

18.03 Hour Exam I Solutions: February 24, 2010

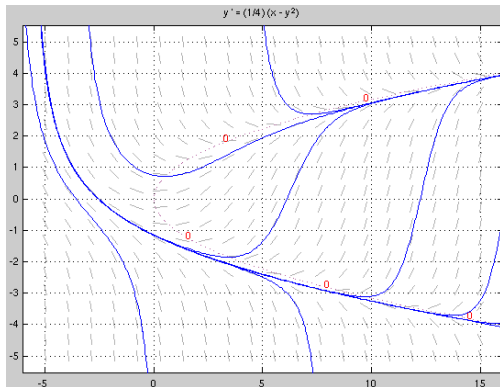
1. (a) $x(t)$ = number of rats at time t ; t measured in years. $\dot{x} = kx$. So $x(t) = x(0)e^{kt}$. $x(0) = x(1) = x(0)e^{kt}$ implies $k = 1$.

(b) $\dot{x} = k \left(1 - \frac{x}{R}\right) x = \left(1 - \frac{x}{1000}\right) x$.

(c) $\dot{x} = \left(1 - \frac{x}{R}\right) x - a$. The pest control people hope for an equilibrium at $x = \frac{3}{4}R$. $\dot{x} = 0$ at equilibrium, so $a = \left(1 - \frac{3}{4}\right)\frac{3}{4}R = \frac{3}{16}R = 375$.

2. (a) The phase line shows unstable critical points at $x = -2$ and $x = 1$ and a stable critical point at $x = 0$. The arrows of time are directed up above 1 and between -2 and 0, and down between 0 and 1 and below -2 .

? (b) There are seven basic solution types: three equilibria; a solution rising above $x = 1$, a solution falling from 1 towards 0, a solution rising from -2 towards 0, and a solution falling away from -2 .



(f) True. After the solution crosses the nullcline, it is “inside” the parabola and its derivative is positive. If it were to cross the nullcline again it would have to cross the upper branch, from below. But the slope of the nullcline is positive, while at the moment of crossing the slope of the solution would have to be zero. So it does not cross again; it stays below the upper branch of the nullcline, which has equation $y = \sqrt{x}$.

3. (a)

k	x_k	y_k	$m_k = x_k + y_k$	hm_k
0	0	1	1	1/2
1	1/2	3/2	4	1
2	1	5/2	7/2	7/4
3	3/2	17/4		

Ans: 17/4.

(b) The equation is $\frac{d}{dt}(tx) = \cos t$, so $tx = \sin t + c$ and $x = \frac{c + \sin t}{t}$. $1 = x(\pi) = \frac{c}{\pi}$ so $c = \pi$ and $x = \frac{\pi + \sin t}{t}$.

4. (a) $\frac{1}{3+2i} = \frac{3-2i}{3^2+2^2}$: $a = \frac{3}{13}$, $b = -\frac{2}{13}$.

(b) $r = |1 - i| = \sqrt{2}$. $\theta = \text{Arg}(1 - i) = -\frac{\pi}{4}$.

(c) $|1 - i| = \sqrt{2}$ and $\text{Arg}(1 - i) = \frac{\pi}{4}$, so $|(1 - i)^8| = (\sqrt{2})^8 = 16$ and $\text{Arg}((1 - i)^8) = 8\frac{\pi}{4} = 2\pi$, so $(1 - i)^8 = 16$: $a = 16$, $b = 0$.

(d) If $(a + bi)^3 = -1$ then $|a + bi|^3 = |(a + bi)^3| = |-1| = 1$ so $|a + bi| = 1$, and $3\text{Arg}(a + bi) = \text{Arg}(-1) = \pi$ (or 3π or 5π) so $\text{Arg}(a + bi) = \frac{\pi}{3}$ or π or $\frac{5\pi}{3}$. The first is the one with positive imaginary part, so $a = \cos \frac{\pi}{3} = \frac{1}{2}$, $b = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

(e) $e^{\ln 2 + i\pi} = e^{\ln 2} e^{i\pi} = 2(-1) = -2$: $a = -2$, $b = 0$.

(f) A, ϕ are the polar coordinates of $(a, b) = (2, -2)$: $A = 2\sqrt{2}$, $\phi = -\frac{\pi}{4}$.

5. (a) Try $x = Ae^{2t}$, so that $\dot{x} = 2Ae^{2t}$ and $e^{2t} = Ae^{2t} + 3Ae^{2t} = 5Ae^{2t}$ so $A = \frac{1}{5}$: $x_p = \frac{1}{5}e^{2t}$. The

transient is ce^{-3t} , so $x = \frac{1}{5}e^{2t} + ce^{-3t}$ is a valid solution for any c as well.

(b) $1 = x(0) = \frac{1}{5} + c$ implies $c = \frac{4}{5}$: this particular solution is $x = \frac{1}{5}e^{2t} + \frac{4}{5}e^{-3t}$

(c) $\dot{z} + 3z = e^{2it}$.

(d) Try $z = Ae^{2it}$: $\dot{z} = 2iAe^{2it}$, so $e^{2it} = \dot{z} + 3z = A(3 + 2i)e^{2it}$. This gives $A = \frac{1}{3+2i}$ and solution $z_p = \frac{1}{3+2i}e^{2it} = \frac{3-2i}{13}(\cos(2t) + i\sin(2t))$, which has real part $x_p = \frac{3}{13}\cos(2t) + \frac{2}{13}\sin(2t)$.

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18.03 Differential Equations

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