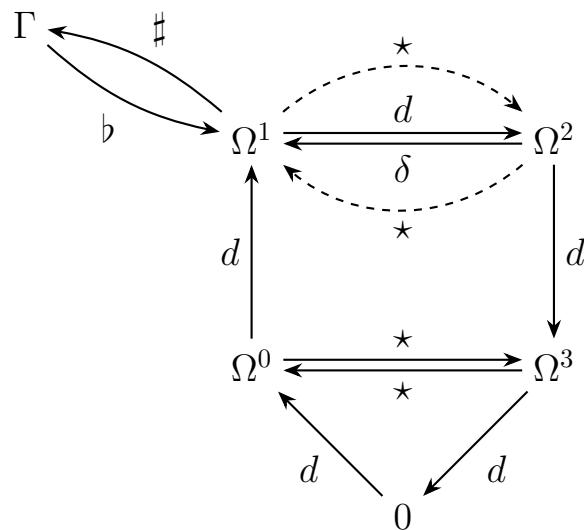
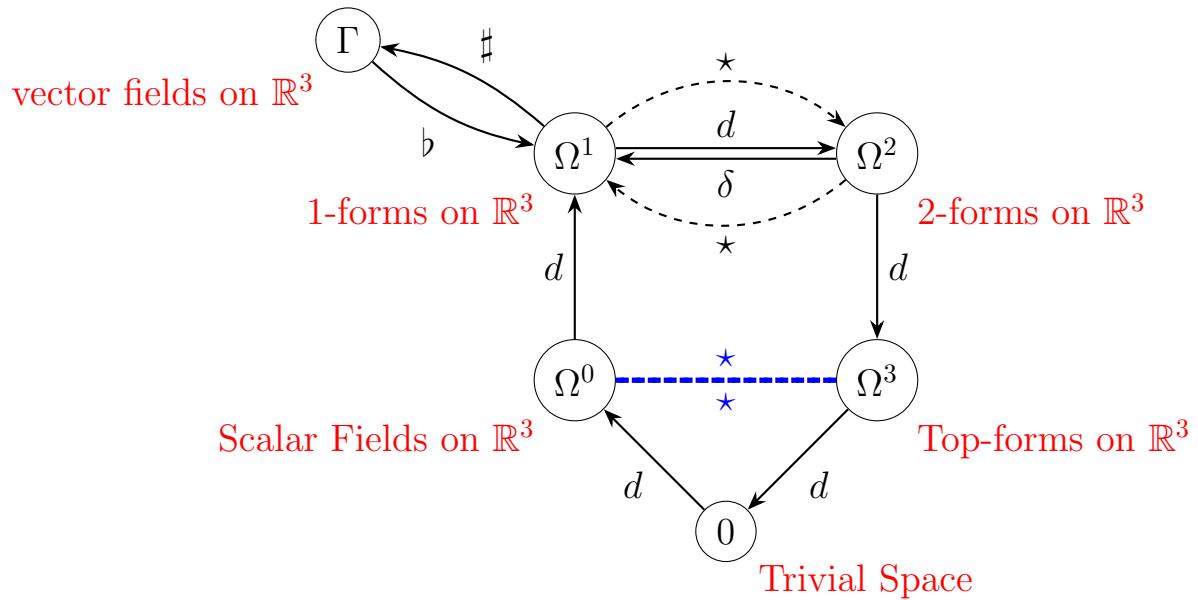


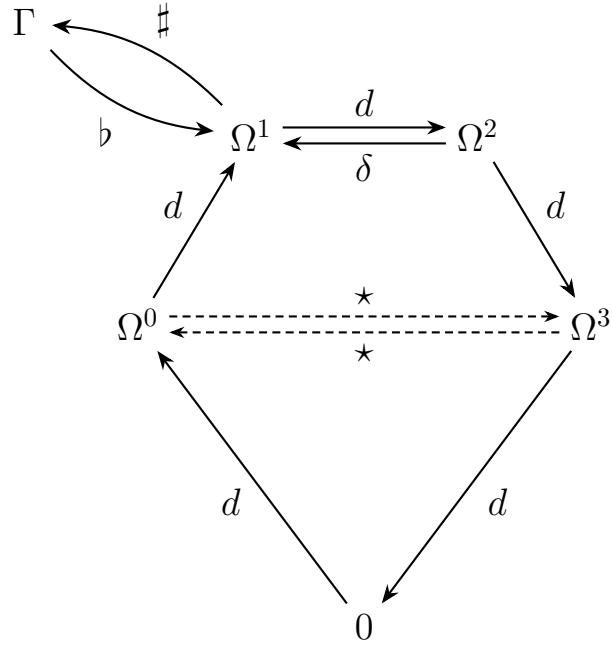
Notes on the Hodge-deRham Complex

John A. Janik

January 29, 2026

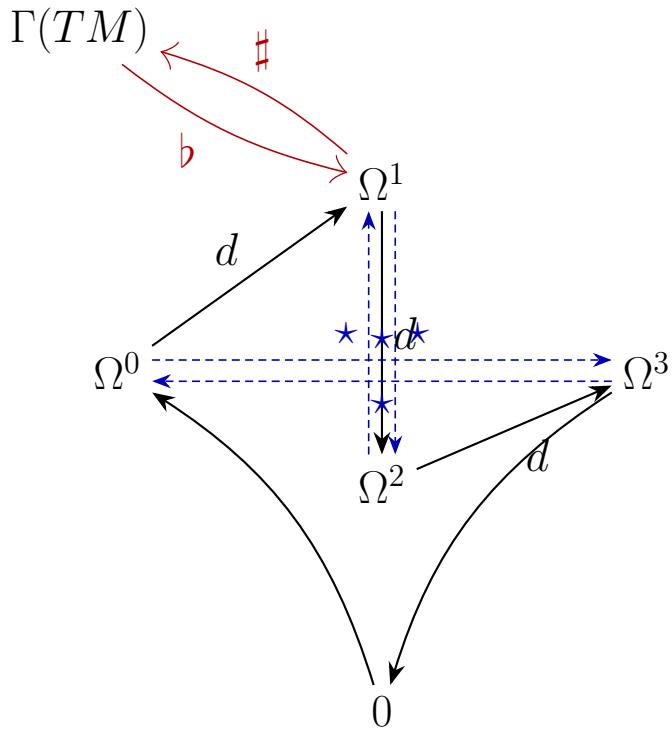


Alternative Layout (Diamond with 0 centered at bottom)



de Rham complex with Hodge duality

Final Version



1 Physical Significance of the ω^2 -Centric Diagram

Placing ω^2 at the geometric center of the diagram changes our perspective to reveal deep physical significance.

Why ω^2 is Central

1. Field Strength Lives in ω^2

In gauge theory, the hierarchy is:

$$\underbrace{\Omega^0}_{\text{gauge function}} \xrightarrow{d} \underbrace{\Omega^1}_{\text{potential } A} \xrightarrow{d} \underbrace{\Omega^2}_{\text{field strength } F} \xrightarrow{d} \underbrace{\Omega^3}_{\text{Bianchi } dF=0}$$

The **physics** (energy, equations of motion, observables) lives at ω^2 :

- Electromagnetic field: $F = dA \in \Omega^2$
- Yang-Mills curvature: $F = dA + A \wedge A \in \Omega^2$
- Riemann curvature: $R^a_b \in \Omega^2(\mathfrak{so}(n))$

2. ω^2 is the "Self-Dual" Level (in 4D)

In 4 dimensions, the Hodge star satisfies:

$$\star : \Omega^2 \rightarrow \Omega^2, \quad \star^2 = +1 \text{ (Euclidean) or } -1 \text{ (Lorentzian)}$$

This allows decomposition into **self-dual** and **anti-self-dual** parts:

$$\Omega^2 = \Omega_+^2 \oplus \Omega_-^2$$

This is central to:

- Instantons ($F = \star F$)
- Twistor theory
- Chiral structure of spinors
- Donaldson theory of 4-manifolds

3. Symplectic Structure

In Hamiltonian mechanics, everything revolves around $\omega \in \Omega^2(T^*M)$:

$$\omega = dp_i \wedge dq^i$$

The symplectic form is a **closed, non-degenerate 2-form**. Phase space geometry is fundamentally ω^2 -geometry.

4. The "Flux" Interpretation

Form	Integrates over	Physical meaning
Ω^0	point	field value
Ω^1	curve	work, circulation
Ω^2	surface	flux
Ω^3	volume	total charge/mass

Flux through surfaces is the natural "middle" concept—it connects local (differential) to global (integral) physics via Stokes' theorem.

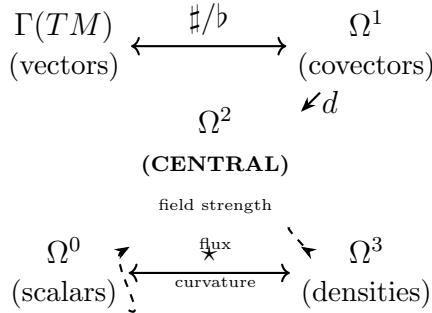
5. In 3D: The Pseudovector Level

In 3 dimensions:

$\star : \Omega^1 \leftrightarrow \Omega^2$	
ω^1 (polar vectors)	ω^2 (axial vectors)
Electric field \mathbf{E}	Magnetic field \mathbf{B}
Velocity \mathbf{v}	Vorticity $\boldsymbol{\omega}$
Force \mathbf{F}	Torque $\boldsymbol{\tau}$
Momentum \mathbf{p}	Angular momentum \mathbf{L}

The $\omega^1 \leftrightarrow \omega^2$ duality is the **polar/axial** or **vector/pseudovector** distinction.

The Diagram's Physical Meaning



Interpretation:

- Ω^0 (left): Potentials, scalar fields, gauge functions
- Ω^3 (right): Densities, sources, charges (integrated quantities)
- Ω^1 (top): The "configuration" level—connections, velocities
- Ω^2 (center): The "dynamics" level—field strengths, curvatures, fluxes
- $\Gamma(TM)$ (upper left): Vectors, related to ω^1 by the metric via musical isomorphisms

The **musical isomorphisms** (\flat and \sharp) require a metric to convert between vectors and covectors. The **Hodge star** also requires a metric and orientation.

2 The de Rham Complex in Vector Calculus Disguise

The de Rham complex on \mathbb{R}^3 , when translated through the metric isomorphisms, becomes:

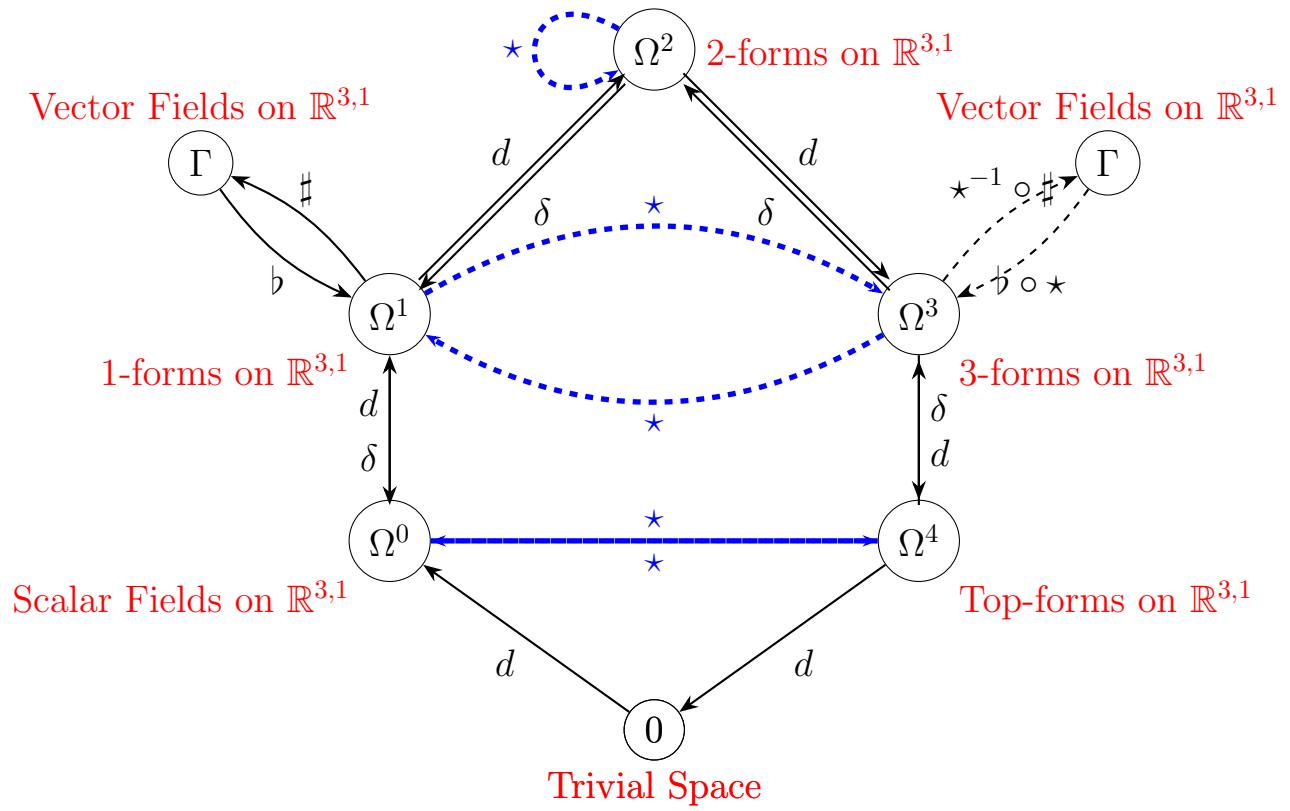
$$0 \longrightarrow C^\infty(\mathbb{R}^3) \xrightarrow{\text{grad}} \mathfrak{X}(\mathbb{R}^3) \xrightarrow{\text{curl}} \mathfrak{X}(\mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\mathbb{R}^3) \longrightarrow 0, \quad (1)$$

where $\mathfrak{X}(\mathbb{R}^3)$ denotes vector fields. The identities $\text{curl} \circ \text{grad} = 0$ and $\text{div} \circ \text{curl} = 0$ are the statement that this is a cochain complex.

$$\begin{array}{ccccccc} \Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \Omega^2 & \xrightarrow{d} & \Omega^3 \\ \uparrow = & & \uparrow \sharp & & \uparrow \star\sharp & & \uparrow \star \\ \text{functions} & \xrightarrow{\nabla} & \text{“vectors”} & \xrightarrow{\nabla \times} & \text{“vectors”} & \xrightarrow{\nabla \cdot} & \text{functions} \end{array}$$

Figure 1: The de Rham complex (top) and its vector calculus disguise (bottom). The vertical arrows are the metric-dependent isomorphisms that obscure the unified structure.

3 Hodge-de Rham for Minkowski Space $\text{CL}(3,1)$



Hodge star in $\mathbb{R}^{3,1}$: $\star^2 = (-1)^{k(4-k)+1}$ on Ω^k

Note: 2-forms decompose into self-dual and anti-self-dual parts