Extending the Multinomial Coefficients to the Continuum

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Abstract

In this paper, we explore a couple of extensions of the multinomial coefficients to the real or complex numbers. This includes examining the basic properties of said coefficients when viewed as continuous functions, as well as an investigation of which combinatorial identities are retained from the discrete case.

1 Introduction

You are probably familiar with the famed *binomial coefficients*, which appear as the coefficients of the polynomial expansion of a binomial raised to some natural power in the Binomial Theorem, and tell us the number of ways to choose k elements from an n element set, when $0 \leq k \leq n$. They are computed as such, for n, k nonnegative integers:

$$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!} & k \leq n \\ 0 & k > n \end{cases}$$

Likewise, you may have heard about their generalization, the *multinomial coefficients*, which appear as the coefficients in the expansion of a *multinomial* raised to some natural power, n. These coefficients give us the number of ways to permute a multiset with n elements, each k_i of them being of the same "type," $1 \leq i \leq m$. They are given the following formula, analogous to the binomial coefficients, with m taken to be a positive integer, and $n, k_1, ..., k_m$ nonnegative integers:

$$\binom{n}{k_1, ..., k_m} = \begin{cases} \frac{n!}{k_1!k_2!...k_m!} & \sum_{i=1}^{m} k_i = n \\ 0 & o.w. \end{cases}$$

It is a natural question to ask whether these coefficients can be extended in a meaningful way to include numbers for which they weren't originally defined, such as the rationals, or even the real and complex numbers.

This is not a new question. Indeed, the idea of extending discrete notions to include numbers from the continuous domain has been around for centuries, with Leonard Euler "discovering" the now-ubiquitous gamma function for this exact purpose– as an extension of the factorial operation to include all real numbers except for the negative integers. (For a history of the gamma function, see [Dav59]).

Much more recently, in [Sal18], the binomial coefficients were extended to include real numbers. Specifically, for a fixed real number $y \notin \mathbb{Z}_{\leq 1}$, the author of that paper used the gamma function to define

$$\binom{y}{x} := \frac{\Gamma(1+y)}{\Gamma(1+x)\Gamma(1+y-x)}, \qquad x \notin \mathbb{Z}_{\leq -1}$$

as the "continuous binomial coefficient" function of a real variable, x. This function nicely encapsulates all of the properties of the binomial coefficients over the naturals, with the obvious improvement of being able to interpolate *between* the integers and speak of the binomial coefficients of *any* real number, barring the negative integers. We seek to do something similar here, this time in the multinomial case.

The outline for the paper is as follows:

- First, we present a couple of suitable definitions for our "extended multinomial coefficients" which still take on the values of the *usual* multinomial coefficients for nonnegative integers, but are also defined to take on values for numbers which are NOT positive integers, and do so in a way which reasonably interpolates between the integer values.

- Next, we look at some of the basic properties of our definitions when viewed as functions of one or more continuous variables. This includes a discussion of limits, continuity, and differentiability, as well as computations of the derivatives for our definitions in terms of the coefficient functions themselves. We also discuss things that DIDN'T work as intended.
- Finally, we present (with proof) extensions of a few well-known combinatorial identities, which carry over from the discrete case and are now defined to work in a more general continuous setting. This culminates with proving an (admittedly *very*) limited case of the "Generalized Multinomial Theorem," which will allow us to expand a finite sum of complex numbers to a *real* power, r, as opposed to a natural power, n. As before, we present a discussion of things that didn't work, and also include a list of topics pertaining to both this and the previous section that we would like to get to work on in the future.

And, **we seek to do all of the above using as few non-elementary techniques as possible!** This means refraining from utilizing mathematics beyond high-school-level algebra and calculus, except where absolutely necessary.

2 Definitions and Background

To begin, we include some definitions for continuous *binomial* coefficients; we use them as inspiration for our definitions in the multinomial case, and benefit from the interplay between them and *our* multinomial definitions in future proofs.

First, we present a well-known definition of binomial coefficients valid for an *arbitrary* parameter, α, and integer k. Note that while technically α can be *any* element of a commutative ring in which all positive integers are invertible [GKP89], here we shall understand α to be real or complex, and define our coefficients in terms of the "falling factorial" function or Pochhammer symbol, written

$$(\alpha)_k = \underbrace{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}_{\text{k factors}}$$

for α and natural k, defining $(\alpha)_0 = 1$ So:

Definition 1. *Let $k \in \mathbb{Z}$ and α arbitrary. Then,*

$$\binom{\alpha}{k} = \begin{cases} 0 & k < 0 \\ \frac{(\alpha)_k}{k!} & k \geq 0 \end{cases}.$$

Note that $\binom{\alpha}{k}$ can also be given in a more 'computationally useful' product form as: $\binom{\alpha}{k} = \prod_{i=1}^{k} \frac{\alpha+1-i}{i}$.

We next present the aforementioned definition in terms of the Γ function, which is also quite common:

Definition 2. *Let $x, y \in \mathbb{C}$ such that $y \notin \mathbb{Z}_{\leq -1}$. Then,*

$$\binom{y}{x} = \frac{\Gamma(y+1)}{\Gamma(x+1)\Gamma(y-x+1)}$$

Notice that for fixed y, these coefficients heavily resemble a damped sinusoid when plotted as a function of x. This idea is explored thoroughly in [Dav21], and while the author would have liked to do something similar for the multinomials, some technical difficulties and a lack of time prevented this. See the Future Work section for more details.

Next, we present our definitions of extensions of the Multinomial coefficients:

Definition 3. *Let α be arbitrary, k be an integer greater than 1, and let $n_1, ..., n_{k-1}$ be nonnegative integers. Then, setting the last entry equal to $\alpha - n$, where $n = \sum_{i=1}^{k-1} n_i$, we define*

$$\binom{\alpha}{n_1, \cdots, n_{k-1}, \alpha - n} = \binom{\alpha}{n_1}\binom{\alpha - n_1}{n_2}\binom{\alpha - n_1 - n_2}{n_3}\cdots\binom{\alpha - n_1 - \cdots - n_{k-2}}{n_{k-1}},$$

where the RHS is a product of binomials using Definition 1. By expanding this product in terms of Definition 1 and applying a quick lemma (immediately below this definition), we can rewrite the RHS of the above as:

$$\binom{\alpha}{n_1, \cdots, n_{k-1}, \alpha - n} = \frac{(\alpha)_n}{n_1! n_2! \cdots n_{k-1}!}.$$

We will use this latter expression as our "working definition" in practice.

Lemma 1. *Let z be arbitrary, and for an integer $k > 1$ let $r_1, r_2, ..., r_{k-1}$ be nonnegative integers. Also, define $r = \sum_{i=1}^{k-1}$. Then,*

$$(z)_r = (z)_{r_1}(z - r_1)_{r_2}(z - r_1 - r_2)_{r_3}\cdots(z - r_1 - r_2 - \cdots - r_{k-2})_{r_{k-1}}.$$

Proof. First, let's observe a simpler case. Suppose a arbitrary, b, c nonnegative integers. Then, we can show that $(a)_{b+c} = (a)_b(a-b)_c$ by simply expanding the LHS and RHS using the falling factorial definition.

Now, for $(z)_r = (z)_{r_1+r_2+...+r_{k-1}}$. If we group together the sum $r_2 + \cdots + r_{k-1}$ and use the above fact, one can see that

$$(z)_{r_1+r_2+...+r_{k-1}} = \underbrace{z(z-1)(z-2)\cdots(z - r_1 + 1)}_{(z)_{r_1}} \cdot (z - r_1)_{r_2+...+r_{k-1}}.$$

This can be done repeatedly, until all Pochhammer symbols are expanded and we see that the resulting product is equal to $(z)_r$. ☐

Next, we present a definition in terms of the gamma function.

Definition 4. *Let k be a positive integer, $x_1, ..., x_k \in \mathbb{C}$ such that $\sum_{i=1}^{k} x_i =: y \notin \mathbb{Z}_{\leq -1}$. Then, we define*

$$\binom{y}{x_1, \cdots, x_k} = \frac{\Gamma(y+1)}{\Gamma(x_1+1)\Gamma(x_2+1)\cdots\Gamma(x_k+1)}.$$

As will be shown, these varying definitions are best suited to different purposes and applications, and we can note the following way to "translate" between them:

Remark 1. *If $\alpha, x \in \mathbb{C}$ and $\alpha, \alpha + x \notin \mathbb{Z}_{\leq -1}$, then we have the following extended definition for the falling factorial* in terms of the Γ function*:*

$$(\alpha)_x = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha - x + 1)}.$$

To show that this retains its usual meaning when x is a natural number, simply use the functional equation for Γ, which is valid everywhere alpha is NOT a negative integer. This gives:

$$\frac{\Gamma(\alpha+1)}{\Gamma(\alpha - x + 1)} = \frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha - x + 1)} = \frac{\alpha(\alpha-1)\Gamma(\alpha-1)}{\Gamma(\alpha - x + 1)} = \cdots = \frac{\alpha(\alpha-1)\cdots(\alpha-(x-1))\cancel{\Gamma(\alpha-(x-1))}}{\cancel{\Gamma(\alpha - x + 1)}} = (\alpha)_x.$$

Thus, for $\alpha \in \mathbb{C}$, $n = n_1 + ... + n_{k-1} \in \mathbb{N}$ such that $\alpha, \alpha + n \notin \mathbb{Z}_{\leq -1}$ we see that our Pochhammer definition corresponds to $\frac{\Gamma(\alpha+1)}{\prod_{i=1}^{k-1}\Gamma(n_i+1)\Gamma(\alpha-n+1)} = \binom{\alpha}{n_1, ..., n_{k-1}, \alpha - n}$ using our Gamma function definition, so the two definitions coincide in this "overlapping" domain.

Moreover, if we "extend" our falling factorial definition to include complex numbers $x_1, ..., x_{k-1}$ which sum to x, and $\alpha, \alpha + x \notin \mathbb{Z}_{\leq -1}$, then in terms of the above extension for $(\alpha)_x$ and complex "factorials" "$x_i!$" $= \Gamma(x_i + 1)$, we have:

$$\frac{(\alpha)_x}{\prod_{i=1}^{k-1}\Gamma(x_i+1)} = \frac{\Gamma(\alpha+1)}{\prod_{i=1}^{k-1}\Gamma(x_i+1)\Gamma(\alpha - x + 1)} = \underbrace{\binom{\alpha}{x_1, ..., x_{k-1}, \alpha - x}}_{\textit{Gamma definition}},$$

so this equivalence goes even further!

3 Basic Analytic Properties

Next, let's move on to some discussion of our two definitions when viewed as functions on $\mathbb{R}$ or $\mathbb{C}$.

3.1 Pochhammer Symbol Def.'n

First, for our Pochhammer / Falling Factorial Definition... Let's restrict α to be real. To aid in discussion, for the duration of this section let us identify our multinomial coefficient of Definition 3 with a function MC, i.e. for a given α and $n_1, ..., n_{k-1}$ let's set:

$$MC(\alpha; n_1, ..., n_{k-1}) := \binom{\alpha}{n_1, \cdots, n_{k-1}, \alpha - n} = \frac{(\alpha)_n}{n_1! n_2! \cdots n_{k-1}!}.$$

For fixed α, this can be thought of as $MC : \mathbb{N}^{k-1} \to \mathbb{R}$, in other words a real-valued multivariate sequence.

Conversely, for fixed $n_1, ..., n_{k-1}$, MC can be thought of as a real-valued function of a real variable, i.e. $MC : \mathbb{R} \to \mathbb{R}$. When speaking of this function of α, we will suppress the fixed parameters $n_1, ..., n_{k-1}$ and simply write $MC(\alpha)$ henceforward. Let's inspect $MC(\alpha)$ a bit more.

First, from the definition one can see that $MC(\alpha)$ is a real-valued polynomial in α of degree n. Indeed, using the generating function for Stirling numbers of the first kind, the polynomial expansion of $MC(\alpha)$ is

$$MC(\alpha) = \frac{1}{n_1! n_2! \cdots n_{k-1}!}[s(n, n)\alpha^n + s(n, n-1)\alpha^{n-1} + \cdots + s(n, 1)\alpha + s(n, 0)].$$

As such, MC is clearly continuous and differentiable for all α.

Thus, it is natural to try to find a formula for its derivative at a point, $MC'(\alpha)$. Especially helpful would be a formula for $MC'(\alpha)$ in terms of $MC(\alpha)$ itself, as doing such might be more useful in application. We next present our original result for the derivative of our function at any $\alpha \in \mathbb{R}$, followed by its derivation:

Computation 1.

$$\frac{d}{d\alpha} MC(\alpha) = \begin{cases} 0 & n = 0 \\ \lim\limits_{\beta \to \alpha} MC(\beta)[\psi(-\beta) - \psi(n - \beta)] & n > 0, \end{cases}$$

where ψ is the digamma function, defined as

$$\psi(z) := \frac{d}{dx} \ln(\Gamma(z)) = \frac{\Gamma'(z)}{\Gamma(z)},$$

z being in the set $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$.

Proof. First, if $n = 0$, then $(\alpha)_n = 1$ regardless of α, so its derivative is 0. Now, consider the case when $n > 0$. Then, $(\alpha)_n = \alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - n + 1)$, so we can take its derivative using the Product Rule to get:

$$[(\alpha)_n]' = (\alpha - 1)(\alpha - 2) \cdots (\alpha - n + 1) + \alpha(\alpha - 2) \cdots (\alpha - n + 1) + \\ + \alpha(\alpha - 1)(\alpha - 3) \cdots (\alpha - n + 1) + \cdots + \alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - n + 2).$$

If we assume $\alpha \notin \{0, 1, 2, ..., n - 1\}$, then we can replace all these terms with $(\alpha)_n$, divided by the factor "missing" from $(\alpha)_n$. So

$$[(\alpha)_n]' = (\alpha)_n \cdot \sum_{k=0}^{n-1} \frac{1}{\alpha - k}, \qquad n \geq 1.$$

Now, this restriction on the value of α is certainly less than desirable. So, how do we get around it? Suppose now that $\alpha = k$, where $k \in \{0, 1, 2, ..., n - 1\}$. Then, taking the derivative using the product rule, notice that all terms containing a factor of $(\alpha - k)$ will vanish, leaving only one term in our derivative:

$$\frac{d}{d\alpha}(\alpha)_n\bigg|_{\alpha=k} = \alpha(\alpha-1)\cdots(\alpha-(k-1))(\alpha-(k+1))\cdots(\alpha-n+1).$$

On the other hand, with this same $\alpha = k$, consider the following limit:

$$\lim_{\beta\to\alpha}(\beta)_n\sum_{k=0}^{n-1}\frac{1}{\beta-k}.$$

Putting aside the question of the limit's existence for a split second, one can formally distribute...

$$= \lim_{\beta\to\alpha}\left(\frac{(\beta)_n}{\beta}+\frac{(\beta)_n}{\beta-1}+\frac{(\beta)_n}{\beta-2}+\cdots+\frac{(\beta)_n}{\beta-n+1}\right)$$

... then separate out the k^{th} term in this limit...

$$= \lim_{\beta\to\alpha}\left(\frac{(\beta)_n}{\beta}+\cdots+\frac{(\beta)_n}{\beta-(k-1)}+\frac{(\beta)_n}{\beta-(k+1)}+\cdots+\frac{(\beta)_n}{\beta-n+1}\right)+\lim_{\beta\to\alpha}\frac{(\beta)_n}{\beta-k}$$

...observing that the limit of all but the rightmost term will vanish, leaving us with:

$$\lim_{\beta\to\alpha}(\beta)_n\sum_{k=0}^{n-1}\frac{1}{\beta-k} = \lim_{\beta\to\alpha}\frac{(\beta)_n}{\beta-k}$$

$$= \alpha(\alpha-1)\cdots(\alpha-(k-1))(\alpha-(k+1))\cdots(\alpha-n+1).$$

Thus, through manipulating symbols, we've shown that this limit *equals* $[(\alpha)_n]'(k)$. The existence of this limit is automatic, since we know that $MC(\alpha)$ is everywhere differentiable and its derivative is continuous, *and* it must equal our limit. Since this limit also clearly equals $MC'(\alpha)$ when α is NOT one of the naturals less than n, we can combine these results together and state our derivative for $MC(\alpha)$ in terms of the continuous extension

$$MC'(\alpha) = \begin{cases} 0 & n = 0 \\ \lim_{\beta\to\alpha} MC(\beta)\sum_{k=0}^{n-1}\frac{1}{\beta-k} & n > 0. \end{cases}$$

A quick substitution involving a difference equation based on the digamma recurrence relation $\psi(x+N)-\psi(x) = \sum_{k=0}^{N-1}\frac{1}{x+k}$ yields the final form of the answer. $\square$

The author also spent a great deal of time trying to find an *antiderivative* for $MC(\alpha)$ in terms of itself, using various methods. However, an answer still remains elusive at this time.

3.2 Gamma Function Def.'n

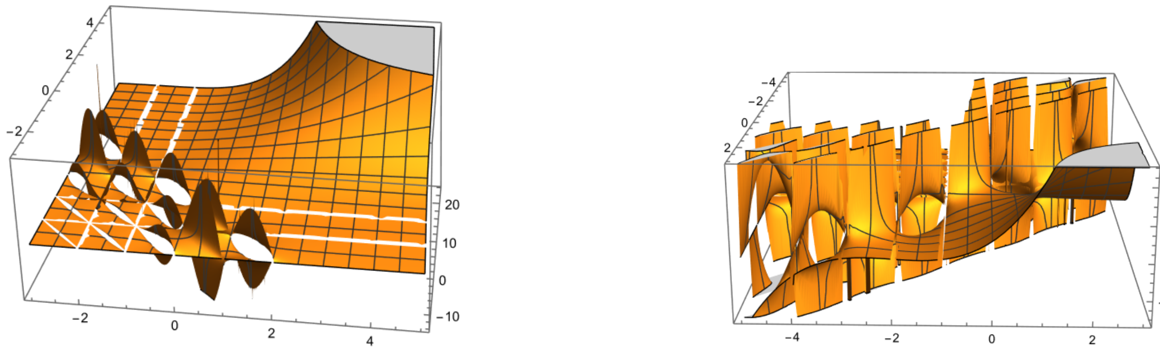
Likewise, we can carry out a similar exploration for our Gamma definition. Recall that if $x_1, ..., x_n \in \mathbb{C}$, then we say that for $y = x_1 + ... + x_n$, $\binom{y}{x_1,...,x_n} = \frac{\Gamma(y+1)}{\Gamma(x_1+1)...\Gamma(x_n+1)}$.

First, consider the Weierstrass definition of $\Gamma(z)$:

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}e^{-z/n}.$$

Through examining this definition, one can show that Γ is *analytic* $\forall z \in \mathbb{C}$ except at $z = 0, -1, -2, ...$, where it has simple poles [WW96]. It is also *holomorphic* everywhere it is defined, since these two terms are synonyms for functions of a single complex variable. Thus, $\Gamma'(z)$ exists for all z.

Moreover, since Γ is *meromorphic* and nonzero everywhere, $\frac{1}{\Gamma(z)}$ is an *entire* function. Hence, restricting $\frac{1}{\Gamma}$ to the *real* numbers, one can see that it is a real-valued function of a real variable which is everywhere continuously differentiable in the real sense, and Γ is differentiable everywhere it's defined.



(a) Looking toward positive octant

(b) Looking toward negative octant

Figure 1: $GMC(x, y)$ viewed from 2 different angles

Now, by a result from the study of calculus on manifolds, (see [Lee00]), the function from $\mathbb{R}^n$ to $\mathbb{R}$ formed from a *product* of differentiable functions on $\mathbb{R}$ will be *total differentiable* on $\mathbb{R}^n$. Thus, $\frac{1}{\Gamma(x_1+1)\cdots\Gamma(x_n+1)}$ is differentiable everywhere on $\mathbb{R}^n$, and $\frac{\Gamma(y+1)}{\Gamma(x_1+1)\cdots\Gamma(x_n+1)}$ is differentiable everywhere $\Gamma(y+1)$ is.

With this out of the way, let's restrict $x_1, ..., x_n$ to be real, and consider our multinomial coefficients as a function of a vector of real numbers, $\vec{x} = (x_1, ..., x_n)$. We denote this function $GMC(\vec{x})$. That is, we are now thinking of our coefficients as the function $GMC : \mathbb{R}^n \to \mathbb{R}$ such that $\sum_{i=1}^n x_i \notin \mathbb{Z}_{\leq -1}$. Views of the graph of this function can be seen from a couple of different angles in Figure 1, at the top of the page.

Now, we can compute partial derivatives of GMC at points where it is differentiable. Note that by symmetry of GMC, we need only do this once.

Computation 2. *Fix $j \in \{1, 2, ..., n\}$. Then,*

$$\frac{\partial}{\partial x_j} GMC(\vec{x}) = GMC(\vec{x})\Big(\psi(y+1) - \psi(x_j+1)\Big),$$

everywhere where GMC is defined.

From this, we can obtain the gradient as:

Computation 3.

$$\nabla GMC(\vec{x}) = GMC(\vec{x})\psi(y+1)\Big(1 - \psi(x_1+1), 1 - \psi(x_2+1), ..., 1 - \psi(x_n+1)\Big),$$

everywhere GMC is defined.

And, the directional derivative can be expressed as:

Computation 4. *If $\vec{x} = (x_1, ..., x_n)$ is a value at which GMC is defined, and $\vec{v} = (v_1, ..., v_n) \in \mathbb{R}^n$, then the directional derivative of GMC along $\vec{v}$ at $\vec{x}$ can be expressed in coordinates as:*

$$D_{\vec{v}}[GMC(\vec{x})] = GMC(\vec{x})\psi(y+1)\left[\sum_{i=1}^{n} v_i \left(1 - \frac{\psi(x_i+1)}{\psi(y+1)}\right)\right].$$

All of these computations are straightforward and are not written down here. Note the apparent symmetry between the partial derivatives of $GMC(\vec{x})$ and the derivative of $MC(\alpha)$ from the last subsection.

Also note that it appears there is nothing stopping us from extending the above results to the complex numbers, for both definitions. While this would be nice, it is beyond the scope of this paper.

Finally, it is worth mentioning a thing or two about the set of points in $\mathbb{R}^n$ on which GMC *isn't* defined; let's call this set $S = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_1 + \cdots + x_n \in \mathbb{Z}_{\leq -1}\}$. We conjecture that this set is the union (over $s \in \mathbb{Z}_{\leq -1}$) of all $n-1$-dimensional hyperplanes which pass through the points

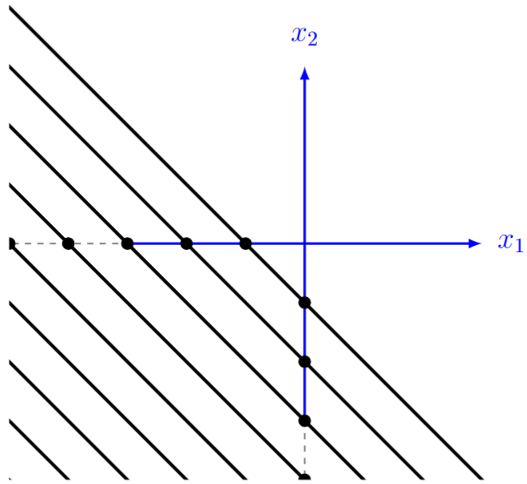


Figure 2: The set of points in $\mathbb{R}^2$ for which our Gamma definition isn't defined

$(-s, 0, ...0), (0, -s, 0, ..., 0), \cdots, (0, ..., 0, -s)$, etc., (the same hyperplane passing through all n of these points). As an illustration, consider Figure 2, which shows the set S for the case when $n = 2$.

We tried, without success, to prove this claim. However, once the claim *is* proven, it would be a simple matter to show that we cannot define a "continuous extension" of our function GMC to include points in S, (similar to what we did for $MC(\alpha)$'s derivative), because the limit as we approach any one of these points does not exist.

4 Combinatorial Identities

We now begin our discussion of the various identities that can be derived from our definitions of the continuous multinomial coefficients. We would prefer, (for somewhat obvious reasons), that our definitions allow us to retain and even "extend" *as many* of the basic combinatorial properties of the usual multinomial coefficients as possible; this was actually somewhat instructive in choosing our definitions; other more general definitions exist, but they might not uphold the following properties, which we shall present as a series of Lemmas:

Lemma 2. *(Generalized Multinomial as Product of Binomials):*
Let $n, n_1, ..., n_k \in \mathbb{C} \setminus \mathbb{Z}_{\leq -1}$, $k \in \mathbb{N}$ greater than 1 such that $\sum_{i=1}^{k} n_i = n$. Then, using our Gamma Definition of the multinomial coefficients,

$$\binom{n}{n_1, n_2, ..., n_k} = \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-\cdots-n_{k-1}}{n_k},$$

where the Binomial coefficients are defined as in Definition 2 or in [Sal18].

Proof. Let $n, n_1, ..., n_k \in \mathbb{C} \setminus \mathbb{Z}_{\leq -1}$, k be a positive integer, with $\sum_{i=1}^{k} n_i = n$. Expanding the RHS, using Definition 2, we have

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-\cdots-n_{k-1}}{n_k}$$

$$= \frac{\Gamma(n+1)}{\Gamma(n_1+1)\cancel{\Gamma(n-n_1+1)}} \cdot \frac{\cancel{\Gamma(n-n_1+1)}}{\Gamma(n_2+1)\cancel{\Gamma(n-n_1-n_2+1)}} \cdot \frac{\cancel{\Gamma(n-n_1-n_2+1)}}{\Gamma(n_3+1)\cancel{\Gamma(n-n_1-n_2-n_3+1)}} \times \cdots$$

$$\times \frac{\cancel{\Gamma(n-n_1-n_2-...-n_{k-1}+1)}}{\Gamma(n_k+1)\underbrace{\Gamma(n-n_1-...-n_{k-1}-n_k+1)}_{=\Gamma(n-n+1)=\Gamma(1)=1}}.$$

Most terms cancel, and we're left with the LHS of the statement of the Lemma. □

Note that Lemma 2 was already used, without calling it that by name, in the *definition* of our multinomial coefficients using the Pochhammer Symbol! So, it obviously holds there.

Lemma 3. *(Generalized Pascal's Identity): Let $n, n_1, ..., n_k \in \mathbb{R}^+$, $k \in \mathbb{N}$ be greater than 1, such that $\sum_{i=1}^{k} n_i = n$. Then, using our Gamma Definition,*

$$\binom{n}{n_1, n_2, \cdots, n_k} = \binom{n-1}{n_1 - 1, n_2, \cdots, n_k} + \binom{n-1}{n_1, n_2 - 1, \cdots, n_k} + \cdots + \binom{n-1}{n_1, n_2, n_3, \cdots, n_k - 1}.$$

Proof. Suppose k is a positive integer, $n, n_1, ..., n_k \in \mathbb{R}^+$ such that $\sum_{i=1}^{k} n_i = n$. Consider the RHS of the statement of the lemma,

$$\binom{n-1}{n_1 - 1, n_2, \cdots, n_k} + \binom{n-1}{n_1, n_2 - 1, \cdots, n_k} + \cdots + \binom{n-1}{n_1, n_2, n_3, \cdots, n_k - 1}.$$

Expanding this in terms of the definition, this equals:

$$\frac{\Gamma(n)}{\Gamma(n_1)\Gamma(n_2+1)\cdots\Gamma(n_k+1)} + \frac{\Gamma(n)}{\Gamma(n_1+1)\Gamma(n_2)\cdots\Gamma(n_k+1)} + \cdots + \frac{\Gamma(n)}{\Gamma(n_1+1)\Gamma(n_2+1)\cdots\Gamma(n_k)}.$$

To obtain a common denominator, note that each term is "missing" a $\Gamma(n_i + 1)$, and also missing all $\Gamma(n_j)$ such that $j \neq i$. Obtaining a common denominator that includes these missing factors, we have:

$$\frac{\Gamma(n)\Gamma(n_1+1)\prod_{i\neq 1, i\leq k}\Gamma(n_i)}{\prod_{i=1}^{k}\Gamma(n_i+1)\prod_{i=1}^{k}\Gamma(n_i)} + \frac{\Gamma(n)\Gamma(n_2+1)\prod_{i\neq 2, i\leq k}\Gamma(n_i)}{\prod_{i=1}^{k}\Gamma(n_i+1)\prod_{i=1}^{k}\Gamma(n_i)} + \cdots + \frac{\Gamma(n)\Gamma(n_k+1)\prod_{i=1}^{k}\Gamma(n_i)}{\prod_{i=1}^{k}\Gamma(n_i+1)\prod_{i=1}^{k-1}\Gamma(n_i)}.$$

Using the recurrence relation $\Gamma(z + 1) = z\Gamma(z)$, which holds everywhere Γ is defined, we have

$$\Gamma(n)\frac{\Gamma(n_1+1)\prod_{i\neq 1, i\leq k}\Gamma(n_i) + \Gamma(n_2+1)\prod_{i\neq 2, i\leq k}\Gamma(n_i) + \cdots + \Gamma(n_k+1)\prod_{i\neq k, i\leq k}\Gamma(n_i)}{\prod_{i=1}^{k} n_i\Gamma(n_i)^2}$$

from which we can cancel...

$$= \Gamma(n)\frac{\frac{\Gamma(n_1+1)}{\Gamma(n_1)} + \frac{\Gamma(n_2+1)}{\Gamma(n_2)} + \cdots + \frac{\Gamma(n_k+1)}{\Gamma(n_k)}}{\prod_{i=1}^{k} n_i\Gamma(n_i)}$$

... use our recurrence relation again, and simplify to obtain

$$\Gamma(n) \cdot \frac{n_1 + n_2 + \cdots n_k}{\prod_{i=1}^{k} n_i\Gamma(n_i)} = \frac{\Gamma(n)n}{\prod_{i=1}^{k}\Gamma(n_i+1)} = \frac{\Gamma(n+1)}{\prod_{i=1}^{k}\Gamma(n_i+1)} = \binom{n}{n_1, n_2, \cdots, n_k}.$$

□

Similar manipulations of our Pochhammer definition have been tried in order to establish a similar result, to no avail.

Next, the author attempted to find a result analogous to the *"Hockey Stick Identity"* from basic combinatorics, which says that if you sum along the diagonals of Pascal's triangle, the result of this sum will equal the entry immediately below and to the right (or left, depending on the direction of the diagonal) of the terminating entry in the sum.

The author of [Jon96] does discuss a multinomial analog of the hockey-stick identity; their discussion is based on counting arguments. Unfortunately, a search of the literature yielded no other results of this sort corresponding to a more *algebraic* approach, which would be of greater usefulness to our study. So, with nowhere to start from, the idea never really got off the ground.

Next, we present an extension of the remarkable *Chu-Vandermonde Identity* to the multinomial case, using our Pochhammer definition. This is due to the author of [Bel14], and we refer the reader to their paper for a proof of this and the next results.

Theorem 1. *(Continuous Multinomial Chu-Vandermonde Convolution):*
Let k be a positive integer, $x, y \in \mathbb{C}$, and $n_1, ..., n_{k-1}$ be nonnegative integers. If we define $n = \sum_{i=1}^{k-1} n_i$ and use the Pochhammer symbol definition, then:

$$\binom{x+y}{n_1, n_2, ..., n_{k-1}, x+y-n} = \sum_{m_1,...,m_{k-1} \geq 0} \binom{x}{m_1, ..., m_{k-1}, x - \sum_j m_j} \times$$
$$\times \binom{z}{n_1 - m_1, ..., n_{k-1} - m_{k-1}, y - n + \sum_j m_j}.$$

The author of that paper also states an *even more* generalized version of the Chu-Vandermonde Convolution, stating that a proof follows from the consideration of the proof of Conjecture 1 for s complex arguments:

Theorem 2. *(Generalized Continuous Multinomial Chu-Vandermonde Convolution):*
Let s, t be positive integers, $x_1, ..., x_s \in \mathbb{C}$, and $n_1, n_2, ..., n_{t-1}$ be nonnegative integers. Set $x = \sum_{j=1}^{s} x_j$, $n = \sum_{j=1}^{t-1} n_j$. Then, we have the following:

$$\binom{x_1 + \cdots + x_s}{n_1, ..., n_{t-1}, x - n} = \sum_{m_{i,j}} \binom{x_1}{m_{1,1}, ..., m_{1,t-1}, x_1 - \sum_j m_{1,j}} \cdots \binom{x_s}{m_{s,1}, ..., m_{s,t-1}, x_s - \sum_j m_{s,j}},$$

where the sum is taken over all $m_{i,j}$, $i \in [s]$, $j \in [t-1]$, such that $m_{1,\ell} + m_{2,\ell} + \cdots + m_{s,\ell} = n_\ell$, $\forall \ell \in [t-1]$.

Finally, we turn to what could be considered the "holy grail" of any naive attempt to generalize the multinomial coefficients to the continuum: finding an extension of the Multinomial Theorem. During this term, the author was able to prove a special case of the the theorem expanding a sum of complex numbers of the form $x_1 + \cdots + x_k$ to a real power, r.

In this version of the Multinomial Theorem, our sum becomes infinite, and our coefficients are replaced with the Pochhammer definition, but the general form stays the same. Here is the statement of the Theorem:

Theorem 3. *(Special Case of GMT for x's $\in \mathbb{R}_{\geq 0}$, $r \in \mathbb{R}$):*
Consider $(x_1 + x_2 + \cdots + x_k)^r$, $r \in \mathbb{R}$, k an integer greater than 1, and $x_j \in \mathbb{R}_{\geq 0}$ for all $j \in [k]$. Suppose $\exists i \in [k]$ such that $x_i > \sum_{\substack{1 \leq j \leq k \\ j \neq i}} x_j$. Then, calling $\sum_{j \neq i} n_j = n$, we have

$$(x_1 + x_2 + \cdots + x_k)^r = \sum_{\substack{(n_1,...,n_{i-1},n_{i+1},...,n_k) \\ n_1,n_2,...,n_k \geq 0}} \binom{r}{n_1, n_2, ..., n_{i-1}, r-n, n_{i+1}, ..., n_k} x_1^{n_1} x_2^{n_2} \cdots x_i^{r-n} \cdots x_k^{n_k},$$

the series on the right converging.

Notice that we are summing over all $k-1$-tuples of positive integers n_j, and that we fill the i^{th} "slot" in the multinomial coefficients –that is, the slot of the maximum x_j– with r minus the sum of the n_j's. We also attach an exponent of $r - n$ to x_i instead of the would-be "n_i." This assignment of our coefficients and exponents is *essential* for the convergence of the series. (We'll discuss this more in a bit).

For now, let's set about proving the theorem... To do so, we must first lay some groundwork, including a lemma which was given as an example in [WW96]. We fill in the details of the proof here.

Lemma 4. *(Generalized Binomial Theorem):*
Let $z, h \in \mathbb{C}$ such that $|z| > |h|$. Then, $\forall r \in \mathbb{R}$, using the Pochhammer definition for the binomial coefficients, the series $\sum_{n=0}^{\infty} \binom{r}{n} z^{r-n} h^n$ is convergent and equal to $(z+h)^r$.

Proof. Let $r \in \mathbb{R}$, and let $f(s) = s^r$ be a complex-valued function of a complex variable, s. Further, suppose $h, z \in \mathbb{C}$ be such that $|h| < |z|$. Then, consider points $z + h$ inside the open disk of radius $|z|$ centered at z in the complex plane. (See Figure 3 at the top of the next page.)

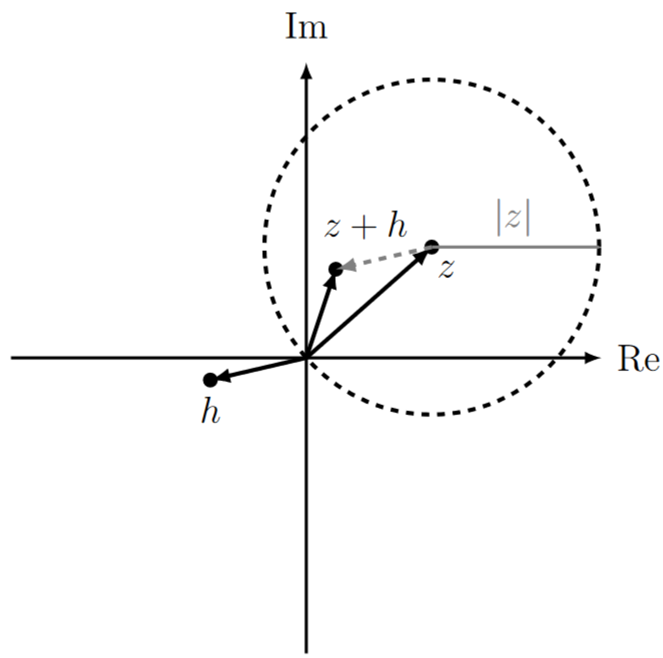


Figure 3: An example circle at z, h, and $z+h$ in $\mathbb{C}$

If we consider $f(z+h)$ *as a function of* h, clearly this function cannot have any singularities within the interior of our disk, because in order for this to happen we would need negative r and also $z+h=0$. But if $z+h=0$, then $|z|=|h|$, which is not the case.

Additionally, clearly f is differentiable everywhere in the disk, since it's a polynomial. Therefore, $f(z+h)$ as a function of h is *analytic* in our circle, *if* $|z|>|h|$. Now, recall Taylor's Theorem: If f is analytic through the disk $|s-s_0|<R$ centered at s_0 of radius R, then $f(s)$ has a power series representation inside the disk:

$$f(s)=\sum_{n=0}^{\infty}\frac{f^{(n)}(s_0)}{n!}(s-s_0)^n, \qquad |s-s_0|<R.$$

Applying Taylor's Theorem to $f(z+h)$ as a function of h, inside our disk of radius $|z|$ centered at z, we obtain the following series with condition of convergence $|h|<|z|$:

$$f(z+h)=f(z)+hf'(z)+\frac{h^2}{2!}f''(z)+\cdots+\frac{h^n}{n!}f^{(n)}(z)+\cdots, \qquad |h|<|z|,$$

or,

$$(z+h)^r=z^r+rz^{r-1}h+\frac{r(r-1)}{2}z^{r-2}h^2+\cdots+\frac{r(r-1)\cdots(r-n+1)}{n!}z^{r-n}h^n+\cdots, \quad |h|<|z|.$$

We recognize this as our Pochhammer coefficient binomial series! Therefore,

$$(z+h)^r=\sum_{n=0}^{\infty}\binom{r}{n}z^{r-n}h^n, \qquad |h|<|z|,$$

as desired! $\square$

Note that we can extend this result a bit, to the case where $|z|=|h|$ and r is *strictly* positive. When this applies, our function of h is still analytic inside the disk, but also on the *boundary*, and we end up with the same Taylor series from last time.

Also note that this proof of the Generalized Binomial Theorem trivially extends to our Γ definition when $r, r+k\in\mathbb{R}\setminus\mathbb{Z}_{\leq -1}$ and $k\in\mathbb{N}$, by comments made in Remark 1.

Now, for the proof of Theorem 3.

Proof. (Proof of Theorem 3, our Generalized Multinomial Coefficients):
Suppose our $x_1, ..., x_k, r$ given as stated in the Theorem. Further, suppose we do have an $i \in [k]$ so that *not only* is x_i strictly greater than all x_j, $j \neq i$, but also $x_i > \sum\limits_{\substack{1 \le j \le k \\ j \neq i}} x_j$.

Without loss of generality, say that $x_i = x_k$. (This will make the notation easier). Defining n to be $n = \sum_{j=1}^{k-1} n_j$, consider the formal iterated sum

$$\sum_{\substack{(n_1,...,n_{k-1}) \\ n_1,n_2,...,n_{k-1} \ge 0}} \binom{r}{n_1, n_2, ..., n_{k-1}, r-n} x_1^{n_1} x_2^{n_2} \cdots x_{k-1}^{n_{k-1}} x_k^{r-n}$$

$$= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_{k-1}=0}^{\infty} \binom{r}{n_1, n_2, ..., n_{k-1}, r-n} x_1^{n_1} x_2^{n_2} \cdots x_k^{r-n}.$$

Note that $r-n$ can be positive, negative, or zero. By Lemma 2, we have

$$\binom{r}{n_1, n_2, ..., n_{k-1}, r-n} = \binom{r}{n_1} \binom{r-n_1}{n_2} \binom{r-n_1-n_2}{n_3} \cdots \binom{r-n_1-n_2-\cdots-n_{k-2}}{n_{k-1}} \cancel{\binom{r-n}{r-n}},$$

so our sum is just

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_{k-1}=0}^{\infty} \binom{r}{n_1} \binom{r-n_1}{n_2} \binom{r-n_1-n_2}{n_3} \cdots \binom{r-n_1-n_2-\cdots-n_{k-2}}{n_{k-1}} x_1^{n_1} x_2^{n_2} \cdots x_k^{r-n}.$$

We can move all the "lower n_j" factors out of the inner sums:

$$= \sum_{n_1=0}^{\infty} \binom{r}{n_1} x_1^{n_1} \sum_{n_2=0}^{\infty} \binom{r-n_1}{n_2} x_2^{n_2} \cdots \sum_{n_{k-1}=0}^{\infty} \binom{r-n_1-n_2-\cdots-n_{k-2}}{n_{k-1}} x_{k-1}^{n_{k-1}} x_k^{(r-n_1-n_2-...-n_{k-2})-n_{k-1}}.$$

Now, focus on the innermost sum. If we combine x's to the power of n_{k-1}, and pull the other x out front, then this inner sum is:

$$x_k^{r-n_1-n_2-...-n_{k-2}} \sum_{n_{k-1}=0}^{\infty} \binom{r-n_1-n_2-\cdots-n_{k-2}}{n_{k-1}} \left(\frac{x_{k-1}}{x_k}\right)^{n_{k-1}} \cdot \underbrace{(1)^{(r-n_1-n_2-...-n_{k-2})-n_{k-1}}}_{\text{(Multiply by 1)}}.$$

Now, note that since $x_k > x_{k-1}$, we know $\frac{x_{k-1}}{x_k} < 1$, so by Lemma 4 (the GBT) we have

$$x_k^{r-n_1-n_2-...-n_{k-2}} \left(1 + \frac{x_{k-1}}{x_k}\right)^{r-n_1-n_2-\cdots-n_{k-2}}.$$

Thus, our total sum is now

$$\sum_{n_1=0}^{\infty} \binom{r}{n_1} x_1^{n_1} \sum_{n_2=0}^{\infty} \binom{r-n_1}{n_2} x_2^{n_2} \cdots$$
$$\sum_{n_{k-2}=0}^{\infty} \binom{r-n_1-n_2-\cdots-n_{k-3}}{n_{k-2}} x_{k-2}^{n_{k-2}} x_k^{r-n_1-\cdots-n_{k-2}} \left(1 + \frac{x_{k-1}}{x_k}\right)^{(r-\cdots-n_{k-3})-n_{k-2}}.$$

Again, focus on the innermost sum:

$$\sum_{n_{k-2}=0}^{\infty} \binom{r-n_1-n_2-\cdots-n_{k-3}}{n_{k-2}} x_{k-2}^{n_{k-2}} x_k^{(r-n_1-\cdots-n_{k-3})-n_{k-2}} \left(1 + \frac{x_{k-1}}{x_k}\right)^{(r-\cdots-n_{k-3})-n_{k-2}},$$

and, again, combine x_{k-2} factors and pull the remaining x_j out, giving

$$x_k^{r-n_1-\cdots-n_{k-3}} \cdot \sum_{n_{k-2}=0}^{\infty} \binom{r-n_1-n_2-\cdots-n_{k-3}}{n_{k-2}} \left(\frac{x_{k-2}}{x_k}\right)^{n_{k-2}} \left(1 + \frac{x_{k-1}}{x_k}\right)^{(r-\cdots-n_{k-3})-n_{k-2}}.$$

Since $\frac{x_{k-2}}{x_k} < 1$ and $(1 + \frac{x_{k-1}}{x_k}) \geq 1$, GBT *again* applies here, so our innermost sum equals

$$x_k^{r-n_1-\cdots-n_{k-3}} \left(1 + \frac{x_{k-1} + x_{k-2}}{x_k}\right)^{r-n_1-\cdots-n_{k-3}}.$$

Clearly, we can continue this process, combining our innermost x_j's, factoring out $x_k^{(\ldots)}$, and applying GBT, until we've exhausted all but the outermost sum.

At this point, we would have:

$$\sum_{n_1=0}^{\infty} \binom{r}{n_1} x_1^{n_1} \cdot x_k^{r-n_1} \cdot \left(1 + \frac{x_2 + x_3 + \cdots + x_{k-1}}{x_k}\right)^{r-n_1},$$

from which we repeat the process once more to get:

$$x_k^r \sum_{n_1=0}^{\infty} \binom{r}{n_1} \underbrace{\left(\frac{x_1}{x_k}\right)^{n_1}}_{|\cdot|<1} \cdot \underbrace{\left(1 + \frac{x_2 + \cdots + x_{k-1}}{x_k}\right)^{r-n_1}}_{|\cdot|\geq 1}$$

$$= x_k^r \left(1 + \frac{x_1 + \cdots + x_{k-1}}{x_k}\right)^r.$$

Now, since $x_k \neq 0$, (by virtue of being *strictly* greater than a sum of nonnegative reals), we can use the distributive property: $\forall s \in \mathbb{R},\ A, B \in \mathbb{C}$:

$$A^s \left(1 + \frac{B}{A}\right)^s = \left(A\left(1 + \frac{B}{A}\right)\right)^s = (A+B)^s,$$

so our sum is

$$= (x_1 + x_2 + \cdots + x_k)^r,$$

as desired. □

While this is nice, this result is far from covering all possible cases. So, the author spent a great deal of time trying various edge cases, as well as poking and prodding at the sum with the goal of extending it to cover a more general case involving the $x_1, ..., x_k$. This involved a great deal of numerical experimentation in Mathematica, to see when the series did and didn't converge to the correct value. Out of this experimentation, the following conjecture was born:

Conjecture 1. *(Behavior of Multinomial Series):*
Consider $(x_1 + x_2 + \cdots + x_k)^r$, $r \in \mathbb{R}$, k an integer greater than 1, and all $x_j \in \mathbb{C}$, $j \in [k]$. Suppose $\exists i \in [k]$ such that $\forall j \neq i$ in $[k]$, $|x_i| > |x_j|$. Call $\sum_{j\neq i} n_j = n$. Then, the formal series using the Gamma Definition given by

$$\sum_{\substack{(n_1,\ldots,n_{i-1},n_{i+1},\ldots,n_k) \\ n_1,n_2,\ldots,n_k \geq 0}} \binom{r}{n_1, n_2, ..., n_{i-1}, r-n, n_{i+1}, ..., n_k} x_1^{n_1} x_2^{n_2} \cdots x_i^{r-n} \cdots x_k^{n_k}$$

will exhibit the following behavior:

- ***Diverge**, if you attach the "$r-n$" exponent to any of the x_j besides x_i, regardless of the sum of the other magnitudes.*
- ***Approach the true value of** $(x_1 + x_2 + \cdots + x_k)^r$ **for some finite number of terms, then diverge**, if you give x_i the right exponent, but $\sum_{j\neq i} |x_j| \geq |x_i|$.*
- ***Converge to the desired value**, if you* have *an x_i such that $|x_i| > \sum_{j\neq i} |x_j|$, and you give x_i the correct exponent. (See Thm. 3).*

*Otherwise, if no such i exists, then for $k > 2$ the sum will **approach** **some** **value dependent upon the x_j's for a finite number of terms, then** **diverge**.*

What this value is in relation to $x_1, ..., x_k$ has still, as of yet, not been ascertained.

5 Future Work

In the interest of brevity, we simply present a list of topics pertaining to this work which we would like to investigate more in the future:

- "Vectorize" our notation: Similar as to what was done in [Zen96] and [Guo10], we wish to restate our results for multinomials in terms of ordered lists of natural numbers. While more of a notational change, this would *greatly* improve the readability of many of the formulas in this paper, and make it easy for the reader to see the relationship between the multinomial and binomial cases.

- Investigate the asymptotic behavior of MC, either as a function of α (which is easy), or as a multivariate sequence in $n_1, ..., n_{k-1}$.

- Look at the asymptotic behavior of $GMC(\vec{x})$ as we travel in different directions away from the origin. Establishing bounds on this growth would also take steps toward proving various results regarding the convergence of the Multinomial Theorem series, another large goal.

- Prove rigorously the total differentiability of $GMC(\vec{x})$ on $\mathbb{R}^n \setminus S$.

- Completely characterize S. Prove that one cannot construct a "continuous extension" of GMC that includes points in S by showing limits don't exist as you approach points in S.

- Find applications: This one is quite self-explanatory. One possible avenue for application is in Probability Theory. Indeed, products of Gamma functions are already used extensively in the density functions of various probability distributions. See the PMF of the "Dirichlet-Multinomial" Distribution as an example.

- Collect all results from [Zen96] and [Guo10] together: These papers present many identities not discussed here, some of them multinomial, some of them binomial. For those which are binomial, we seek to find an extension to the multinomial case for real and complex arguments. For those which are multinomial, it would be nice to simplify the proofs and gather all these results together in one place, to give a "wide view" on which identities can be generalized.

- Prove the behavior in Conjecture 1. And/or, find further generalizations of Theorem 3.

- Find *more* definitions for our coefficients which "reasonably interpolate" between the discrete case values. Especially interesting would be finding something similar to [Dav21] involving series of trig-like functions.

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