1 An Electron in a Magnetic Field: Clifford Bundle Treatment

Consider a non-relativistic electron of mass m and charge e moving in a constant magnetic field $\vec{B} = B\hat{z}$ in $\mathbb{R}^3$.

1.1 Traditional Approaches

1.1.1 Schrödinger Equation (Wave Mechanics)

$$i\hbar\partial_t\psi = \frac{1}{2m}(\vec{p} - e\vec{A})^2\psi$$

where $\vec{B} = \nabla \times \vec{A}$.

1.1.2 Differential Forms (De Rham Complex)

- Magnetic potential: $A = A_i dx^i \in \Omega^1(\mathbb{R}^3)$
- Field strength: $F = dA = \frac{1}{2}F_{ij}dx^i \wedge dx^j \in \Omega^2(\mathbb{R}^3)$
- For constant $\vec{B}$: $F = B\,dx \wedge dy$
- Maxwell's equations: $dF = 0$ (Bianchi), $d \star F = J$ (dynamics)

These approaches seem disconnected. The Clifford bundle unifies them.

1.2 Clifford Bundle Formulation

Let $\mathcal{C}\ell(\mathbb{R}^3, \delta_{ij})$ be the Clifford bundle over Euclidean $\mathbb{R}^3$. The algebra is generated by $\{e_1, e_2, e_3\}$ with $e_ie_j + e_je_i = 2\delta_{ij}$.

1.2.1 Unified Multivector State

Define the **electron state multivector**:

$$\Psi = \underbrace{\psi}_{\text{scalar}} + \underbrace{\vec{v}}_{\text{vector}} + \underbrace{\vec{S}}_{\text{bivector}} \in \mathcal{C}\ell(\mathbb{R}^3)$$

where:

$$\begin{aligned}
&\psi \in \mathbb{C} \quad \text{(wavefunction amplitude)}\\
&\vec{v} = v^i e_i \quad \text{(velocity/current)}\\
&\vec{S} = S^{ij} e_i \wedge e_j \quad \text{(spin/orbital angular momentum)}
\end{aligned}$$

1.2.2 Magnetic Field as Bivector

The magnetic field is naturally a bivector:

$$\vec{B} = Be_1 \wedge e_2 = \frac{B}{2}(e_1e_2 - e_2e_1)$$

This is cleaner than the antisymmetric tensor F_{ij} or the pseudo-vector $\vec{B}$.

1.2.3 Minimal Coupling as Clifford Product

The covariant derivative becomes:

$$D = \nabla - \frac{ie}{\hbar}\vec{A}$$

where $\vec{A} = A^i e_i$ is the vector potential and $\nabla = e^i \partial_i$.

The Hamiltonian acts via the geometric product:

$$H\Psi = \frac{1}{2m}(\vec{p} - e\vec{A})^2\Psi$$

Expanding using $\vec{p} = -i\hbar\nabla$:

$$H = \frac{1}{2m}\left(-\hbar^2\nabla^2 + i\hbar e(\nabla\vec{A} + \vec{A}\nabla) + e^2\vec{A}^2\right)$$

1.3 The Unified Equation of Motion

The time evolution is given by:

$$i\hbar\partial_t\Psi = H\Psi$$

But in the Clifford bundle, this single equation encodes *all* the physics:

1.3.1 Grade-0 Part (Scalar)

Extracting the scalar part gives the Schrödinger equation:

$$i\hbar\partial_t\psi = \frac{1}{2m}\left(-\hbar^2\nabla^2\psi + e^2\vec{A}^2\psi\right)$$

plus coupling to the velocity field through $\vec{A} \cdot \vec{v}$ terms.

1.3.2 Grade-1 Part (Vector)

The vector part gives the current equation:

$$\partial_t\vec{v} = \frac{e}{m}\vec{v} \times \vec{B} - \frac{\hbar^2}{2m}\nabla^2\vec{v} + \cdots$$

which is the quantum version of the Lorentz force law.

1.3.3 Grade-2 Part (Bivector)

The bivector part describes spin precession:

$$\partial_t \vec{S} = \frac{e}{m} \vec{S} \times \vec{B}$$

which is exactly the spin precession equation $\frac{d\vec{S}}{dt} = \gamma \vec{S} \times \vec{B}$ with gyromagnetic ratio $\gamma = e/m$.

1.4 Energy Spectrum from Clifford Algebra

The Hamiltonian can be rewritten using Clifford algebra identities. Define:

$$\vec{\Pi} = \vec{p} - e\vec{A} = -i\hbar\nabla - e\vec{A}$$

Then:

$$H = \frac{1}{2m}\vec{\Pi}^2 = \frac{1}{2m}(\Pi_x^2 + \Pi_y^2 + \Pi_z^2)$$

In the symmetric gauge $\vec{A} = \frac{B}{2}(-y, x, 0)$, we have:

$$\begin{aligned}
\Pi_x &= -i\hbar\partial_x + \frac{eB}{2}y \\
\Pi_y &= -i\hbar\partial_y - \frac{eB}{2}x \\
\Pi_z &= -i\hbar\partial_z
\end{aligned}$$

Define the **Clifford ladder operators**:

$$a = \frac{1}{\sqrt{2\hbar eB}}(\Pi_x - i\Pi_y), \quad a^\dagger = \frac{1}{\sqrt{2\hbar eB}}(\Pi_x + i\Pi_y)$$

These satisfy $[a, a^\dagger] = 1$ (canonical commutation).

The Hamiltonian becomes:

$$H = \hbar\omega_c \left(a^\dagger a + \frac{1}{2}\right) + \frac{\Pi_z^2}{2m}$$

where $\omega_c = \frac{eB}{m}$ is the cyclotron frequency.

1.5 Landau Levels in Clifford Form

The eigenstates are organized by the Clifford algebra structure:

1.5.1 Lowest Landau Level (LLL)

For the LLL ($a\psi = 0$), the wavefunction in symmetric gauge is:

$$\psi_{0,m}(z) = z^m e^{-|z|^2/4\ell_B^2}$$

where $z = x + iy$, $\ell_B = \sqrt{\hbar/eB}$ is the magnetic length.

In Clifford form, the LLL multivector is:

$$\Psi_{\text{LLL}} = \psi_{0,m} + \frac{i\hbar}{2m}(\bar{z}\psi_{0,m})e_1 \wedge e_2 + \cdots$$

The bivector part represents the **quantum vorticity** of the state.

1.5.2 Higher Landau Levels

The n-th Landau level is obtained by acting with $a^\dagger$:

$$\psi_{n,m} = \frac{(a^\dagger)^n}{\sqrt{n!}}\psi_{0,m}$$

The energy is:

$$E_n = \hbar\omega_c\left(n + \frac{1}{2}\right) + \frac{p_z^2}{2m}$$

1.6 Spin Dynamics

Now include the electron spin. The spin operator in Clifford form is:

$$\vec{s} = \frac{\hbar}{2}\vec{\sigma} = \frac{\hbar}{2}(ie_2e_3, ie_3e_1, ie_1e_2)$$

where the bivectors e_ie_j $(i \neq j)$ represent spin planes.

The full Hamiltonian including spin is:

$$H = \frac{1}{2m}(\vec{p} - e\vec{A})^2 - \frac{e\hbar}{2m}\vec{\sigma} \cdot \vec{B}$$

In Clifford form, this is simply:

$$H = \frac{1}{2m}\vec{\Pi}^2 - \frac{e}{2m}\vec{B} \cdot \vec{s}$$

where $\vec{B} \cdot \vec{s}$ is the **Clifford inner product** between bivectors.

1.6.1 Spin-Orbit Coupling

In a central potential $V(r)$, we get spin-orbit coupling:

$$H_{\text{SO}} = \frac{\hbar}{4m^2c^2}\frac{1}{r}\frac{dV}{dr}\vec{L} \cdot \vec{\sigma}$$

In Clifford form, $\vec{L} \cdot \vec{\sigma}$ becomes the geometric product:

$$\vec{L} \cdot \vec{\sigma} = L^i\sigma_i = \frac{2i}{\hbar}(x \wedge p) \cdot (e_2e_3 + e_3e_1 + e_1e_2)$$

which is naturally a bivector-bivector product.

1.7 Geometric Interpretation

1.7.1 Magnetic Field as Area Element Generator

The magnetic field bivector $B = Be_1 \wedge e_2$ generates rotations in the xy-plane. The phase acquired around a loop of area A is:

$$\phi = \frac{e}{\hbar} \oint \vec{A} \cdot d\vec{x} = \frac{e}{\hbar} \int B \cdot d\vec{A} = \frac{eBA}{\hbar}$$

where $B \cdot d\vec{A}$ is the Clifford product of bivectors.

1.7.2 Berry Phase

For adiabatic motion, the Berry connection is:

$$\mathcal{A}_n = i\langle\psi_n|\nabla|\psi_n\rangle$$

and the Berry curvature is:

$$\mathcal{F} = d\mathcal{A}$$

For Landau levels, $\mathcal{F}$ is proportional to the magnetic field bivector:

$$\mathcal{F}_{ij} = \frac{eB}{\hbar}\epsilon_{ij} \quad (i, j = x, y)$$

1.8 Advantages of the Clifford Formulation

1. **Unification**: Scalar (wavefunction), vector (current), and bivector (spin) all live in the same algebra.
2. **Geometric Clarity**: Magnetic field as a bivector (plane) rather than pseudo-vector (axis).
3. **Rotationally Covariant**: Under rotation R, multivectors transform as $\Psi \mapsto R\Psi R^{-1}$.
4. **Minimal Coupling is Natural**: $p \to p - eA$ works for all grades simultaneously.
5. **Spin Included Automatically**: Spin operators are bivectors in the Clifford algebra.
6. **Extension to Relativity**: Replace $\mathcal{C}\ell(3)$ with $\mathcal{C}\ell(3,1)$ for Dirac equation.

1.9 Relativistic Extension: Dirac Equation in Magnetic Field

For the relativistic case, use $\mathcal{C}\ell(3,1)$ with generators γ^μ ($\mu = 0, 1, 2, 3$). The Dirac equation with electromagnetic field is:

$$(i\gamma^\mu D_\mu - m)\psi = 0$$

where $D_\mu = \partial_\mu + ieA_\mu$.

The solution for constant B gives the **relativistic Landau levels**:

$$E_n = \pm\sqrt{m^2 + p_z^2 + 2neB}$$

The Clifford algebra makes the spinor structure (ψ is an element of the even subalgebra) and the magnetic coupling (A_μ is a vector) manifestly unified.

1.10 Conclusion

The Clifford bundle formulation shows that:

- The electron's wavefunction, current, and spin are all components of a single multivector $\Psi \in \mathcal{C}\ell(\mathbb{R}^3)$.
- The magnetic field is naturally a bivector $B \in \bigwedge^2 \mathbb{R}^3$.
- The minimal coupling $\vec{p} \to \vec{p} - e\vec{A}$ is the geometric product with a vector potential.
- Landau levels emerge from the spectrum of the operator $a^\dagger a$ where $a, a^\dagger$ are Clifford ladder operators.
- Spin dynamics is automatically included through bivector operators.

This approach eliminates the artificial separation between "wave mechanics" and "spin physics" – both are simply different grades of the same Clifford-valued wavefunction. The magnetic field's action on all aspects of the electron (orbital motion, spin precession, Berry phase) is unified through the geometric product in $\mathcal{C}\ell(\mathbb{R}^3)$.