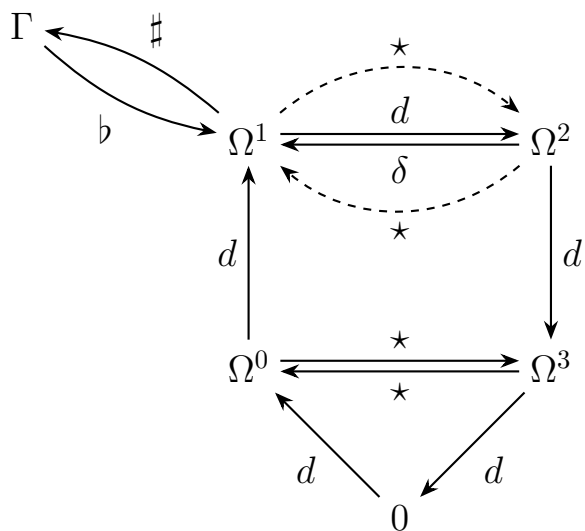
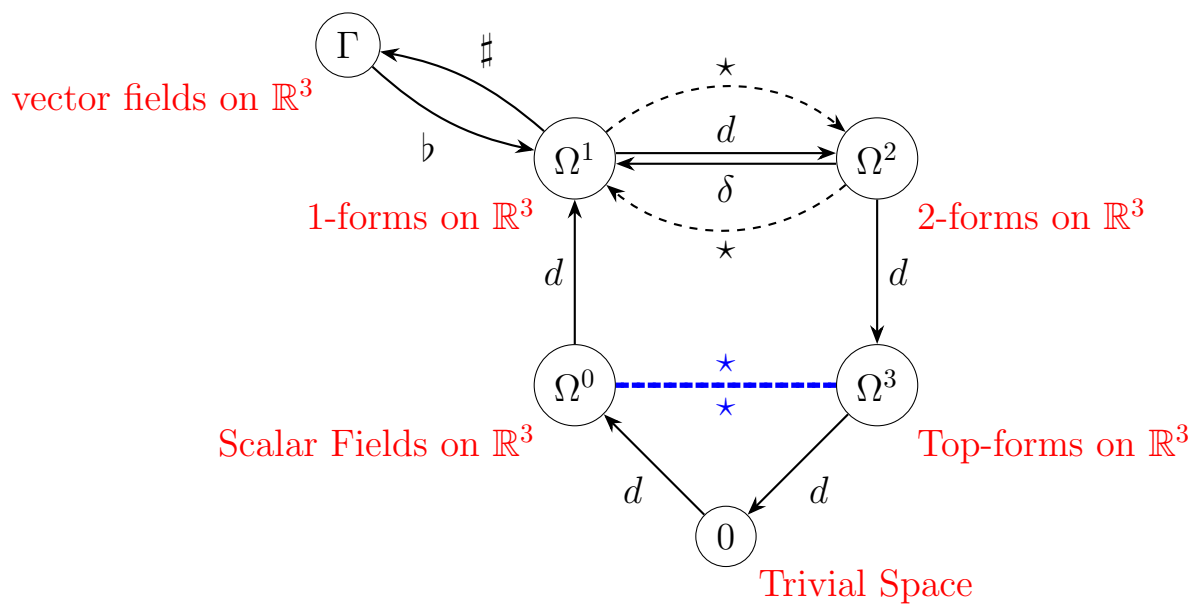


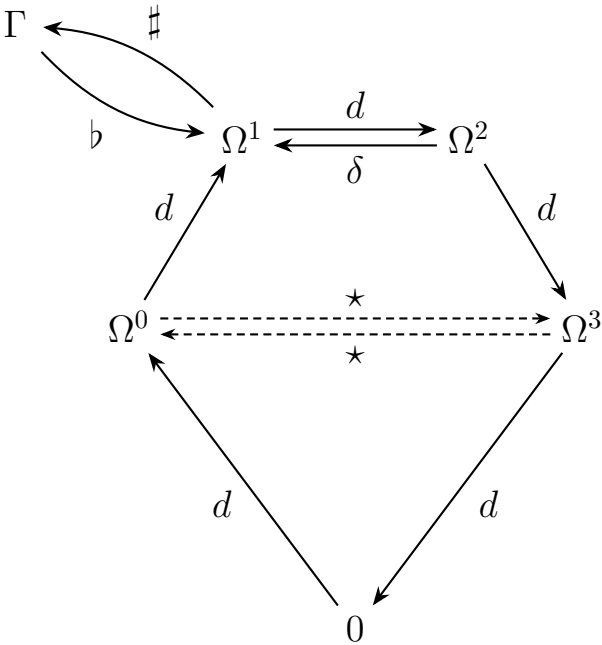
# Notes on the Hodge-deRham Complex

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January 29, 2026

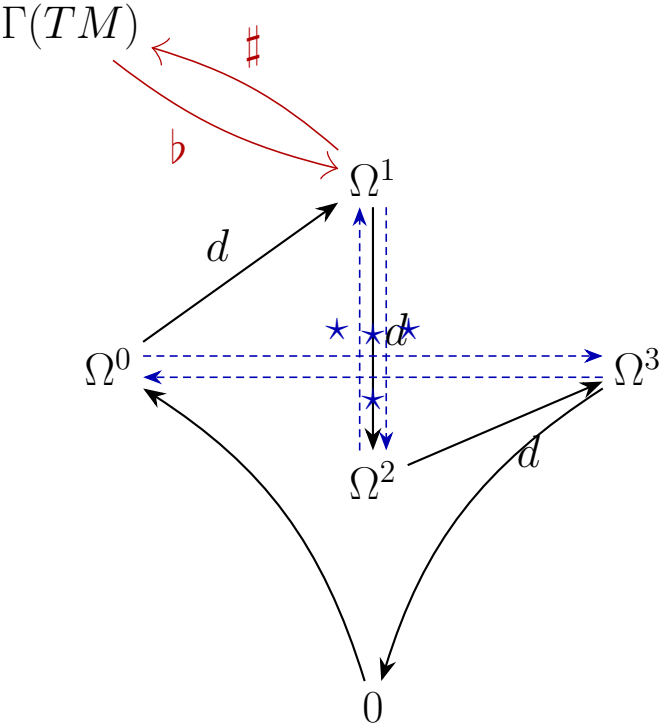


Alternative Layout (Diamond with 0 centered at bottom)



*de Rham complex with Hodge duality*

Final Version



# 1 Physical Significance of the $\omega^2$ -Centric Diagram

Placing  $\omega^2$  at the geometric center of the diagram changes our perspective to reveal deep physical significance.

## Why $\omega^2$ is Central

### 1. Field Strength Lives in $\omega^2$

In gauge theory, the hierarchy is:

$$\underbrace{\Omega^0}_{\text{gauge function}} \xrightarrow{d} \underbrace{\Omega^1}_{\text{potential } A} \xrightarrow{d} \underbrace{\Omega^2}_{\text{field strength } F} \xrightarrow{d} \underbrace{\Omega^3}_{\text{Bianchi } dF=0}$$

The **physics** (energy, equations of motion, observables) lives at  $\omega^2$ :

- Electromagnetic field:  $F = dA \in \Omega^2$
- Yang-Mills curvature:  $F = dA + A \wedge A \in \Omega^2$
- Riemann curvature:  $R^a_b \in \Omega^2(\mathfrak{so}(n))$

### 2. $\omega^2$ is the "Self-Dual" Level (in 4D)

In 4 dimensions, the Hodge star satisfies:

$$\star : \Omega^2 \rightarrow \Omega^2, \quad \star^2 = +1 \text{ (Euclidean) or } -1 \text{ (Lorentzian)}$$

This allows decomposition into **self-dual** and **anti-self-dual** parts:

$$\Omega^2 = \Omega^2_+ \oplus \Omega^2_-$$

This is central to:

- Instantons ( $F = \star F$ )
- Twistor theory
- Chiral structure of spinors
- Donaldson theory of 4-manifolds

### 3. Symplectic Structure

In Hamiltonian mechanics, everything revolves around  $\omega \in \Omega^2(T^*M)$ :

$$\omega = dp_i \wedge dq^i$$

The symplectic form is a **closed, non-degenerate 2-form**. Phase space geometry is fundamentally  $\omega^2$ -geometry.

#### 4. The "Flux" Interpretation

Form	Integrates over	Physical meaning
$\Omega^0$	point	field value
$\Omega^1$	curve	work, circulation
$\Omega^2$	<b>surface</b>	<b>flux</b>
$\Omega^3$	volume	total charge/mass

Flux through surfaces is the natural "middle" concept—it connects local (differential) to global (integral) physics via Stokes' theorem.

#### 5. In 3D: The Pseudovector Level

In 3 dimensions:

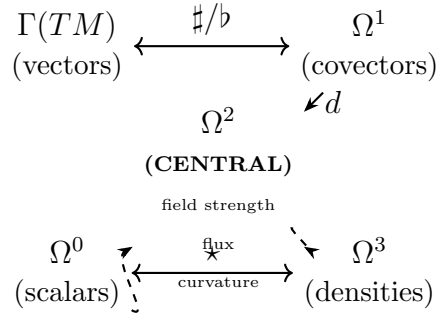
$$\star : \Omega^1 \leftrightarrow \Omega^2$$

$\omega^1$ (polar vectors)	$\omega^2$ (axial vectors)
Electric field $\mathbf{E}$	Magnetic field $\mathbf{B}$
Velocity $\mathbf{v}$	Vorticity $\boldsymbol{\omega}$
Force $\mathbf{F}$	Torque $\boldsymbol{\tau}$
Momentum $\mathbf{p}$	Angular momentum $\mathbf{L}$

The  $\omega^1 \leftrightarrow \omega^2$  duality is the **polar/axial** or **vector/pseudovector** distinction.

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#### The Diagram's Physical Meaning



#### Interpretation:

- $\Omega^0$  (left): Potentials, scalar fields, gauge functions
- $\Omega^3$  (right): Densities, sources, charges (integrated quantities)
- $\Omega^1$  (top): The "configuration" level—connections, velocities
- $\Omega^2$  (center): The "dynamics" level—field strengths, curvatures, fluxes
- $\Gamma(TM)$  (upper left): Vectors, related to  $\omega^1$  by the metric via musical isomorphisms

The **musical isomorphisms** ( $b$  and  $\sharp$ ) require a metric to convert between vectors and covectors. The **Hodge star** also requires a metric and orientation.

## 2 The de Rham Complex in Vector Calculus Disguise

The de Rham complex on  $\mathbb{R}^3$ , when translated through the metric isomorphisms, becomes:

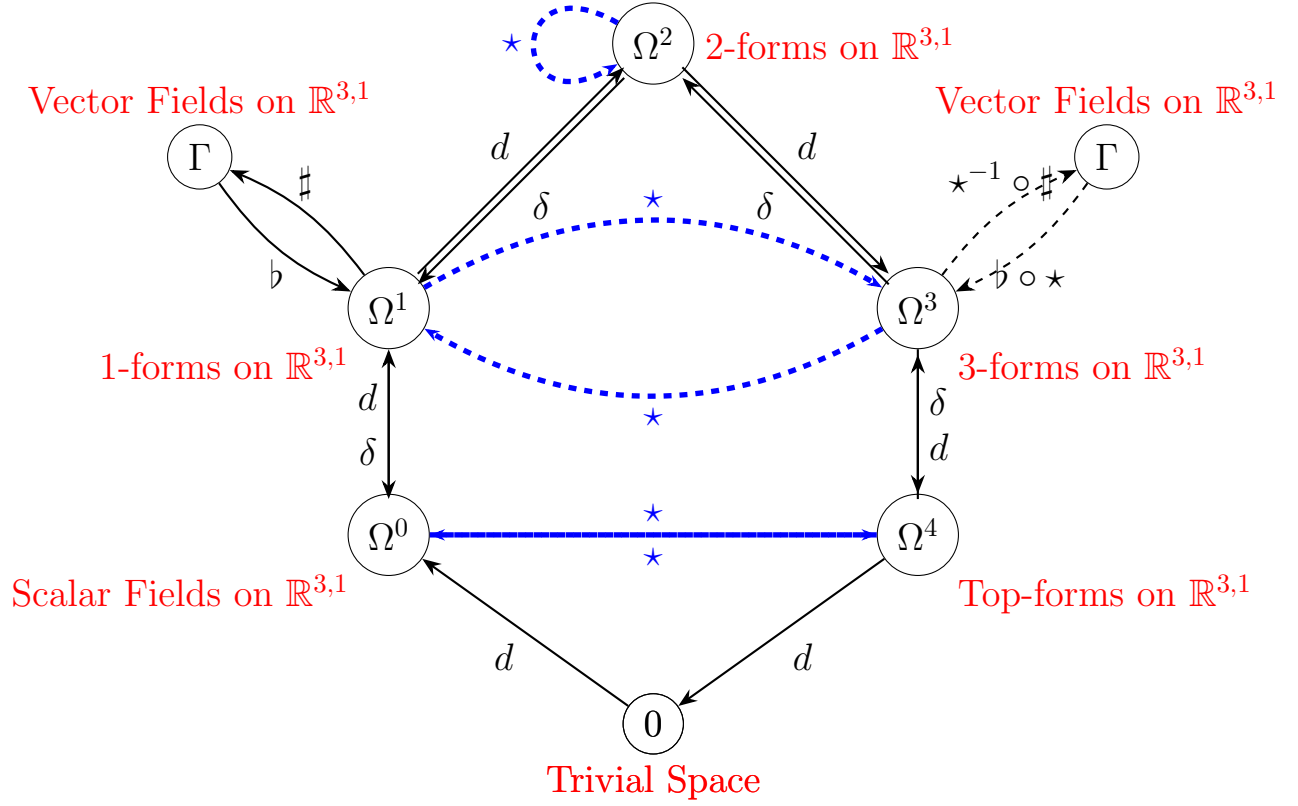
$$0 \longrightarrow C^\infty(\mathbb{R}^3) \xrightarrow{\text{grad}} \mathfrak{X}(\mathbb{R}^3) \xrightarrow{\text{curl}} \mathfrak{X}(\mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\mathbb{R}^3) \longrightarrow 0, \quad (1)$$

where  $\mathfrak{X}(\mathbb{R}^3)$  denotes vector fields. The identities  $\text{curl} \circ \text{grad} = 0$  and  $\text{div} \circ \text{curl} = 0$  are the statement that this is a cochain complex.

$$\begin{array}{ccccccc}
 \Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \Omega^2 & \xrightarrow{d} & \Omega^3 \\
 \uparrow = & & \uparrow \sharp & & \uparrow \star \sharp & & \uparrow \star \\
 \text{functions} & \xrightarrow{\nabla} & \text{"vectors"} & \xrightarrow{\nabla \times} & \text{"vectors"} & \xrightarrow{\nabla \cdot} & \text{functions}
 \end{array}$$

Figure 1: The de Rham complex (top) and its vector calculus disguise (bottom). The vertical arrows are the metric-dependent isomorphisms that obscure the unified structure.

### 3 Hodge-de Rham for Minkowski Space $\text{CL}(3,1)$



Hodge star in  $\mathbb{R}^{3,1}$ :  $\star^2 = (-1)^{k(4-k)+1}$  on  $\Omega^k$

Note: 2-forms decompose into self-dual and anti-self-dual parts