

Section II Solutions

2A-1a) This is true because D^2 , pD , and multiplication by q are all linear operators:

$$q(y_1 + y_2) = qy_1 + qy_2 \quad (1)$$

$$pD(y_1 + y_2) = p(Dy_1 + Dy_2) \\ = pDy_1 + pDy_2 \quad (2)$$

$$\therefore D^2(y_1 + y_2) = D^2y_1 + D^2y_2 \quad (3)$$

Adding ①, ②, ③ gives

$$L(y_1 + y_2) = Ly_1 + Ly_2$$

The proof for $L(cy_1) = cLy_1$ is similar.

b) (i) $Ly_h = 0$ since y_h solves the eqn $Ly = 0$
 $Ly_p = r$ since y_p solves the original eqn.

Adding, using linearity of L : $L(y_h + y_p) = r \quad \therefore y_h + y_p$ is a soln.

(ii) if y_1 is any soln, then

$$L(y_1 - y_p) = Ly_1 - Ly_p = r - r = 0$$

$$\therefore y_1 - y_p = y_h \text{ (a soln of } Ly = 0)$$

$$\therefore y_1 = y_h + y_p.$$

Parts (i) + (ii) together show all solns are of the form $y_h + y_p$.

2A-2a)
$$\begin{cases} y = c_1 e^x + c_2 e^{2x} \\ y' = c_1 e^x + 2c_2 e^{2x} \\ y'' = c_1 e^x + 4c_2 e^{2x} \end{cases} \Rightarrow \begin{cases} y' - y = c_2 e^{2x} \\ y'' - y' = 2c_2 e^{2x} \end{cases}$$

$$\therefore y'' - y' = 2(y' - y)$$

 or:
$$y'' - 3y' + 2y = 0$$

b) The question is whether we can find values for c_1, c_2 such that

$$c_1 e^{x_0} + c_2 e^{2x_0} = y_0$$

$$c_1 e^{x_0} + 2c_2 e^{2x_0} = y_0'$$

These equations can be solved (by Cramer's rule) for c_1, c_2 provided that $\begin{vmatrix} e^{x_0} & e^{2x_0} \\ e^{x_0} & 2e^{2x_0} \end{vmatrix} \neq 0$.
 (coefficient determinant)

But this det = $e^{3x_0} \neq 0$ for any x_0 . ✓

2A-3 a) $y = c_1 x + c_2 x^2$
 $y' = c_1 + 2c_2 x$
 $y'' = 2c_2$
 You want to eliminate c_1, c_2 .
 One way —:

$$\begin{cases} c_2 = y''/2 \text{ from last eqn} \\ c_1 = y' - y''x \text{ from 2nd + 3rd eqn.} \end{cases}$$

Substitute into 1st eqn, get

$$y = (y' - y''x)x + \frac{y''}{2}x^2,$$

which by algebra becomes

$$x^2 y'' - 2xy' + 2y = 0$$

b) all solns $y = c_1 x + c_2 x^2$ satisfy $y(0) = 0$

c) This theorem requires that when eqn is written $y'' + p(x)y' + q(x)y = 0$, that p, q be continuous functions. But here, the ODE in standard form is

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0;$$

coefficients are discontinuous at $x=0$.

2A-4 a) Suppose y_1 is a solution to $y'' + p(x)y' + q(x)y = 0$ ①

tangent to x -axis at the pt. x_0 .

Then $y_1(x_0) = 0$

$$y_1'(x_0) = 0.$$

But $y_2(x) \equiv 0$ is another soln to ① with this same property:

$$y_2(x_0) = 0$$

$$y_2'(x_0) = 0$$

\therefore by the uniqueness theorem,

$$y_1 \equiv y_2 \text{ for all } x,$$

$$\text{i.e., } y_1 \equiv 0.$$

b) $y = x^2$
 $y' = 2x$
 $y'' = 2$
 $\therefore xy'' - y' = 0$
 is such an equation
 or: $y'' - \frac{1}{x}y' = 0$

Part (a) is not contradicted, since the coefficient $\frac{1}{x}$ is discontinuous at $x=0$.

2A-5 a) $W(e^{m_1 x}, e^{m_2 x}) = \begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} \end{vmatrix}$
 $= (m_2 - m_1) e^{(m_1 + m_2)x}$;

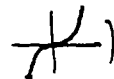
Since $e^x \neq 0$ for all x , this is never 0
 if $m_1 \neq m_2$. \therefore functions are lin. ind.

b) $W(e^{mx}, xe^{mx}) = \begin{vmatrix} e^{mx} & xe^{mx} \\ me^{mx} & mx e^{mx} + e^{mx} \end{vmatrix}$

$= e^{2mx} \neq 0$ for any x .

(This holds true even if $m=0$).

\therefore the functions are lin. indept.

2A-6 (The graph of $x|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$ )

a) If $x \geq 0$, $W = \begin{vmatrix} x^2 & x^2 \\ 2x & 2x \end{vmatrix} \equiv 0$

if $x \leq 0$, $W = \begin{vmatrix} x^2 & -x^2 \\ 2x & -2x \end{vmatrix} \equiv 0$

b) Suppose they were linearly dependent on an interval (a, b) containing 0, that is, suppose there are c_1, c_2 such that

$c_1 y_1 + c_2 y_2 = 0$ for all $x \in (a, b)$.


Then if $x \geq 0$, $y_1 = y_2$, $\therefore c_1 = -c_2$

if $x < 0$, $y_1 = -y_2$, $\therefore c_1 = c_2$

Thus $c_1 = 0$ and $c_2 = 0$, so that

y_1 and y_2 are not lin. dep't on (a, b) .

Since $y_2' = 2x$ for $x > 0$,
 $y_2' = -2x$ for $x < 0$

graph of y_2' is 

Thus y_2'' does not exist at $x=0$,
 so it cannot be the solution to a
 2nd order equation $y'' + p(x)y' + q(x)y = 0$
 on the interval (a, b) containing 0.

Thus thm in the book ($W \equiv 0 \Rightarrow$ solns are lin. dep't)
 is not contradicted.

2A-7 a) This can be done directly, by
 differentiating $y_1 y_2' - y_1' y_2$. (*see below)

An elegant way to do it is to use
 the formula for differentiating a
 determinant: diff. one row at a time, then add:

$\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}' = \begin{vmatrix} u_1' & u_2' \\ v_1 & v_2 \end{vmatrix} + \begin{vmatrix} u_1 & u_2 \\ v_1' & v_2' \end{vmatrix}$

(this works for det's. of any size e).

Applying this to the Wronskian:

$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}' = \begin{vmatrix} y_1' & y_2' \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix};$

since y_1 and y_2 solve $y'' = -py' - qy$,
 we get the above right-hand det.

$= \begin{vmatrix} y_1 & y_2 & y_2 \\ -py_1' - qy_1 & -py_2' - qy_2 & -py_2' - qy_2 \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ -py_1' - qy_2' \end{vmatrix}$

(adding $q \cdot$ (1st row) to 2nd doesn't
 change value of the determinant)

$= -p \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -pW.$

(* you also have to use that y_1, y_2
 are solns, i.e., that

$y_1'' = -py_1' - qy_1, \quad y_2'' = -py_2' - qy_2$).

b) From part (a), if $p(x) = 0$,
 then $\frac{dW}{dx} = 0$, so $W(y_1, y_2) = C$.

c) $y'' + k^2 y = 0$ Here $p = 0$

$W(\cos kx, \sin kx)$

$= \begin{vmatrix} \cos kx & \sin kx \\ -k \sin kx & k \cos kx \end{vmatrix}$

$= k(\cos^2 kx + \sin^2 kx)$

$= k$, a constant.

2B-1

$$\begin{aligned} \text{a) } y_2 &= ue^x \\ x-2] \quad y_2' &= u'e^x + ue^x \\ y_2'' &= u''e^x + 2u'e^x + ue^x \end{aligned}$$

Multiply second row by -2 and add:

$$y_2'' - 2y_2' + y_2 = u''e^x \quad (\text{all other terms cancel out})$$

If y_2 is a soln to the ODE, the left-hand side must be 0. Therefore we must have

$$u''e^x = 0$$

so

$$u'' = 0,$$

$$\therefore u = ax + b$$

and

$$\therefore y_2 = (ax + b)e^x$$

Any of these for which $a \neq 0$ gives a second solution - for ex., $y_2 = xe^x$.

$$\text{b) From II/7a, } \frac{dW}{dx} = -pW = 2W$$

$$\therefore W(y_1, y_2) = ce^{2x}, \quad c \neq 0$$

$$\text{But } W(y_1, y_2) = \begin{vmatrix} e^x & y_2 \\ e^x & y_2' \end{vmatrix}$$

Equating these two expressions for W ,

$$e^x(y_2' - y_2) = ce^{2x}$$

$$\therefore y_2' - y_2 = ce^x$$

(c can have any $\neq 0$ value)

Solving this ODE gives (it's a linear equation)

$$y_2 = e^x(x + c_1) \quad \text{as a family of second solutions.}$$

$$\text{c) } y_2 = e^x \int \frac{1}{e^{2x}} e^{-\int -2dx} dx$$

$$= e^x \int 1 \cdot dx = e^x(x + c)$$

[more generally: $e^{\int 2dx} = e^{2x+c_1}$

$$\therefore y_2 = e^x \int (e^{-c_1}) dx \quad \text{put } c_2 = e^{-c_1}$$

$$= e^x(c_2 x + c)]$$

d) All the solutions are the same - the most general form is

$$y_2 = e^x(c_1 x + c_2), \quad \text{with } c_1 \neq 0$$

(if $c_1 = 0$, we just get y_1 back)

2B-2

$$W(y_1, y_2) = \begin{vmatrix} e^x & e^x(ax+b) \\ e^x & e^x(ax+b) + ae^x \end{vmatrix}$$

$$= ae^{2x}, \neq 0 \text{ if } a \neq 0.$$

[This shows it for the special equation only].

In general:

$$W[y_1, y_2] = y_1 y_2' - y_2 y_1'$$

$$y_2 = y_1 \int \frac{1}{y_1^2} e^{-\int p dx} dx$$

$$\begin{aligned} \therefore y_2' &= y_1' \int \frac{1}{y_1^2} e^{-\int p dx} dx + y_1 \cdot \frac{1}{y_1^2} e^{-\int p dx} \\ &= y_1' y_2 / y_1 + \frac{1}{y_1} e^{-\int p dx} \end{aligned}$$

$$\begin{aligned} \therefore W(y_1, y_2) &= y_1' y_2 + e^{-\int p dx} - y_1' y_2 \\ &= e^{-\int p dx} \neq 0 \end{aligned}$$

[Note that this same formula for the Wronskian follows from II/7a].

2B-3

let $y_2 = x \cdot u$, so that

$$y_2' = u + xu', \quad y_2'' = 2u' + xu''$$

Substituting into $x^2 y'' + 2xy' - 2y = 0$ gives after cancellation and dividing by x^2 :

$$xu'' + 4u' = 0 \quad \text{Put } v = u'$$

$$x \frac{dv}{dx} + 4v = 0 \quad \text{or } \boxed{\frac{dv}{v} = -\frac{4dx}{x}}$$

$$\text{Solving, } v = \frac{c}{x^4}, \quad \text{or } u' = \frac{c}{x^4}$$

$$\therefore u = \frac{c}{-3x^3} + c_0 = \frac{c_1}{x^3} + c_0$$

$$\therefore \boxed{y_2 = \frac{c_1}{x^2} + c_0 x}, \quad \text{a second sol'n (if } c_1 \neq 0)$$

[can also use the general formula given in II/8c]

2B-4

Using the general formula [II/8c]:

$$\text{Find: } e^{-\int p dx} \quad \int p dx = \int \frac{-2x}{1-x^2} dx = \ln(1-x^2)$$

$$\leftarrow = \frac{1}{1-x^2}$$

$$\therefore \int \frac{1}{x^2} e^{-\int p dx} = \int \frac{dx}{x^2(1-x^2)}$$

we do this by partial fractions \rightarrow (cont'd)

2B-4

(cont'd)

$$\frac{1}{x^2(1-x^2)} = \frac{1}{x^2(1-x)(1+x)}$$

$$= \frac{1}{x^2} + \frac{1/2}{1-x} + \frac{1/2}{1+x}$$

$$\therefore \int \frac{dx}{x^2(1-x^2)} = -\frac{1}{x} + \frac{1}{2} \ln(1-x) + \frac{1}{2} \ln(1+x)$$

$$= -\frac{1}{x} + \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$$

$$\therefore y_2 = y_1 \int \frac{1}{y_1^2} e^{-\int p dx} = \boxed{-1 + \frac{x}{2} \ln \frac{1+x}{1-x}}$$

\uparrow
 $= x$

The general solution is now
 $C_1 y_1 + C_2 y_2$

or $\boxed{C_1 x + C_2 \left(-1 + \frac{x}{2} \ln \frac{1+x}{1-x} \right)}$

2C-1

a) Char eqn: $\lambda^2 - 3\lambda + 2 = 0$
 or $(\lambda-1)(\lambda-2) = 0$

roots: $\lambda = 1, 2$

$$\therefore \boxed{y = C_1 e^x + C_2 e^{2x}}$$

b) Char eqn: $r^2 + 2r - 3 = 0$
 $(r+3)(r-1) = 0$

$\therefore y = C_1 e^x + C_2 e^{-3x}$ Put in initial conditions:

$y(0)=1 \Rightarrow C_1 + C_2 = 1$

$y'(0)=1 \Rightarrow C_1 - 3C_2 = -1$ } solve for C_1, C_2

$C_1 = 1/2, C_2 = 1/2$

$$\therefore \boxed{y = \frac{1}{2} e^x + \frac{1}{2} e^{-3x}}$$

c) Char. eqn $r^2 + 2r + 2 = 0$

By quad. formula: $r = -1 \pm i$

$$\therefore y = e^{-x} (C_1 \cos x + C_2 \sin x)$$

[using as y_1, y_2 the real + imaginary

parts of the ex. soln $y = e^{(1+i)x}$
 $= e^x (\cos x + i \sin x)$]

2C-1

d) Char. eqn: $r^2 - 2r + 5 = 0$

By quad. formula: $r = 1 \pm 2i$

Gen'l soln: $y = e^x (C_1 \cos 2x + C_2 \sin 2x)$

Putting in initial condns (you'll have to find y' first!)

$y(0)=1 \Rightarrow C_1=1$

$y'(0)=1 \Rightarrow C_1 + 2C_2 = -1, \therefore C_2 = -1$

so $y = e^x (\cos 2x - \sin 2x)$

e) Char. eqn: $r^2 - 4r + 4 = 0$

or $(r-2)^2 = 0; r=2$ double root

$\therefore y = e^{2x} (C_1 x + C_2)$

is the general solution. Put in initial conditions:

$y(0)=1 \Rightarrow C_2=1$

$y'(0)=1 \Rightarrow 2C_2 + C_1 = 1, \therefore C_1 = -1$

so sol'n is: $y = (1-x)e^{2x}$

2C-2

$$W = \begin{vmatrix} e^{ax} \cos bx & e^{ax} \sin bx \\ e^{ax}(a \cos bx - b \sin bx) & e^{ax}(a \sin bx + b \cos bx) \end{vmatrix}$$

$$= \begin{vmatrix} e^{ax} \cos bx & e^{ax} \sin bx \\ -e^{ax} b \sin bx & e^{ax} (b \cos bx) \end{vmatrix}$$

(by subtracting $a \cdot (1^{st} \text{ row})$ from 2^{nd} row);

$= e^{2ax} (b \cos^2 bx + b \sin^2 bx) = e^{2ax} \cdot b$

$\neq 0$ if $\boxed{b \neq 0}$ (no restriction on a)

2C-3

Char. eqn: $r^2 + cr + 4 = 0$

roots: $r = \frac{-c \pm \sqrt{c^2 - 16}}{2}$

a) has oscillatory solns $\Leftrightarrow r$ is complex
 (so soln has sin + cos terms);

$\Leftrightarrow c^2 - 16 < 0$, or $\boxed{-4 < c < 4}$

b) if the solutions oscillate, above shows
 that $r = -\frac{c}{2} \pm i\beta$ ($\beta \neq 0$)

and solns are $y = e^{-\frac{cx}{2}} (C_1 \cos \beta x + C_2 \sin \beta x)$.

Damped oscillations $\Leftrightarrow c > 0$ (so $y \rightarrow 0$ as $t \rightarrow \infty$)

$\therefore \boxed{0 < c < 4}$ is condition.

2C-4

a) [use y' for $\frac{dy}{dx}$, \dot{y} for $\frac{dy}{dt}$]

We have $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$ $x = e^t$
 $\frac{dx}{dt} = e^t, \frac{dt}{dx} = e^{-t}$

$$\begin{aligned}\therefore y' &= \dot{y} e^{-t} \\ y'' &= \frac{d}{dt}(\dot{y} e^{-t}) \cdot \frac{dt}{dx} \\ &= (\ddot{y} e^{-t} - \dot{y} e^{-t}) e^{-t} \\ &= (\ddot{y} - \dot{y}) e^{-2t}\end{aligned}$$

Substituting into the ODE:

$$\begin{aligned}x^2 y'' + p x y' + q y &= 0 \text{ becomes} \\ (\ddot{y} - \dot{y}) + p \dot{y} + q y &= 0\end{aligned}$$

b) $p = q = 1$, so we get $\ddot{y} + y = 0$, whose solution are $y = c_1 \cos t + c_2 \sin t$
 $x = e^t$ } gives $y = c_1 \cos(\ln x) + c_2 \sin(\ln x)$
 $\therefore t = \ln x$

2C-5

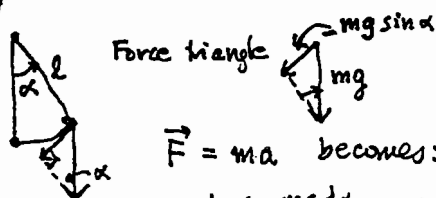
Char. eqn is $Mr^2 + cr + k = 0$

For critical damping, it should have two equal roots; by quadratic formula

$$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2M}, \therefore \boxed{c^2 - 4mk = 0} \text{ is condition}$$

(when $c^2 - 4mk < 0$, get oscillations).

2C-6



$$\begin{aligned}-mg \sin \alpha - m c \frac{d\alpha}{dt} &= m l \frac{d^2 \alpha}{dt^2} \\ (\text{grav.}) & (\text{air res.})\end{aligned}$$

$$\therefore \ddot{\alpha} + \frac{c}{l} \dot{\alpha} + \frac{g}{l} \sin \alpha = 0 \quad \text{If } \alpha \text{ small, } \sin \alpha \approx \alpha$$

If undamped, $c = 0$, get approx.

$$\ddot{\alpha} + \frac{g}{l} \alpha = 0 \quad \left[\begin{array}{l} \text{char eqn is} \\ r^2 + g/l = 0 \end{array} \right]$$

$$\therefore \text{Solns are } y = c_1 \cos \sqrt{\frac{g}{l}} t + c_2 \sin \sqrt{\frac{g}{l}} t$$

$$\text{The period} = \frac{2\pi}{\sqrt{g/l}} = 2\pi \sqrt{\frac{l}{g}}$$

(so as length increases, so does the period;
 on the moon, it swings slower (bigger period)
 (less g))

2C-7

a) $a + bx + ce^x$ b) $a \cos 2x + b \sin 2x$

c) $ax \cos 2x + bx \sin 2x$

d) $ax^2 e^x$ (1 is a double root of the char. eqn)

e) $ae^{-x} + bxe^{2x}$ (2 is a root of char. eqn)

f) $(ax^3 + bx^2)e^{3x}$ (3 is double root of char. eqn)

2C-8

b) $y_h = a_1 \cos 2x + a_2 \sin 2x$

To find y_p , use undet. coefficients:

$$y_p = c_1 \cos x + c_2 \sin x \quad \left[\times 4 \text{ (mult. factor)} \right]$$

$$\therefore y_p'' = -c_1 \cos x - c_2 \sin x \quad \left[\begin{array}{l} \text{and add:} \\ \text{LHS is by hypothesis:} \end{array} \right]$$

$$2 \cos x = 3c_1 \cos x + 3c_2 \sin x \quad y_p'' + 4y_p = 2 \cos x$$

$$\therefore c_1 = 2/3, \quad c_2 = 0$$

$$\text{So } y = a_1 \cos 2x + a_2 \sin 2x + \frac{2}{3} \cos x$$

$$y(0) = 0 \Rightarrow a_1 + 2/3 = 0 \quad \therefore a_1 = -2/3$$

$$y'(0) = 1 \Rightarrow 2a_2 = 1 \quad a_2 = 1/2$$

2C-8

a) $y_h = a_1 e^x + a_2 e^{5x}$, as usual.

Try $y_p = c x e^x$ $\left[\begin{array}{l} \times 5 \\ \times -6 \end{array} \right] \left\{ \begin{array}{l} \text{multiply} \\ \text{factor} \end{array} \right\}$

$$\therefore y_p' = c e^x (x+1) \quad \left[\times -6 \right]$$

$$y_p'' = c e^x (x+2) \quad \text{then add:}$$

$$e^x = e^x (-6c + 2c) + x e^x (5c - 6c + c)$$

$$\therefore -4c = 1$$

$$c = -1/4$$

$$y = a_1 e^x + a_2 e^{5x} - \frac{1}{4} x e^x$$

c) Char eqn: $r^2 + r + 1 = 0$, $r = \frac{-1 \pm \sqrt{-3}}{2}$

$$\therefore y_h = e^{-x/2} (a_1 \cos \frac{\sqrt{3}}{2} x + a_2 \sin \frac{\sqrt{3}}{2} x)$$

Try $y_p = c_1 x e^x + c_2 e^x$

$$y_p' = c_1 e^x (x+1) + c_2 e^x \quad \text{Add the eqns:}$$

$$y_p'' = c_1 e^x (x+2) + c_2 e^x$$

$$2x e^x = 3c_1 x e^x + (3c_1 + 3c_2) e^x$$

$$\therefore c_1 = 2/3, \quad c_2 = -2/3$$

$$y = e^{-x/2} (a_1 \cos \frac{\sqrt{3}}{2} x + a_2 \sin \frac{\sqrt{3}}{2} x) + \frac{2}{3} e^x (x-1)$$

2C-8

d) $y_h = a_1 e^x + a_2 e^{-x}$

Try: $y_p = c_1 x^2 + c_2 x + c_3$ L-1

$y_p'' = 2c_1$ Add:

$x^2 = -c_1 x^2 + c_2 x + 2c_1 - c_3$

$\therefore c_1 = -1, c_2 = 0, 2c_1 - c_3 = 0$
 $c_3 = -2$

$y = a_1 e^x + a_2 e^{-x} - x^2 - 2$

$y(0) = 0 \Rightarrow a_1 - a_2 = 2$

$y'(0) = -1 \Rightarrow a_1 - a_2 = -1$

solving, $a = 1/2, a_2 = 3/2$

$\therefore y = \frac{1}{2} e^x + \frac{3}{2} e^{-x} - x^2 - 2$

2C-9

a) Write the ODE as $Ly = r$,

where L is the linear operator

$L = D^2 + pD + q$

• By hypothesis,

$Ly_1 = r_1$ (i.e., y_1 is a solution to $Ly = r_1$)

$Ly_2 = r_2$ (similarly)

Adding, $L(y_1 + y_2) = r_1 + r_2$

(using the linearity of L : $L(y_1 + y_2) = Ly_1 + Ly_2$)

$\therefore y_1 + y_2$ solves $Ly = r_1 + r_2$.

b) First consider $y'' + 2y' + 2y = 2x$

Try $y_1 = c_1 x + c_2$ L-2

$y_1' = c_1$ L-2

$y_1'' = 0$ Add

$2x = 2c_1 x + (2c_2 + 2c_1)$

$\therefore c_1 = 1, c_2 = -1$ $y_1 = x - 1$

Then: $y'' + 2y' + 2y = \cos x$

Try $y_2 = a_1 \cos x + a_2 \sin x$ L-2

$y_2' = -a_1 \sin x + a_2 \cos x$ L-2

$y_2'' = -a_1 \cos x - a_2 \sin x$ Add

$\cos x = \cos x (2a_1 + 2a_2 - a_1)$
 $+ \sin x (2a_2 - 2a_1 - a_2)$

$\therefore \begin{cases} a_1 + 2a_2 = 1 \\ -2a_1 + a_2 = 0 \end{cases} \therefore \begin{cases} a_2 = 2/5 \\ a_1 = 1/5 \end{cases} \therefore y_2 = \frac{1}{5} \cos x + \frac{2}{5} \sin x$

2C-10

a) $R = 0, E = 0$

Eqn is $Lq'' + \frac{q}{C} = 0$ or $q'' + \frac{q}{LC} = 0$

Solving as usual,

$q = c_1 \cos \frac{1}{\sqrt{LC}} t + c_2 \sin \frac{1}{\sqrt{LC}} t$

Period is $2\pi\sqrt{LC}$ ($= 2\pi/\text{frequency}$)
 frequency = $1/\sqrt{LC}$

b) Char. eqn is $Lr^2 + Rr + \frac{1}{C} = 0$

roots: $r = \frac{-R \pm \sqrt{R^2 - 4LC}}{2L}$

oscillates if $R^2 - \frac{4L}{C} < 0$

c) $Li'' + \frac{i}{C} = \omega E_0 \cos \omega t$

Solns of homog. eqn are

$i = a_1 \cos \frac{1}{\sqrt{LC}} t + a_2 \sin \frac{1}{\sqrt{LC}} t$

The particular soln i_p will have form $c_1 \cos \omega t + c_2 \sin \omega t$ unless $\omega = \frac{1}{\sqrt{LC}}$, in which case it will be $c_1 t \cos \omega t + c_2 t \sin \omega t$, which gets large as $t \rightarrow \infty$.

Thus if $\omega \approx \frac{1}{\sqrt{LC}}$, solns will be large in amplitude
 \therefore this is ω_0

The advantage of this method (divide and conquer!) is that we don't have to assume
 $y_p = d_1 x + d_2 + d_3 \cos x + d_4 \sin x$, which would give 4 equations in 4 unknowns to solve.

Using part (a), the particular solution to $y'' + 2y' + 2y = 2x + \cos x$

is $y = y_1 + y_2 = x - 1 + \frac{1}{5} \cos x + \frac{2}{5} \sin x$

2D-1

a) $y_h = C_1 \cos x + C_2 \sin x$, as usual.

$$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$\text{Let } y_p = u_1 y_1 + u_2 y_2$$

The equations for variation of pars.

$$\begin{aligned} \text{are: } u_1' \cos x + u_2' \sin x &= 0 \\ u_1'(-\sin x) + u_2' \cos x &= \tan x \end{aligned}$$

Either by elimination, or by Cramer's rule, we get as sol'n: (the denom. is $W(y_1, y_2)$)

$$u_1' = \frac{-y_2 f(x)}{W(y_1, y_2)} = \frac{-\sin x \tan x}{1} = -\sin x \sec x \quad (\text{so it can be integrated})$$

$$u_2' = \frac{y_1 f(x)}{W(y_1, y_2)} = \cos x \tan x = \sin x \quad (\text{from tables})$$

$$\therefore u_1 = \sin x - \ln|\sec x + \tan x|$$

$$u_2 = -\cos x$$

$$\therefore y_p = (\sin x - \ln|\sec x + \tan x|) \cos x - \cos x \sin x$$

$$\text{de, } \boxed{y_p = -\cos x (\ln|\sec x + \tan x|)}$$

b) Two indept solns of the assoc. homog. eqn

$$\text{are: } y_1 = e^x, y_2 = e^{-3x} \quad (\text{method as usual})$$

$$W(y_1, y_2) = -4e^{-2x} = \begin{vmatrix} e^x & e^{-3x} \\ e^x & -3e^{-3x} \end{vmatrix}$$

$$y_p = u_1 y_1 + u_2 y_2 \quad (*)$$

The eqns for variation of parameters are:

$$u_1' e^x + u_2' e^{-3x} = 0 \quad \leftarrow f(x),$$

$$u_1' e^x - 3u_2' e^{-3x} = e^{-x} \quad (\text{from orig. eqn})$$

Solve them by elimination, or by Cramer's rule; following the latter, we get as sol'n

$$u_1' = \frac{-y_2 f(x)}{W} = \frac{1}{4} e^{-2x}$$

$$u_2' = \frac{y_1 f(x)}{W} = \frac{e^x \cdot e^{-x}}{-4e^{-2x}} = -\frac{1}{4} e^{2x}$$

$$\therefore u_1 = -\frac{1}{8} e^{-2x}, \quad u_2 = -\frac{1}{8} e^{2x}$$

$$\text{and so } y_p = -\frac{1}{8} e^{-2x} \cdot e^x - \frac{1}{8} e^{2x} \cdot e^{-3x}, \quad \text{by } (*)$$

$$\text{or: } \boxed{y_p = -\frac{1}{4} e^{-x}}$$

c) Two indept solns of the assoc. homog. eqn are: $y_1 = \cos 2x, y_2 = \sin 2x$ (by the usual method)

$$W(y_1, y_2) = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

$$\text{Let } y_p = u_1 y_1 + u_2 y_2$$

$$\begin{aligned} \text{Then } u_1' \cos 2x + u_2' \sin 2x &= 0 \\ u_1'(-2\sin 2x) + u_2'(2\cos 2x) &= \sec^2 2x \end{aligned}$$

are the eqns for the method of var. of pars.

Solving them by elimination, or by Cramer's rule:

$$u_1' = \frac{-y_2 f(x)}{W} = \frac{-\sin 2x}{2 \cos^2 2x}$$

$$u_2' = \frac{y_1 f(x)}{W} = \frac{\cos 2x}{2 \cos^2 2x} = \frac{\sec 2x}{2}$$

Integrating,

$$u_1 = -\frac{1}{4} \cdot \frac{1}{\cos 2x}$$

$$u_2 = \frac{1}{4} \ln|\sec 2x + \tan 2x|$$

$$\therefore \boxed{y_p = -\frac{1}{4} + \frac{1}{4} \ln|\sec x + \tan x| \cdot \sin 2x}$$

2D-2

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -\frac{1}{x}, \quad \text{after some calculation.}$$

$$y_p = u_1 y_1 + u_2 y_2$$

Equations for method of var. of pars. are:

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \quad \leftarrow \text{note: the ODE must be written } y'' + \frac{1}{x} y' + (-1)y = \frac{\cos x}{\sqrt{x}} \\ u_1' y_1' + u_2' y_2' &= \frac{\cos x}{\sqrt{x}} \end{aligned}$$

Solving these by Cramer's rule:

$$u_1' = \frac{-y_2 f(x)}{W} = \cos^2 x$$

$$u_2' = \frac{y_1 f(x)}{W} = -\sin x \cos x$$

$$\therefore u_1 = \frac{x}{2} + \frac{\sin 2x}{4}, \quad u_2 = \frac{\cos 2x}{4}$$

and so (using identities):

$$y_p = \frac{\sin x}{\sqrt{x}} \left(\frac{x}{2} + \frac{2 \sin x \cos x}{4} \right) + \frac{\cos x}{\sqrt{x}} \left(\frac{\cos^2 x - \sin^2 x}{4} \right)$$

$$\text{so } y_p = \frac{x \sin x}{2\sqrt{x}} + \frac{1}{4} \frac{\cos x}{\sqrt{x}}$$

(The term $\frac{1}{4} \frac{\cos x}{\sqrt{x}}$ is part of the general soln $y = y_p + C_1 \frac{\cos x}{\sqrt{x}} + C_2 \frac{\sin x}{\sqrt{x}}$; so it can be omitted.)

$$\boxed{y_p = \frac{\sqrt{x} \sin x}{2}} \quad \text{is the best answer)$$

2D-3

a) Let y_1, y_2 be ^{indep't} solutions of the associated homogeneous equation.

$$y_p = u_1 y_1 + u_2 y_2, \quad W = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

and the eqns for the method of var. of pars. are

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' y_1' + u_2' y_2' = f(x)$$

Solving by Cramer's rule gives

$$u_1' = \frac{-y_2(x)f(x)}{W[y_1(x), y_2(x)]}, \quad u_2' = \frac{y_1(x)f(x)}{W[y_1(x), y_2(x)]}$$

so that (use definite integrals so as to get a definite function)

$$u_1(x) = \int_a^x \frac{-y_2(t)f(t)}{W[y_1(t), y_2(t)]} dt, \quad u_2(x) = \int_a^x \frac{y_1(t)f(t)}{W[y_1(t), y_2(t)]} dt$$

Thus: $y_p(x) = u_1(x) \cdot y_1(x) + u_2(x) y_2(x) -$

we can put $y_1(x)$ and $y_2(x)$ inside the integral sign because they are "constants" — the integration is with respect to t , not x ; then we can add the integrands. The result is:

$$y_p = \int_a^x \frac{-y_1(x)y_2(t) + y_2(x)y_1(t)}{W[y_1(t), y_2(t)]} \cdot f(t) dt$$

$$\text{or } y_p = \int_a^x \frac{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1(x) & y_2(x) \end{vmatrix}}{W(y_1(t), y_2(t))} f(t) dt$$

b) The arbitrary constants of integration — call them a_1 and a_2 , — will change u_1 and u_2 by an additive constant:

$$u_1 + a_1, \quad u_2 + a_2$$

leading to the particular soln:

$$y_p = (u_1 + a_1)y_1 + (u_2 + a_2)y_2$$

$$(*) \quad y_p = \boxed{u_1 y_1 + u_2 y_2} + a_1 y_1 + a_2 y_2$$

The boxed part is the particular solution of part (a); the part added on is in the general soln y_h to the associated homog. eqn, hence the particular soln $(*)$ is just as good a particular soln as the previous one.

2D-4

It depends on the ODE form — (it must be linear!)

Undetermined coefficients

requires

① The ODE is linear, with constant coefficients

② The inhomogeneous term $f(x)$ has a special form: a sum of terms of the form

$$(\text{polynomial}) \cdot e^{ax} \cdot \begin{cases} \sin bx \\ \cos bx \end{cases}$$

\uparrow can be 1 \uparrow a can be 0 \uparrow b can be 0

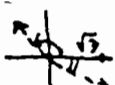
If the coeffs. are not constant, or $f(x)$ is not of the above form, you must use variation of parameters to find y_p .

Drawback: you must be able to find y_1, y_2 first — i.e., solve the assoc. homog. eqn.

(Note that finding y_p by undet.

coeffs. does not require you to solve for y_1, y_2 first (unless you are unlucky and $f(x)$ is a soln of the assoc. homog. eqn — but you can always test this without solving the eqn.)

Notes: Solutions

2E-1  $-1+i = \sqrt{2} e^{i3\pi/4}$
 $\sqrt{3}-i = 2 e^{-i\pi/6}$

2E-2 $\frac{1-i}{1+i} = \frac{(1-i)^2}{(1+i)(1-i)} = \frac{-2i}{2} = -i$

Other way:

$$1-i = \sqrt{2} e^{-i\pi/4}$$

$$1+i = \sqrt{2} e^{i\pi/4}$$

$$\therefore \frac{1-i}{1+i} = \frac{\sqrt{2}}{\sqrt{2}} \cdot e^{i(-\pi/4 - \pi/4)} = e^{-i\pi/2} = -i$$

2E-4 $z = a+bi$, $w = c+di$
 $zw = (ac-bd) + i(ad+bc)$
 $\therefore \overline{zw} = (ac-bd) - i(ad+bc)$
 $\overline{z}\overline{w} = (a-bi)(c-di) = (ac-bd) - i(ad+bc)$

2E-7 $(1-i)^4 = 1 + 4(-i) + 6(-i)^2 + 4(-i)^3 + (-i)^4$
 $= 1 - 4i + 6(-1) + 4(i) + 1 = -4$

By DeMoivre:

$1-i = \sqrt{2} e^{-i\pi/4}$ **be elegant**

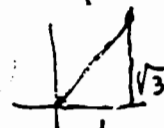
$(1-i)^4 = (\sqrt{2})^4 e^{-i\pi} = 4 \cdot (-1) = -4$

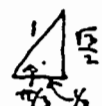
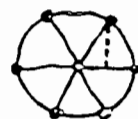
b) $(1+i\sqrt{3})^3 = 1 + 3(i\sqrt{3}) + 3(i\sqrt{3})^2 + (i\sqrt{3})^3$

$$= 1 + 3i\sqrt{3} + 3 \cdot -3 + i^3 3\sqrt{3}$$

$$= -8 + i(3\sqrt{3} - 3\sqrt{3}) = -8$$

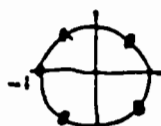
By polar form:

 $1+i\sqrt{3} = 2 e^{i\pi/3}$
 $(1+i\sqrt{3})^3 = 8 e^{i\pi} = -8$



2E-9 The sixth roots of 1 are $e^{i\frac{2\pi k}{6}}$ where $k=0,1,2,\dots,5$
 get: $1, -1, \frac{\pm 1 \pm i\sqrt{3}}{2}$

2E-10 $\sqrt[4]{16} = 2 \cdot \sqrt[4]{-1}$



The 4th roots of -1 are on the picture: $\frac{\pm 1 \pm i}{\sqrt{2}}$

$\therefore \sqrt{2} \cdot (\pm 1 \pm i)$ are the roots
 really? $\checkmark x^4 + 16 = 0$

2E-14 $\sin^4 x = \left(\frac{e^{ix} - e^{-ix}}{2i} \right)^4$; by bin. thm, this
 $= \frac{1}{16} (e^{4ix} - 4e^{3ix}e^{-ix} + 6e^{2ix}e^{-2ix} - 4e^{ix}e^{-3ix} + e^{-4ix})$
 $= \frac{1}{16} (e^{4ix} + e^{-4ix}) - \frac{4}{16} (e^{2ix} + e^{-2ix}) + \frac{6}{16} \cdot 1$
 $= \frac{1}{8} \cos 4x - \frac{1}{2} \cos 2x + \frac{3}{8}$

Since $\sin^4 x$ is an even function, the answer should not contain the odd functions $\sin 4x, \sin 2x$.

2E-15 $e^{(2+i)x} = e^{2x}(\cos x + i \sin x)$

So $e^{2x} \sin x = \text{Im } e^{(2+i)x}$

$\int e^{(2+i)x} dx = \frac{1}{2+i} e^{(2+i)x}$; $\frac{1}{2+i} \cdot \frac{2-i}{2-i} = \frac{2-i}{5}$
 $= \frac{2-i}{5} (e^{2x} \cos x + i e^{2x} \sin x)$

We want just the imaginary part:

$\therefore \int e^{2x} \sin x dx = e^{2x} \left(\frac{2}{5} \sin x - \frac{1}{5} \cos x \right)$

2E-16 $e^{ix} = \cos x + i \sin x$

$e^{-ix} = \cos x - i \sin x$

Adding: $\frac{e^{ix} + e^{-ix}}{2} = \cos x$

Subtract: $\frac{e^{ix} - e^{-ix}}{2i} = \sin x$

since $\cos(-x) = \cos x$
 $\sin(-x) = -\sin x$

2F-1

a) $D^2 + 2D + 2 = 0$ has roots $-1 \pm i$

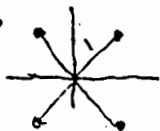
$$\therefore y = e^{2x}(c_1 + c_2x + c_3x^2) + e^{-x}(c_4 \cos x + c_5 \sin x)$$

$$b) D^8 - 2D^4 + 1 = (D^4 - 1)^2 = [(D^2 - 1)(D^2 + 1)]^2 \\ = (D - 1)^2(D + 1)^2(D^2 + 1)^2$$

$$\therefore y = e^x(c_1 + c_2x) + e^{-x}(c_3 + c_4x) + \cos x(c_5 + c_6x) + \sin x(c_7 + c_8x)$$

c) Characteristic eqn is $z^4 + 1 = 0$
Roots are $\sqrt[4]{-1}$

$$\frac{1 \pm i}{\sqrt{2}} \text{ and } \frac{-1 \pm i}{\sqrt{2}}$$



letting $a = 1/\sqrt{2}$, get \therefore

$$y = e^{ax}(c_1 \cos ax + c_2 \sin ax) + e^{-ax}(c_3 \cos ax + c_4 \sin ax)$$

d) Char. eqn is $z^4 - 8z^2 + 16 = 0$

which factors as

$$(z^2 - 4)^2 \text{ or } (z + 2)^2(z - 2)^2$$

\therefore has double roots at $2, -2$

so

$$y = c_1 e^{2x} + c_2 x e^{2x} + c_3 e^{-2x} + c_4 x e^{-2x}$$

$$e) y = c_1 e^x + c_2 e^{-x} + e^{x/2}(c_3 \cos \frac{\sqrt{2}}{2}x + c_4 \sin \frac{\sqrt{2}}{2}x) + e^{-x/2}(c_5 \cos \frac{\sqrt{2}}{2}x + c_6 \sin \frac{\sqrt{2}}{2}x)$$

[using roots as given in soln to 2F-9]

$$f) y = e^{\sqrt{2}x}(c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x) + e^{-\sqrt{2}x}(c_3 \cos \sqrt{2}x + c_4 \sin \sqrt{2}x)$$

2F-2

$$y''' - 16y = 0$$

characteristic equation

$$z^3 - 16 = 0$$

roots: $2, 2i, -2, -2i$

(one real root is 2 , so the others are all of the form $2\sqrt[3]{i}$, where

$$\sqrt[3]{i} = 1, i, -i$$

from roots, general soln is

$$y = c_1 e^{2x} + c_2 e^{-2x} + c_3 \sin 2x + c_4 \cos 2x$$

Putting in conditions:

$$c_1 = 0 \text{ since } |y(x)| < K \text{ for all } x > 0$$

$$(|c_1 e^{2x}| \rightarrow \infty \text{ unless } c_1 = 0)$$

$$y(0) = 0 \Rightarrow c_2 + c_4 = 0 \therefore c_4 = -c_2$$

$$y'(0) = 0 \Rightarrow -2c_2 + 2c_3 = 0 \therefore c_3 = c_2$$

\therefore soln is - so far -

$$y = c_2(e^{-2x} + \sin 2x - \cos 2x)$$

finally

$$y(\pi) = 1 \Rightarrow c_2(e^{-2\pi} - 1) = 1$$

$$\therefore c_2 = \frac{1}{e^{-2\pi} - 1}$$

2F-3

a) $z^3 - z^2 + 2z - 2 = 0$ is char. eqn.

1 is a root, $\therefore z - 1$ is factor

$$\text{get } (z - 1)(z^2 + 2) \text{ roots: } 1, i\sqrt{2}, -i\sqrt{2}$$

$$y = c_1 e^x + c_2 \cos \sqrt{2}x + c_3 \sin \sqrt{2}x$$

$$b) z^3 + z^2 - 2 = 0 = (z - 1)(z^2 + 2z + 2)$$

roots $1, -1 \pm i$

$$\therefore y = c_1 e^x + c_2 e^{-x} \cos x + c_3 e^{-x} \sin x$$

$$c) (D^3 - 2D - 4) = (D - 2)(D^2 + 2D + 2)$$

$$\therefore y = c_1 e^{2x} + e^{-x}(c_2 \cos x + c_3 \sin x) \text{ roots are } -1 \pm i$$

2F-4

$$d) x^4 + 2x^2 + 4 = 0; \therefore x^2 = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 4}}{2} = -1 \pm \sqrt{3} = -1 \pm \sqrt{3}i$$

changing to polar representation: $= 2e^{i\pi/3}, 2e^{i4\pi/3}$

$$\therefore x = \sqrt{2}e^{i\pi/3}, \sqrt{2}e^{i4\pi/3} \text{ (square roots of the first)} \\ = \sqrt{2}e^{i\pi/3}, \sqrt{2}e^{i5\pi/3} \text{ (" " " " " other)}$$

\nwarrow and \nearrow are conjugates

d) \searrow

using therefore just $\sqrt{2}e^{i\pi/3}$ and $\sqrt{2}e^{i5\pi/3}$:

$$\sqrt{2}e^{i\pi/3} = \sqrt{2}(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}) = \sqrt{2}(\frac{1}{2} + i\frac{\sqrt{3}}{2}) ; \text{ similarly, get } \sqrt{2}(\frac{1}{2} - i\frac{\sqrt{3}}{2})$$

$$\text{leading to: } y = e^{\sqrt{2}x}(c_1 \cos \frac{\sqrt{2}}{2}x + c_2 \sin \frac{\sqrt{2}}{2}x) + e^{-\sqrt{2}x}(c_3 \cos \frac{\sqrt{2}}{2}x + c_4 \sin \frac{\sqrt{2}}{2}x)$$

2F-4

$$x_1'' + 2x_1 - x_2 = 0$$

$$x_2'' + x_2 - x_1 = 0$$

Eliminate x_1 by solving for x_1 :

$$x_1 = x_2'' + x_2$$

substitute into first equation:

$$(x_2'' + x_2)'' + 2(x_2'' + x_2) - x_2 = 0$$

$$\text{or } x_2'''' + 3x_2'' + x_2 = 0$$

$$\text{char. eqn: } z^4 + 3z^2 + 1 = 0$$

as quadratic eqn in z^2 : solve, get

$$z^2 = \frac{-3 \pm \sqrt{5}}{2} : \text{ both nos. are } \underline{\text{neg.}}, + \underline{\text{negative}}$$

$$\therefore z^2 = -a^2, z^2 = -b^2 \quad \text{call them } -a^2, -b^2$$

$$z = \pm ia, z = \pm ib$$

$$\text{so } x_2 = c_1 \cos at + c_2 \sin at + c_3 \cos bt + c_4 \sin bt$$

2F-5

$$\begin{aligned} D^2 e^{2x} \cos x &= e^{2x} (D+2)^2 \cos x \\ &= e^{2x} (D^2 + 4D + 4) \cos x \\ &= e^{2x} (3 \cos x - 4 \sin x) \end{aligned}$$

2F-6

a) By (12) in notes, (see Example 2)

$$y_p = \frac{4}{r+1} e^x = 2e^x$$

$$\text{b) } (D^3 + D^2 - D + 2)y = 2e^{ix}$$

$$\therefore y_p = \frac{2e^{ix}}{i^3 + i^2 - i + 2} = \frac{2(1+2i)}{(1+2i)(1+2i)} e^{ix}$$

$$\therefore y_p = \frac{2+4i}{5} (\cos x + i \sin x) \quad \therefore \text{Re}(y_p) = \frac{2 \cos x - 4 \sin x}{5}$$

$$\text{c) } (D^2 - 2D + 4)y = e^{(1+i)x}$$

$$(1+i)^2 - 2(1+i) + 4 = 2 \quad \therefore y_p = \frac{e^{(1+i)x}}{2}$$

$$\text{Re}(y_p) = \frac{1}{2} e^x \cos x$$

$$\text{d) } D^2 - 6D + 9 = (D-3)^2 \quad \therefore y_p = cx^2 e^{3x}$$

$$\begin{aligned} (D-3)^2 y_p &= ce^{3x} D^2 x^2 \quad (\text{by exp-shift rule}) \\ &= 2ce^{3x} = e^{3x} \quad (\text{from the ODE}) \end{aligned}$$

$$\therefore c = 1/2, \quad y_p = \frac{1}{2} x^2 e^{3x}$$

2F-7

$$(D+a)e^{-ax}u = e^{-ax}Du = f(x)$$

$$\therefore Du = e^{ax}f(x), \quad u = \int e^{ax}f(x)dx$$

$$y_p = e^{-ax} \int e^{ax}f(x)dx$$

2G-1

$$y'' + 2y' + cy = 0$$

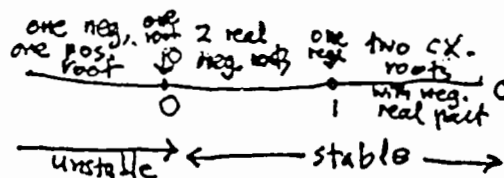
Char. eqn is:

$$r^2 + 2r + c = 0$$

By quadratic formula:

$$\text{roots} = \frac{-2 \pm \sqrt{4-4c}}{2}$$

$$= -1 \pm \sqrt{1-c}$$

**2G-2**

$$r^2 + \frac{b}{a}r + \frac{c}{a} = (r-r_1)(r-r_2)$$

$$\therefore \frac{b}{a} = -(r_1 + r_2)$$

$$\frac{c}{a} = r_1 r_2$$

$$\text{Real case: } r_1, r_2 < 0 \Rightarrow \begin{aligned} b/a &> 0 \\ c/a &> 0 \end{aligned}$$

$\therefore a, b, c$ have same sign.

Complex case:

$$r_1 = \alpha + i\beta, \quad \alpha < 0 \Rightarrow \frac{b}{a} = -2\alpha > 0$$

$$r_2 = \alpha - i\beta, \quad \frac{c}{a} = \alpha^2 + \beta^2 > 0$$

2G-3

Assume $a, b, c > 0$ (if not, multiply DE through by -1).

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{are the roots.}$$

$$\text{If roots are real, } \frac{-b - \sqrt{b^2 - 4ac}}{2a} < 0$$

and $-b + \sqrt{b^2 - 4ac} < 0$, therefore (since $b^2 - 4ac < b^2$).

$$\text{If roots are complex, } \frac{-b}{2a} < 0$$

\therefore in both cases, the char. roots have negative real part.

2H-1

$$y'' - k^2 y = 0, \quad y(0) = 0, \quad y'(0) = 1$$

$$y_c = c_1 e^{kx} + c_2 e^{-kx}$$

Soln to IVP is

$$w(t) = \frac{e^{kx} - e^{-kx}}{2k} = \frac{\sinh kx}{k}$$

2H-3a

By Example 2 (p. 2),

$$w(x) = x e^{-2x}$$

therefore

$$y(x) = \int_0^x \underbrace{(x-t) e^{-2(x-t)}}_{w(x-t)} \cdot \underbrace{e^{-2t}}_{f(t)} dt$$

$$= e^{-2x} \int_0^x (x-t) dt$$

$$= e^{-2x} \left(xt - \frac{t^2}{2} \right)_0^x = \boxed{\frac{x^2}{2} e^{-2x}}$$

By undetermined coeffs, since

$$y_c = e^{-2x} (c_1 + c_2 x), \text{ try } c x^2 e^{-2x}$$

$$(D+2)^2 c e^{-2x} x^2 = c e^{-2x} D^2 x^2$$

$$= c e^{-2x} \cdot 2$$

From the ODE, $\checkmark e^{-2x} \cdot 2 = e^{-2x}$, $\boxed{c = \frac{1}{2}} \checkmark$

2H-4

a) By Leibniz:

$$\phi'(x) = \frac{d}{dx} \int_0^x (2x+3t)^2 dt =$$

$$= (5x)^2 + \int_0^x 2 \cdot (2x+3t) \cdot 2 dt$$

$$= (5x)^2 + 4 \left(2xt + \frac{3t^2}{2} \right) \Big|_0^x = (5x)^2 + 14x^2$$

$$= \boxed{39x^2} \checkmark$$

b) Directly:

$$\phi(x) = \frac{1}{9} (2x+3t)^3 \Big|_0^x = \frac{1}{9} (5x)^3 - (2x)^3$$

$$\text{So } \phi'(x) = 39x^2 \checkmark$$

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18.03 Differential Equations

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