling the temperature of the chip, as a function of time measured in minutes, if the temperature in the environment is a constant 20°C. Your equation will have a constant in it which can't be determined from the data given so far.

(b) What is the general solution of this equation? (This will still involve the unknown constant).

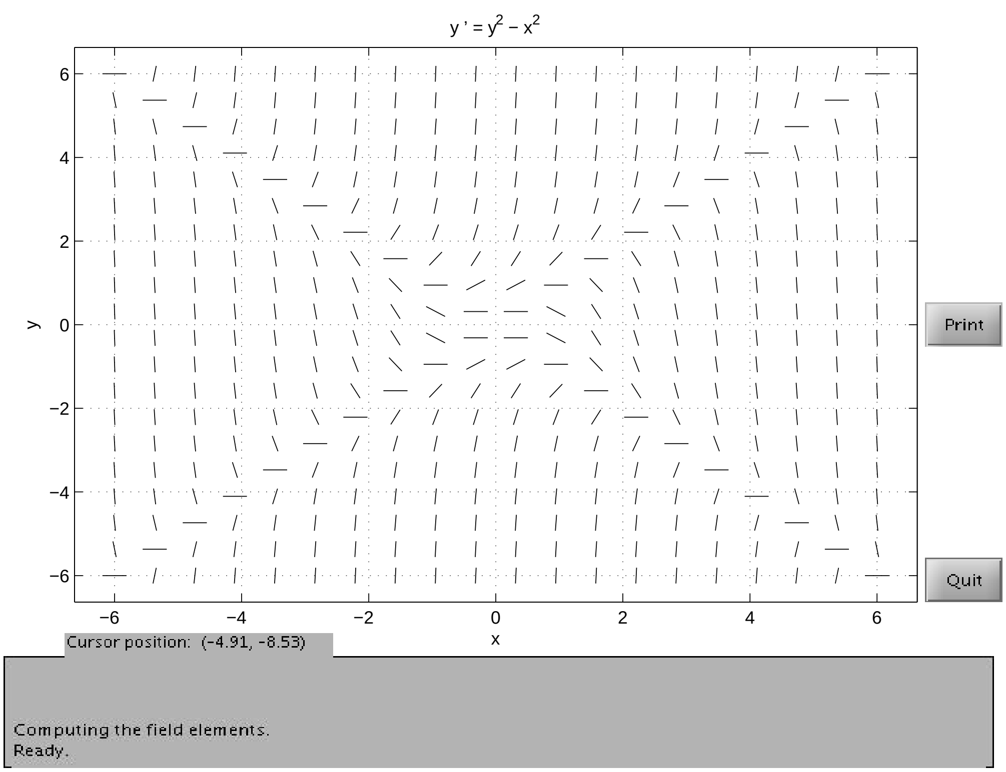
(c) It is observed that if the chip is powered down at $t = 0$ at a temperature of 70°C in a room at 20°C, its temperature at $t = 10$ minutes is 60°C. Use this new information to complete the determination of the differential equation.

2. Estimate $y(2.2)$ where y is the solution of the differential equation $y' = y^2 - x^2$ with $y(2) = 0$, using Euler's method with step size 0.1.

3. This problem concerns the differential equation $y' = y^2 - x^2$. Part of its direction field is shown below.

(a) On the diagram, sketch and label the isoclines for slope $m = -4$, $m = 0$, and $m = 4$.

(b) On the diagram, sketch the graph of the solution of the equation with $y(2) = 0$.



(c) Estimate the value $y(100)$ of the solution with $y(2) = 0$. Is your estimate too large or too small?

(d) A certain solution y has a local extremum at $x = -1$. What can you say about $y(-1)$? Is the extremum a maximum or a minimum? For full credit, make a relevant calculation, rather than merely relying on the picture.

4. (a) Find the general solution of $t\dot{x} + 2x = t^2$.

(b) Find a sinusoidal solution to the differential equation $\dot{x} + 2x = \cos(2t)$. Express your answer as a sum of sines and cosines. You may use any method to find this solution.

5. (a) Express each of the cube roots of $-8i$ first in the form $Ae^{i\theta}$ and then in the form $a + bi$.

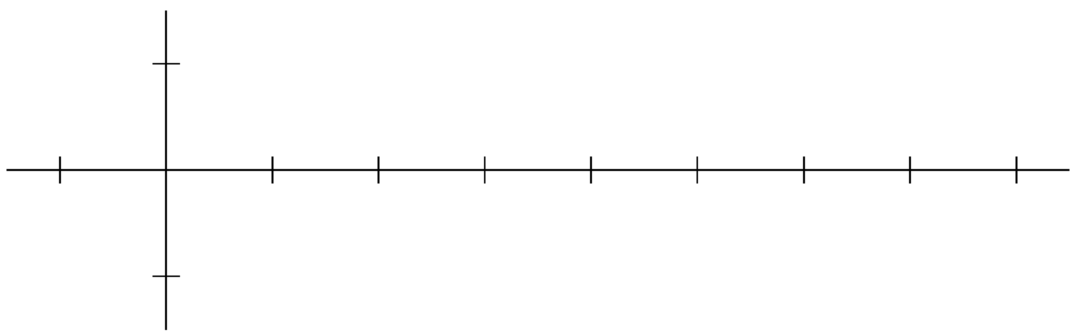
(b)–(e) relate to the sinusoidal function $f(t) = -\cos(\frac{\pi}{2}t) - \sin(\frac{\pi}{2}t)$.

(b) Find positive real numbers A and ϕ such that $f(t) = A\cos(\frac{\pi}{2}t - \phi)$.

(c) What is the period P of this sinusoidal function?

(d) What is the time lag t_0 of this sinusoidal function?

(e) Please sketch a graph of this function below, marking on the diagram A, P, and t_0.



6. This problem concerns the autonomous equation $\dot{y} = y^3 - y$.

(a) Sketch the phase line for this equation. Mark on it all critical points. Label each critical point as stable, unstable, or neither.

(b) Sketch some solutions for this equation, enough so that for any b between -2 and $+2$ you show a solution y such that $y(t) = b$ for some t.

(c) Determine where points of inflection occur in solutions to this equation. (A function $f(t)$ has a point of inflection at (a, b) if $f(a) = b$ and $f''(a) = 0$.)

Solutions

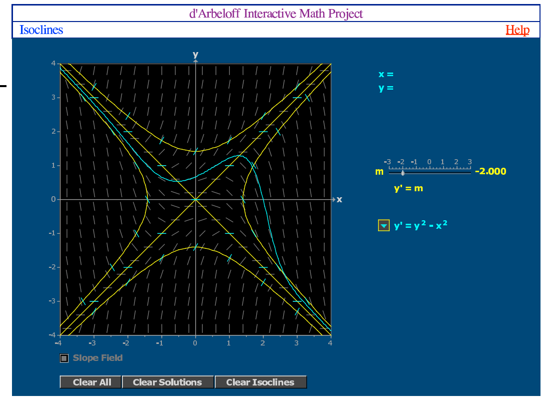
1. (a) Let $x(t)$ be the temperature of the chip in degrees C. $\dot{x} = (20 - x)$, or $\dot{x} + kx = 20$.

(b) $x = 20 + Ce^{-kt}$.

(c) The data gives $70 = x(0) = 20 + C$, so $C = 50$ and $60 = x(10) = 20 + 50e^{-10k}$, so $-10k = \ln(40) - \ln(50)$ or $k = (\ln(50) - \ln(40))/10$.

2.

n	x_n	y_n	$A_n = y_n^2 - x_n^2$	$hA_n = A_n$
0	2	0	-4.0	-0.4
1	2.1	-0.4	$0.16 - 4.41 = -4.25$	-0.425
2	2.2	-0.825		



3. (a) The nullcline is the pair of crossed lines $y = \pm 1$. The $m = 2$ isocline is the upper/lower hyperbola; the $m = -2$ isocline is the left/right hyperbola.

(c) The graphed solution is trapped by the funnel having the nullcline as its lower fence and the $m = -2$ isocline as the upper fence, $y(100)$ is very near to -100, but slightly larger.

(d) Extrema occur when $\dot{y} = 0$; that is, along the nullcline. $\ddot{y} = 2y\dot{y} - 2x$, which is $-2x$ on the nullcline. At $x = -1$ this is positive, so we have a minimum.

4. (a) Multiply through by t: $\dfrac{d}{dt}(t^2 x) = t^3$. Thus $t^2 x = t^4/4 + c$ so $x = t^2/4 + c/t^2$.

(b) First solve the complex-valued equation $\dot{z} + 2z = e^{2it}$. One way to solve this is to try $z_p = Ae^{2it}$ and solving for A: $(2i + 2)A = 1$, or $A = \frac{1}{2+2i} = \frac{2-2i}{8} = \frac{1-i}{4}$. The real part of $\frac{1-i}{4}e^{2it}$ is $x_p = \frac{1}{4}(\cos(2t) + \sin(2t))$.

5. (a) $-8i = 8e^{3\pi i/2}$ so its cube roots all have modulus equal to the postive number whose cube is 8, namely 2. The arguments are $\frac{\pi}{2}$, $\frac{\pi}{2} + \frac{2\pi}{3} = \frac{7\pi}{6}$, and $\frac{\pi}{2} + \frac{4\pi}{3} = \frac{11\pi}{6}$, so the roots are $2e^{\pi i/2} = 2i$, $2e^{7\pi i/6} = -\sqrt{3} - i$, and $2e^{11\pi i/6} = \sqrt{3} - i$.

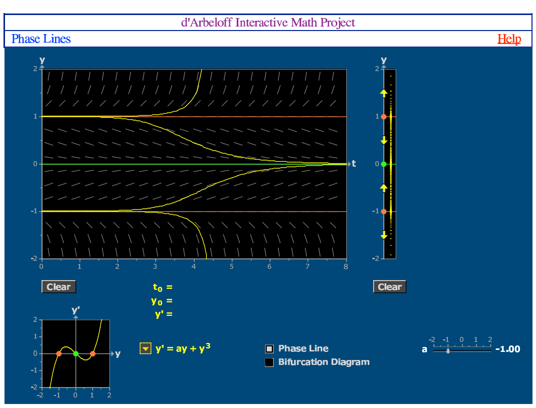
(b) The point $(a, b) = (-1, -1)$ has polar coordinates $A = \sqrt{2}$ and $\phi = 5\pi/4$.

(c) $P = \frac{2\pi}{\omega} = \frac{2\pi}{\pi/2} = 4$.

(d) $t_0 = \phi/\omega = \frac{5\pi}{4}/\frac{\pi}{2} = \frac{5}{2}$.

(e) Amplitude $\sqrt{2}$, period 4, trough at 1/2, peak at 5/2.

6. (a), (b)



(c) $\ddot{y} = 3y^2\dot{y} - \dot{y} = (3y^2 - 1)\dot{y}$ is zero if either $\dot{y} = 0$—that is, $y = 0$ or $y = \pm 1$—or $3y^2 - 1 = 0$—that is, $y = \pm 1/\sqrt{3}$.