PARTIAL DERIVATIVES

Notation and Terminology: given a function $f(x,y)$;

- *partial derivative* of f *with respect to* x is denoted by
$$\frac{\partial f}{\partial x}(x,y) \equiv f_x(x,y) \equiv D_x f(x,y) \equiv f_1;$$
- *partial derivative* of f *with respect to* y is denoted by
$$\frac{\partial f}{\partial y}(x,y) \equiv f_y(x,y) \equiv D_y f(x,y) \equiv f_2.$$

Definitions: given a function $f(x,y)$;

- definition for $f_x(x,y)$:
$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h};$$
- definition for $f_y(x,y)$:
$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}.$$

Determination of f_x and f_y:

- to find $f_x(x,y)$: keeping y constant, take x derivative;
- to find $f_y(x,y)$: keeping x constant, take y derivative.

Graphical Interpretation of f_x and f_y:

- $f_x(a,b)$ is slope of tangent line in x direction for the surface $z = f(x,y)$ at $f(a,b)$;
- $f_y(a,b)$ is slope of tangent line in y direction for the surface $z = f(x,y)$ at $f(a,b)$.

PARTIAL DERIVATIVES CONTINUED

Applications to Implicit Differentiation

Functions of $n > 2$ Variables: given $f(\mathbf{x}) = f(x_1, \ldots, x_n)$

- notation and terminology: the *partial derivative* of f with respect to x_i is denoted by

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) \equiv f_{x_i}(\mathbf{x}) \equiv D_{x_i} f(\mathbf{x}) \equiv f_i(\mathbf{x});$$

definition:

$$f_{x_i}(\mathbf{x}) = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(\mathbf{x})}{h}.$$

Higher Derivatives: given $f(x, y)$ defined on a domain D;

- notation: second partials are denoted by

$$\frac{\partial^2 f}{\partial x^2}(x, y) \equiv \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) \equiv (f_x)_x \equiv f_{11};$$

$$\frac{\partial^2 f}{\partial y \partial x}(x, y) \equiv \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) \equiv f_{xy} \equiv (f_x)_y \equiv f_{12};$$

$$\frac{\partial^2 f}{\partial x \partial y}(x, y) \equiv \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) \equiv f_{yx} \equiv (f_y)_x \equiv f_{21};$$

$$\frac{\partial^2 f}{\partial y^2}(x, y) \equiv \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) \equiv (f_y)_y \equiv f_{22}.$$

- similar notation for functions with > 2 variables.
- **Clairaut's Theorem**: if $(a, b) \in D$, and f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

- Applications to Partial Differential Equations