Continuous Multinomial Coefficients

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For MATH 496, Honors Seminar

GOALS:

- Find suitable definitions which "reasonably" interpolate between the values of the discrete multinomial coefficients as well as preserve their nice properties.
- Investigate the properties of our coefficients when viewed as functions of one or more continuous variables... (continuity, differentiability, and computing derivatives...)
- Find and prove various extensions of familiar combinatorial identities.
- And, on top of all this, **we seek to do all of the above using as few non-elementary techniques as possible!**

Outline

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Falling Factorial Def.'n – Binomial

Pochhammer Symbol: For arbitrary α and $k \in \mathbb{N}$, we say $(\alpha)_k = \underbrace{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}_{\text{k factors}}$, and define $(\alpha)_0 = 1$.

Definition

(Binomial Coefficients using the Falling Factorial):
Let $k \in \mathbb{Z}$ and $\alpha \in \mathbb{C}$. Then,

$$\binom{\alpha}{k} = \begin{cases} 0 & k < 0 \\ 1 & k = 0 \\ \frac{(\alpha)_k}{k!} & k > 0 \end{cases}.$$

Gamma Function Def'n. – Binomial

Definition

(Binomial Coefficients using the Gamma Function):
Let $x, y \in \mathbb{C}$ such that $y \notin \mathbb{Z}_{\leq -1}$. Then,

$$\binom{y}{x} = \frac{\Gamma(y+1)}{\Gamma(x+1)\Gamma(y-x+1)}.$$

- See [Sal18].

"Quick Fact"

Lemma

Let z be arbitrary, and for an integer $k > 1$ let $n_1, n_2, ..., n_{k-1} \in \mathbb{N} \cup \{0\}$. Also, define $n = \sum_{i=1}^{k-1} n_i$. Then,

$$(z)_n = (z)_{n_1}(z - n_1)_{n_2}(z - n_1 - n_2)_{n_3} \cdots (z - n_1 - n_2 - \cdots - n_{k-2})_{n_{k-1}}.$$

Proof.

- Simple case: $(a)_{b+c} = (a)_b(a - b)_c$

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- Now, for $(z)_n = (z)_{n_1+n_2+...+n_{k-1}}$:

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- Group together $n_2 + \cdots + n_{k-1}$, use above fact, see:

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Proof.

- Simple case: $(a)_{b+c} = (a)_b(a - b)_c$
- Now, for $(z)_n = (z)_{n_1+n_2+...+n_{k-1}}$:
- Group together $n_2 + \cdots + n_{k-1}$, use above fact, see:

$$(z)_{n_1+n_2+...+n_{k-1}} = \underbrace{z(z-1)(z-2)\cdots(z-n_1+1)}_{(z)_{n_1}} \cdot (z-n_1)_{n_2+...+n_{k-1}}.$$

- Rinse, Repeat!

$\square$

Falling Factorial Def.'n – Multinomial

Definition

(Multinomial Coefficients using the Falling Factorial): *Let α be arbitrary, k be an integer greater than 1, and let $n_1, ..., n_{k-1} \in \mathbb{N} \cup \{0\}$. Then, setting the last entry equal to $\alpha - n$, where $n = \sum_{i=1}^{k-1} n_i$, we say*

$$\binom{\alpha}{n_1, \cdots, n_{k-1}, \alpha - n} = \frac{(\alpha)_n}{n_1! n_2! \cdots n_{k-1}!}.$$

(Originally, I defined this as a product of binomials, but the equation was too long to fit on this slide).

Gamma Function Def.'n – Multinomial

Definition

(Multinomial Coefficients using the Gamma Function):
Let $k \in \mathbb{N} \setminus \{0\}$, $x_1, \ldots, x_k \in \mathbb{C}$ *such that* $\sum_{i=1}^{k} x_i =: y \notin \mathbb{Z}_{\leq -1}$. *Then, we define*

$$\binom{y}{x_1, \cdots, x_k} = \frac{\Gamma(y+1)}{\Gamma(x_1+1)\Gamma(x_2+1)\cdots\Gamma(x_k+1)}.$$

- These varying definitions are best suited to different purposes.
- Note that there are ways of translating between them.

Outline

1. Motivation / Definitions
2. Basic Analytic Properties
3. Some Combinatorial Identities
4. Future Work

Pochhammer Def.'n

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- ...becomes $MC(\alpha)$ when $n_1, ..., n_{k-1}$ are fixed.
- $MC(\alpha)$ is a real-valued polynomial in α of degree n:

$$MC(\alpha) = \frac{1}{n_1! n_2! \cdots n_{k-1}!} \sum_{k=0}^{n} s(n,k)\alpha^k$$

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- As such, MC is clearly continuous and differentiable for all α.

Derivative of $MC(\alpha)$

Computation

$$\frac{d}{d\alpha}MC(\alpha) = \begin{cases} 0 & n = 0 \\ \lim_{\beta\to\alpha} MC(\beta)[\psi(-\beta) - \psi(n-\beta)] & n > 0, \end{cases}$$

where ψ is the digamma function.

Derivative of $MC(\alpha)$

Proof:

- First, if $n = 0$, then $(\alpha)_n = 1$ regardless of α, so its derivative is 0.
- Now, consider the case when $n > 0$. Then, $(\alpha)_n = \alpha(\alpha - 1)(\alpha - 2)\cdots(\alpha - n + 1)$, so we can take its derivative using the Product Rule to get:

$$\begin{aligned}[(\alpha)_n]' = &(\alpha - 1)(\alpha - 2)\cdots(\alpha - n + 1) + \\ &+ \alpha(\alpha - 2)\cdots(\alpha - n + 1) + \\ &+ \alpha(\alpha - 1)(\alpha - 3)\cdots(\alpha - n + 1) + \cdots \\ &+ \alpha(\alpha - 1)(\alpha - 2)\cdots(\alpha - n + 2).\end{aligned}$$

Derivative of $MC(\alpha)$

- If we assume $\alpha \notin \{0, 1, 2, ..., n-1\}$, then we can replace all these terms with $(\alpha)_n$, divided by the "missing" factor. So:
- $$[(\alpha)_n]' = (\alpha)_n \cdot \sum_{k=0}^{n-1} \frac{1}{\alpha - k}, \qquad n \geq 1.$$

Derivative of $MC(\alpha)$

- Suppose now that $\alpha = k$, where $k \in \{0, 1, 2, ..., n-1\}$.
- Then, using Product Rule, we're left with: (...)

$$[(\alpha)_n]' = \alpha(\alpha-1)\cdots(\alpha-(k-1))(\alpha-(k+1))\cdots(\alpha-n+1).$$

- On the other hand, with this same $\alpha = k$, consider the following limit:

$$\lim_{\beta\to\alpha}(\beta)_n \sum_{k=0}^{n-1} \frac{1}{\beta-k}.$$

Derivative of $MC(\alpha)$

- Putting aside the question of the limit's existence for a second, one can formally distribute...

$$= \lim_{\beta \to \alpha} \left(\frac{(\beta)_n}{\beta} + \frac{(\beta)_n}{\beta - 1} + \frac{(\beta)_n}{\beta - 2} + \cdots + \frac{(\beta)_n}{\beta - n + 1} \right)$$

- ... then separate out the k^{th} term in this limit...

$$= \lim_{\beta \to \alpha} \left(\frac{(\beta)_n}{\beta} + \cdots + \frac{(\beta)_n}{\beta - (k-1)} + \frac{(\beta)_n}{\beta - (k+1)} + \cdots + \right.$$
$$\left. + \frac{(\beta)_n}{\beta - n + 1} \right) + \lim_{\beta \to \alpha} \frac{(\beta)_n}{\beta - k}$$

Derivative of $MC(\alpha)$

- ...observing that the limit of all but the rightmost term will vanish, leaving us with:

$$\lim_{\beta\to\alpha}(\beta)_n \sum_{k=0}^{n-1} \frac{1}{\beta - k} = \lim_{\beta\to\alpha} \frac{(\beta)_n}{\beta - k}$$

- $$= \alpha(\alpha - 1)\cdots(\alpha - (k-1))(\alpha - (k+1))\cdots(\alpha - n + 1).$$
- Thus, we've shown that this limit equals $[(\alpha)_n]'(k)$, if it exists.
- But existence is automatic, since we know that $MC(\alpha)$ (and hence $(\alpha)_n$)) is everywhere differentiable and its derivative is continuous!

Derivative of $MC(\alpha)$

- Thus we can combine these cases together, yielding

$$MC'(\alpha) = \begin{cases} 0 & n = 0 \\ \lim_{\beta\to\alpha} MC(\beta) \sum_{k=0}^{n-1} \frac{1}{\beta - k} & n > 0. \end{cases}$$

- Quick substitution using $\psi(x+N) - \psi(x) = \sum_{k=0}^{N-1} \frac{1}{x+k}$ gives us final answer. $\square$

Gamma Function Definition

Likewise, we can carry out a similar exploration for our Gamma definition...

- Let's restrict $x_1, ..., x_n \in \mathbb{R}$. I.e., let's view our coefficients as a function $GMC : \mathbb{R}^n \to \mathbb{R}$.

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- Let's restrict $x_1, ..., x_n \in \mathbb{R}$. I.e., let's view our coefficients as a function $GMC : \mathbb{R}^n \to \mathbb{R}$.
- Recall the definition. If $\vec{x} = (x_1, ..., x_n)$, clearly $GMC(\vec{x})$ exists wherever $y = x_1 + \cdots + x_n \notin \mathbb{Z}_{\leq -1}$.

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- SIDE NOTE: The set, S, in $\mathbb{R}^n$ on which GMC is not defined: (...)

Picture Time!

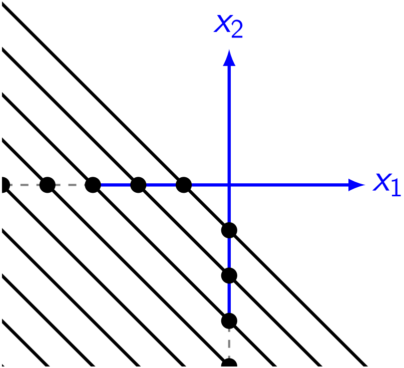
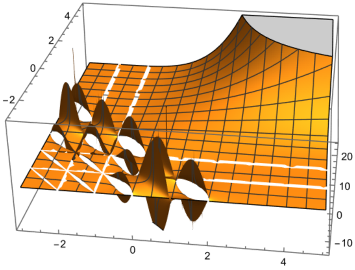
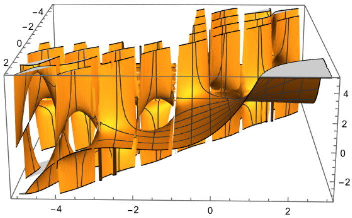


Figure: The set of points in $\mathbb{R}^2$ for which our Gamma definition isn't defined

A view of GMC for the $n = 2$ case:



(a) Facing the positive direction



(b) Facing the negative direction

Figure: $GMC(x, y)$ viewed from 2 different angles

- Now, how to show that GMC is differentiable where it's defined...

Gamma Function Definition

- Consider the Weierstrass product definition of Γ:

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} e^{-z/n}.$$

- Stepping back into $\mathbb{C}$ for a minute, Γ is *holomorphic* everywhere it is defined, (so everywhere except for $z = 0, -1, -2, ...$), and has a set of isolated poles, so is *meromorphic*, and has no zeros, therefore $\frac{1}{\Gamma}$ is an *entire* function. [WW96]
- All that to say, $\Gamma'(x)$ exists and is continuous whenever x is *not* a nonpositive integer, and $\frac{1}{\Gamma(x)}$ is everywhere differentiable. [BR]
- Hence, by [Lee00], the function on $\mathbb{R}^n$ formed by taking the product of $1/\Gamma$'s is total differentiable on $\mathbb{R}^n$, and
- $\frac{\Gamma(y+1)}{\Gamma(x_1+1)...\Gamma(x_n+1)}$ is total differentiable everywhere $\Gamma(y+1)$ is. (Apologies for the abuse of notation.)
- So, we can take derivatives!

Derivatives of $GMC(\vec{x})$

- **Partial Derivatives:** Note that by symmetry of GMC, we need only do this once.

Computation

Fix $j \in \{1, 2, ..., n\}$. Then,

$$\frac{\partial}{\partial x_j} GMC(\vec{x}) = GMC(\vec{x})\Big(\psi(y+1) - \psi(x_j+1)\Big),$$

everywhere where GMC is defined.

- **Gradient:**

Computation

$$\nabla GMC(\vec{x}) = GMC(\vec{x})\psi(y+1)\Big(1-\psi(x_1+1), 1-\psi(x_2+1), ..., 1-\psi(x_n+1)\Big),$$

everywhere GMC is defined.

- **Directional Derivative:**

Computation

If $\vec{x} = (x_1, ..., x_n)$ *is a value at which* GMC *is defined, and* $\vec{v} = (v_1, ..., v_n) \in \mathbb{R}^n$*, then the directional derivative of* GMC *along* $\vec{v}$ *at* $\vec{x}$ *can be expressed in coordinates as:*

$$D_{\vec{v}}[GMC(\vec{x})] = GMC(\vec{x})\psi(y+1)\left[\sum_{i=1}^{n} v_i\left(1 - \frac{\psi(x_i+1)}{\psi(y+1)}\right)\right].$$

Outline

Multinomial as Product of Binomial

Lemma

(<u>Generalized Multinomial as Product of Binomials</u>):
Let $n, n_1, ..., n_k \in \mathbb{C} \setminus \mathbb{Z}_{\leq -1}$, $k \in \mathbb{N}$ greater than 1 such that $\sum_{i=1}^{k} n_i = n$. Then, using our Gamma Definition of the multinomial coefficients,

$$\binom{n}{n_1, n_2, ..., n_k} = \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-\cdots-n_{k-1}}{n_k},$$

where the Binomial coefficients are Gamma Definition.

Note that this Lemma was already used, without calling it that by name, in the *definition* of our multinomial coefficients using the Pochhammer Symbol! So, it obviously holds there.

Pascal's Identity

Lemma

(Generalized Pascal's Identity):
Let $n, n_1, ..., n_k \in \mathbb{R}^+$, $k \in \mathbb{N}$ *be greater than* 1, *such that* $\sum_{i=1}^k n_i = n$.
Then, using our Gamma Definition,

$$\binom{n}{n_1, n_2, \cdots, n_k} = \binom{n-1}{n_1 - 1, n_2, \cdots, n_k} + \binom{n-1}{n_1, n_2 - 1, \cdots, n_k} + \cdots + \binom{n-1}{n_1, n_2, n_3, \cdots, n_k - 1}.$$

Chu-Vandermonde Convolution

Theorem

(Continuous Multinomial Chu-Vandermonde Convolution):
Let $k \in \mathbb{N} \setminus \{0\}$, $x, y \in \mathbb{C}$, *and* $n_1, \ldots, n_{k-1} \in \mathbb{N} \cup \{0\}$. *If we define* $n = \sum_{i=1}^{k-1} n_i$ *and use the Pochhammer symbol definition, then:*

$$\binom{x+y}{n_1, n_2, \ldots, n_{k-1}, x+y-n} = \sum_{m_1,\ldots,m_{k-1}\geq 0} \binom{x}{m_1, \ldots, m_{k-1}, x - \sum_j m_j} \cdot \binom{z}{n_1 - m_1, \ldots, n_{k-1} - m_{k-1}, y - n + \sum_j m_j}$$

.

- Due to [Bel14], but...

GENERALIZED Chu-Vandermonde Convolution

Theorem

(<u>Generalized Continuous Multinomial Chu-Vandermonde Convolution</u>): Let $s, t \in \mathbb{N} \setminus \{0\}$, $x_1, \ldots, x_s \in \mathbb{C}$, *and* $n_1, n_2, \ldots, n_{t-1} \in \mathbb{N} \cup \{0\}$. *Set* $x = \sum_{j=1}^{s} x_j$, $n = \sum_{j=1}^{t-1} n_j$. *Then, we have the following:*

$$\binom{x_1 + \cdots + x_s}{n_1, \ldots, n_{t-1}, x - n} = \sum_{m_{i,j}} \binom{x_1}{m_{1,1}, \ldots, m_{1,t-1}, x_1 - \sum_j m_{1,j}} \cdots \binom{x_s}{m_{s,1}, \ldots, m_{s,t-1}, x_s - \sum_j m_{s,j}},$$

where the sum is taken over all $m_{i,j}$, $i \in [s]$, $j \in [t-1]$, *such that* $m_{1,\ell} + m_{2,\ell} + \cdots + m_{s,\ell} = n_\ell$, $\forall \ell \in [t-1]$.

Now, the "Holy Grail"... Generalized Multinomial Theorem

Theorem

(Special Case of GMT for x's $\in \mathbb{R}_{\geq 0}$, $r \in \mathbb{R}$):
Consider $(x_1 + x_2 + \cdots + x_k)^r$, $r \in \mathbb{R}$, *k a positive integer, and* $x_j \in \mathbb{R}_{\geq 0}$ *for all* $j \in [k]$. *Suppose* $\exists i \in [k]$ *such that* $x_i > \sum_{\substack{z \leq j \leq k \\ j \neq i}} x_j$. *Then, calling*

$\sum_{j \neq i} n_j = n$, *we have*

$$(x_1 + x_2 + \cdots + x_k)^r =$$

$$\sum_{\substack{(n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_k) \\ n1, n_2, \ldots, n_k \geq 0}} \binom{r}{n_1, n_2, \ldots, n_{i-1}, r-n, n_{i+1}, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_i^{r-n} \cdots x_k^{n_k},$$

the series converging.

- We are summing over all $k-1$-tuples of positive integers n_j.
- We fill the i^{th} "slot" in the multinomial coefficients –that is, the slot of the maximum x_j– with r minus the sum of the n_j's.
- We also attach an exponent of $r-n$ to x_i instead of the would-be "n_i." This assignment of our coefficients and exponents is *essential* for the convergence of the series, which we present as a conjecture:

Conjecture

(<u>Behavior of Multinomial Series</u>):
Consider $(x_1 + x_2 + \cdots + x_k)^r$, $r \in \mathbb{R}$, k *a positive integer, and all* $x_j \in \mathbb{C}$, $j \in [k]$. *Suppose* $\exists i \in [k]$ *such that* $\forall j \neq i$ *in* $[k]$, $|x_i| > |x_j|$. *Call* $\sum_{j \neq i} n_j = n$. *Then, the formal series using the Gamma Definition given by*

$$\sum_{\substack{(n_1,\ldots,n_{i-1},n_{i+1},\ldots,n_k) \\ n1,n_2,\ldots,n_k \geq 0}} \binom{r}{n_1, n_2, \ldots, n_{i-1}, r-n, n_{i+1}, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_i^{r-n} \cdots x_k^{n_k}$$

will exhibit the following behavior:
(...)

Conjecture: Behavior of our Multinomial Series (...)

Conjecture

(...)

- **Diverge**, *if you attach the "$r - n$" exponent to any of the x_j <u>besides</u> x_i, regardless of the sum of the other magnitudes.*
- **Approach the true value of $(x_1 + x_2 + \cdots + x_k)^r$ for some finite number of terms, then diverge**, *if you give x_i the right exponent, but $\sum_{j \neq i} |x_j| \geq |x_i|$.*
- **Converge to the desired value**, *if you have an x_i such that $|x_i| > \sum_{j \neq i} |x_j|$, and you give x_i the correct exponent. (See Thm. 1).*

Otherwise, if no such i exists, then for $k > 2$ the sum will **approach some value dependent upon the x_j's for a finite number of terms, then diverge**.

Generalized Binomial Theorem (GBT)

- Now, let's set about proving the special case...
- To do so, we'll need to prove a result from [WW96]:

Lemma

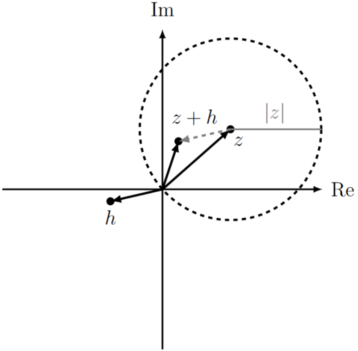
(Generalized Binomial Theorem):
Let $z, h \in \mathbb{C}$ such that $|z| > |h|$. Then, $\forall r \in \mathbb{R}$, using the Pochhammer definition for the binomial coefficients,

$$(z+h)^r = \sum_{n=0}^{\infty} \binom{r}{n} z^{r-n} h^n,$$

the series on the right converging.

Proof.:

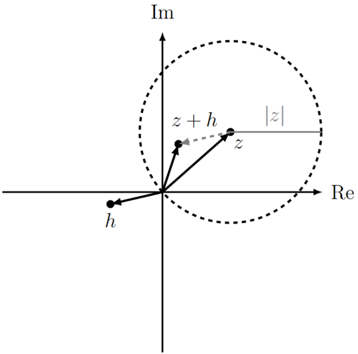
- Let $r \in \mathbb{R}$, and let $f(s) = s^r$ be a complex-valued function of a complex variable.



GBT Proof 1

Proof.:

- Let $r \in \mathbb{R}$, and let $f(s) = s^r$ be a complex-valued function of a complex variable.
- Further, suppose $h, z \in \mathbb{C}$ be such that $|h| < |z|$.



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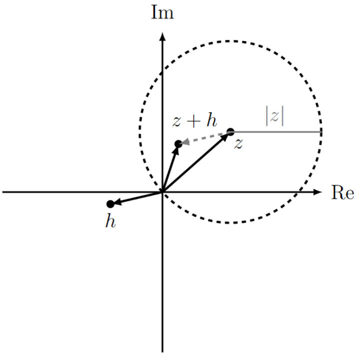
- Then, consider points $z + h$ inside the open disk of radius $|z|$ centered at z in the complex plane.



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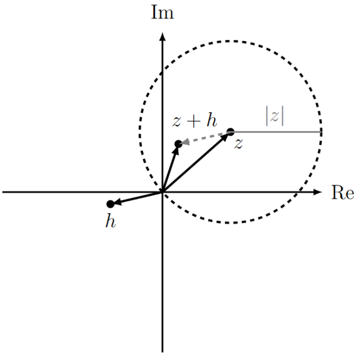
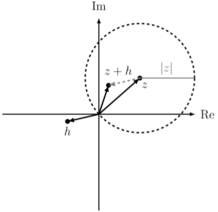
- Then, consider points $z + h$ inside the open disk of radius $|z|$ centered at z in the complex plane.
- If we consider $f(z + h)$ *as a function of* h, clearly this function cannot have any singularities within the interior of our disk, because in order for this to happen we would need $r < 0$ and $z + h = 0$. But $z + h = 0 \implies |z| = |h|$.



GBT Proof 2

- Additionally, f is differentiable everywhere in the disk... (polynomial)!



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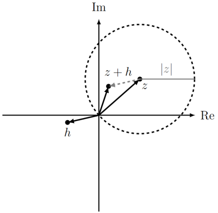
- Thus, $f(z+h)$ as a function of h is *analytic* in our circle, *if* $|z| > |h|$.



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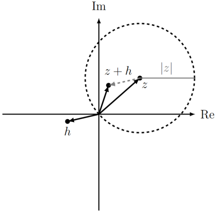

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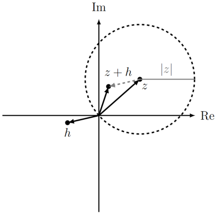

Theorem

If f is analytic through the disk $|s-s_0| < R$ centered at s_0 of radius R, then $f(s)$ has a power series representation inside the disk:

$$f(s) = \sum_{n=0}^{\infty} \frac{f^{(n)}(s_0)}{n!}(s-s_0)^n, \qquad |s-s_0| < R.$$

GBT Proof 3

- Taking $f(z+h)$ on our disk and substituting it into Taylor's Theorem, we have:

$$f(z+h) = f(z) + hf'(z) + \frac{h^2}{2!}f''(z) + \cdots + \frac{h^n}{n!}f^{(n)}(z) + \cdots, \qquad |h| < |z|,$$

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- We recognize this as our Pochhammer coefficient binomial series!
- Thus $(z+h)^r = \sum_{n=0}^{\infty} \binom{r}{n} z^{r-n} h^n, \qquad |h| < |z|$, as desired! $\square$

GBT Notes

- Note that we can extend this result a bit, to the case where $|z| = |h|$ and r is *strictly* positive. When this applies, our function of h is still analytic inside the disk, but also on the *boundary*, and we end up with the same Taylor series from last time.
- Also note that this proof of the Generalized Binomial Theorem trivially extends to our Γ definition when $r, r+k \in \mathbb{R} \setminus \mathbb{Z}_{\leq -1}$, $k \in \mathbb{N}$, by comments made in Remark 1.

Now, onto the special case...

Recall...

Theorem

(Special Case of GMT for x's$\in \mathbb{R}_{\geq 0}$*,* $r \in \mathbb{R}$*):*
Consider $(x_1 + x_2 + \cdots + x_k)^r$*,* $r \in \mathbb{R}$*,* k *a positive integer, and* $x_j \in \mathbb{R}_{\geq 0}$ *for all* $j \in [k]$*. Suppose* $\exists i \in [k]$ *such that* $x_i > \sum_{\substack{z \leq j \leq k \\ j \neq i}} x_j$*. Then, calling*

$\sum_{j \neq i} n_j = n$*, we have*

$$(x_1 + x_2 + \cdots + x_k)^r =$$

$$\sum_{\substack{(n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_k) \\ n1, n_2, \ldots, n_k \geq 0}} \binom{r}{n_1, n_2, \ldots, n_{i-1}, r-n, n_{i+1}, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_i^{r-n} \cdots x_k^{n_k},$$

the series converging.

GMT Proof 1

Proof.:

- $r \in \mathbb{R}$, k a positive integer, and $x_j \in \mathbb{R}_{\geq 0}$ for all $j \in [k]$.
- Further, suppose we do have an $i \in [k]$ so that *not only* is x_i strictly greater than all x_j, $j \neq i$, but also $x_i > \sum\limits_{\substack{z \leq j \leq k \\ j \neq i}} x_j$.
- Without loss of generality, say that $x_i = x_k$. (This will make the notation easier).
- Defining n to be $n = \sum_{j=1}^{k-1} n_j$, consider the formal sum

$$\sum_{\substack{(n_1, \dots, n_{k-1}) \\ n1, n_2, \dots, n_{k-1} \geq 0}} \binom{r}{n_1, n_2, \dots, n_{k-1}} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_{k-1}}$$

-

$$= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_{k-1}=0}^{\infty} \binom{r}{n_1, n_2, \dots, n_{k-1}, r-n} x_1^{n_1} x_2^{n_2} \cdots x_k^{r-n}.$$

GMT Proof 2

- By Lemma 2, we have

$$\binom{r}{n_1, n_2, \ldots, n_{k-1}, r-n} = \binom{r}{n_1}\binom{r-n_1}{n_2}\binom{r-n_1-n_2}{n_3}\cdots\times$$
$$\times\binom{r-n_1-n_2-\cdots-n_{k-2}}{n_{k-1}}\cancel{\binom{r-n}{r-n}},$$

- So our sum is just:

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_{k-1}=0}^{\infty}\binom{r}{n_1}\binom{r-n_1}{n_2}\binom{r-n_1-n_2}{n_3}\cdots\times$$
$$\times\binom{r-n_1-n_2-\cdots-n_{k-2}}{n_{k-1}}x_1^{n_1}x_2^{n_2}\cdots x_k^{r-n}.$$

- We can move all the "lower n_j" factors out of the inner sums:

$$\sum_{n_1=0}^{\infty} \binom{r}{n_1} x_1^{n_1} \sum_{n_2=0}^{\infty} \binom{r-n_1}{n_2} x_2^{n_2} \cdots$$
$$\sum_{n_{k-1}=0}^{\infty} \binom{r-n_1-n_2-\cdots-n_{k-2}}{n_{k-1}} x_{k-1}^{n_{k-1}} x_k^{(r-n_1-n_2-\ldots-n_{k-2})-n_{k-1}}.$$

- Now, focus on the innermost sum. Combine all x's of power n_{k-1}, pull the rest out front:

$$x_k^{r-n_1-n_2-\ldots-n_{k-2}} \sum_{n_{k-1}=0}^{\infty} \binom{r-n_1-n_2-\cdots-n_{k-2}}{n_{k-1}} \times$$
$$\times \left(\frac{x_{k-1}}{x_k}\right)^{n_{k-1}} \cdot \underbrace{(1)^{(r-n_1-n_2-\ldots-n_{k-2})-n_{k-1}}}_{\text{(Multiply by 1)}}.$$

GMT Proof 4

- Now, note that since $x_k > x_{k-1}$, $\frac{x_{k-1}}{x_k} < 1$, so by Lemma 4 (the GBT) we have

$$x_k^{r-n_1-n_2-\ldots-n_{k-2}}\left(1+\frac{x_{k-1}}{x_k}\right)^{r-n_1-n_2-\cdots-n_{k-2}}.$$

- Thus, our total sum is now

$$\sum_{n_1=0}^{\infty}\binom{r}{n_1}x_1^{n_1}\sum_{n_2=0}^{\infty}\binom{r-n_1}{n_2}x_2^{n_2}\cdots$$
$$\sum_{n_{k-2}=0}^{\infty}\binom{r-n_1-n_2-\cdots-n_{k-3}}{n_{k-2}}x_{k-2}^{n_{k-2}}\cdot$$
$$\cdot\, x_k^{r-n_1-\cdots-n_{k-2}}\left(1+\frac{x_{k-1}}{x_k}\right)^{(r-\cdots-n_{k-3})-n_{k-2}}.$$

GMT Proof 5

- Again, focus on the innermost sum:

$$\sum_{n_{k-2}=0}^{\infty} \binom{r-n_1-n_2-\cdots-n_{k-3}}{n_{k-2}} x_{k-2}^{n_{k-2}} \cdot$$
$$\cdot\, x_k^{(r-n_1-\cdots-n_{k-3})-n_{k-2}} \left(1+\frac{x_{k-1}}{x_k}\right)^{(r-\cdots-n_{k-3})-n_{k-2}},$$

- Again, combine x_{k-2} factors and pull the remaining x_j out, giving

$$x_k^{r-n_1-\cdots-n_{k-3}} \cdot \sum_{n_{k-2}=0}^{\infty} \binom{r-n_1-n_2-\cdots-n_{k-3}}{n_{k-2}} \times$$
$$\times \left(\frac{x_{k-2}}{x_k}\right)^{n_{k-2}} \left(1+\frac{x_{k-1}}{x_k}\right)^{(r-\cdots-n_{k-3})-n_{k-2}}.$$

GMT Proof 6

- Since $\frac{x_k - 2}{x_k} < 1$ and $(1 + \frac{x_{k-1}}{x_k}) \geq 1$, GBT *again* applies here, so

$$x_k^{r-n_1-\cdots-n_{k-3}} \left(1 + \frac{x_{k-1} + x_{k-2}}{x_k}\right)^{r-n_1-\cdots-n_{k-3}}.$$

- Clearly, we can continue this process until we've reached the outermost sum!
- At this point, we'll have:

$$\sum_{n_1=0}^{\infty} \binom{r}{n_1} x_1^{n_1} \cdot x_k^{r-n_1} \cdot \left(1 + \frac{x_2 + x_3 + \cdots + x_{k-1}}{x_k}\right)^{r-n_1},$$

GMT Proof 7

- … But even now, we can repeat the process once more (I'm too lazy to type it), and end up with:

$$x_k^r \cdot \left(1 + \frac{x_1 + \cdots + x_{k-1}}{x_k}\right)^r.$$

- Now, since $x_k \neq 0$, (by virtue of being *strictly* greater than a sum of nonnegative reals), we can use the distributive property: $\forall s \in \mathbb{R}$, $A, B \in \mathbb{C}$:

$$A^s \left(1 + \frac{B}{A}\right)^s = \left(A\left(1 + \frac{B}{A}\right)\right)^s = (A + B)^s,$$

- … so our sum is

$$= (x_1 + x_2 + \cdots + x_k)^r,$$

as desired. $\square$

Cool, so...

While it is nice, this result is far from covering all possible cases. There is a lot more work that can be done with this –and other parts of the project– which brings us to...

Outline

Future Work:

Things I tried to do and failed:

- Find antiderivative for $MC(\alpha)$
- "Re-Prove" Chu-Vandermonde convolution conjecture / find an alternate proof (either counting or algebraic)
- "Hockey-Stick" identity

Things I would like to do:

- "Vectorize" our notation like in [Zen96], [Guo10]
- Look at limit behavior of $GMC(\vec{x})$ as we travel in different directions away from the origin
- NOT handwave proof of total differentiability of $GMC(\vec{x})$ on $\mathbb{R}^n \setminus S$
- Completely characterize S. Prove that one cannot construct a "continuous extension" of GMC that includes points in S by showing limits don't exist as you approach them.
- **Application...Probability?**
- More Idents.: Rothe-Hagen, Abel, Jensen's, Graham-Knuth-Patashnik, Mohanty-Handa. (Again, see [Zen96], [Guo10]).

Sources

Hac'ene Belbachir.
A combinatorial contribution to the multinomial chu-vandermonde convolution.
Les annales RECITS, 1:27–32, 2014.

Victor J. W. Guo.
Simple proofs of jensen's, chu's, mohanty-handa's, and graham-knuth-patashnik's identities, 2010.

John M. Lee.
Introduction to smooth manifolds.
2000.

David Salwinski.
The continuous binomial coefficient: An elementary approach.
The American Mathematical Monthly, 125:231–244, 03 2018.

E. T. Whittaker and G. N. Watson.
A Course of Modern Analysis.