Sets

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The Sage Development Team

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CONTENTS

CHAPTER

ONE

SET CONSTRUCTIONS

1.1 Cartesian products

AUTHORS:

- Nicolas Thiery (2010-03): initial version

class sage.sets.cartesian_product.**CartesianProduct**(*sets*, *category*, *flatten=False*)

Bases: UniqueRepresentation, Parent

A class implementing a raw data structure for Cartesian products of sets (and elements thereof). See cartesian_product for how to construct full fledged Cartesian products.

EXAMPLES:

```
sage: G = cartesian_product([GF(5), Permutations(10)])
sage: G.cartesian_factors()
(Finite Field of size 5, Standard permutations of 10)
sage: G.cardinality()
18144000
sage: G.random_element()     # random
(1, [4, 7, 6, 5, 10, 1, 3, 2, 8, 9])
sage: G.category()
Join of Category of finite monoids
    and Category of Cartesian products of monoids
    and Category of Cartesian products of finite enumerated sets
```

```
>>> from sage.all import *
>>> G = cartesian_product([GF(Integer(5)), Permutations(Integer(10))])
>>> G.cartesian_factors()
(Finite Field of size 5, Standard permutations of 10)
>>> G.cardinality()
18144000
>>> G.random_element()     # random
(1, [4, 7, 6, 5, 10, 1, 3, 2, 8, 9])
>>> G.category()
Join of Category of finite monoids
    and Category of Cartesian products of monoids
    and Category of Cartesian products of finite enumerated sets
```

_cartesian_product_of_elements(*elements*)

Return the Cartesian product of the given elements.

This implements `Sets.CartesianProducts.ParentMethods._cartesian_product_of_elements()`.
INPUT:

- `elements` – an iterable (e.g. tuple, list) with one element of each Cartesian factor of `self`

Warning

This is meant as a fast low-level method. In particular, no coercion is attempted. When coercion or sanity checks are desirable, please use instead `self(elements)` or `self._element_constructor_(elements)`.

EXAMPLES:

```
sage: S1 = Sets().example()
sage: S2 = InfiniteEnumeratedSets().example()
sage: C = cartesian_product([S2, S1, S2])
sage: C._cartesian_product_of_elements([S2.an_element(), S1.an_element(), S2.
↪an_element()])
(42, 47, 42)
```

```
>>> from sage.all import *
>>> S1 = Sets().example()
>>> S2 = InfiniteEnumeratedSets().example()
>>> C = cartesian_product([S2, S1, S2])
>>> C._cartesian_product_of_elements([S2.an_element(), S1.an_element(), S2.an_
↪element()])
(42, 47, 42)
```

class Element

Bases: `ElementWrapperCheckWrappedClass`

cartesian_factors()

Return the tuple of elements that compose this element.

EXAMPLES:

```
sage: A = cartesian_product([ZZ, RR])
sage: A((1, 1.23)).cartesian_factors()                                    #␣
↪needs sage.rings.real_mpfr
(1, 1.23000000000000)
sage: type(_)
<... 'tuple'>
```

```
>>> from sage.all import *
>>> A = cartesian_product([ZZ, RR])
>>> A((Integer(1), RealNumber('1.23'))).cartesian_factors()                ␣
↪                    # needs sage.rings.real_mpfr
(1, 1.23000000000000)
>>> type(_)
<... 'tuple'>
```

cartesian_projection(i)

Return the projection of `self` on the i-th factor of the Cartesian product, as per `Sets.CartesianProducts.ElementMethods.cartesian_projection()`.

INPUT:

- `i` – the index of a factor of the Cartesian product

EXAMPLES:

```
sage: C = Sets().CartesianProducts().example(); C
The Cartesian product of (Set of prime numbers (basic implementation), An␣
↪example of an infinite enumerated set: the nonnegative integers, An␣
↪example of a finite enumerated set: {1,2,3})
sage: x = C.an_element(); x
(47, 42, 1)
sage: x.cartesian_projection(1)
42
```

```
>>> from sage.all import *
>>> C = Sets().CartesianProducts().example(); C
The Cartesian product of (Set of prime numbers (basic implementation), An␣
↪example of an infinite enumerated set: the nonnegative integers, An␣
↪example of a finite enumerated set: {1,2,3})
>>> x = C.an_element(); x
(47, 42, 1)
>>> x.cartesian_projection(Integer(1))
42
```

wrapped_class

alias of `tuple`

an_element()

EXAMPLES:

```
sage: C = Sets().CartesianProducts().example(); C
The Cartesian product of (Set of prime numbers (basic implementation),
 An example of an infinite enumerated set: the nonnegative integers,
 An example of a finite enumerated set: {1,2,3})
sage: C.an_element()
(47, 42, 1)
```

```
>>> from sage.all import *
>>> C = Sets().CartesianProducts().example(); C
The Cartesian product of (Set of prime numbers (basic implementation),
 An example of an infinite enumerated set: the nonnegative integers,
 An example of a finite enumerated set: {1,2,3})
>>> C.an_element()
(47, 42, 1)
```

cartesian_factors()

Return the Cartesian factors of `self`.

See also

`Sets.CartesianProducts.ParentMethods.cartesian_factors()`.

EXAMPLES:

```
sage: cartesian_product([QQ, ZZ, ZZ]).cartesian_factors()
(Rational Field, Integer Ring, Integer Ring)
```

```
>>> from sage.all import *
>>> cartesian_product([QQ, ZZ, ZZ]).cartesian_factors()
(Rational Field, Integer Ring, Integer Ring)
```

cartesian_projection(*i*)

Return the natural projection onto the i-th Cartesian factor of `self` as per `Sets.CartesianProducts.ParentMethods.cartesian_projection()`.

INPUT:

- `i` – the index of a Cartesian factor of `self`

EXAMPLES:

```
sage: C = Sets().CartesianProducts().example(); C
The Cartesian product of (Set of prime numbers (basic implementation), An␣
↪example of an infinite enumerated set: the nonnegative integers, An example␣
↪of a finite enumerated set: {1,2,3})
sage: x = C.an_element(); x
(47, 42, 1)
sage: pi = C.cartesian_projection(1)
sage: pi(x)
42

sage: C.cartesian_projection('hey')
Traceback (most recent call last):
...
ValueError: i (=hey) must be in {0, 1, 2}
```

```
>>> from sage.all import *
>>> C = Sets().CartesianProducts().example(); C
The Cartesian product of (Set of prime numbers (basic implementation), An␣
↪example of an infinite enumerated set: the nonnegative integers, An example␣
↪of a finite enumerated set: {1,2,3})
>>> x = C.an_element(); x
(47, 42, 1)
>>> pi = C.cartesian_projection(Integer(1))
>>> pi(x)
42

>>> C.cartesian_projection('hey')
Traceback (most recent call last):
...
ValueError: i (=hey) must be in {0, 1, 2}
```

construction()

Return the construction functor and its arguments for this Cartesian product.

OUTPUT:

A pair whose first entry is a Cartesian product functor and its second entry is a list of the Cartesian factors.

EXAMPLES:

```
sage: cartesian_product([ZZ, QQ]).construction()
(The cartesian_product functorial construction,
 (Integer Ring, Rational Field))
```

```
>>> from sage.all import *
>>> cartesian_product([ZZ, QQ]).construction()
(The cartesian_product functorial construction,
 (Integer Ring, Rational Field))
```

1.2 Families

A Family is an associative container which models a family $(f_i)_{i \in I}$. Then, `f[i]` returns the element of the family indexed by `i`. Whenever available, set and combinatorial class operations (counting, iteration, listing) on the family are induced from those of the index set. Families should be created through the *Family()* function.

AUTHORS:

- Nicolas Thiery (2008-02): initial release
- Florent Hivert (2008-04): various fixes, cleanups and improvements.

class sage.sets.family.**AbstractFamily**

Bases: Parent

The abstract class for family.

Any family belongs to a class which inherits from *AbstractFamily*.

hidden_keys()

Return the hidden keys of the family, if any.

EXAMPLES:

```
sage: f = Family({3: 'a', 4: 'b', 7: 'd'})
sage: f.hidden_keys()
[]
```

```
>>> from sage.all import *
>>> f = Family({Integer(3): 'a', Integer(4): 'b', Integer(7): 'd'})
>>> f.hidden_keys()
[]
```

inverse_family()

Return the inverse family, with keys and values exchanged. This presumes that there are no duplicate values in `self`.

This default implementation is not lazy and therefore will only work with not too big finite families. It is also cached for the same reason:

```
sage: Family({3: 'a', 4: 'b', 7: 'd'}).inverse_family()
Finite family {'a': 3, 'b': 4, 'd': 7}

sage: Family((3,4,7)).inverse_family()
Finite family {3: 0, 4: 1, 7: 2}
```

```
>>> from sage.all import *
>>> Family({Integer(3): 'a', Integer(4): 'b', Integer(7): 'd'}).inverse_
↪family()
Finite family {'a': 3, 'b': 4, 'd': 7}

>>> Family((Integer(3),Integer(4),Integer(7))).inverse_family()
Finite family {3: 0, 4: 1, 7: 2}
```

items()

Return an iterator for key-value pairs.

A key can only appear once, but if the function is not injective, values may appear multiple times.

EXAMPLES:

```
sage: f = Family([-2, -1, 0, 1, 2], abs)
sage: list(f.items())
[(-2, 2), (-1, 1), (0, 0), (1, 1), (2, 2)]
```

```
>>> from sage.all import *
>>> f = Family([-Integer(2), -Integer(1), Integer(0), Integer(1), Integer(2)],
↪ abs)
>>> list(f.items())
[(-2, 2), (-1, 1), (0, 0), (1, 1), (2, 2)]
```

keys()

Return the keys of the family.

EXAMPLES:

```
sage: f = Family({3: 'a', 4: 'b', 7: 'd'})
sage: sorted(f.keys())
[3, 4, 7]
```

```
>>> from sage.all import *
>>> f = Family({Integer(3): 'a', Integer(4): 'b', Integer(7): 'd'})
>>> sorted(f.keys())
[3, 4, 7]
```

map(f, *name=None*)

Return the family $(f(\texttt{self}[i]))_{i\in I}$, where I is the index set of `self`.

EXAMPLES:

```
sage: f = Family({3: 'a', 4: 'b', 7: 'd'})
sage: g = f.map(lambda x: x+'1')
sage: list(g)
['a1', 'b1', 'd1']
```

```
>>> from sage.all import *
>>> f = Family({Integer(3): 'a', Integer(4): 'b', Integer(7): 'd'})
>>> g = f.map(lambda x: x+'1')
>>> list(g)
['a1', 'b1', 'd1']
```

values()

Return the elements (values) of this family.

EXAMPLES:

```
sage: f = Family(["c", "a", "b"], lambda x: x + x)
sage: sorted(f.values())
['aa', 'bb', 'cc']
```

```
>>> from sage.all import *
>>> f = Family(["c", "a", "b"], lambda x: x + x)
>>> sorted(f.values())
['aa', 'bb', 'cc']
```

zip(*f*, *other*, *name=None*)

Given two families with same index set I (and same hidden keys if relevant), returns the family $(f(self[i], other[i]))_{i \in I}$

Todo

generalize to any number of families and merge with map?

EXAMPLES:

```
sage: f = Family({3: 'a', 4: 'b', 7: 'd'})
sage: g = Family({3: '1', 4: '2', 7: '3'})
sage: h = f.zip(lambda x,y: x+y, g)
sage: list(h)
['a1', 'b2', 'd3']
```

```
>>> from sage.all import *
>>> f = Family({Integer(3): 'a', Integer(4): 'b', Integer(7): 'd'})
>>> g = Family({Integer(3): '1', Integer(4): '2', Integer(7): '3'})
>>> h = f.zip(lambda x,y: x+y, g)
>>> list(h)
['a1', 'b2', 'd3']
```

class sage.sets.family.**EnumeratedFamily**(*enumset*)

Bases: *LazyFamily*

EnumeratedFamily turns an enumerated set `c` into a family indexed by the set $\{0, \ldots, |c| - 1\}$ (or NN if $|c|$ is countably infinite).

Instances should be created via the *Family()* factory. See its documentation for examples and tests.

cardinality()

Return the number of elements in `self`.

EXAMPLES:

```
sage: from sage.sets.family import EnumeratedFamily
sage: f = EnumeratedFamily(Permutations(3))
sage: f.cardinality()
6
```

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```
sage: f = Family(NonNegativeIntegers())
sage: f.cardinality()
+Infinity
```

```
>>> from sage.all import *
>>> from sage.sets.family import EnumeratedFamily
>>> f = EnumeratedFamily(Permutations(Integer(3)))
>>> f.cardinality()
6

>>> f = Family(NonNegativeIntegers())
>>> f.cardinality()
+Infinity
```

sage.sets.family.**Family**(*indices*, *function=None*, *hidden_keys=[]*, *hidden_function=None*, *lazy=False*, *name=None*)

A Family is an associative container which models a family $(f_i)_{i \in I}$. Then, `f[i]` returns the element of the family indexed by i. Whenever available, set and combinatorial class operations (counting, iteration, listing) on the family are induced from those of the index set.

There are several available implementations (classes) for different usages; Family serves as a factory, and will create instances of the appropriate classes depending on its arguments.

INPUT:

- `indices` – the indices for the family
- `function` – (optional) the function f applied to all visible indices; the default is the identity function
- `hidden_keys` – (optional) a list of hidden indices that can be accessed through `my_family[i]`
- `hidden_function` – (optional) a function for the hidden indices
- `lazy` – boolean (default: `False`); whether the family is lazily created or not; if the indices are infinite, then this is automatically made `True`
- `name` – (optional) the name of the function; only used when the family is lazily created via a function

EXAMPLES:

In its simplest form, a list $l = [l_0, l_1, \ldots, l_\ell]$ or a tuple by itself is considered as the family $(l_i)_{i \in I}$ where I is the set $\{0, \ldots, \ell\}$ where ℓ is `len(l) - 1`. So `Family(l)` returns the corresponding family:

```
sage: f = Family([1,2,3])
sage: f
Family (1, 2, 3)
sage: f = Family((1,2,3))
sage: f
Family (1, 2, 3)
```

```
>>> from sage.all import *
>>> f = Family([Integer(1),Integer(2),Integer(3)])
>>> f
Family (1, 2, 3)
>>> f = Family((Integer(1),Integer(2),Integer(3)))
```

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```
>>> f
Family (1, 2, 3)
```

Instead of a list you can as well pass any iterable object:

```
sage: f = Family(2*i+1 for i in [1,2,3])
sage: f
Family (3, 5, 7)
```

```
>>> from sage.all import *
>>> f = Family(Integer(2)*i+Integer(1) for i in [Integer(1),Integer(2),
↪Integer(3)])
>>> f
Family (3, 5, 7)
```

A family can also be constructed from a dictionary `t`. The resulting family is very close to `t`, except that the elements of the family are the values of `t`. Here, we define the family $(f_i)_{i\in\{3,4,7\}}$ with $f_3 = a$, $f_4 = b$, and $f_7 = d$:

```
sage: f = Family({3: 'a', 4: 'b', 7: 'd'})
sage: f
Finite family {3: 'a', 4: 'b', 7: 'd'}
sage: f[7]
'd'
sage: len(f)
3
sage: list(f)
['a', 'b', 'd']
sage: [ x for x in f ]
['a', 'b', 'd']
sage: f.keys()
[3, 4, 7]
sage: 'b' in f
True
sage: 'e' in f
False
```

```
>>> from sage.all import *
>>> f = Family({Integer(3): 'a', Integer(4): 'b', Integer(7): 'd'})
>>> f
Finite family {3: 'a', 4: 'b', 7: 'd'}
>>> f[Integer(7)]
'd'
>>> len(f)
3
>>> list(f)
['a', 'b', 'd']
>>> [ x for x in f ]
['a', 'b', 'd']
>>> f.keys()
[3, 4, 7]
>>> 'b' in f
```

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```
True
>>> 'e' in f
False
```

A family can also be constructed by its index set I and a function f, as in $(f(i))_{i \in I}$:

```
sage: f = Family([3,4,7], lambda i: 2*i)
sage: f
Finite family {3: 6, 4: 8, 7: 14}
sage: f.keys()
[3, 4, 7]
sage: f[7]
14
sage: list(f)
[6, 8, 14]
sage: [x for x in f]
[6, 8, 14]
sage: len(f)
3
```

```
>>> from sage.all import *
>>> f = Family([Integer(3),Integer(4),Integer(7)], lambda i: Integer(2)*i)
>>> f
Finite family {3: 6, 4: 8, 7: 14}
>>> f.keys()
[3, 4, 7]
>>> f[Integer(7)]
14
>>> list(f)
[6, 8, 14]
>>> [x for x in f]
[6, 8, 14]
>>> len(f)
3
```

By default, all images are computed right away, and stored in an internal dictionary:

```
sage: f = Family((3,4,7), lambda i: 2*i)
sage: f
Finite family {3: 6, 4: 8, 7: 14}
```

```
>>> from sage.all import *
>>> f = Family((Integer(3),Integer(4),Integer(7)), lambda i: Integer(2)*i)
>>> f
Finite family {3: 6, 4: 8, 7: 14}
```

Note that this requires all the elements of the list to be hashable. One can ask instead for the images $f(i)$ to be computed lazily, when needed:

```
sage: f = Family([3,4,7], lambda i: 2*i, lazy=True)
sage: f
Lazy family (<lambda>(i))_{i in [3, 4, 7]}
```

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```
sage: f[7]
14
sage: list(f)
[6, 8, 14]
sage: [x for x in f]
[6, 8, 14]
```

```
>>> from sage.all import *
>>> f = Family([Integer(3),Integer(4),Integer(7)], lambda i: Integer(2)*i,␣
↪lazy=True)
>>> f
Lazy family (<lambda>(i))_{i in [3, 4, 7]}
>>> f[Integer(7)]
14
>>> list(f)
[6, 8, 14]
>>> [x for x in f]
[6, 8, 14]
```

This allows in particular for modeling infinite families:

```
sage: f = Family(ZZ, lambda i: 2*i, lazy=True)
sage: f
Lazy family (<lambda>(i))_{i in Integer Ring}
sage: f.keys()
Integer Ring
sage: f[1]
2
sage: f[-5]
-10
sage: i = iter(f)
sage: next(i), next(i), next(i), next(i), next(i)
(0, 2, -2, 4, -4)
```

```
>>> from sage.all import *
>>> f = Family(ZZ, lambda i: Integer(2)*i, lazy=True)
>>> f
Lazy family (<lambda>(i))_{i in Integer Ring}
>>> f.keys()
Integer Ring
>>> f[Integer(1)]
2
>>> f[-Integer(5)]
-10
>>> i = iter(f)
>>> next(i), next(i), next(i), next(i), next(i)
(0, 2, -2, 4, -4)
```

Note that the `lazy` keyword parameter is only needed to force laziness. Usually it is automatically set to a correct default value (ie: `False` for finite data structures and `True` for enumerated sets:

```
sage: f == Family(ZZ, lambda i: 2*i)
True
```

```
>>> from sage.all import *
>>> f == Family(ZZ, lambda i: Integer(2)*i)
True
```

Beware that for those kind of families len(f) is not supposed to work. As a replacement, use the .cardinality() method:

```
sage: f = Family(Permutations(3), attrcall("to_lehmer_code"))
sage: list(f)
[[0, 0, 0], [0, 1, 0], [1, 0, 0], [1, 1, 0], [2, 0, 0], [2, 1, 0]]
sage: f.cardinality()
6
```

```
>>> from sage.all import *
>>> f = Family(Permutations(Integer(3)), attrcall("to_lehmer_code"))
>>> list(f)
[[0, 0, 0], [0, 1, 0], [1, 0, 0], [1, 1, 0], [2, 0, 0], [2, 1, 0]]
>>> f.cardinality()
6
```

Caveat: Only certain families with lazy behavior can be pickled. In particular, only functions that work with Sage's pickle_function and unpickle_function (in sage.misc.fpickle) will correctly unpickle. The following two work:

```
sage: f = Family(Permutations(3), lambda p: p.to_lehmer_code()); f
Lazy family (<lambda>(i))_{i in Standard permutations of 3}
sage: f == loads(dumps(f))
True

sage: f = Family(Permutations(3), attrcall("to_lehmer_code")); f
Lazy family (i.to_lehmer_code())_{i in Standard permutations of 3}
sage: f == loads(dumps(f))
True
```

```
>>> from sage.all import *
>>> f = Family(Permutations(Integer(3)), lambda p: p.to_lehmer_code()); f
Lazy family (<lambda>(i))_{i in Standard permutations of 3}
>>> f == loads(dumps(f))
True

>>> f = Family(Permutations(Integer(3)), attrcall("to_lehmer_code")); f
Lazy family (i.to_lehmer_code())_{i in Standard permutations of 3}
>>> f == loads(dumps(f))
True
```

But this one does not:

```
sage: def plus_n(n): return lambda x: x+n
sage: f = Family([1,2,3], plus_n(3), lazy=True); f
Lazy family (<lambda>(i))_{i in [1, 2, 3]}
```

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```
sage: f == loads(dumps(f))
Traceback (most recent call last):
...
ValueError: Cannot pickle code objects from closures
```

```
>>> from sage.all import *
>>> def plus_n(n): return lambda x: x+n
>>> f = Family([Integer(1),Integer(2),Integer(3)], plus_n(Integer(3)), lazy=True);
↪ f
Lazy family (<lambda>(i))_{i in [1, 2, 3]}
>>> f == loads(dumps(f))
Traceback (most recent call last):
...
ValueError: Cannot pickle code objects from closures
```

Finally, it can occasionally be useful to add some hidden elements in a family, which are accessible as `f[i]`, but do not appear in the keys or the container operations:

```
sage: f = Family([3,4,7], lambda i: 2*i, hidden_keys=[2])
sage: f
Finite family {3: 6, 4: 8, 7: 14}
sage: f.keys()
[3, 4, 7]
sage: f.hidden_keys()
[2]
sage: f[7]
14
sage: f[2]
4
sage: list(f)
[6, 8, 14]
sage: [x for x in f]
[6, 8, 14]
sage: len(f)
3
```

```
>>> from sage.all import *
>>> f = Family([Integer(3),Integer(4),Integer(7)], lambda i: Integer(2)*i, hidden_
↪keys=[Integer(2)])
>>> f
Finite family {3: 6, 4: 8, 7: 14}
>>> f.keys()
[3, 4, 7]
>>> f.hidden_keys()
[2]
>>> f[Integer(7)]
14
>>> f[Integer(2)]
4
>>> list(f)
[6, 8, 14]
>>> [x for x in f]
```

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```
[6, 8, 14]
>>> len(f)
3
```

The following example illustrates when the function is actually called:

```
sage: def compute_value(i):
....:     print('computing 2*'+str(i))
....:     return 2*i
sage: f = Family([3,4,7], compute_value, hidden_keys=[2])
computing 2*3
computing 2*4
computing 2*7
sage: f
Finite family {3: 6, 4: 8, 7: 14}
sage: f.keys()
[3, 4, 7]
sage: f.hidden_keys()
[2]
sage: f[7]
14
sage: f[2]
computing 2*2
4
sage: f[2]
4
sage: list(f)
[6, 8, 14]
sage: [x for x in f]
[6, 8, 14]
sage: len(f)
3
```

```
>>> from sage.all import *
>>> def compute_value(i):
...     print('computing 2*'+str(i))
...     return Integer(2)*i
>>> f = Family([Integer(3),Integer(4),Integer(7)], compute_value, hidden_
↪keys=[Integer(2)])
computing 2*3
computing 2*4
computing 2*7
>>> f
Finite family {3: 6, 4: 8, 7: 14}
>>> f.keys()
[3, 4, 7]
>>> f.hidden_keys()
[2]
>>> f[Integer(7)]
14
>>> f[Integer(2)]
computing 2*2
```

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```
4
>>> f[Integer(2)]
4
>>> list(f)
[6, 8, 14]
>>> [x for x in f]
[6, 8, 14]
>>> len(f)
3
```

Here is a close variant where the function for the hidden keys is different from that for the other keys:

```
sage: f = Family([3,4,7], lambda i: 2*i, hidden_keys=[2], hidden_function =␣
↪lambda i: 3*i)
sage: f
Finite family {3: 6, 4: 8, 7: 14}
sage: f.keys()
[3, 4, 7]
sage: f.hidden_keys()
[2]
sage: f[7]
14
sage: f[2]
6
sage: list(f)
[6, 8, 14]
sage: [x for x in f]
[6, 8, 14]
sage: len(f)
3
```

```
>>> from sage.all import *
>>> f = Family([Integer(3),Integer(4),Integer(7)], lambda i: Integer(2)*i, hidden_
↪keys=[Integer(2)], hidden_function = lambda i: Integer(3)*i)
>>> f
Finite family {3: 6, 4: 8, 7: 14}
>>> f.keys()
[3, 4, 7]
>>> f.hidden_keys()
[2]
>>> f[Integer(7)]
14
>>> f[Integer(2)]
6
>>> list(f)
[6, 8, 14]
>>> [x for x in f]
[6, 8, 14]
>>> len(f)
3
```

Family accept finite and infinite EnumeratedSets as input:

```
sage: f = Family(FiniteEnumeratedSet([1,2,3]))
sage: f
Family (1, 2, 3)
sage: f = Family(NonNegativeIntegers())
sage: f
Family (Non negative integers)
```

```
>>> from sage.all import *
>>> f = Family(FiniteEnumeratedSet([Integer(1),Integer(2),Integer(3)]))
>>> f
Family (1, 2, 3)
>>> f = Family(NonNegativeIntegers())
>>> f
Family (Non negative integers)
```

```
sage: f = Family(FiniteEnumeratedSet([3,4,7]), lambda i: 2*i)
sage: f
Finite family {3: 6, 4: 8, 7: 14}
sage: f.keys()
{3, 4, 7}
sage: f[7]
14
sage: list(f)
[6, 8, 14]
sage: [x for x in f]
[6, 8, 14]
sage: len(f)
3
```

```
>>> from sage.all import *
>>> f = Family(FiniteEnumeratedSet([Integer(3),Integer(4),Integer(7)]), lambda i:␣
↪Integer(2)*i)
>>> f
Finite family {3: 6, 4: 8, 7: 14}
>>> f.keys()
{3, 4, 7}
>>> f[Integer(7)]
14
>>> list(f)
[6, 8, 14]
>>> [x for x in f]
[6, 8, 14]
>>> len(f)
3
```

class sage.sets.family.**FiniteFamily**

Bases: *AbstractFamily*

A *FiniteFamily* is an associative container which models a finite family $(f_i)_{i \in I}$. Its elements f_i are therefore its values. Instances should be created via the *Family()* factory. See its documentation for examples and tests.

EXAMPLES:

We define the family $(f_i)_{i \in \{3,4,7\}}$ with $f_3 = a$, $f_4 = b$, and $f_7 = d$:

```
sage: from sage.sets.family import FiniteFamily
sage: f = FiniteFamily({3: 'a', 4: 'b', 7: 'd'})
```

```
>>> from sage.all import *
>>> from sage.sets.family import FiniteFamily
>>> f = FiniteFamily({Integer(3): 'a', Integer(4): 'b', Integer(7): 'd'})
```

Individual elements are accessible as in a usual dictionary:

```
sage: f[7]
'd'
```

```
>>> from sage.all import *
>>> f[Integer(7)]
'd'
```

And the other usual dictionary operations are also available:

```
sage: len(f)
3
sage: f.keys()
[3, 4, 7]
```

```
>>> from sage.all import *
>>> len(f)
3
>>> f.keys()
[3, 4, 7]
```

However f behaves as a container for the f_i's:

```
sage: list(f)
['a', 'b', 'd']
sage: [ x for x in f ]
['a', 'b', 'd']
```

```
>>> from sage.all import *
>>> list(f)
['a', 'b', 'd']
>>> [ x for x in f ]
['a', 'b', 'd']
```

The order of the elements can be specified using the `keys` optional argument:

```
sage: f = FiniteFamily({"a": "aa", "b": "bb", "c" : "cc" }, keys = ["c", "a", "b
↪"])
sage: list(f)
['cc', 'aa', 'bb']
```

```
>>> from sage.all import *
>>> f = FiniteFamily({"a": "aa", "b": "bb", "c" : "cc" }, keys = ["c", "a", "b"])
>>> list(f)
['cc', 'aa', 'bb']
```

cardinality()

Return the number of elements in `self`.

EXAMPLES:

```
sage: from sage.sets.family import FiniteFamily
sage: f = FiniteFamily({3: 'a', 4: 'b', 7: 'd'})
sage: f.cardinality()
3
```

```
>>> from sage.all import *
>>> from sage.sets.family import FiniteFamily
>>> f = FiniteFamily({Integer(3): 'a', Integer(4): 'b', Integer(7): 'd'})
>>> f.cardinality()
3
```

has_key(k)

Return whether `k` is a key of `self`.

EXAMPLES:

```
sage: Family({"a":1, "b":2, "c":3}).has_key("a")
True
sage: Family({"a":1, "b":2, "c":3}).has_key("d")
False
```

```
>>> from sage.all import *
>>> Family({"a":Integer(1), "b":Integer(2), "c":Integer(3)}).has_key("a")
True
>>> Family({"a":Integer(1), "b":Integer(2), "c":Integer(3)}).has_key("d")
False
```

keys()

Return the index set of this family.

EXAMPLES:

```
sage: f = Family(["c", "a", "b"], lambda x: x+x)
sage: f.keys()
['c', 'a', 'b']
```

```
>>> from sage.all import *
>>> f = Family(["c", "a", "b"], lambda x: x+x)
>>> f.keys()
['c', 'a', 'b']
```

values()

Return the elements of this family.

EXAMPLES:

```
sage: f = Family(["c", "a", "b"], lambda x: x+x)
sage: f.values()
['cc', 'aa', 'bb']
```

```
>>> from sage.all import *
>>> f = Family(["c", "a", "b"], lambda x: x+x)
>>> f.values()
['cc', 'aa', 'bb']
```

class sage.sets.family.**FiniteFamilyWithHiddenKeys**(*dictionary*, *hidden_keys*, *hidden_function*, *keys=None*)

Bases: *FiniteFamily*

A close variant of *FiniteFamily* where the family contains some hidden keys whose corresponding values are computed lazily (and remembered). Instances should be created via the *Family()* factory. See its documentation for examples and tests.

Caveat: Only instances of this class whose functions are compatible with sage.misc.fpickle can be pickled.

hidden_keys()

Return self's hidden keys.

EXAMPLES:

```
sage: f = Family([3,4,7], lambda i: 2*i, hidden_keys=[2])
sage: f.hidden_keys()
[2]
```

```
>>> from sage.all import *
>>> f = Family([Integer(3),Integer(4),Integer(7)], lambda i: Integer(2)*i,␣
↪hidden_keys=[Integer(2)])
>>> f.hidden_keys()
[2]
```

class sage.sets.family.**LazyFamily**(*set*, *function*, *name=None*)

Bases: *AbstractFamily*

A LazyFamily(I, f) is an associative container which models the (possibly infinite) family $(f(i))_{i \in I}$.

Instances should be created via the *Family()* factory. See its documentation for examples and tests.

cardinality()

Return the number of elements in self.

EXAMPLES:

```
sage: from sage.sets.family import LazyFamily
sage: f = LazyFamily([3,4,7], lambda i: 2*i)
sage: f.cardinality()
3
sage: l = LazyFamily(NonNegativeIntegers(), lambda i: 2*i)
sage: l.cardinality()
+Infinity
```

```
>>> from sage.all import *
>>> from sage.sets.family import LazyFamily
>>> f = LazyFamily([Integer(3),Integer(4),Integer(7)], lambda i: Integer(2)*i)
>>> f.cardinality()
3
```

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```
>>> l = LazyFamily(NonNegativeIntegers(), lambda i: Integer(2)*i)
>>> l.cardinality()
+Infinity
```

keys()

Return `self`'s keys.

EXAMPLES:

```
sage: from sage.sets.family import LazyFamily
sage: f = LazyFamily([3,4,7], lambda i: 2*i)
sage: f.keys()
[3, 4, 7]
```

```
>>> from sage.all import *
>>> from sage.sets.family import LazyFamily
>>> f = LazyFamily([Integer(3),Integer(4),Integer(7)], lambda i: Integer(2)*i)
>>> f.keys()
[3, 4, 7]
```

class sage.sets.family.**TrivialFamily**(*enumeration*)

Bases: *AbstractFamily*

TrivialFamily turns a list/tuple c into a family indexed by the set $\{0, \ldots, |c| - 1\}$.

Instances should be created via the *Family()* factory. See its documentation for examples and tests.

cardinality()

Return the number of elements in `self`.

EXAMPLES:

```
sage: from sage.sets.family import TrivialFamily
sage: f = TrivialFamily([3,4,7])
sage: f.cardinality()
3
```

```
>>> from sage.all import *
>>> from sage.sets.family import TrivialFamily
>>> f = TrivialFamily([Integer(3),Integer(4),Integer(7)])
>>> f.cardinality()
3
```

keys()

Return `self`'s keys.

EXAMPLES:

```
sage: from sage.sets.family import TrivialFamily
sage: f = TrivialFamily([3,4,7])
sage: f.keys()
[0, 1, 2]
```

```
>>> from sage.all import *
>>> from sage.sets.family import TrivialFamily
>>> f = TrivialFamily([Integer(3),Integer(4),Integer(7)])
>>> f.keys()
[0, 1, 2]
```

map(*f*, *name=None*)

Return the family $(f(\texttt{self}[i]))_{i\in I}$, where I is the index set of `self`.

The result is again a *TrivialFamily*.

EXAMPLES:

```
sage: from sage.sets.family import TrivialFamily
sage: f = TrivialFamily(['a', 'b', 'd'])
sage: g = f.map(lambda x: x + '1'); g
Family ('a1', 'b1', 'd1')
```

```
>>> from sage.all import *
>>> from sage.sets.family import TrivialFamily
>>> f = TrivialFamily(['a', 'b', 'd'])
>>> g = f.map(lambda x: x + '1'); g
Family ('a1', 'b1', 'd1')
```

1.3 Sets

AUTHORS:

- William Stein (2005) - first version
- William Stein (2006-02-16) - large number of documentation and examples; improved code
- Mike Hansen (2007-3-25) - added differences and symmetric differences; fixed operators
- Florent Hivert (2010-06-17) - Adapted to categories
- Nicolas M. Thiery (2011-03-15) - Added subset and superset methods
- Julian Rueth (2013-04-09) - Collected common code in *Set_object_binary*, fixed `__hash__`.

`sage.sets.set.`**Set**(*X=None*, *category=None*)

Create the underlying set of `X`.

If `X` is a list, tuple, Python set, or `X.is_finite()` is `True`, this returns a wrapper around Python's enumerated immutable `frozenset` type with extra functionality. Otherwise it returns a more formal wrapper.

If you need the functionality of mutable sets, use Python's builtin set type.

EXAMPLES:

```
sage: # needs sage.rings.finite_rings
sage: X = Set(GF(9, 'a'))
sage: X
{0, 1, 2, a, a + 1, a + 2, 2*a, 2*a + 1, 2*a + 2}
sage: type(X)
<class 'sage.sets.set.Set_object_enumerated_with_category'>
sage: Y = X.union(Set(QQ))
```

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```
sage: Y
Set-theoretic union of
 {0, 1, 2, a, a + 1, a + 2, 2*a, 2*a + 1, 2*a + 2} and
 Set of elements of Rational Field
sage: type(Y)
<class 'sage.sets.set.Set_object_union_with_category'>
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> X = Set(GF(Integer(9), 'a'))
>>> X
{0, 1, 2, a, a + 1, a + 2, 2*a, 2*a + 1, 2*a + 2}
>>> type(X)
<class 'sage.sets.set.Set_object_enumerated_with_category'>
>>> Y = X.union(Set(QQ))
>>> Y
Set-theoretic union of
 {0, 1, 2, a, a + 1, a + 2, 2*a, 2*a + 1, 2*a + 2} and
 Set of elements of Rational Field
>>> type(Y)
<class 'sage.sets.set.Set_object_union_with_category'>
```

Usually sets can be used as dictionary keys.

```
sage: # needs sage.symbolic
sage: d = {Set([2*I, 1 + I]): 10}
sage: d                   # key is randomly ordered
{{I + 1, 2*I}: 10}
sage: d[Set([1+I,2*I])]
10
sage: d[Set((1+I,2*I))]
10
```

```
>>> from sage.all import *
>>> # needs sage.symbolic
>>> d = {Set([Integer(2)*I, Integer(1) + I]): Integer(10)}
>>> d                   # key is randomly ordered
{{I + 1, 2*I}: 10}
>>> d[Set([Integer(1)+I,Integer(2)*I])]
10
>>> d[Set((Integer(1)+I,Integer(2)*I))]
10
```

The original object is often forgotten.

```
sage: v = [1,2,3]
sage: X = Set(v)
sage: X
{1, 2, 3}
sage: v.append(5)
sage: X
{1, 2, 3}
```

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```
sage: 5 in X
False
```

```
>>> from sage.all import *
>>> v = [Integer(1),Integer(2),Integer(3)]
>>> X = Set(v)
>>> X
{1, 2, 3}
>>> v.append(Integer(5))
>>> X
{1, 2, 3}
>>> Integer(5) in X
False
```

Set also accepts iterators, but be careful to only give *finite* sets:

```
sage: sorted(Set(range(1,6)))
[1, 2, 3, 4, 5]
sage: sorted(Set(list(range(1,6))))
[1, 2, 3, 4, 5]
sage: sorted(Set(iter(range(1,6))))
[1, 2, 3, 4, 5]
```

```
>>> from sage.all import *
>>> sorted(Set(range(Integer(1),Integer(6))))
[1, 2, 3, 4, 5]
>>> sorted(Set(list(range(Integer(1),Integer(6)))))
[1, 2, 3, 4, 5]
>>> sorted(Set(iter(range(Integer(1),Integer(6)))))
[1, 2, 3, 4, 5]
```

We can also create sets from different types:

```
sage: sorted(Set([Sequence([3,1], immutable=True), 5, QQ, Partition([3,1,1])]),␣
↪key=str)    # needs sage.combinat
[5, Rational Field, [3, 1, 1], [3, 1]]
```

```
>>> from sage.all import *
>>> sorted(Set([Sequence([Integer(3),Integer(1)], immutable=True), Integer(5), QQ,
↪ Partition([Integer(3),Integer(1),Integer(1)])]), key=str)    # needs sage.
↪combinat
[5, Rational Field, [3, 1, 1], [3, 1]]
```

Sets with unhashable objects work, but with less functionality:

```
sage: A = Set([QQ, (3, 1), 5])  # hashable
sage: sorted(A.list(), key=repr)
[(3, 1), 5, Rational Field]
sage: type(A)
<class 'sage.sets.set.Set_object_enumerated_with_category'>
sage: B = Set([QQ, [3, 1], 5])  # unhashable
sage: sorted(B.list(), key=repr)
```

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```
Traceback (most recent call last):
...
AttributeError: 'Set_object_with_category' object has no attribute 'list'...
sage: type(B)
<class 'sage.sets.set.Set_object_with_category'>
```

```
>>> from sage.all import *
>>> A = Set([QQ, (Integer(3), Integer(1)), Integer(5)])  # hashable
>>> sorted(A.list(), key=repr)
[(3, 1), 5, Rational Field]
>>> type(A)
<class 'sage.sets.set.Set_object_enumerated_with_category'>
>>> B = Set([QQ, [Integer(3), Integer(1)], Integer(5)])  # unhashable
>>> sorted(B.list(), key=repr)
Traceback (most recent call last):
...
AttributeError: 'Set_object_with_category' object has no attribute 'list'...
>>> type(B)
<class 'sage.sets.set.Set_object_with_category'>
```

class sage.sets.set.**Set_add_sub_operators**

Bases: object

Mix-in class providing the operators __add__ and __sub__.

The operators delegate to the methods union and intersection, which need to be implemented by the class.

class sage.sets.set.**Set_base**

Bases: object

Abstract base class for sets, not necessarily parents.

difference(*X*)

Return the set difference self - X.

EXAMPLES:

```
sage: X = Set(ZZ).difference(Primes())
sage: 4 in X
True
sage: 3 in X
False

sage: 4/1 in X
True

sage: X = Set(GF(9,'b')).difference(Set(GF(27,'c'))); X                    #␣
↪needs sage.rings.finite_rings
{0, 1, 2, b, b + 1, b + 2, 2*b, 2*b + 1, 2*b + 2}

sage: X = Set(GF(9,'b')).difference(Set(GF(27,'b'))); X                    #␣
↪needs sage.rings.finite_rings
{0, 1, 2, b, b + 1, b + 2, 2*b, 2*b + 1, 2*b + 2}
```

```
>>> from sage.all import *
>>> X = Set(ZZ).difference(Primes())
>>> Integer(4) in X
True
>>> Integer(3) in X
False

>>> Integer(4)/Integer(1) in X
True

>>> X = Set(GF(Integer(9),'b')).difference(Set(GF(Integer(27),'c'))); X
↪              # needs sage.rings.finite_rings
{0, 1, 2, b, b + 1, b + 2, 2*b, 2*b + 1, 2*b + 2}

>>> X = Set(GF(Integer(9),'b')).difference(Set(GF(Integer(27),'b'))); X
↪              # needs sage.rings.finite_rings
{0, 1, 2, b, b + 1, b + 2, 2*b, 2*b + 1, 2*b + 2}
```

intersection(*X*)

Return the intersection of `self` and `X`.

EXAMPLES:

```
sage: X = Set(ZZ).intersection(Primes())
sage: 4 in X
False
sage: 3 in X
True

sage: 2/1 in X
True

sage: X = Set(GF(9,'b')).intersection(Set(GF(27,'c'))); X                    #␣
↪needs sage.rings.finite_rings
{}

sage: X = Set(GF(9,'b')).intersection(Set(GF(27,'b'))); X                    #␣
↪needs sage.rings.finite_rings
{}
```

```
>>> from sage.all import *
>>> X = Set(ZZ).intersection(Primes())
>>> Integer(4) in X
False
>>> Integer(3) in X
True

>>> Integer(2)/Integer(1) in X
True

>>> X = Set(GF(Integer(9),'b')).intersection(Set(GF(Integer(27),'c'))); X
↪              # needs sage.rings.finite_rings
{}
```

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```
>>> X = Set(GF(Integer(9),'b')).intersection(Set(GF(Integer(27),'b'))); X
↪              # needs sage.rings.finite_rings
{}
```

symmetric_difference(*X*)

Return the symmetric difference of `self` and `X`.

EXAMPLES:

```
sage: X = Set([1,2,3]).symmetric_difference(Set([3,4]))
sage: X
{1, 2, 4}
```

```
>>> from sage.all import *
>>> X = Set([Integer(1),Integer(2),Integer(3)]).symmetric_
↪difference(Set([Integer(3),Integer(4)]))
>>> X
{1, 2, 4}
```

union(*X*)

Return the union of `self` and `X`.

EXAMPLES:

```
sage: Set(QQ).union(Set(ZZ))
Set-theoretic union of
 Set of elements of Rational Field and
 Set of elements of Integer Ring
sage: Set(QQ) + Set(ZZ)
Set-theoretic union of
 Set of elements of Rational Field and
 Set of elements of Integer Ring
sage: X = Set(QQ).union(Set(GF(3))); X
Set-theoretic union of
 Set of elements of Rational Field and
 {0, 1, 2}
sage: 2/3 in X
True
sage: GF(3)(2) in X                                                    #␣
↪needs sage.libs.pari
True
sage: GF(5)(2) in X
False
sage: sorted(Set(GF(7)) + Set(GF(3)), key=int)
[0, 0, 1, 1, 2, 2, 3, 4, 5, 6]
```

```
>>> from sage.all import *
>>> Set(QQ).union(Set(ZZ))
Set-theoretic union of
 Set of elements of Rational Field and
 Set of elements of Integer Ring
```

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```
>>> Set(QQ) + Set(ZZ)
Set-theoretic union of
 Set of elements of Rational Field and
 Set of elements of Integer Ring
>>> X = Set(QQ).union(Set(GF(Integer(3)))); X
Set-theoretic union of
 Set of elements of Rational Field and
 {0, 1, 2}
>>> Integer(2)/Integer(3) in X
True
>>> GF(Integer(3))(Integer(2)) in X
↪              # needs sage.libs.pari
True
>>> GF(Integer(5))(Integer(2)) in X
False
>>> sorted(Set(GF(Integer(7))) + Set(GF(Integer(3))), key=int)
[0, 0, 1, 1, 2, 2, 3, 4, 5, 6]
```

class sage.sets.set.**Set_boolean_operators**

Bases: object

Mix-in class providing the Boolean operators __or__, __and__, __xor__.

The operators delegate to the methods union, intersection, and symmetric_difference, which need to be implemented by the class.

class sage.sets.set.**Set_object**(*X*, *category=None*)

Bases: Set_generic, *Set_base*, *Set_boolean_operators*, *Set_add_sub_operators*

A set attached to an almost arbitrary object.

EXAMPLES:

```
sage: K = GF(19)
sage: Set(K)
{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18}
sage: S = Set(K)

sage: latex(S)
\left\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\right\}
sage: TestSuite(S).run()

sage: latex(Set(ZZ))
\Bold{Z}
```

```
>>> from sage.all import *
>>> K = GF(Integer(19))
>>> Set(K)
{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18}
>>> S = Set(K)

>>> latex(S)
\left\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\right\}
>>> TestSuite(S).run()
```

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```
>>> latex(Set(ZZ))
\Bold{Z}
```

cardinality()

Return the cardinality of this set, which is either an integer or `Infinity`.

EXAMPLES:

```
sage: Set(ZZ).cardinality()
+Infinity
sage: Primes().cardinality()
+Infinity
sage: Set(GF(5)).cardinality()
5
sage: Set(GF(5^2,'a')).cardinality()                                    #␣
↪needs sage.rings.finite_rings
25
```

```
>>> from sage.all import *
>>> Set(ZZ).cardinality()
+Infinity
>>> Primes().cardinality()
+Infinity
>>> Set(GF(Integer(5))).cardinality()
5
>>> Set(GF(Integer(5)**Integer(2),'a')).cardinality()                   ␣
↪               # needs sage.rings.finite_rings
25
```

is_empty()

Return boolean representing emptiness of the set.

OUTPUT: `True` if the set is empty, `False` otherwise

EXAMPLES:

```
sage: Set([]).is_empty()
True
sage: Set([0]).is_empty()
False
sage: Set([1..100]).is_empty()
False
sage: Set(SymmetricGroup(2).list()).is_empty()                          #␣
↪needs sage.groups
False
sage: Set(ZZ).is_empty()
False
```

```
>>> from sage.all import *
>>> Set([]).is_empty()
True
>>> Set([Integer(0)]).is_empty()
```

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```
False
>>> Set((ellipsis_range(Integer(1),Ellipsis,Integer(100)))).is_empty()
False
>>> Set(SymmetricGroup(Integer(2)).list()).is_empty()
↪     # needs sage.groups
False
>>> Set(ZZ).is_empty()
False
```

is_finite()

Return `True` if `self` is finite.

EXAMPLES:

```
sage: Set(QQ).is_finite()
False
sage: Set(GF(250037)).is_finite()
↪needs sage.rings.finite_rings
True
sage: Set(Integers(2^1000000)).is_finite()
True
sage: Set([1,'a',ZZ]).is_finite()
True
```

```
>>> from sage.all import *
>>> Set(QQ).is_finite()
False
>>> Set(GF(Integer(250037))).is_finite()
↪     # needs sage.rings.finite_rings
True
>>> Set(Integers(Integer(2)**Integer(1000000))).is_finite()
True
>>> Set([Integer(1),'a',ZZ]).is_finite()
True
```

object()

Return underlying object.

EXAMPLES:

```
sage: X = Set(QQ)
sage: X.object()
Rational Field
sage: X = Primes()
sage: X.object()
Set of all prime numbers: 2, 3, 5, 7, ...
```

```
>>> from sage.all import *
>>> X = Set(QQ)
>>> X.object()
Rational Field
>>> X = Primes()
```

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```
>>> X.object()
Set of all prime numbers: 2, 3, 5, 7, ...
```

subsets(*size=None*)

Return the `Subsets` object representing the subsets of a set. If size is specified, return the subsets of that size.

EXAMPLES:

```
sage: X = Set([1, 2, 3])
sage: list(X.subsets())
[{}, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}]
sage: list(X.subsets(2))
[{1, 2}, {1, 3}, {2, 3}]
```

```
>>> from sage.all import *
>>> X = Set([Integer(1), Integer(2), Integer(3)])
>>> list(X.subsets())
[{}, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}]
>>> list(X.subsets(Integer(2)))
[{1, 2}, {1, 3}, {2, 3}]
```

subsets_lattice()

Return the lattice of subsets ordered by containment.

EXAMPLES:

```
sage: X = Set([1,2,3])
sage: X.subsets_lattice()                                                   #␣
↪needs sage.graphs
Finite lattice containing 8 elements
sage: Y = Set()
sage: Y.subsets_lattice()                                                   #␣
↪needs sage.graphs
Finite lattice containing 1 elements
```

```
>>> from sage.all import *
>>> X = Set([Integer(1),Integer(2),Integer(3)])
>>> X.subsets_lattice()                                                     #␣
↪needs sage.graphs
Finite lattice containing 8 elements
>>> Y = Set()
>>> Y.subsets_lattice()                                                     #␣
↪needs sage.graphs
Finite lattice containing 1 elements
```

class sage.sets.set.**Set_object_binary**(*X*, *Y*, *op*, *latex_op*, *category=None*)

Bases: *Set_object*

An abstract common base class for sets defined by a binary operation (ex. *Set_object_union*, *Set_object_intersection*, *Set_object_difference*, and *Set_object_symmetric_difference*).

INPUT:

- `X`, `Y` – sets, the operands to `op`

- `op` – string describing the binary operation
- `latex_op` – string used for rendering this object in LaTeX

EXAMPLES:

```
sage: X = Set(QQ^2)                                                          #␣
↪needs sage.modules
sage: Y = Set(ZZ)
sage: from sage.sets.set import Set_object_binary
sage: S = Set_object_binary(X, Y, "union", "\\cup"); S                       #␣
↪needs sage.modules
Set-theoretic union of
 Set of elements of Vector space of dimension 2 over Rational Field and
 Set of elements of Integer Ring
```

```
>>> from sage.all import *
>>> X = Set(QQ**Integer(2))                                                   ␣
↪      # needs sage.modules
>>> Y = Set(ZZ)
>>> from sage.sets.set import Set_object_binary
>>> S = Set_object_binary(X, Y, "union", "\\cup"); S                        #␣
↪needs sage.modules
Set-theoretic union of
 Set of elements of Vector space of dimension 2 over Rational Field and
 Set of elements of Integer Ring
```

class sage.sets.set.**Set_object_difference**(*X*, *Y*, *category=None*)

Bases: *Set_object_binary*

Formal difference of two sets.

is_finite()

Return whether this set is finite.

EXAMPLES:

```
sage: X = Set(range(10))
sage: Y = Set(range(-10,5))
sage: Z = Set(QQ)
sage: X.difference(Y).is_finite()
True
sage: X.difference(Z).is_finite()
True
sage: Z.difference(X).is_finite()
False
sage: Z.difference(Set(ZZ)).is_finite()
Traceback (most recent call last):
...
NotImplementedError
```

```
>>> from sage.all import *
>>> X = Set(range(Integer(10)))
>>> Y = Set(range(-Integer(10),Integer(5)))
>>> Z = Set(QQ)
```

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```
>>> X.difference(Y).is_finite()
True
>>> X.difference(Z).is_finite()
True
>>> Z.difference(X).is_finite()
False
>>> Z.difference(Set(ZZ)).is_finite()
Traceback (most recent call last):
...
NotImplementedError
```

class sage.sets.set.**Set_object_enumerated**(*X*, *category=None*)

Bases: *Set_object*

A finite enumerated set.

cardinality()

Return the cardinality of `self`.

EXAMPLES:

```
sage: Set([1,1]).cardinality()
1
```

```
>>> from sage.all import *
>>> Set([Integer(1),Integer(1)]).cardinality()
1
```

difference(*other*)

Return the set difference `self - other`.

EXAMPLES:

```
sage: X = Set([1,2,3,4])
sage: Y = Set([1,2])
sage: X.difference(Y)
{3, 4}
sage: Z = Set(ZZ)
sage: W = Set([2.5, 4, 5, 6])
sage: W.difference(Z)                                                    #␣
↪needs sage.rings.real_mpfr
{2.50000000000000}
```

```
>>> from sage.all import *
>>> X = Set([Integer(1),Integer(2),Integer(3),Integer(4)])
>>> Y = Set([Integer(1),Integer(2)])
>>> X.difference(Y)
{3, 4}
>>> Z = Set(ZZ)
>>> W = Set([RealNumber('2.5'), Integer(4), Integer(5), Integer(6)])
>>> W.difference(Z)                                                      #␣
↪needs sage.rings.real_mpfr
{2.50000000000000}
```

frozenset()

Return the Python frozenset object associated to this set, which is an immutable set (hence hashable).

EXAMPLES:

```
sage: # needs sage.rings.finite_rings
sage: X = Set(GF(8,'c'))
sage: X
{0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1}
sage: s = X.set(); s
{0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1}
sage: hash(s)
Traceback (most recent call last):
...
TypeError: unhashable type: 'set'
sage: s = X.frozenset(); s
frozenset({0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1})

sage: hash(s) != hash(tuple(X.set()))                                       #␣
↪needs sage.rings.finite_rings
True

sage: type(s)                                                               #␣
↪needs sage.rings.finite_rings
<... 'frozenset'>
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> X = Set(GF(Integer(8),'c'))
>>> X
{0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1}
>>> s = X.set(); s
{0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1}
>>> hash(s)
Traceback (most recent call last):
...
TypeError: unhashable type: 'set'
>>> s = X.frozenset(); s
frozenset({0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1})

>>> hash(s) != hash(tuple(X.set()))                                         #␣
↪needs sage.rings.finite_rings
True

>>> type(s)                                                                 #␣
↪needs sage.rings.finite_rings
<... 'frozenset'>
```

intersection(*other*)

Return the intersection of `self` and `other`.

EXAMPLES:

```
sage: X = Set(GF(8,'c'))                                                    #␣
↪needs sage.rings.finite_rings
sage: Y = Set([GF(8,'c').0, 1, 2, 3])                                       #␣
↪needs sage.rings.finite_rings
sage: sorted(X.intersection(Y), key=str)                                    #␣
↪needs sage.rings.finite_rings
[1, c]
```

```
>>> from sage.all import *
>>> X = Set(GF(Integer(8),'c'))                                              ␣
↪     # needs sage.rings.finite_rings
>>> Y = Set([GF(Integer(8),'c').gen(0), Integer(1), Integer(2), Integer(3)]) ␣
↪                                    # needs sage.rings.finite_rings
>>> sorted(X.intersection(Y), key=str)                                      #␣
↪needs sage.rings.finite_rings
[1, c]
```

is_finite()

Return `True` as this is a finite set.

EXAMPLES:

```
sage: Set(GF(19)).is_finite()
True
```

```
>>> from sage.all import *
>>> Set(GF(Integer(19))).is_finite()
True
```

issubset(*other*)

Return whether `self` is a subset of `other`.

INPUT:

- `other` – a finite Set

EXAMPLES:

```
sage: X = Set([1,3,5])
sage: Y = Set([0,1,2,3,5,7])
sage: X.issubset(Y)
True
sage: Y.issubset(X)
False
sage: X.issubset(X)
True
```

```
>>> from sage.all import *
>>> X = Set([Integer(1),Integer(3),Integer(5)])
>>> Y = Set([Integer(0),Integer(1),Integer(2),Integer(3),Integer(5),
↪Integer(7)])
>>> X.issubset(Y)
True
>>> Y.issubset(X)
```

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```
False
>>> X.issubset(X)
True
```

issuperset(*other*)

Return whether `self` is a superset of `other`.

INPUT:

- `other` – a finite Set

EXAMPLES:

```
sage: X = Set([1,3,5])
sage: Y = Set([0,1,2,3,5])
sage: X.issuperset(Y)
False
sage: Y.issuperset(X)
True
sage: X.issuperset(X)
True
```

```
>>> from sage.all import *
>>> X = Set([Integer(1),Integer(3),Integer(5)])
>>> Y = Set([Integer(0),Integer(1),Integer(2),Integer(3),Integer(5)])
>>> X.issuperset(Y)
False
>>> Y.issuperset(X)
True
>>> X.issuperset(X)
True
```

list()

Return the elements of `self`, as a list.

EXAMPLES:

```
sage: # needs sage.rings.finite_rings
sage: X = Set(GF(8,'c'))
sage: X
{0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1}
sage: X.list()
[0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1]
sage: type(X.list())
<... 'list'>
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> X = Set(GF(Integer(8),'c'))
>>> X
{0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1}
>>> X.list()
[0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1]
```

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```
>>> type(X.list())
<... 'list'>
```

Todo

FIXME: What should be the order of the result? That of `self.object()`? Or the order given by `set(self.object())`? Note that `__getitem__()` is currently implemented in term of this list method, which is really inefficient …

random_element()

Return a random element in this set.

EXAMPLES:

```
sage: Set([1,2,3]).random_element() # random
2
```

```
>>> from sage.all import *
>>> Set([Integer(1),Integer(2),Integer(3)]).random_element() # random
2
```

set()

Return the Python set object associated to this set.

Python has a notion of finite set, and often Sage sets have an associated Python set. This function returns that set.

EXAMPLES:

```
sage: # needs sage.rings.finite_rings
sage: X = Set(GF(8,'c'))
sage: X
{0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1}
sage: X.set()
{0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1}
sage: type(X.set())
<... 'set'>
sage: type(X)
<class 'sage.sets.set.Set_object_enumerated_with_category'>
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> X = Set(GF(Integer(8),'c'))
>>> X
{0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1}
>>> X.set()
{0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1}
>>> type(X.set())
<... 'set'>
>>> type(X)
<class 'sage.sets.set.Set_object_enumerated_with_category'>
```

symmetric_difference(*other*)

Return the symmetric difference of `self` and `other`.

EXAMPLES:

```
sage: X = Set([1,2,3,4])
sage: Y = Set([1,2])
sage: X.symmetric_difference(Y)
{3, 4}
sage: Z = Set(ZZ)
sage: W = Set([2.5, 4, 5, 6])
sage: U = W.symmetric_difference(Z)
sage: 2.5 in U
True
sage: 4 in U
False
sage: V = Z.symmetric_difference(W)
sage: V == U
True
sage: 2.5 in V
True
sage: 6 in V
False
```

```
>>> from sage.all import *
>>> X = Set([Integer(1),Integer(2),Integer(3),Integer(4)])
>>> Y = Set([Integer(1),Integer(2)])
>>> X.symmetric_difference(Y)
{3, 4}
>>> Z = Set(ZZ)
>>> W = Set([RealNumber('2.5'), Integer(4), Integer(5), Integer(6)])
>>> U = W.symmetric_difference(Z)
>>> RealNumber('2.5') in U
True
>>> Integer(4) in U
False
>>> V = Z.symmetric_difference(W)
>>> V == U
True
>>> RealNumber('2.5') in V
True
>>> Integer(6) in V
False
```

union(*other*)

Return the union of `self` and `other`.

EXAMPLES:

```
sage: # needs sage.rings.finite_rings
sage: X = Set(GF(8,'c'))
sage: Y = Set([GF(8,'c').0, 1, 2, 3])
sage: X
{0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1}
```

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```
sage: sorted(Y)
[1, 2, 3, c]
sage: sorted(X.union(Y), key=str)
[0, 1, 2, 3, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1]
```

```
>>> from sage.all import *
>>> # needs sage.rings.finite_rings
>>> X = Set(GF(Integer(8),'c'))
>>> Y = Set([GF(Integer(8),'c').gen(0), Integer(1), Integer(2), Integer(3)])
>>> X
{0, 1, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1}
>>> sorted(Y)
[1, 2, 3, c]
>>> sorted(X.union(Y), key=str)
[0, 1, 2, 3, c, c + 1, c^2, c^2 + 1, c^2 + c, c^2 + c + 1]
```

class sage.sets.set.**Set_object_intersection**(*X*, *Y*, *category=None*)

Bases: *Set_object_binary*

Formal intersection of two sets.

is_finite()

Return whether this set is finite.

EXAMPLES:

```
sage: X = Set(IntegerRange(100))
sage: Y = Set(ZZ)
sage: X.intersection(Y).is_finite()
True
sage: Y.intersection(X).is_finite()
True
sage: Y.intersection(Set(QQ)).is_finite()
Traceback (most recent call last):
...
NotImplementedError
```

```
>>> from sage.all import *
>>> X = Set(IntegerRange(Integer(100)))
>>> Y = Set(ZZ)
>>> X.intersection(Y).is_finite()
True
>>> Y.intersection(X).is_finite()
True
>>> Y.intersection(Set(QQ)).is_finite()
Traceback (most recent call last):
...
NotImplementedError
```

class sage.sets.set.**Set_object_symmetric_difference**(*X*, *Y*, *category=None*)

Bases: *Set_object_binary*

Formal symmetric difference of two sets.

is_finite()

Return whether this set is finite.

EXAMPLES:

```
sage: X = Set(range(10))
sage: Y = Set(range(-10,5))
sage: Z = Set(QQ)
sage: X.symmetric_difference(Y).is_finite()
True
sage: X.symmetric_difference(Z).is_finite()
False
sage: Z.symmetric_difference(X).is_finite()
False
sage: Z.symmetric_difference(Set(ZZ)).is_finite()
Traceback (most recent call last):
...
NotImplementedError
```

```
>>> from sage.all import *
>>> X = Set(range(Integer(10)))
>>> Y = Set(range(-Integer(10),Integer(5)))
>>> Z = Set(QQ)
>>> X.symmetric_difference(Y).is_finite()
True
>>> X.symmetric_difference(Z).is_finite()
False
>>> Z.symmetric_difference(X).is_finite()
False
>>> Z.symmetric_difference(Set(ZZ)).is_finite()
Traceback (most recent call last):
...
NotImplementedError
```

class sage.sets.set.**Set_object_union**(*X*, *Y*, *category=None*)

Bases: *Set_object_binary*

A formal union of two sets.

cardinality()

Return the cardinality of this set.

EXAMPLES:

```
sage: X = Set(GF(3)).union(Set(GF(2)))
sage: X
{0, 1, 2, 0, 1}
sage: X.cardinality()
5

sage: X = Set(GF(3)).union(Set(ZZ))
sage: X.cardinality()
+Infinity
```

```
>>> from sage.all import *
>>> X = Set(GF(Integer(3))).union(Set(GF(Integer(2))))
>>> X
{0, 1, 2, 0, 1}
>>> X.cardinality()
5

>>> X = Set(GF(Integer(3))).union(Set(ZZ))
>>> X.cardinality()
+Infinity
```

is_finite()

Return whether this set is finite.

EXAMPLES:

```
sage: X = Set(range(10))
sage: Y = Set(range(-10,0))
sage: Z = Set(Primes())
sage: X.union(Y).is_finite()
True
sage: X.union(Z).is_finite()
False
```

```
>>> from sage.all import *
>>> X = Set(range(Integer(10)))
>>> Y = Set(range(-Integer(10),Integer(0)))
>>> Z = Set(Primes())
>>> X.union(Y).is_finite()
True
>>> X.union(Z).is_finite()
False
```

sage.sets.set.**has_finite_length**(*obj*)

Return `True` if `obj` is known to have finite length.

This is mainly meant for pure Python types, so we do not call any Sage-specific methods.

EXAMPLES:

```
sage: from sage.sets.set import has_finite_length
sage: has_finite_length(tuple(range(10)))
True
sage: has_finite_length(list(range(10)))
True
sage: has_finite_length(set(range(10)))
True
sage: has_finite_length(iter(range(10)))
False
sage: has_finite_length(GF(17^127))                                             #␣
↪needs sage.rings.finite_rings
True
sage: has_finite_length(ZZ)
False
```

```
>>> from sage.all import *
>>> from sage.sets.set import has_finite_length
>>> has_finite_length(tuple(range(Integer(10))))
True
>>> has_finite_length(list(range(Integer(10))))
True
>>> has_finite_length(set(range(Integer(10))))
True
>>> has_finite_length(iter(range(Integer(10))))
False
>>> has_finite_length(GF(Integer(17)**Integer(127)))
↪               # needs sage.rings.finite_rings
True
>>> has_finite_length(ZZ)
False
```

1.4 Disjoint-set data structure

The main entry point is *DisjointSet()* which chooses the appropriate type to return. For more on the data structure, see *DisjointSet()*.

This module defines a class for mutable partitioning of a set, which cannot be used as a key of a dictionary, a vertex of a graph, etc. For immutable partitioning see SetPartition.

AUTHORS:

- Sébastien Labbé (2008) - Initial version.
- Sébastien Labbé (2009-11-24) - Pickling support
- Sébastien Labbé (2010-01) - Inclusion into sage (Issue #6775).
- Giorgos Mousa (2024-04-22): Optimize

EXAMPLES:

Disjoint set of integers from 0 to n - 1:

```
sage: s = DisjointSet(6)
sage: s
{{0}, {1}, {2}, {3}, {4}, {5}}
sage: s.union(2, 4)
sage: s.union(1, 3)
sage: s.union(5, 1)
sage: s
{{0}, {1, 3, 5}, {2, 4}}
sage: s.find(3)
1
sage: s.find(5)
1
sage: list(map(s.find, range(6)))
[0, 1, 2, 1, 2, 1]
```

```
>>> from sage.all import *
>>> s = DisjointSet(Integer(6))
```

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```
>>> s
{{0}, {1}, {2}, {3}, {4}, {5}}
>>> s.union(Integer(2), Integer(4))
>>> s.union(Integer(1), Integer(3))
>>> s.union(Integer(5), Integer(1))
>>> s
{{0}, {1, 3, 5}, {2, 4}}
>>> s.find(Integer(3))
1
>>> s.find(Integer(5))
1
>>> list(map(s.find, range(Integer(6))))
[0, 1, 2, 1, 2, 1]
```

Disjoint set of hashables objects:

```
sage: d = DisjointSet('abcde')
sage: d
{{'a'}, {'b'}, {'c'}, {'d'}, {'e'}}
sage: d.union('a', 'b')
sage: d.union('b', 'c')
sage: d.union('c', 'd')
sage: d
{{'a', 'b', 'c', 'd'}, {'e'}}
sage: d.find('c')
'a'
```

```
>>> from sage.all import *
>>> d = DisjointSet('abcde')
>>> d
{{'a'}, {'b'}, {'c'}, {'d'}, {'e'}}
>>> d.union('a', 'b')
>>> d.union('b', 'c')
>>> d.union('c', 'd')
>>> d
{{'a', 'b', 'c', 'd'}, {'e'}}
>>> d.find('c')
'a'
```

sage.sets.disjoint_set.**DisjointSet**(*arg*)

Construct a disjoint set where each element of `arg` is in its own set. If `arg` is an integer, then the disjoint set returned is made of the integers from `0` to `arg - 1`.

A disjoint-set data structure (sometimes called union-find data structure) is a data structure that keeps track of a partitioning of a set into a number of separate, nonoverlapping sets. It performs two operations:

- *`find()`* – Determine which set a particular element is in.
- *`union()`* – Combine or merge two sets into a single set.

REFERENCES:

- Wikipedia article Disjoint-set_data_structure

INPUT:

- `arg` – nonnegative integer or an iterable of hashable objects

EXAMPLES:

From a nonnegative integer:

```
sage: DisjointSet(5)
{{0}, {1}, {2}, {3}, {4}}
```

```
>>> from sage.all import *
>>> DisjointSet(Integer(5))
{{0}, {1}, {2}, {3}, {4}}
```

From an iterable:

```
sage: DisjointSet('abcde')
{{'a'}, {'b'}, {'c'}, {'d'}, {'e'}}
sage: DisjointSet(range(6))
{{0}, {1}, {2}, {3}, {4}, {5}}
sage: DisjointSet(['yi', 45, 'cheval'])
{{'cheval'}, {'yi'}, {45}}
```

```
>>> from sage.all import *
>>> DisjointSet('abcde')
{{'a'}, {'b'}, {'c'}, {'d'}, {'e'}}
>>> DisjointSet(range(Integer(6)))
{{0}, {1}, {2}, {3}, {4}, {5}}
>>> DisjointSet(['yi', Integer(45), 'cheval'])
{{'cheval'}, {'yi'}, {45}}
```

From a set partition (see Issue #38693):

```
sage: SP = SetPartition(DisjointSet(5))
sage: DisjointSet(SP)
{{0}, {1}, {2}, {3}, {4}}
sage: DisjointSet(SP) == DisjointSet(5)
True
```

```
>>> from sage.all import *
>>> SP = SetPartition(DisjointSet(Integer(5)))
>>> DisjointSet(SP)
{{0}, {1}, {2}, {3}, {4}}
>>> DisjointSet(SP) == DisjointSet(Integer(5))
True
```

class sage.sets.disjoint_set.**DisjointSet_class**

Bases: `SageObject`

Common class and methods for *DisjointSet_of_integers* and *DisjointSet_of_hashables*.

cardinality()

Return the number of elements in `self`, *not* the number of subsets.

EXAMPLES:

```
sage: d = DisjointSet(5)
sage: d.cardinality()
5
sage: d.union(2, 4)
sage: d.cardinality()
5
sage: d = DisjointSet(range(5))
sage: d.cardinality()
5
sage: d.union(2, 4)
sage: d.cardinality()
5
```

```
>>> from sage.all import *
>>> d = DisjointSet(Integer(5))
>>> d.cardinality()
5
>>> d.union(Integer(2), Integer(4))
>>> d.cardinality()
5
>>> d = DisjointSet(range(Integer(5)))
>>> d.cardinality()
5
>>> d.union(Integer(2), Integer(4))
>>> d.cardinality()
5
```

number_of_subsets()

Return the number of subsets in `self`.

EXAMPLES:

```
sage: d = DisjointSet(5)
sage: d.number_of_subsets()
5
sage: d.union(2, 4)
sage: d.number_of_subsets()
4
sage: d = DisjointSet(range(5))
sage: d.number_of_subsets()
5
sage: d.union(2, 4)
sage: d.number_of_subsets()
4
```

```
>>> from sage.all import *
>>> d = DisjointSet(Integer(5))
>>> d.number_of_subsets()
5
>>> d.union(Integer(2), Integer(4))
>>> d.number_of_subsets()
4
>>> d = DisjointSet(range(Integer(5)))
```

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```
>>> d.number_of_subsets()
5
>>> d.union(Integer(2), Integer(4))
>>> d.number_of_subsets()
4
```

class sage.sets.disjoint_set.**DisjointSet_of_hashables**

Bases: *DisjointSet_class*

Disjoint set of hashables.

EXAMPLES:

```
sage: d = DisjointSet('abcde')
sage: d
{{'a'}, {'b'}, {'c'}, {'d'}, {'e'}}
sage: d.union('a', 'c')
sage: d
{{'a', 'c'}, {'b'}, {'d'}, {'e'}}
sage: d.find('a')
'a'
```

```
>>> from sage.all import *
>>> d = DisjointSet('abcde')
>>> d
{{'a'}, {'b'}, {'c'}, {'d'}, {'e'}}
>>> d.union('a', 'c')
>>> d
{{'a', 'c'}, {'b'}, {'d'}, {'e'}}
>>> d.find('a')
'a'
```

element_to_root_dict()

Return the dictionary where the keys are the elements of `self` and the values are their representative inside a list.

EXAMPLES:

```
sage: d = DisjointSet(range(5))
sage: d.union(2, 3)
sage: d.union(4, 1)
sage: e = d.element_to_root_dict()
sage: sorted(e.items())
[(0, 0), (1, 4), (2, 2), (3, 2), (4, 4)]
sage: WordMorphism(e)                                                    #␣
↪needs sage.combinat
WordMorphism: 0->0, 1->4, 2->2, 3->2, 4->4
```

```
>>> from sage.all import *
>>> d = DisjointSet(range(Integer(5)))
>>> d.union(Integer(2), Integer(3))
>>> d.union(Integer(4), Integer(1))
>>> e = d.element_to_root_dict()
```

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```
>>> sorted(e.items())
[(0, 0), (1, 4), (2, 2), (3, 2), (4, 4)]
>>> WordMorphism(e)                                                    #␣
↪needs sage.combinat
WordMorphism: 0->0, 1->4, 2->2, 3->2, 4->4
```

find(*e*)

Return the representative of the set that `e` currently belongs to.

INPUT:

- `e` – element in `self`

EXAMPLES:

```
sage: e = DisjointSet(range(5))
sage: e.union(4, 2)
sage: e
{{0}, {1}, {2, 4}, {3}}
sage: e.find(2)
4
sage: e.find(4)
4
sage: e.union(1,3)
sage: e
{{0}, {1, 3}, {2, 4}}
sage: e.find(1)
1
sage: e.find(3)
1
sage: e.union(3, 2)
sage: e
{{0}, {1, 2, 3, 4}}
sage: [e.find(i) for i in range(5)]
[0, 1, 1, 1, 1]
sage: e.find(5)
Traceback (most recent call last):
...
KeyError: 5
```

```
>>> from sage.all import *
>>> e = DisjointSet(range(Integer(5)))
>>> e.union(Integer(4), Integer(2))
>>> e
{{0}, {1}, {2, 4}, {3}}
>>> e.find(Integer(2))
4
>>> e.find(Integer(4))
4
>>> e.union(Integer(1),Integer(3))
>>> e
{{0}, {1, 3}, {2, 4}}
>>> e.find(Integer(1))
```

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```
1
>>> e.find(Integer(3))
1
>>> e.union(Integer(3), Integer(2))
>>> e
{{0}, {1, 2, 3, 4}}
>>> [e.find(i) for i in range(Integer(5))]
[0, 1, 1, 1, 1]
>>> e.find(Integer(5))
Traceback (most recent call last):
...
KeyError: 5
```

make_set (*new_elt=None*)

Add a new element into a new set containing only the new element.

According to Wikipedia article Disjoint-set_data_structure#Making_new_sets the `make_set` operation adds a new element into a new set containing only the new element. The new set is added at the end of `self`.

INPUT:

- `new_elt` – (optional) element to add. If $None$, then an integer is added.

EXAMPLES:

```
sage: e = DisjointSet('abcde')
sage: e.union('d', 'c')
sage: e.union('c', 'e')
sage: e.make_set('f')
sage: e
{{'a'}, {'b'}, {'c', 'd', 'e'}, {'f'}}
sage: e.union('f', 'b')
sage: e
{{'a'}, {'b', 'f'}, {'c', 'd', 'e'}}
sage: e.make_set('e'); e
{{'a'}, {'b', 'f'}, {'c', 'd', 'e'}}
sage: e.make_set(); e
{{'a'}, {'b', 'f'}, {'c', 'd', 'e'}, {6}}
```

```
>>> from sage.all import *
>>> e = DisjointSet('abcde')
>>> e.union('d', 'c')
>>> e.union('c', 'e')
>>> e.make_set('f')
>>> e
{{'a'}, {'b'}, {'c', 'd', 'e'}, {'f'}}
>>> e.union('f', 'b')
>>> e
{{'a'}, {'b', 'f'}, {'c', 'd', 'e'}}
>>> e.make_set('e'); e
{{'a'}, {'b', 'f'}, {'c', 'd', 'e'}}
>>> e.make_set(); e
{{'a'}, {'b', 'f'}, {'c', 'd', 'e'}, {6}}
```

root_to_elements_dict()

Return the dictionary where the keys are the roots of `self` and the values are the elements in the same set.

EXAMPLES:

```
sage: d = DisjointSet(range(5))
sage: d.union(2, 3)
sage: d.union(4, 1)
sage: e = d.root_to_elements_dict()
sage: sorted(e.items())
[(0, [0]), (2, [2, 3]), (4, [1, 4])]
```

```
>>> from sage.all import *
>>> d = DisjointSet(range(Integer(5)))
>>> d.union(Integer(2), Integer(3))
>>> d.union(Integer(4), Integer(1))
>>> e = d.root_to_elements_dict()
>>> sorted(e.items())
[(0, [0]), (2, [2, 3]), (4, [1, 4])]
```

to_digraph()

Return the current digraph of `self` where (a, b) is an oriented edge if b is the parent of a.

EXAMPLES:

```
sage: d = DisjointSet(range(5))
sage: d.union(2, 3)
sage: d.union(4, 1)
sage: d.union(3, 4)
sage: d
{{0}, {1, 2, 3, 4}}
sage: g = d.to_digraph()
sage: g                                                                     #␣
↪needs sage.graphs
Looped digraph on 5 vertices
sage: g.edges(sort=True)                                                    #␣
↪needs sage.graphs
[(0, 0, None), (1, 2, None), (2, 2, None), (3, 2, None), (4, 2, None)]
```

```
>>> from sage.all import *
>>> d = DisjointSet(range(Integer(5)))
>>> d.union(Integer(2), Integer(3))
>>> d.union(Integer(4), Integer(1))
>>> d.union(Integer(3), Integer(4))
>>> d
{{0}, {1, 2, 3, 4}}
>>> g = d.to_digraph()
>>> g                                                                       #␣
↪needs sage.graphs
Looped digraph on 5 vertices
>>> g.edges(sort=True)                                                      #␣
↪needs sage.graphs
[(0, 0, None), (1, 2, None), (2, 2, None), (3, 2, None), (4, 2, None)]
```

The result depends on the ordering of the union:

```
sage: d = DisjointSet(range(5))
sage: d.union(1, 2)
sage: d.union(1, 3)
sage: d.union(1, 4)
sage: d
{{0}, {1, 2, 3, 4}}
sage: d.to_digraph().edges(sort=True)                                    #␣
↪needs sage.graphs
[(0, 0, None), (1, 1, None), (2, 1, None), (3, 1, None), (4, 1, None)]
```

```
>>> from sage.all import *
>>> d = DisjointSet(range(Integer(5)))
>>> d.union(Integer(1), Integer(2))
>>> d.union(Integer(1), Integer(3))
>>> d.union(Integer(1), Integer(4))
>>> d
{{0}, {1, 2, 3, 4}}
>>> d.to_digraph().edges(sort=True)                                      #␣
↪needs sage.graphs
[(0, 0, None), (1, 1, None), (2, 1, None), (3, 1, None), (4, 1, None)]
```

union(*e*, *f*)

Combine the set of `e` and the set of `f` into one.

All elements in those two sets will share the same representative that can be retrieved using *find()*.

INPUT:

- `e` – element in `self`
- `f` – element in `self`

EXAMPLES:

```
sage: e = DisjointSet('abcde')
sage: e
{{'a'}, {'b'}, {'c'}, {'d'}, {'e'}}
sage: e.union('a', 'b')
sage: e
{{'a', 'b'}, {'c'}, {'d'}, {'e'}}
sage: e.union('c', 'e')
sage: e
{{'a', 'b'}, {'c', 'e'}, {'d'}}
sage: e.union('b', 'e')
sage: e
{{'a', 'b', 'c', 'e'}, {'d'}}
sage: e.union('a', 2**10)
Traceback (most recent call last):
...
KeyError: 1024
```

```
>>> from sage.all import *
>>> e = DisjointSet('abcde')
>>> e
{{'a'}, {'b'}, {'c'}, {'d'}, {'e'}}
```

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```
>>> e.union('a', 'b')
>>> e
{{'a', 'b'}, {'c'}, {'d'}, {'e'}}
>>> e.union('c', 'e')
>>> e
{{'a', 'b'}, {'c', 'e'}, {'d'}}
>>> e.union('b', 'e')
>>> e
{{'a', 'b', 'c', 'e'}, {'d'}}
>>> e.union('a', Integer(2)**Integer(10))
Traceback (most recent call last):
...
KeyError: 1024
```

class sage.sets.disjoint_set.**DisjointSet_of_integers**

Bases: *DisjointSet_class*

Disjoint set of integers from `0` to `n-1`.

EXAMPLES:

```
sage: d = DisjointSet(5)
sage: d
{{0}, {1}, {2}, {3}, {4}}
sage: d.union(2, 4)
sage: d.union(0, 2)
sage: d
{{0, 2, 4}, {1}, {3}}
sage: d.find(2)
2
```

```
>>> from sage.all import *
>>> d = DisjointSet(Integer(5))
>>> d
{{0}, {1}, {2}, {3}, {4}}
>>> d.union(Integer(2), Integer(4))
>>> d.union(Integer(0), Integer(2))
>>> d
{{0, 2, 4}, {1}, {3}}
>>> d.find(Integer(2))
2
```

element_to_root_dict()

Return the dictionary where the keys are the elements of `self` and the values are their representative inside a list.

EXAMPLES:

```
sage: d = DisjointSet(5)
sage: d.union(2, 3)
sage: d.union(4, 1)
sage: e = d.element_to_root_dict()
sage: e
```

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```
{0: 0, 1: 4, 2: 2, 3: 2, 4: 4}
sage: WordMorphism(e)                                                       #␣
↪needs sage.combinat
WordMorphism: 0->0, 1->4, 2->2, 3->2, 4->4
```

```
>>> from sage.all import *
>>> d = DisjointSet(Integer(5))
>>> d.union(Integer(2), Integer(3))
>>> d.union(Integer(4), Integer(1))
>>> e = d.element_to_root_dict()
>>> e
{0: 0, 1: 4, 2: 2, 3: 2, 4: 4}
>>> WordMorphism(e)                                                         #␣
↪needs sage.combinat
WordMorphism: 0->0, 1->4, 2->2, 3->2, 4->4
```

find(*i*)

Return the representative of the set that `i` currently belongs to.

INPUT:

- `i` – element in `self`

EXAMPLES:

```
sage: e = DisjointSet(5)
sage: e.union(4, 2)
sage: e
{{0}, {1}, {2, 4}, {3}}
sage: e.find(2)
4
sage: e.find(4)
4
sage: e.union(1, 3)
sage: e
{{0}, {1, 3}, {2, 4}}
sage: e.find(1)
1
sage: e.find(3)
1
sage: e.union(3, 2)
sage: e
{{0}, {1, 2, 3, 4}}
sage: [e.find(i) for i in range(5)]
[0, 1, 1, 1, 1]
sage: e.find(2**10)
Traceback (most recent call last):
...
ValueError: i must be between 0 and 4 (1024 given)
```

```
>>> from sage.all import *
>>> e = DisjointSet(Integer(5))
>>> e.union(Integer(4), Integer(2))
```

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```
>>> e
{{0}, {1}, {2, 4}, {3}}
>>> e.find(Integer(2))
4
>>> e.find(Integer(4))
4
>>> e.union(Integer(1), Integer(3))
>>> e
{{0}, {1, 3}, {2, 4}}
>>> e.find(Integer(1))
1
>>> e.find(Integer(3))
1
>>> e.union(Integer(3), Integer(2))
>>> e
{{0}, {1, 2, 3, 4}}
>>> [e.find(i) for i in range(Integer(5))]
[0, 1, 1, 1, 1]
>>> e.find(Integer(2)**Integer(10))
Traceback (most recent call last):
...
ValueError: i must be between 0 and 4 (1024 given)
```

Note

This method performs input checks. To avoid them you may directly use `OP_find()`.

make_set()

Add a new element into a new set containing only the new element.

According to Wikipedia article Disjoint-set_data_structure#Making_new_sets the `make_set` operation adds a new element into a new set containing only the new element. The new set is added at the end of `self`.

EXAMPLES:

```
sage: d = DisjointSet(5)
sage: d.union(1, 2)
sage: d.union(0, 1)
sage: d.make_set()
sage: d
{{0, 1, 2}, {3}, {4}, {5}}
sage: d.find(1)
1
```

```
>>> from sage.all import *
>>> d = DisjointSet(Integer(5))
>>> d.union(Integer(1), Integer(2))
>>> d.union(Integer(0), Integer(1))
>>> d.make_set()
>>> d
{{0, 1, 2}, {3}, {4}, {5}}
```

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```
>>> d.find(Integer(1))
1
```

root_to_elements_dict()

Return the dictionary where the keys are the roots of `self` and the values are the elements in the same set as the root.

EXAMPLES:

```
sage: d = DisjointSet(5)
sage: sorted(d.root_to_elements_dict().items())
[(0, [0]), (1, [1]), (2, [2]), (3, [3]), (4, [4])]
sage: d.union(2, 3)
sage: sorted(d.root_to_elements_dict().items())
[(0, [0]), (1, [1]), (2, [2, 3]), (4, [4])]
sage: d.union(3, 0)
sage: sorted(d.root_to_elements_dict().items())
[(1, [1]), (2, [0, 2, 3]), (4, [4])]
sage: d
{{0, 2, 3}, {1}, {4}}
```

```
>>> from sage.all import *
>>> d = DisjointSet(Integer(5))
>>> sorted(d.root_to_elements_dict().items())
[(0, [0]), (1, [1]), (2, [2]), (3, [3]), (4, [4])]
>>> d.union(Integer(2), Integer(3))
>>> sorted(d.root_to_elements_dict().items())
[(0, [0]), (1, [1]), (2, [2, 3]), (4, [4])]
>>> d.union(Integer(3), Integer(0))
>>> sorted(d.root_to_elements_dict().items())
[(1, [1]), (2, [0, 2, 3]), (4, [4])]
>>> d
{{0, 2, 3}, {1}, {4}}
```

to_digraph()

Return the current digraph of `self` where (a, b) is an oriented edge if b is the parent of a.

EXAMPLES:

```
sage: d = DisjointSet(5)
sage: d.union(2, 3)
sage: d.union(4, 1)
sage: d.union(3, 4)
sage: d
{{0}, {1, 2, 3, 4}}
sage: g = d.to_digraph(); g                                                 #␣
↪needs sage.graphs
Looped digraph on 5 vertices
sage: g.edges(sort=True)                                                    #␣
↪needs sage.graphs
[(0, 0, None), (1, 2, None), (2, 2, None), (3, 2, None), (4, 2, None)]
```

```
>>> from sage.all import *
>>> d = DisjointSet(Integer(5))
>>> d.union(Integer(2), Integer(3))
>>> d.union(Integer(4), Integer(1))
>>> d.union(Integer(3), Integer(4))
>>> d
{{0}, {1, 2, 3, 4}}
>>> g = d.to_digraph(); g                                                    #␣
↪needs sage.graphs
Looped digraph on 5 vertices
>>> g.edges(sort=True)                                                       #␣
↪needs sage.graphs
[(0, 0, None), (1, 2, None), (2, 2, None), (3, 2, None), (4, 2, None)]
```

The result depends on the ordering of the union:

```
sage: d = DisjointSet(5)
sage: d.union(1, 2)
sage: d.union(1, 3)
sage: d.union(1, 4)
sage: d
{{0}, {1, 2, 3, 4}}
sage: d.to_digraph().edges(sort=True)                                       #␣
↪needs sage.graphs
[(0, 0, None), (1, 1, None), (2, 1, None), (3, 1, None), (4, 1, None)]
```

```
>>> from sage.all import *
>>> d = DisjointSet(Integer(5))
>>> d.union(Integer(1), Integer(2))
>>> d.union(Integer(1), Integer(3))
>>> d.union(Integer(1), Integer(4))
>>> d
{{0}, {1, 2, 3, 4}}
>>> d.to_digraph().edges(sort=True)                                          #␣
↪needs sage.graphs
[(0, 0, None), (1, 1, None), (2, 1, None), (3, 1, None), (4, 1, None)]
```

union(*i*, *j*)

Combine the set of `i` and the set of `j` into one.

All elements in those two sets will share the same representative that can be retrieved using *find()*.

INPUT:

- `i` – element in `self`
- `j` – element in `self`

EXAMPLES:

```
sage: d = DisjointSet(5)
sage: d
{{0}, {1}, {2}, {3}, {4}}
sage: d.union(0, 1)
sage: d
```

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```
{{0, 1}, {2}, {3}, {4}}
sage: d.union(2, 4)
sage: d
{{0, 1}, {2, 4}, {3}}
sage: d.union(1, 4)
sage: d
{{0, 1, 2, 4}, {3}}
sage: d.union(1, 5)
Traceback (most recent call last):
...
ValueError: j must be between 0 and 4 (5 given)
```

```
>>> from sage.all import *
>>> d = DisjointSet(Integer(5))
>>> d
{{0}, {1}, {2}, {3}, {4}}
>>> d.union(Integer(0), Integer(1))
>>> d
{{0, 1}, {2}, {3}, {4}}
>>> d.union(Integer(2), Integer(4))
>>> d
{{0, 1}, {2, 4}, {3}}
>>> d.union(Integer(1), Integer(4))
>>> d
{{0, 1, 2, 4}, {3}}
>>> d.union(Integer(1), Integer(5))
Traceback (most recent call last):
...
ValueError: j must be between 0 and 4 (5 given)
```

Note

This method performs input checks. To avoid them you may directly use `OP_join()`.

1.5 Disjoint union of enumerated sets

AUTHORS:

- Florent Hivert (2009-07/09): initial implementation.
- Florent Hivert (2010-03): classcall related stuff.
- Florent Hivert (2010-12): fixed facade element construction.

class sage.sets.disjoint_union_enumerated_sets.**DisjointUnionEnumeratedSets**(*family*, *facade=True*, *keepkey=False*, *category=None*)

Bases: `UniqueRepresentation`, `Parent`

A class for disjoint unions of enumerated sets.

INPUT:

- `family` – list (or iterable or family) of enumerated sets
- `keepkey` – boolean
- `facade` – boolean

This models the enumerated set obtained by concatenating together the specified ordered sets. The latter are supposed to be pairwise disjoint; otherwise, a multiset is created.

The argument `family` can be a list, a tuple, a dictionary, or a family. If it is not a family it is first converted into a family (see *sage.sets.family.Family()*).

Experimental options:

By default, there is no way to tell from which set of the union an element is generated. The option `keepkey=True` keeps track of those by returning pairs `(key, el)` where `key` is the index of the set to which `el` belongs. When this option is specified, the enumerated sets need not be disjoint anymore.

With the option `facade=False` the elements are wrapped in an object whose parent is the disjoint union itself. The wrapped object can then be recovered using the `value` attribute.

The two options can be combined.

The names of those options is imperfect, and subject to change in future versions. Feedback welcome.

EXAMPLES:

The input can be a list or a tuple of FiniteEnumeratedSets:

```
sage: U1 = DisjointUnionEnumeratedSets((
....:       FiniteEnumeratedSet([1,2,3]),
....:       FiniteEnumeratedSet([4,5,6])))
sage: U1
Disjoint union of Family ({1, 2, 3}, {4, 5, 6})
sage: U1.list()
[1, 2, 3, 4, 5, 6]
sage: U1.cardinality()
6
```

```
>>> from sage.all import *
>>> U1 = DisjointUnionEnumeratedSets((
...       FiniteEnumeratedSet([Integer(1),Integer(2),Integer(3)]),
...       FiniteEnumeratedSet([Integer(4),Integer(5),Integer(6)])))
>>> U1
Disjoint union of Family ({1, 2, 3}, {4, 5, 6})
>>> U1.list()
[1, 2, 3, 4, 5, 6]
>>> U1.cardinality()
6
```

The input can also be a dictionary:

```
sage: U2 = DisjointUnionEnumeratedSets({1: FiniteEnumeratedSet([1,2,3]),
....:                                   2: FiniteEnumeratedSet([4,5,6])})
sage: U2
Disjoint union of Finite family {1: {1, 2, 3}, 2: {4, 5, 6}}
sage: U2.list()
```

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```
[1, 2, 3, 4, 5, 6]
sage: U2.cardinality()
6
```

```
>>> from sage.all import *
>>> U2 = DisjointUnionEnumeratedSets({Integer(1): FiniteEnumeratedSet([Integer(1),
↪Integer(2),Integer(3)]),
...                                Integer(2): FiniteEnumeratedSet([Integer(4),
↪Integer(5),Integer(6)])})
>>> U2
Disjoint union of Finite family {1: {1, 2, 3}, 2: {4, 5, 6}}
>>> U2.list()
[1, 2, 3, 4, 5, 6]
>>> U2.cardinality()
6
```

However in that case the enumeration order is not specified.

In general the input can be any family:

```
sage: # needs sage.combinat
sage: U3 = DisjointUnionEnumeratedSets(
....:     Family([2,3,4], Permutations, lazy=True))
sage: U3
Disjoint union of Lazy family
 (<class 'sage.combinat.permutation.Permutations'>(i))_{i in [2, 3, 4]}
sage: U3.cardinality()
32
sage: it = iter(U3)
sage: [next(it), next(it), next(it), next(it), next(it), next(it)]
[[1, 2], [2, 1], [1, 2, 3], [1, 3, 2], [2, 1, 3], [2, 3, 1]]
sage: U3.unrank(18)
[2, 4, 1, 3]
```

```
>>> from sage.all import *
>>> # needs sage.combinat
>>> U3 = DisjointUnionEnumeratedSets(
...     Family([Integer(2),Integer(3),Integer(4)], Permutations, lazy=True))
>>> U3
Disjoint union of Lazy family
 (<class 'sage.combinat.permutation.Permutations'>(i))_{i in [2, 3, 4]}
>>> U3.cardinality()
32
>>> it = iter(U3)
>>> [next(it), next(it), next(it), next(it), next(it), next(it)]
[[1, 2], [2, 1], [1, 2, 3], [1, 3, 2], [2, 1, 3], [2, 3, 1]]
>>> U3.unrank(Integer(18))
[2, 4, 1, 3]
```

This allows for infinite unions:

```
sage: # needs sage.combinat
```

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```
sage: U4 = DisjointUnionEnumeratedSets(
....:     Family(NonNegativeIntegers(), Permutations))
sage: U4
Disjoint union of Lazy family
 (<class 'sage.combinat.permutation.Permutations'>(i))_{i in Non negative␣
↪integers}
sage: U4.cardinality()
+Infinity
sage: it = iter(U4)
sage: [next(it), next(it), next(it), next(it), next(it), next(it)]
[[], [1], [1, 2], [2, 1], [1, 2, 3], [1, 3, 2]]
sage: U4.unrank(18)
[2, 3, 1, 4]
```

```
>>> from sage.all import *
>>> # needs sage.combinat
>>> U4 = DisjointUnionEnumeratedSets(
...     Family(NonNegativeIntegers(), Permutations))
>>> U4
Disjoint union of Lazy family
 (<class 'sage.combinat.permutation.Permutations'>(i))_{i in Non negative␣
↪integers}
>>> U4.cardinality()
+Infinity
>>> it = iter(U4)
>>> [next(it), next(it), next(it), next(it), next(it), next(it)]
[[], [1], [1, 2], [2, 1], [1, 2, 3], [1, 3, 2]]
>>> U4.unrank(Integer(18))
[2, 3, 1, 4]
```

Warning

Beware that some of the operations assume in that case that infinitely many of the enumerated sets are non empty.

Experimental options

We demonstrate the `keepkey` option:

```
sage: # needs sage.combinat
sage: Ukeep = DisjointUnionEnumeratedSets(
....:             Family(list(range(4)), Permutations), keepkey=True)
sage: it = iter(Ukeep)
sage: [next(it) for i in range(6)]
[(0, []), (1, [1]), (2, [1, 2]), (2, [2, 1]), (3, [1, 2, 3]), (3, [1, 3, 2])]
sage: type(next(it)[1])
<class 'sage.combinat.permutation.StandardPermutations_n_with_category.element_
↪class'>
```

```
>>> from sage.all import *
>>> # needs sage.combinat
>>> Ukeep = DisjointUnionEnumeratedSets(
...             Family(list(range(Integer(4))), Permutations), keepkey=True)
>>> it = iter(Ukeep)
>>> [next(it) for i in range(Integer(6))]
[(0, []), (1, [1]), (2, [1, 2]), (2, [2, 1]), (3, [1, 2, 3]), (3, [1, 3, 2])]
>>> type(next(it)[Integer(1)])
<class 'sage.combinat.permutation.StandardPermutations_n_with_category.element_
↪class'>
```

We now demonstrate the `facade` option:

```
sage: # needs sage.combinat
sage: UNoFacade = DisjointUnionEnumeratedSets(
....:                 Family(list(range(4)), Permutations), facade=False)
sage: it = iter(UNoFacade)
sage: [next(it) for i in range(6)]
[[], [1], [1, 2], [2, 1], [1, 2, 3], [1, 3, 2]]
sage: el = next(it); el
[2, 1, 3]
sage: type(el)
<... 'sage.structure.element_wrapper.ElementWrapper'>
sage: el.parent() == UNoFacade
True
sage: elv = el.value; elv
[2, 1, 3]
sage: type(elv)
<class 'sage.combinat.permutation.StandardPermutations_n_with_category.element_
↪class'>
```

```
>>> from sage.all import *
>>> # needs sage.combinat
>>> UNoFacade = DisjointUnionEnumeratedSets(
...                 Family(list(range(Integer(4))), Permutations), facade=False)
>>> it = iter(UNoFacade)
>>> [next(it) for i in range(Integer(6))]
[[], [1], [1, 2], [2, 1], [1, 2, 3], [1, 3, 2]]
>>> el = next(it); el
[2, 1, 3]
>>> type(el)
<... 'sage.structure.element_wrapper.ElementWrapper'>
>>> el.parent() == UNoFacade
True
>>> elv = el.value; elv
[2, 1, 3]
>>> type(elv)
<class 'sage.combinat.permutation.StandardPermutations_n_with_category.element_
↪class'>
```

The elements `el` of the disjoint union are simple wrapped elements. So to access the methods, you need to do `el.value`:

```
sage: el[0]                                                                 #␣
↪needs sage.combinat
Traceback (most recent call last):
...
TypeError: 'sage.structure.element_wrapper.ElementWrapper' object is not␣
↪subscriptable

sage: el.value[0]                                                           #␣
↪needs sage.combinat
2
```

```
>>> from sage.all import *
>>> el[Integer(0)]                                                          ␣
↪     # needs sage.combinat
Traceback (most recent call last):
...
TypeError: 'sage.structure.element_wrapper.ElementWrapper' object is not␣
↪subscriptable

>>> el.value[Integer(0)]                                                   ␣
↪     # needs sage.combinat
2
```

Possible extensions: the current enumeration order is not suitable for unions of infinite enumerated sets (except possibly for the last one). One could add options to specify alternative enumeration orders (anti-diagonal, round robin, …) to handle this case.

Inheriting from `DisjointUnionEnumeratedSets`

There are two different use cases for inheriting from *DisjointUnionEnumeratedSets*: writing a parent which happens to be a disjoint union of some known parents, or writing generic disjoint unions for some particular classes of sage.categories.enumerated_sets.EnumeratedSets.

- In the first use case, the input of the `__init__` method is most likely different from that of *DisjointUnionEnumeratedSets*. Then, one simply writes the `__init__` method as usual:

```
sage: class MyUnion(DisjointUnionEnumeratedSets):
....:   def __init__(self):
....:       DisjointUnionEnumeratedSets.__init__(self,
....:            Family([1,2], Permutations))
sage: pp = MyUnion()
sage: pp.list()
[[1], [1, 2], [2, 1]]
```

```
>>> from sage.all import *
>>> class MyUnion(DisjointUnionEnumeratedSets):
...   def __init__(self):
...       DisjointUnionEnumeratedSets.__init__(self,
...            Family([Integer(1),Integer(2)], Permutations))
>>> pp = MyUnion()
>>> pp.list()
[[1], [1, 2], [2, 1]]
```

In case the `__init__()` method takes optional arguments, or does some normalization on them, a specific method `__classcall_private__` is required (see the documentation of `UniqueRepresentation`).

- In the second use case, the input of the `__init__` method is the same as that of *`DisjointUnionEnumeratedSets`*; one therefore wants to inherit the `__classcall_private__()` method as well, which can be achieved as follows:

```
sage: class UnionOfSpecialSets(DisjointUnionEnumeratedSets):
....:     __classcall_private__ = staticmethod(DisjointUnionEnumeratedSets.__
↪classcall_private__)
sage: psp = UnionOfSpecialSets(Family([1,2], Permutations))
sage: psp.list()
[[1], [1, 2], [2, 1]]
```

```
>>> from sage.all import *
>>> class UnionOfSpecialSets(DisjointUnionEnumeratedSets):
...     __classcall_private__ = staticmethod(DisjointUnionEnumeratedSets.__
↪classcall_private__)
>>> psp = UnionOfSpecialSets(Family([Integer(1),Integer(2)], Permutations))
>>> psp.list()
[[1], [1, 2], [2, 1]]
```

Element()

an_element()

Return an element of this disjoint union, as per `Sets.ParentMethods.an_element()`.

EXAMPLES:

```
sage: U4 = DisjointUnionEnumeratedSets(
....:           Family([3, 5, 7], Permutations))
sage: U4.an_element()
[1, 2, 3]
```

```
>>> from sage.all import *
>>> U4 = DisjointUnionEnumeratedSets(
...           Family([Integer(3), Integer(5), Integer(7)], Permutations))
>>> U4.an_element()
[1, 2, 3]
```

cardinality()

Return the cardinality of this disjoint union.

EXAMPLES:

For finite disjoint unions, the cardinality is computed by summing the cardinalities of the enumerated sets:

```
sage: U = DisjointUnionEnumeratedSets(Family([0,1,2,3], Permutations))
sage: U.cardinality()
10
```

```
>>> from sage.all import *
>>> U = DisjointUnionEnumeratedSets(Family([Integer(0),Integer(1),Integer(2),
↪Integer(3)], Permutations))
```

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```
>>> U.cardinality()
10
```

For infinite disjoint unions, this makes the assumption that the result is infinite:

```
sage: U = DisjointUnionEnumeratedSets(
....:          Family(NonNegativeIntegers(), Permutations))
sage: U.cardinality()
+Infinity
```

```
>>> from sage.all import *
>>> U = DisjointUnionEnumeratedSets(
...          Family(NonNegativeIntegers(), Permutations))
>>> U.cardinality()
+Infinity
```

Warning

As pointed out in the main documentation, it is possible to construct examples where this is incorrect:

```
sage: U = DisjointUnionEnumeratedSets(
....:          Family(NonNegativeIntegers(), lambda x: []))
sage: U.cardinality()  # Should be 0!
+Infinity
```

```
>>> from sage.all import *
>>> U = DisjointUnionEnumeratedSets(
...          Family(NonNegativeIntegers(), lambda x: []))
>>> U.cardinality()  # Should be 0!
+Infinity
```

1.6 Enumerated set from iterator

EXAMPLES:

We build a set from the iterator `graphs` that returns a canonical representative for each isomorphism class of graphs:

```
sage: # needs sage.graphs
sage: from sage.sets.set_from_iterator import EnumeratedSetFromIterator
sage: E = EnumeratedSetFromIterator(
....:   graphs,
....:   name='Graphs',
....:   category=InfiniteEnumeratedSets(),
....:   cache=True)
sage: E
Graphs
sage: E.unrank(0)
Graph on 0 vertices
sage: E.unrank(4)
Graph on 3 vertices
```

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```
sage: E.cardinality()
+Infinity
sage: E.category()
Category of facade infinite enumerated sets
```

```
>>> from sage.all import *
>>> # needs sage.graphs
>>> from sage.sets.set_from_iterator import EnumeratedSetFromIterator
>>> E = EnumeratedSetFromIterator(
...   graphs,
...   name='Graphs',
...   category=InfiniteEnumeratedSets(),
...   cache=True)
>>> E
Graphs
>>> E.unrank(Integer(0))
Graph on 0 vertices
>>> E.unrank(Integer(4))
Graph on 3 vertices
>>> E.cardinality()
+Infinity
>>> E.category()
Category of facade infinite enumerated sets
```

The module also provides decorator for functions and methods:

```
sage: from sage.sets.set_from_iterator import set_from_function
sage: @set_from_function
....: def f(n): return xsrange(n)
sage: f(3)
{0, 1, 2}
sage: f(5)
{0, 1, 2, 3, 4}
sage: f(100)
{0, 1, 2, 3, 4, ...}

sage: from sage.sets.set_from_iterator import set_from_method
sage: class A:
....:     @set_from_method
....:     def f(self, n):
....:         return xsrange(n)
sage: a = A()
sage: a.f(3)
{0, 1, 2}
sage: f(100)
{0, 1, 2, 3, 4, ...}
```

```
>>> from sage.all import *
>>> from sage.sets.set_from_iterator import set_from_function
>>> @set_from_function
... def f(n): return xsrange(n)
>>> f(Integer(3))
```

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```
{0, 1, 2}
>>> f(Integer(5))
{0, 1, 2, 3, 4}
>>> f(Integer(100))
{0, 1, 2, 3, 4, ...}

>>> from sage.sets.set_from_iterator import set_from_method
>>> class A:
...     @set_from_method
...     def f(self, n):
...         return xsrange(n)
>>> a = A()
>>> a.f(Integer(3))
{0, 1, 2}
>>> f(Integer(100))
{0, 1, 2, 3, 4, ...}
```

class sage.sets.set_from_iterator.**Decorator**

Bases: object

Abstract class that manage documentation and sources of the wrapped object.

The method needs to be stored in the attribute `self.f`

class sage.sets.set_from_iterator.**DummyExampleForPicklingTest**

Bases: object

Class example to test pickling with the decorator *set_from_method*.

> **Warning**
>
> This class is intended to be used in doctest only.

EXAMPLES:

```
sage: from sage.sets.set_from_iterator import DummyExampleForPicklingTest
sage: DummyExampleForPicklingTest().f()
{10, 11, 12, 13, 14, ...}
```

```
>>> from sage.all import *
>>> from sage.sets.set_from_iterator import DummyExampleForPicklingTest
>>> DummyExampleForPicklingTest().f()
{10, 11, 12, 13, 14, ...}
```

f()

Return the set between `self.start` and `self.stop`.

EXAMPLES:

```
sage: from sage.sets.set_from_iterator import DummyExampleForPicklingTest
sage: d = DummyExampleForPicklingTest()
sage: d.f()
{10, 11, 12, 13, 14, ...}
```

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```
sage: d.start = 4
sage: d.stop = 200
sage: d.f()
{4, 5, 6, 7, 8, ...}
```

```
>>> from sage.all import *
>>> from sage.sets.set_from_iterator import DummyExampleForPicklingTest
>>> d = DummyExampleForPicklingTest()
>>> d.f()
{10, 11, 12, 13, 14, ...}
>>> d.start = Integer(4)
>>> d.stop = Integer(200)
>>> d.f()
{4, 5, 6, 7, 8, ...}
```

start = 10

stop = 100

class sage.sets.set_from_iterator.**EnumeratedSetFromIterator**(*f*, *args=None*, *kwds=None*, *name=None*, *category=None*, *cache=False*)

Bases: Parent

A class for enumerated set built from an iterator.

INPUT:

- f – a function that returns an iterable from which the set is built from
- args – tuple; arguments to be sent to the function f
- kwds – dictionary; keywords to be sent to the function f
- name – an optional name for the set
- category – (default: None) an optional category for that enumerated set. If you know that your iterator will stop after a finite number of steps you should set it as FiniteEnumeratedSets, conversely if you know that your iterator will run over and over you should set it as InfiniteEnumeratedSets.
- cache – boolean (default: False); whether or not use a cache mechanism for the iterator. If True, then the function f is called only once.

EXAMPLES:

```
sage: from sage.sets.set_from_iterator import EnumeratedSetFromIterator
sage: E = EnumeratedSetFromIterator(graphs, args=(7,)); E                    #␣
↪needs sage.graphs
{Graph on 7 vertices, Graph on 7 vertices, Graph on 7 vertices,
 Graph on 7 vertices, Graph on 7 vertices, ...}
sage: E.category()                                                           #␣
↪needs sage.graphs
Category of facade enumerated sets
```

```
>>> from sage.all import *
>>> from sage.sets.set_from_iterator import EnumeratedSetFromIterator
```

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```
>>> E = EnumeratedSetFromIterator(graphs, args=(Integer(7),)); E
↪     # needs sage.graphs
{Graph on 7 vertices, Graph on 7 vertices, Graph on 7 vertices,
 Graph on 7 vertices, Graph on 7 vertices, ...}
>>> E.category()
↪needs sage.graphs
Category of facade enumerated sets
```

The same example with a cache and a custom name:

```
sage: E = EnumeratedSetFromIterator(graphs, args=(8,), cache=True,
↪needs sage.graphs
....:                               name="Graphs with 8 vertices",
....:                               category=FiniteEnumeratedSets()); E
Graphs with 8 vertices
sage: E.unrank(3)
↪needs sage.graphs
Graph on 8 vertices
sage: E.category()
↪needs sage.graphs
Category of facade finite enumerated sets
```

```
>>> from sage.all import *
>>> E = EnumeratedSetFromIterator(graphs, args=(Integer(8),), cache=True,
↪     # needs sage.graphs
...                               name="Graphs with 8 vertices",
...                               category=FiniteEnumeratedSets()); E
Graphs with 8 vertices
>>> E.unrank(Integer(3))
↪     # needs sage.graphs
Graph on 8 vertices
>>> E.category()
↪needs sage.graphs
Category of facade finite enumerated sets
```

Note

In order to make the `TestSuite` works, the elements of the set should have parents.

clear_cache()

Clear the cache.

EXAMPLES:

```
sage: from itertools import count
sage: from sage.sets.set_from_iterator import EnumeratedSetFromIterator
sage: E = EnumeratedSetFromIterator(count, args=(1,), cache=True)
sage: e1 = E._cache; e1
lazy list [1, 2, 3, ...]
sage: E.clear_cache()
sage: E._cache
```

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```
lazy list [1, 2, 3, ...]
sage: e1 is E._cache
False
```

```
>>> from sage.all import *
>>> from itertools import count
>>> from sage.sets.set_from_iterator import EnumeratedSetFromIterator
>>> E = EnumeratedSetFromIterator(count, args=(Integer(1),), cache=True)
>>> e1 = E._cache; e1
lazy list [1, 2, 3, ...]
>>> E.clear_cache()
>>> E._cache
lazy list [1, 2, 3, ...]
>>> e1 is E._cache
False
```

is_parent_of(*x*)

Test whether `x` is in `self`.

If the set is infinite, only the answer `True` should be expected in finite time.

EXAMPLES:

```
sage: from sage.sets.set_from_iterator import EnumeratedSetFromIterator
sage: P = Partitions(12, min_part=2, max_part=5)                              #␣
↪needs sage.combinat
sage: E = EnumeratedSetFromIterator(P.__iter__)                               #␣
↪needs sage.combinat
sage: P([5,5,2]) in E                                                         #␣
↪needs sage.combinat
True
```

```
>>> from sage.all import *
>>> from sage.sets.set_from_iterator import EnumeratedSetFromIterator
>>> P = Partitions(Integer(12), min_part=Integer(2), max_part=Integer(5))    ␣
↪                        # needs sage.combinat
>>> E = EnumeratedSetFromIterator(P.__iter__)                                #␣
↪needs sage.combinat
>>> P([Integer(5),Integer(5),Integer(2)]) in E                               ␣
↪                        # needs sage.combinat
True
```

unrank(*i*)

Return the element at position `i`.

EXAMPLES:

```
sage: # needs sage.graphs
sage: from sage.sets.set_from_iterator import EnumeratedSetFromIterator
sage: E = EnumeratedSetFromIterator(graphs, args=(8,), cache=True)
sage: F = EnumeratedSetFromIterator(graphs, args=(8,), cache=False)
sage: E.unrank(2)
Graph on 8 vertices
```

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```
sage: E.unrank(2) == F.unrank(2)
True
```

```
>>> from sage.all import *
>>> # needs sage.graphs
>>> from sage.sets.set_from_iterator import EnumeratedSetFromIterator
>>> E = EnumeratedSetFromIterator(graphs, args=(Integer(8),), cache=True)
>>> F = EnumeratedSetFromIterator(graphs, args=(Integer(8),), cache=False)
>>> E.unrank(Integer(2))
Graph on 8 vertices
>>> E.unrank(Integer(2)) == F.unrank(Integer(2))
True
```

class sage.sets.set_from_iterator.**EnumeratedSetFromIterator_function_decorator**(*f=None*, *name=None*, ***options*)

Bases: *Decorator*

Decorator for *EnumeratedSetFromIterator*.

Name could be string or a function `(args, kwds) -> string`.

Warning

If you are going to use this with the decorator `cached_function()`, you must place the `@cached_function` first. See the example below.

EXAMPLES:

```
sage: from sage.sets.set_from_iterator import set_from_function
sage: @set_from_function
....: def f(n):
....:     for i in range(n):
....:         yield i**2 + i + 1
sage: f(3)
{1, 3, 7}
sage: f(100)
{1, 3, 7, 13, 21, ...}
```

```
>>> from sage.all import *
>>> from sage.sets.set_from_iterator import set_from_function
>>> @set_from_function
... def f(n):
...     for i in range(n):
...         yield i**Integer(2) + i + Integer(1)
>>> f(Integer(3))
{1, 3, 7}
>>> f(Integer(100))
{1, 3, 7, 13, 21, ...}
```

To avoid ambiguity, it is always better to use it with a call which provides optional global initialization for the call to *EnumeratedSetFromIterator*:

```
sage: @set_from_function(category=InfiniteEnumeratedSets())
....: def Fibonacci():
....:     a = 1; b = 2
....:     while True:
....:         yield a
....:         a, b = b, a + b
sage: F = Fibonacci(); F
{1, 2, 3, 5, 8, ...}
sage: F.cardinality()
+Infinity
```

```
>>> from sage.all import *
>>> @set_from_function(category=InfiniteEnumeratedSets())
... def Fibonacci():
...     a = Integer(1); b = Integer(2)
...     while True:
...         yield a
...         a, b = b, a + b
>>> F = Fibonacci(); F
{1, 2, 3, 5, 8, ...}
>>> F.cardinality()
+Infinity
```

A simple example with many options:

```
sage: @set_from_function(name="From %(m)d to %(n)d",
....:                    category=FiniteEnumeratedSets())
....: def f(m, n): return xsrange(m, n + 1)
sage: E = f(3,10); E
From 3 to 10
sage: E.list()
[3, 4, 5, 6, 7, 8, 9, 10]
sage: E = f(1,100); E
From 1 to 100
sage: E.cardinality()
100
sage: f(n=100, m=1) == E
True
```

```
>>> from sage.all import *
>>> @set_from_function(name="From %(m)d to %(n)d",
...                    category=FiniteEnumeratedSets())
... def f(m, n): return xsrange(m, n + Integer(1))
>>> E = f(Integer(3),Integer(10)); E
From 3 to 10
>>> E.list()
[3, 4, 5, 6, 7, 8, 9, 10]
>>> E = f(Integer(1),Integer(100)); E
From 1 to 100
>>> E.cardinality()
100
>>> f(n=Integer(100), m=Integer(1)) == E
True
```

An example which mixes together *set_from_function()* and cached_method():

```
sage: @cached_function
....: @set_from_function(name="Graphs on %(n)d vertices",
....:                    category=FiniteEnumeratedSets(), cache=True)
....: def Graphs(n): return graphs(n)
sage: Graphs(10)                                                            #␣
↪needs sage.graphs
Graphs on 10 vertices
sage: Graphs(10).unrank(0)                                                  #␣
↪needs sage.graphs
Graph on 10 vertices
sage: Graphs(10) is Graphs(10)                                              #␣
↪needs sage.graphs
True
```

```
>>> from sage.all import *
>>> @cached_function
... @set_from_function(name="Graphs on %(n)d vertices",
...                    category=FiniteEnumeratedSets(), cache=True)
... def Graphs(n): return graphs(n)
>>> Graphs(Integer(10))                                                      ␣
↪     # needs sage.graphs
Graphs on 10 vertices
>>> Graphs(Integer(10)).unrank(Integer(0))                                   ␣
↪              # needs sage.graphs
Graph on 10 vertices
>>> Graphs(Integer(10)) is Graphs(Integer(10))                               ␣
↪              # needs sage.graphs
True
```

The @cached_function must go first:

```
sage: @set_from_function(name="Graphs on %(n)d vertices",
....:                    category=FiniteEnumeratedSets(), cache=True)
....: @cached_function
....: def Graphs(n): return graphs(n)
sage: Graphs(10)                                                            #␣
↪needs sage.graphs
Graphs on 10 vertices
sage: Graphs(10).unrank(0)                                                  #␣
↪needs sage.graphs
Graph on 10 vertices
sage: Graphs(10) is Graphs(10)                                              #␣
↪needs sage.graphs
False
```

```
>>> from sage.all import *
>>> @set_from_function(name="Graphs on %(n)d vertices",
...                    category=FiniteEnumeratedSets(), cache=True)
... @cached_function
... def Graphs(n): return graphs(n)
>>> Graphs(Integer(10))                                                      ␣
```

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```
↪     # needs sage.graphs
Graphs on 10 vertices
>>> Graphs(Integer(10)).unrank(Integer(0))                                    ␣
↪              # needs sage.graphs
Graph on 10 vertices
>>> Graphs(Integer(10)) is Graphs(Integer(10))                                ␣
↪              # needs sage.graphs
False
```

class sage.sets.set_from_iterator.**EnumeratedSetFromIterator_method_caller**(*inst*, *f*, *name=None*, ***options*)

Bases: *Decorator*

Caller for decorated method in class.

INPUT:

- `inst` – an instance of a class
- `f` – a method of a class of `inst` (and not of the instance itself)
- `name` – (optional) either a string (which may contains substitution rules from argument or a function `args, kwds -> string`
- `options` – any option accepted by *EnumeratedSetFromIterator*

class sage.sets.set_from_iterator.**EnumeratedSetFromIterator_method_decorator**(*f=None*, ***options*)

Bases: `object`

Decorator for enumerated set built from a method.

INPUT:

- `f` – (optional) function from which are built the enumerated sets at each call
- `name` – (optional) string (which may contains substitution rules from argument) or a function `(args,kwds) -> string`
- any option accepted by *EnumeratedSetFromIterator*.

EXAMPLES:

```
sage: from sage.sets.set_from_iterator import set_from_method
sage: class A():
....:     def n(self): return 12
....:     @set_from_method
....:     def f(self): return xsrange(self.n())
sage: a = A()
sage: print(a.f.__class__)
<class 'sage.sets.set_from_iterator.EnumeratedSetFromIterator_method_caller'>
sage: a.f()
{0, 1, 2, 3, 4, ...}
sage: A.f(a)
{0, 1, 2, 3, 4, ...}
```

```
>>> from sage.all import *
>>> from sage.sets.set_from_iterator import set_from_method
>>> class A():
...     def n(self): return Integer(12)
...     @set_from_method
...     def f(self): return xsrange(self.n())
>>> a = A()
>>> print(a.f.__class__)
<class 'sage.sets.set_from_iterator.EnumeratedSetFromIterator_method_caller'>
>>> a.f()
{0, 1, 2, 3, 4, ...}
>>> A.f(a)
{0, 1, 2, 3, 4, ...}
```

A more complicated example with a parametrized name:

```
sage: class B():
....:     @set_from_method(name="Graphs(%(n)d)",
....:                      category=FiniteEnumeratedSets())
....:     def graphs(self, n): return graphs(n)
sage: b = B()
sage: G3 = b.graphs(3); G3
Graphs(3)
sage: G3.cardinality()                                                  #␣
↪needs sage.graphs
4
sage: G3.category()
Category of facade finite enumerated sets
sage: B.graphs(b, 3)
Graphs(3)
```

```
>>> from sage.all import *
>>> class B():
...     @set_from_method(name="Graphs(%(n)d)",
...                      category=FiniteEnumeratedSets())
...     def graphs(self, n): return graphs(n)
>>> b = B()
>>> G3 = b.graphs(Integer(3)); G3
Graphs(3)
>>> G3.cardinality()                                                    #␣
↪needs sage.graphs
4
>>> G3.category()
Category of facade finite enumerated sets
>>> B.graphs(b, Integer(3))
Graphs(3)
```

And a last example with a name parametrized by a function:

```
sage: class D():
....:     def __init__(self, name): self.name = str(name)
....:     def __str__(self): return self.name
....:     @set_from_method(name=lambda self, n: str(self) * n,
```

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```
....:                       category=FiniteEnumeratedSets())
....:     def subset(self, n):
....:         return xsrange(n)
sage: d = D('a')
sage: E = d.subset(3); E
aaa
sage: E.list()
[0, 1, 2]
sage: F = d.subset(n=10); F
aaaaaaaaaa
sage: F.list()
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]
```

```
>>> from sage.all import *
>>> class D():
...     def __init__(self, name): self.name = str(name)
...     def __str__(self): return self.name
...     @set_from_method(name=lambda self, n: str(self) * n,
...                      category=FiniteEnumeratedSets())
...     def subset(self, n):
...         return xsrange(n)
>>> d = D('a')
>>> E = d.subset(Integer(3)); E
aaa
>>> E.list()
[0, 1, 2]
>>> F = d.subset(n=Integer(10)); F
aaaaaaaaaa
>>> F.list()
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]
```

Todo

It is not yet possible to use `set_from_method` in conjunction with `cached_method`.

`sage.sets.set_from_iterator.`**`set_from_function`**

alias of *EnumeratedSetFromIterator_function_decorator*

`sage.sets.set_from_iterator.`**`set_from_method`**

alias of *EnumeratedSetFromIterator_method_decorator*

1.7 Finite Enumerated Sets

class `sage.sets.finite_enumerated_set.`**`FiniteEnumeratedSet`**(*elements*)

Bases: `UniqueRepresentation`, `Parent`

A class for finite enumerated set.

Returns the finite enumerated set with elements in `elements` where `element` is any (finite) iterable object.

The main purpose is to provide a variant of `list` or `tuple`, which is a parent with an interface consistent with `EnumeratedSets` and has unique representation. The list of the elements is expanded in memory.

EXAMPLES:

```
sage: S = FiniteEnumeratedSet([1, 2, 3])
sage: S
{1, 2, 3}
sage: S.list()
[1, 2, 3]
sage: S.cardinality()
3
sage: S.random_element()  # random
1
sage: S.first()
1
sage: S.category()
Category of facade finite enumerated sets
sage: TestSuite(S).run()
```

```
>>> from sage.all import *
>>> S = FiniteEnumeratedSet([Integer(1), Integer(2), Integer(3)])
>>> S
{1, 2, 3}
>>> S.list()
[1, 2, 3]
>>> S.cardinality()
3
>>> S.random_element()  # random
1
>>> S.first()
1
>>> S.category()
Category of facade finite enumerated sets
>>> TestSuite(S).run()
```

Note that being an enumerated set, the result depends on the order:

```
sage: S1 = FiniteEnumeratedSet((1, 2, 3))
sage: S1
{1, 2, 3}
sage: S1.list()
[1, 2, 3]
sage: S1 == S
True
sage: S2 = FiniteEnumeratedSet((2, 1, 3))
sage: S2 == S
False
```

```
>>> from sage.all import *
>>> S1 = FiniteEnumeratedSet((Integer(1), Integer(2), Integer(3)))
>>> S1
{1, 2, 3}
>>> S1.list()
[1, 2, 3]
>>> S1 == S
```

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```
True
>>> S2 = FiniteEnumeratedSet((Integer(2), Integer(1), Integer(3)))
>>> S2 == S
False
```

As an abuse, repeated entries in `elements` are allowed to model multisets:

```
sage: S1 = FiniteEnumeratedSet((1, 2, 1, 2, 2, 3))
sage: S1
{1, 2, 1, 2, 2, 3}
```

```
>>> from sage.all import *
>>> S1 = FiniteEnumeratedSet((Integer(1), Integer(2), Integer(1), Integer(2),␣
↪Integer(2), Integer(3)))
>>> S1
{1, 2, 1, 2, 2, 3}
```

Finally, the elements are not aware of their parent:

```
sage: S.first().parent()
Integer Ring
```

```
>>> from sage.all import *
>>> S.first().parent()
Integer Ring
```

an_element()

cardinality()

first()

Return the first element of the enumeration or raise an `EmptySetError` if the set is empty.

EXAMPLES:

```
sage: S = FiniteEnumeratedSet('abc')
sage: S.first()
'a'
```

```
>>> from sage.all import *
>>> S = FiniteEnumeratedSet('abc')
>>> S.first()
'a'
```

index(x)

Return the index of `x` in this finite enumerated set.

EXAMPLES:

```
sage: S = FiniteEnumeratedSet(['a','b','c'])
sage: S.index('b')
1
```

```
>>> from sage.all import *
>>> S = FiniteEnumeratedSet(['a','b','c'])
>>> S.index('b')
1
```

is_parent_of(*x*)

last()

Return the last element of the iteration or raise an `EmptySetError` if the set is empty.

EXAMPLES:

```
sage: S = FiniteEnumeratedSet([0,'a',1.23, 'd'])
sage: S.last()
'd'
```

```
>>> from sage.all import *
>>> S = FiniteEnumeratedSet([Integer(0),'a',RealNumber('1.23'), 'd'])
>>> S.last()
'd'
```

list()

random_element()

Return a random element.

EXAMPLES:

```
sage: S = FiniteEnumeratedSet('abc')
sage: S.random_element()    # random
'b'
```

```
>>> from sage.all import *
>>> S = FiniteEnumeratedSet('abc')
>>> S.random_element()    # random
'b'
```

rank(*x*)

Return the index of `x` in this finite enumerated set.

EXAMPLES:

```
sage: S = FiniteEnumeratedSet(['a','b','c'])
sage: S.index('b')
1
```

```
>>> from sage.all import *
>>> S = FiniteEnumeratedSet(['a','b','c'])
>>> S.index('b')
1
```

unrank(*i*)

Return the element at position `i`.

EXAMPLES:

```
sage: S = FiniteEnumeratedSet([1,'a',-51])
sage: S[0], S[1], S[2]
(1, 'a', -51)
sage: S[3]
Traceback (most recent call last):
...
IndexError: tuple index out of range
sage: S[-1], S[-2], S[-3]
(-51, 'a', 1)
sage: S[-4]
Traceback (most recent call last):
...
IndexError: index out of range
```

```
>>> from sage.all import *
>>> S = FiniteEnumeratedSet([Integer(1),'a',-Integer(51)])
>>> S[Integer(0)], S[Integer(1)], S[Integer(2)]
(1, 'a', -51)
>>> S[Integer(3)]
Traceback (most recent call last):
...
IndexError: tuple index out of range
>>> S[-Integer(1)], S[-Integer(2)], S[-Integer(3)]
(-51, 'a', 1)
>>> S[-Integer(4)]
Traceback (most recent call last):
...
IndexError: index out of range
```

1.8 Recursively Enumerated Sets

A set S is called recursively enumerable if there is an algorithm that enumerates the members of S. We consider here the recursively enumerated sets that are described by some `seeds` and a successor function `successors`. The successor function may have some structure (symmetric, graded, forest) or not. The elements of a set having a symmetric, graded or forest structure can be enumerated uniquely without keeping all of them in memory. Many kinds of iterators are provided in this module: depth first search, breadth first search and elements of given depth.

See Wikipedia article Recursively_enumerable_set.

See the documentation of *RecursivelyEnumeratedSet()* below for the description of the inputs.

AUTHORS:

- Sébastien Labbé, April 2014, at Sage Days 57, Cernay-la-ville

EXAMPLES:

No hypothesis on the structure

What we mean by "no hypothesis" is that the set is not known to be a forest, symmetric, or graded. However, it may have other structure, such as not containing an oriented cycle, that does not help with the enumeration.

In this example, the seed is 0 and the successor function is either `+2` or `+3`. This is the set of nonnegative linear combinations of 2 and 3:

```
sage: succ = lambda a:[a+2,a+3]
sage: C = RecursivelyEnumeratedSet([0], succ)
sage: C
A recursively enumerated set (breadth first search)
```

```
>>> from sage.all import *
>>> succ = lambda a:[a+Integer(2),a+Integer(3)]
>>> C = RecursivelyEnumeratedSet([Integer(0)], succ)
>>> C
A recursively enumerated set (breadth first search)
```

Breadth first search:

```
sage: it = C.breadth_first_search_iterator()
sage: [next(it) for _ in range(10)]
[0, 2, 3, 4, 5, 6, 7, 8, 9, 10]
```

```
>>> from sage.all import *
>>> it = C.breadth_first_search_iterator()
>>> [next(it) for _ in range(Integer(10))]
[0, 2, 3, 4, 5, 6, 7, 8, 9, 10]
```

Depth first search:

```
sage: it = C.depth_first_search_iterator()
sage: [next(it) for _ in range(10)]
[0, 3, 6, 9, 12, 15, 18, 21, 24, 27]
```

```
>>> from sage.all import *
>>> it = C.depth_first_search_iterator()
>>> [next(it) for _ in range(Integer(10))]
[0, 3, 6, 9, 12, 15, 18, 21, 24, 27]
```

Symmetric structure

The origin `(0, 0)` as seed and the upper, lower, left and right lattice point as successor function. This function is symmetric since p is a successor of q if and only if q is a successor or p:

```
sage: succ = lambda a: [(a[0]-1,a[1]), (a[0],a[1]-1), (a[0]+1,a[1]), (a[0],a[1]+1)]
sage: seeds = [(0,0)]
sage: C = RecursivelyEnumeratedSet(seeds, succ, structure='symmetric', enumeration=
↪'depth')
sage: C
A recursively enumerated set with a symmetric structure (depth first search)
```

```
>>> from sage.all import *
>>> succ = lambda a: [(a[Integer(0)]-Integer(1),a[Integer(1)]), (a[Integer(0)],
↪a[Integer(1)]-Integer(1)), (a[Integer(0)]+Integer(1),a[Integer(1)]), (a[Integer(0)],
↪a[Integer(1)]+Integer(1))]
>>> seeds = [(Integer(0),Integer(0))]
>>> C = RecursivelyEnumeratedSet(seeds, succ, structure='symmetric', enumeration=
↪'depth')
```

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```
>>> C
A recursively enumerated set with a symmetric structure (depth first search)
```

In this case, depth first search is the default enumeration for iteration:

```
sage: it_depth = iter(C)
sage: [next(it_depth) for _ in range(10)]
[(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (0, 6), (0, 7), (0, 8), (0, 9)]
```

```
>>> from sage.all import *
>>> it_depth = iter(C)
>>> [next(it_depth) for _ in range(Integer(10))]
[(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (0, 6), (0, 7), (0, 8), (0, 9)]
```

Breadth first search:

```
sage: it_breadth = C.breadth_first_search_iterator()
sage: [next(it_breadth) for _ in range(13)]
[(0, 0),
 (-1, 0), (0, -1), (1, 0), (0, 1),
 (-2, 0), (-1, -1), (-1, 1), (0, -2), (1, -1), (2, 0), (1, 1), (0, 2)]
```

```
>>> from sage.all import *
>>> it_breadth = C.breadth_first_search_iterator()
>>> [next(it_breadth) for _ in range(Integer(13))]
[(0, 0),
 (-1, 0), (0, -1), (1, 0), (0, 1),
 (-2, 0), (-1, -1), (-1, 1), (0, -2), (1, -1), (2, 0), (1, 1), (0, 2)]
```

Levels (elements of given depth):

```
sage: sorted(C.graded_component(0))
[(0, 0)]
sage: sorted(C.graded_component(1))
[(-1, 0), (0, -1), (0, 1), (1, 0)]
sage: sorted(C.graded_component(2))
[(-2, 0), (-1, -1), (-1, 1), (0, -2), (0, 2), (1, -1), (1, 1), (2, 0)]
```

```
>>> from sage.all import *
>>> sorted(C.graded_component(Integer(0)))
[(0, 0)]
>>> sorted(C.graded_component(Integer(1)))
[(-1, 0), (0, -1), (0, 1), (1, 0)]
>>> sorted(C.graded_component(Integer(2)))
[(-2, 0), (-1, -1), (-1, 1), (0, -2), (0, 2), (1, -1), (1, 1), (2, 0)]
```

Graded structure

Identity permutation as seed and `permutohedron_succ` as successor function:

```
sage: succ = attrcall("permutohedron_succ")
sage: seed = [Permutation([1..5])]
```

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```
sage: R = RecursivelyEnumeratedSet(seed, succ, structure='graded')
sage: R
A recursively enumerated set with a graded structure (breadth first search)
```

```
>>> from sage.all import *
>>> succ = attrcall("permutohedron_succ")
>>> seed = [Permutation((ellipsis_range(Integer(1),Ellipsis,Integer(5))))]
>>> R = RecursivelyEnumeratedSet(seed, succ, structure='graded')
>>> R
A recursively enumerated set with a graded structure (breadth first search)
```

Depth first search iterator:

```
sage: it_depth = R.depth_first_search_iterator()
sage: [next(it_depth) for _ in range(5)]
[[1, 2, 3, 4, 5],
 [1, 2, 3, 5, 4],
 [1, 2, 5, 3, 4],
 [1, 2, 5, 4, 3],
 [1, 5, 2, 4, 3]]
```

```
>>> from sage.all import *
>>> it_depth = R.depth_first_search_iterator()
>>> [next(it_depth) for _ in range(Integer(5))]
[[1, 2, 3, 4, 5],
 [1, 2, 3, 5, 4],
 [1, 2, 5, 3, 4],
 [1, 2, 5, 4, 3],
 [1, 5, 2, 4, 3]]
```

Breadth first search iterator:

```
sage: it_breadth = R.breadth_first_search_iterator()
sage: [next(it_breadth) for _ in range(5)]
[[1, 2, 3, 4, 5],
 [2, 1, 3, 4, 5],
 [1, 3, 2, 4, 5],
 [1, 2, 4, 3, 5],
 [1, 2, 3, 5, 4]]
```

```
>>> from sage.all import *
>>> it_breadth = R.breadth_first_search_iterator()
>>> [next(it_breadth) for _ in range(Integer(5))]
[[1, 2, 3, 4, 5],
 [2, 1, 3, 4, 5],
 [1, 3, 2, 4, 5],
 [1, 2, 4, 3, 5],
 [1, 2, 3, 5, 4]]
```

Elements of given depth iterator:

```
sage: sorted(R.elements_of_depth_iterator(9))
[[4, 5, 3, 2, 1], [5, 3, 4, 2, 1], [5, 4, 2, 3, 1], [5, 4, 3, 1, 2]]
sage: list(R.elements_of_depth_iterator(10))
[[5, 4, 3, 2, 1]]
```

```
>>> from sage.all import *
>>> sorted(R.elements_of_depth_iterator(Integer(9)))
[[4, 5, 3, 2, 1], [5, 3, 4, 2, 1], [5, 4, 2, 3, 1], [5, 4, 3, 1, 2]]
>>> list(R.elements_of_depth_iterator(Integer(10)))
[[5, 4, 3, 2, 1]]
```

Graded components (set of elements of the same depth):

```
sage: # needs sage.combinat
sage: sorted(R.graded_component(0))
[[1, 2, 3, 4, 5]]
sage: sorted(R.graded_component(1))
[[1, 2, 3, 5, 4], [1, 2, 4, 3, 5], [1, 3, 2, 4, 5], [2, 1, 3, 4, 5]]
sage: sorted(R.graded_component(9))
[[4, 5, 3, 2, 1], [5, 3, 4, 2, 1], [5, 4, 2, 3, 1], [5, 4, 3, 1, 2]]
sage: sorted(R.graded_component(10))
[[5, 4, 3, 2, 1]]
```

```
>>> from sage.all import *
>>> # needs sage.combinat
>>> sorted(R.graded_component(Integer(0)))
[[1, 2, 3, 4, 5]]
>>> sorted(R.graded_component(Integer(1)))
[[1, 2, 3, 5, 4], [1, 2, 4, 3, 5], [1, 3, 2, 4, 5], [2, 1, 3, 4, 5]]
>>> sorted(R.graded_component(Integer(9)))
[[4, 5, 3, 2, 1], [5, 3, 4, 2, 1], [5, 4, 2, 3, 1], [5, 4, 3, 1, 2]]
>>> sorted(R.graded_component(Integer(10)))
[[5, 4, 3, 2, 1]]
```

Forest structure (Example 1)

The set of words over the alphabet $\{a, b\}$ can be generated from the empty word by appending the letter a or b as a successor function. This set has a forest structure:

```
sage: seeds = ['']
sage: succ = lambda w: [w+'a', w+'b']
sage: C = RecursivelyEnumeratedSet(seeds, succ, structure='forest')
sage: C
An enumerated set with a forest structure
```

```
>>> from sage.all import *
>>> seeds = ['']
>>> succ = lambda w: [w+'a', w+'b']
>>> C = RecursivelyEnumeratedSet(seeds, succ, structure='forest')
>>> C
An enumerated set with a forest structure
```

Depth first search iterator:

```
sage: it = C.depth_first_search_iterator()
sage: [next(it) for _ in range(6)]
['', 'a', 'aa', 'aaa', 'aaaa', 'aaaaa']
```

```
>>> from sage.all import *
>>> it = C.depth_first_search_iterator()
>>> [next(it) for _ in range(Integer(6))]
['', 'a', 'aa', 'aaa', 'aaaa', 'aaaaa']
```

Breadth first search iterator:

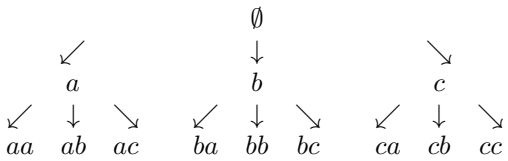
```
sage: it = C.breadth_first_search_iterator()
sage: [next(it) for _ in range(6)]
['', 'a', 'b', 'aa', 'ab', 'ba']
```

```
>>> from sage.all import *
>>> it = C.breadth_first_search_iterator()
>>> [next(it) for _ in range(Integer(6))]
['', 'a', 'b', 'aa', 'ab', 'ba']
```

This example was provided by Florent Hivert.

How to define a set using those classes?

Only two things are necessary to define a set using a *RecursivelyEnumeratedSet* object (the other classes being very similar):



For the previous example, the two necessary pieces of information are:

- the initial element `""`;
- the function:

  ```
  lambda x: [x + letter for letter in ['a', 'b', 'c']
  ```

This would actually describe an **infinite** set, as such rules describe "all words" on 3 letters. Hence, it is a good idea to replace the function by:

```
lambda x: [x + letter for letter in ['a', 'b', 'c']] if len(x) < 2 else []
```

or even:

```
sage: def children(x):
....:     if len(x) < 2:
....:         for letter in ['a', 'b', 'c']:
....:             yield x+letter
```

```
>>> from sage.all import *
>>> def children(x):
```

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```
...     if len(x) < Integer(2):
...         for letter in ['a', 'b', 'c']:
...             yield x+letter
```

We can then create the *RecursivelyEnumeratedSet* object with either:

```
sage: S = RecursivelyEnumeratedSet([''],
....:     lambda x: [x + letter for letter in ['a', 'b', 'c']]
....:               if len(x) < 2 else [],
....:     structure='forest', enumeration='depth',
....:     category=FiniteEnumeratedSets())
sage: S.list()
['', 'a', 'aa', 'ab', 'ac', 'b', 'ba', 'bb', 'bc', 'c', 'ca', 'cb', 'cc']
```

```
>>> from sage.all import *
>>> S = RecursivelyEnumeratedSet([''],
...     lambda x: [x + letter for letter in ['a', 'b', 'c']]
...               if len(x) < Integer(2) else [],
...     structure='forest', enumeration='depth',
...     category=FiniteEnumeratedSets())
>>> S.list()
['', 'a', 'aa', 'ab', 'ac', 'b', 'ba', 'bb', 'bc', 'c', 'ca', 'cb', 'cc']
```

or:

```
sage: S = RecursivelyEnumeratedSet([''], children,
....:     structure='forest', enumeration='depth',
....:     category=FiniteEnumeratedSets())
sage: S.list()
['', 'a', 'aa', 'ab', 'ac', 'b', 'ba', 'bb', 'bc', 'c', 'ca', 'cb', 'cc']
```

```
>>> from sage.all import *
>>> S = RecursivelyEnumeratedSet([''], children,
...     structure='forest', enumeration='depth',
...     category=FiniteEnumeratedSets())
>>> S.list()
['', 'a', 'aa', 'ab', 'ac', 'b', 'ba', 'bb', 'bc', 'c', 'ca', 'cb', 'cc']
```

Forest structure (Example 2)

This example was provided by Florent Hivert.

Here is a little more involved example. We want to iterate through all permutations of a given set S. One solution is to take elements of S one by one and insert them at every position. So a node of the generating tree contains two pieces of information:

- the list `lst` of already inserted element;
- the set `st` of the yet to be inserted element.

We want to generate a permutation only if `st` is empty (leaves on the tree). Also suppose for the sake of the example, that instead of list we want to generate tuples. This selection of some nodes and final mapping of a function to the element is done by the `post_process = f` argument. The convention is that the generated elements are the `s := f(n)`, except when `s` not `None` when no element is generated at all. Here is the code:

```
sage: def children(node):
....:     (lst, st) = node
....:     st = set(st) # make a copy
....:     if st:
....:        el = st.pop()
....:        for i in range(len(lst) + 1):
....:            yield (lst[0:i] + [el] + lst[i:], st)
sage: list(children(([1,2], {3,7,9})))
[([9, 1, 2], {3, 7}), ([1, 9, 2], {3, 7}), ([1, 2, 9], {3, 7})]
sage: def post_process(node):
....:     (l, s) = node
....:     return tuple(l) if not s else None
sage: S = RecursivelyEnumeratedSet( [([], {1,3,6,8})],
....:     children, post_process=post_process,
....:     structure='forest', enumeration='depth',
....:     category=FiniteEnumeratedSets())
sage: S.list()
[(6, 3, 1, 8), (3, 6, 1, 8), (3, 1, 6, 8), (3, 1, 8, 6), (6, 1, 3, 8),
 (1, 6, 3, 8), (1, 3, 6, 8), (1, 3, 8, 6), (6, 1, 8, 3), (1, 6, 8, 3),
 (1, 8, 6, 3), (1, 8, 3, 6), (6, 3, 8, 1), (3, 6, 8, 1), (3, 8, 6, 1),
 (3, 8, 1, 6), (6, 8, 3, 1), (8, 6, 3, 1), (8, 3, 6, 1), (8, 3, 1, 6),
 (6, 8, 1, 3), (8, 6, 1, 3), (8, 1, 6, 3), (8, 1, 3, 6)]
sage: S.cardinality()
24
```

```
>>> from sage.all import *
>>> def children(node):
...     (lst, st) = node
...     st = set(st) # make a copy
...     if st:
...        el = st.pop()
...        for i in range(len(lst) + Integer(1)):
...            yield (lst[Integer(0):i] + [el] + lst[i:], st)
>>> list(children(([Integer(1),Integer(2)], {Integer(3),Integer(7),Integer(9)})))
[([9, 1, 2], {3, 7}), ([1, 9, 2], {3, 7}), ([1, 2, 9], {3, 7})]
>>> def post_process(node):
...     (l, s) = node
...     return tuple(l) if not s else None
>>> S = RecursivelyEnumeratedSet( [([], {Integer(1),Integer(3),Integer(6),Integer(8)}
↪)],
...     children, post_process=post_process,
...     structure='forest', enumeration='depth',
...     category=FiniteEnumeratedSets())
>>> S.list()
[(6, 3, 1, 8), (3, 6, 1, 8), (3, 1, 6, 8), (3, 1, 8, 6), (6, 1, 3, 8),
 (1, 6, 3, 8), (1, 3, 6, 8), (1, 3, 8, 6), (6, 1, 8, 3), (1, 6, 8, 3),
 (1, 8, 6, 3), (1, 8, 3, 6), (6, 3, 8, 1), (3, 6, 8, 1), (3, 8, 6, 1),
 (3, 8, 1, 6), (6, 8, 3, 1), (8, 6, 3, 1), (8, 3, 6, 1), (8, 3, 1, 6),
 (6, 8, 1, 3), (8, 6, 1, 3), (8, 1, 6, 3), (8, 1, 3, 6)]
>>> S.cardinality()
24
```

sage.sets.recursively_enumerated_set.**RecursivelyEnumeratedSet**(*seeds*, *successors*, *structure=None*, *enumeration=None*, *max_depth=None*, *post_process=None*, *facade=None*, *category=None*)

Return a recursively enumerated set.

A set S is called recursively enumerable if there is an algorithm that enumerates the members of S. We consider here the recursively enumerated sets that are described by some `seeds` and a successor function `successors`.

Let U be a set and `successors` : $U \to 2^U$ be a successor function associating to each element of U a subset of U. Let `seeds` be a subset of U. Let $S \subseteq U$ be the set of elements of U that can be reached from a seed by applying recursively the `successors` function. This class provides different kinds of iterators (breadth first, depth first, elements of given depth, etc.) for the elements of S.

See Wikipedia article Recursively_enumerable_set.

INPUT:

- `seeds` – list (or iterable) of hashable objects
- `successors` – function (or callable) returning a list (or iterable) of hashable objects
- `structure` – string (default: `None`); structure of the set, possible values are:
  - `None` – nothing is known about the structure of the set
  - `'forest'` – if the `successors` function generates a *forest*, that is, each element can be reached uniquely from a seed
  - `'graded'` – if the `successors` function is *graded*, that is, all paths from a seed to a given element have equal length
  - `'symmetric'` – if the relation is *symmetric*, that is, `y in successors(x)` if and only if `x in successors(y)`
- `enumeration` – `'depth'`, `'breadth'`, `'naive'` or `None` (default: `None`); the default enumeration for the `__iter__` function
- `max_depth` – integer (default: `float("inf")`); limit the search to a certain depth, currently works only for breadth first search
- `post_process` – (default: `None`) for forest only
- `facade` – (default: `None`)
- `category` – (default: `None`)

EXAMPLES:

A recursive set with no other information:

```
sage: f = lambda a: [a+3, a+5]
sage: C = RecursivelyEnumeratedSet([0], f)
sage: C
A recursively enumerated set (breadth first search)
sage: it = iter(C)
sage: [next(it) for _ in range(10)]
[0, 3, 5, 6, 8, 10, 9, 11, 13, 15]
```

```
>>> from sage.all import *
>>> f = lambda a: [a+Integer(3), a+Integer(5)]
>>> C = RecursivelyEnumeratedSet([Integer(0)], f)
>>> C
A recursively enumerated set (breadth first search)
>>> it = iter(C)
>>> [next(it) for _ in range(Integer(10))]
[0, 3, 5, 6, 8, 10, 9, 11, 13, 15]
```

A recursive set with a forest structure:

```
sage: f = lambda a: [2*a,2*a+1]
sage: C = RecursivelyEnumeratedSet([1], f, structure='forest'); C
An enumerated set with a forest structure
sage: it = C.depth_first_search_iterator()
sage: [next(it) for _ in range(7)]
[1, 2, 4, 8, 16, 32, 64]
sage: it = C.breadth_first_search_iterator()
sage: [next(it) for _ in range(7)]
[1, 2, 3, 4, 5, 6, 7]
```

```
>>> from sage.all import *
>>> f = lambda a: [Integer(2)*a,Integer(2)*a+Integer(1)]
>>> C = RecursivelyEnumeratedSet([Integer(1)], f, structure='forest'); C
An enumerated set with a forest structure
>>> it = C.depth_first_search_iterator()
>>> [next(it) for _ in range(Integer(7))]
[1, 2, 4, 8, 16, 32, 64]
>>> it = C.breadth_first_search_iterator()
>>> [next(it) for _ in range(Integer(7))]
[1, 2, 3, 4, 5, 6, 7]
```

A recursive set given by a symmetric relation:

```
sage: f = lambda a: [a-1,a+1]
sage: C = RecursivelyEnumeratedSet([10, 15], f, structure='symmetric')
sage: C
A recursively enumerated set with a symmetric structure (breadth first search)
sage: it = iter(C)
sage: [next(it) for _ in range(7)]
[10, 15, 9, 11, 14, 16, 8]
```

```
>>> from sage.all import *
>>> f = lambda a: [a-Integer(1),a+Integer(1)]
>>> C = RecursivelyEnumeratedSet([Integer(10), Integer(15)], f, structure=
↪'symmetric')
>>> C
A recursively enumerated set with a symmetric structure (breadth first search)
>>> it = iter(C)
>>> [next(it) for _ in range(Integer(7))]
[10, 15, 9, 11, 14, 16, 8]
```

A recursive set given by a graded relation:

```
sage: # needs sage.symbolic
sage: def f(a):
....:     return [a + 1, a + I]
sage: C = RecursivelyEnumeratedSet([0], f, structure='graded'); C
A recursively enumerated set with a graded structure (breadth first search)
sage: it = iter(C)
sage: [next(it) for _ in range(7)]
[0, 1, I, 2, I + 1, 2*I, 3]
```

```
>>> from sage.all import *
>>> # needs sage.symbolic
>>> def f(a):
...     return [a + Integer(1), a + I]
>>> C = RecursivelyEnumeratedSet([Integer(0)], f, structure='graded'); C
A recursively enumerated set with a graded structure (breadth first search)
>>> it = iter(C)
>>> [next(it) for _ in range(Integer(7))]
[0, 1, I, 2, I + 1, 2*I, 3]
```

Warning

If you do not set a good structure, you might obtain bad results, like elements generated twice:

```
sage: f = lambda a: [a-1,a+1]
sage: C = RecursivelyEnumeratedSet([0], f, structure='graded')
sage: it = iter(C)
sage: [next(it) for _ in range(7)]
[0, -1, 1, -2, 0, 2, -3]
```

```
>>> from sage.all import *
>>> f = lambda a: [a-Integer(1),a+Integer(1)]
>>> C = RecursivelyEnumeratedSet([Integer(0)], f, structure='graded')
>>> it = iter(C)
>>> [next(it) for _ in range(Integer(7))]
[0, -1, 1, -2, 0, 2, -3]
```

class sage.sets.recursively_enumerated_set.**RecursivelyEnumeratedSet_forest**(*roots=None*, *children=None*, *post_process=None*, *algorithm='depth'*, *facade=None*, *category=None*)

Bases: `Parent`

The enumerated set of the nodes of the forest having the given `roots`, and where `children(x)` returns the children of the node `x` of the forest.

See also `sage.combinat.backtrack.GenericBacktracker`, *RecursivelyEnumeratedSet_graded*, and *RecursivelyEnumeratedSet_symmetric*.

INPUT:

- `roots` – list (or iterable)
- `children` – a function returning a list (or iterable, or iterator)
- `post_process` – a function defined over the nodes of the forest (default: no post processing)
- `algorithm` – `'depth'` or `'breadth'` (default: `'depth'`)
- `category` – a category (default: `EnumeratedSets`)

The option `post_process` allows for customizing the nodes that are actually produced. Furthermore, if `f(x)` returns `None`, then `x` won't be output at all.

EXAMPLES:

We construct the set of all binary sequences of length at most three, and list them:

```
sage: from sage.sets.recursively_enumerated_set import RecursivelyEnumeratedSet_
↪forest
sage: S = RecursivelyEnumeratedSet_forest( [[]],
....:     lambda l: [l + [0], l + [1]] if len(l) < 3 else [],
....:     category=FiniteEnumeratedSets())
sage: S.list()
[[],
 [0], [0, 0], [0, 0, 0], [0, 0, 1], [0, 1], [0, 1, 0], [0, 1, 1],
 [1], [1, 0], [1, 0, 0], [1, 0, 1], [1, 1], [1, 1, 0], [1, 1, 1]]
```

```
>>> from sage.all import *
>>> from sage.sets.recursively_enumerated_set import RecursivelyEnumeratedSet_
↪forest
>>> S = RecursivelyEnumeratedSet_forest( [[]],
...     lambda l: [l + [Integer(0)], l + [Integer(1)]] if len(l) < Integer(3)␣
↪else [],
...     category=FiniteEnumeratedSets())
>>> S.list()
[[],
 [0], [0, 0], [0, 0, 0], [0, 0, 1], [0, 1], [0, 1, 0], [0, 1, 1],
 [1], [1, 0], [1, 0, 0], [1, 0, 1], [1, 1], [1, 1, 0], [1, 1, 1]]
```

`RecursivelyEnumeratedSet_forest` needs to be explicitly told that the set is finite for the following to work:

```
sage: S.category()
Category of finite enumerated sets
sage: S.cardinality()
15
```

```
>>> from sage.all import *
>>> S.category()
Category of finite enumerated sets
>>> S.cardinality()
15
```

We proceed with the set of all lists of letters in `0,1,2` without repetitions, ordered by increasing length (i.e. using a breadth first search through the tree):

```
sage: from sage.sets.recursively_enumerated_set import RecursivelyEnumeratedSet_
↪forest
sage: tb = RecursivelyEnumeratedSet_forest( [[]],
....:       lambda l: [l + [i] for i in range(3) if i not in l],
....:       algorithm = 'breadth',
....:       category=FiniteEnumeratedSets())
sage: tb[0]
[]
sage: tb.cardinality()
16
sage: list(tb)
[[],
 [0], [1], [2],
 [0, 1], [0, 2], [1, 0], [1, 2], [2, 0], [2, 1],
 [0, 1, 2], [0, 2, 1], [1, 0, 2], [1, 2, 0], [2, 0, 1], [2, 1, 0]]
```

```
>>> from sage.all import *
>>> from sage.sets.recursively_enumerated_set import RecursivelyEnumeratedSet_
↪forest
>>> tb = RecursivelyEnumeratedSet_forest( [[]],
...       lambda l: [l + [i] for i in range(Integer(3)) if i not in l],
...       algorithm = 'breadth',
...       category=FiniteEnumeratedSets())
>>> tb[Integer(0)]
[]
>>> tb.cardinality()
16
>>> list(tb)
[[],
 [0], [1], [2],
 [0, 1], [0, 2], [1, 0], [1, 2], [2, 0], [2, 1],
 [0, 1, 2], [0, 2, 1], [1, 0, 2], [1, 2, 0], [2, 0, 1], [2, 1, 0]]
```

For infinite sets, this option should be set carefully to ensure that all elements are actually generated. The following example builds the set of all ordered pairs (i, j) of nonnegative integers such that $j \leq 1$:

```
sage: from sage.sets.recursively_enumerated_set import RecursivelyEnumeratedSet_
↪forest
sage: I = RecursivelyEnumeratedSet_forest([(0,0)],
....:                  lambda l: [(l[0]+1, l[1]), (l[0], 1)]
....:                            if l[1] == 0 else [(l[0], l[1]+1)])
```

```
>>> from sage.all import *
>>> from sage.sets.recursively_enumerated_set import RecursivelyEnumeratedSet_
↪forest
>>> I = RecursivelyEnumeratedSet_forest([(Integer(0),Integer(0))],
...                  lambda l: [(l[Integer(0)]+Integer(1), l[Integer(1)]),␣
↪(l[Integer(0)], Integer(1))]
...                            if l[Integer(1)] == Integer(0) else␣
↪[(l[Integer(0)], l[Integer(1)]+Integer(1))])
```

With a depth first search, only the elements of the form $(i, 0)$ are generated:

```
sage: depth_search = I.depth_first_search_iterator()
sage: [next(depth_search) for i in range(7)]
[(0, 0), (1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (6, 0)]
```

```
>>> from sage.all import *
>>> depth_search = I.depth_first_search_iterator()
>>> [next(depth_search) for i in range(Integer(7))]
[(0, 0), (1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (6, 0)]
```

Using instead breadth first search gives the usual anti-diagonal iterator:

```
sage: breadth_search = I.breadth_first_search_iterator()
sage: [next(breadth_search) for i in range(15)]
[(0, 0),
 (1, 0), (0, 1),
 (2, 0), (1, 1), (0, 2),
 (3, 0), (2, 1), (1, 2), (0, 3),
 (4, 0), (3, 1), (2, 2), (1, 3), (0, 4)]
```

```
>>> from sage.all import *
>>> breadth_search = I.breadth_first_search_iterator()
>>> [next(breadth_search) for i in range(Integer(15))]
[(0, 0),
 (1, 0), (0, 1),
 (2, 0), (1, 1), (0, 2),
 (3, 0), (2, 1), (1, 2), (0, 3),
 (4, 0), (3, 1), (2, 2), (1, 3), (0, 4)]
```

Deriving subclasses

The class of a parent A may derive from *RecursivelyEnumeratedSet_forest* so that A can benefit from enumeration tools. As a running example, we consider the problem of enumerating integers whose binary expansion have at most three nonzero digits. For example, $3 = 2^1 + 2^0$ has two nonzero digits. $15 = 2^3 + 2^2 + 2^1 + 2^0$ has four nonzero digits. In fact, 15 is the smallest integer which is not in the enumerated set.

To achieve this, we use `RecursivelyEnumeratedSet_forest` to enumerate binary tuples with at most three nonzero digits, apply a post processing to recover the corresponding integers, and discard tuples finishing by zero.

A first approach is to pass the `roots` and `children` functions as arguments to `RecursivelyEnumeratedSet_forest.__init__()`:

```
sage: from sage.sets.recursively_enumerated_set import RecursivelyEnumeratedSet_
↪forest
sage: class A(UniqueRepresentation, RecursivelyEnumeratedSet_forest):
....:     def __init__(self):
....:         RecursivelyEnumeratedSet_forest.__init__(self, [()],
....:             lambda x: [x + (0,), x + (1,)] if sum(x) < 3 else [],
....:             lambda x: sum(x[i]*2^i for i in range(len(x)))
....:                         if sum(x) != 0 and x[-1] != 0 else None,
....:             algorithm='breadth',
....:             category=InfiniteEnumeratedSets())
sage: MyForest = A(); MyForest
An enumerated set with a forest structure
```

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```
sage: MyForest.category()
Category of infinite enumerated sets
sage: p = iter(MyForest)
sage: [next(p) for i in range(30)]
[1, 2, 3, 4, 6, 5, 7, 8, 12, 10, 14, 9, 13, 11, 16, 24,
 20, 28, 18, 26, 22, 17, 25, 21, 19, 32, 48, 40, 56, 36]
```

```
>>> from sage.all import *
>>> from sage.sets.recursively_enumerated_set import RecursivelyEnumeratedSet_
↪forest
>>> class A(UniqueRepresentation, RecursivelyEnumeratedSet_forest):
...     def __init__(self):
...         RecursivelyEnumeratedSet_forest.__init__(self, [()],
...             lambda x: [x + (Integer(0),), x + (Integer(1),)] if sum(x) <␣
↪Integer(3) else [],
...             lambda x: sum(x[i]*Integer(2)**i for i in range(len(x)))
...                       if sum(x) != Integer(0) and x[-Integer(1)] !=␣
↪Integer(0) else None,
...             algorithm='breadth',
...             category=InfiniteEnumeratedSets())
>>> MyForest = A(); MyForest
An enumerated set with a forest structure
>>> MyForest.category()
Category of infinite enumerated sets
>>> p = iter(MyForest)
>>> [next(p) for i in range(Integer(30))]
[1, 2, 3, 4, 6, 5, 7, 8, 12, 10, 14, 9, 13, 11, 16, 24,
 20, 28, 18, 26, 22, 17, 25, 21, 19, 32, 48, 40, 56, 36]
```

An alternative approach is to implement `roots` and `children` as methods of the subclass (in fact they could also be attributes of A). Namely, `A.roots()` must return an iterable containing the enumeration generators, and `A.children(x)` must return an iterable over the children of x. Optionally, A can have a method or attribute such that `A.post_process(x)` returns the desired output for the node `x` of the tree:

```
sage: from sage.sets.recursively_enumerated_set import RecursivelyEnumeratedSet_
↪forest
sage: class A(UniqueRepresentation, RecursivelyEnumeratedSet_forest):
....:     def __init__(self):
....:         RecursivelyEnumeratedSet_forest.__init__(self, algorithm='breadth',
....:                                  category=InfiniteEnumeratedSets())
....:     def roots(self):
....:         return [()]
....:     def children(self, x):
....:         if sum(x) < 3:
....:             return [x + (0,), x + (1,)]
....:         else:
....:             return []
....:     def post_process(self, x):
....:         if sum(x) == 0 or x[-1] == 0:
....:             return None
....:         else:
....:             return sum(x[i]*2^i for i in range(len(x)))
```

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```
sage: MyForest = A(); MyForest
An enumerated set with a forest structure
sage: MyForest.category()
Category of infinite enumerated sets
sage: p = iter(MyForest)
sage: [next(p) for i in range(30)]
[1, 2, 3, 4, 6, 5, 7, 8, 12, 10, 14, 9, 13, 11, 16, 24,
 20, 28, 18, 26, 22, 17, 25, 21, 19, 32, 48, 40, 56, 36]
```

```
>>> from sage.all import *
>>> from sage.sets.recursively_enumerated_set import RecursivelyEnumeratedSet_
↪forest
>>> class A(UniqueRepresentation, RecursivelyEnumeratedSet_forest):
...     def __init__(self):
...         RecursivelyEnumeratedSet_forest.__init__(self, algorithm='breadth',
...                                 category=InfiniteEnumeratedSets())
...     def roots(self):
...         return [()]
...     def children(self, x):
...         if sum(x) < Integer(3):
...             return [x + (Integer(0),), x + (Integer(1),)]
...         else:
...             return []
...     def post_process(self, x):
...         if sum(x) == Integer(0) or x[-Integer(1)] == Integer(0):
...             return None
...         else:
...             return sum(x[i]*Integer(2)**i for i in range(len(x)))
>>> MyForest = A(); MyForest
An enumerated set with a forest structure
>>> MyForest.category()
Category of infinite enumerated sets
>>> p = iter(MyForest)
>>> [next(p) for i in range(Integer(30))]
[1, 2, 3, 4, 6, 5, 7, 8, 12, 10, 14, 9, 13, 11, 16, 24,
 20, 28, 18, 26, 22, 17, 25, 21, 19, 32, 48, 40, 56, 36]
```

Warning

A *RecursivelyEnumeratedSet_forest* instance is picklable if and only if the input functions are themselves picklable. This excludes anonymous or interactively defined functions:

```
sage: def children(x):
....:     return [x + 1]
sage: S = RecursivelyEnumeratedSet_forest([1], children,␣
↪category=InfiniteEnumeratedSets())
sage: dumps(S)
Traceback (most recent call last):
...
PicklingError: Can't pickle <...function...>: attribute lookup ... failed
```

```
>>> from sage.all import *
>>> def children(x):
...     return [x + Integer(1)]
>>> S = RecursivelyEnumeratedSet_forest([Integer(1)], children,␣
↪category=InfiniteEnumeratedSets())
>>> dumps(S)
Traceback (most recent call last):
...
PicklingError: Can't pickle <...function...>: attribute lookup ... failed
```

Let us now fake `children` being defined in a Python module:

```
sage: import __main__
sage: __main__.children = children
sage: S = RecursivelyEnumeratedSet_forest([1], children,␣
↪category=InfiniteEnumeratedSets())
sage: loads(dumps(S))
An enumerated set with a forest structure
```

```
>>> from sage.all import *
>>> import __main__
>>> __main__.children = children
>>> S = RecursivelyEnumeratedSet_forest([Integer(1)], children,␣
↪category=InfiniteEnumeratedSets())
>>> loads(dumps(S))
An enumerated set with a forest structure
```

breadth_first_search_iterator()

Return a breadth first search iterator over the elements of `self`.

EXAMPLES:

```
sage: from sage.sets.recursively_enumerated_set import␣
↪RecursivelyEnumeratedSet_forest
sage: f = RecursivelyEnumeratedSet_forest([[]],
....:                   lambda l: [l+[0], l+[1]] if len(l) < 3 else [])
sage: list(f.breadth_first_search_iterator())
[[], [0], [1], [0, 0], [0, 1], [1, 0], [1, 1], [0, 0, 0], [0, 0, 1], [0, 1,␣
↪0], [0, 1, 1], [1, 0, 0], [1, 0, 1], [1, 1, 0], [1, 1, 1]]
sage: S = RecursivelyEnumeratedSet_forest([(0,0)],
....: lambda x : [(x[0], x[1]+1)] if x[1] != 0 else [(x[0]+1,0), (x[0],1)],
....: post_process = lambda x: x if ((is_prime(x[0]) and is_prime(x[1])) and␣
↪((x[0] - x[1]) == 2)) else None)
sage: p = S.breadth_first_search_iterator()
sage: [next(p), next(p), next(p), next(p), next(p), next(p), next(p)]
[(5, 3), (7, 5), (13, 11), (19, 17), (31, 29), (43, 41), (61, 59)]
```

```
>>> from sage.all import *
>>> from sage.sets.recursively_enumerated_set import RecursivelyEnumeratedSet_
↪forest
>>> f = RecursivelyEnumeratedSet_forest([[]],
...                   lambda l: [l+[Integer(0)], l+[Integer(1)]] if len(l) <␣
```

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```
↪Integer(3) else [])
>>> list(f.breadth_first_search_iterator())
[[], [0], [1], [0, 0], [0, 1], [1, 0], [1, 1], [0, 0, 0], [0, 0, 1], [0, 1,␣
↪0], [0, 1, 1], [1, 0, 0], [1, 0, 1], [1, 1, 0], [1, 1, 1]]
>>> S = RecursivelyEnumeratedSet_forest([(Integer(0),Integer(0))],
... lambda x : [(x[Integer(0)], x[Integer(1)]+Integer(1))] if x[Integer(1)] !
↪= Integer(0) else [(x[Integer(0)]+Integer(1),Integer(0)), (x[Integer(0)],
↪Integer(1))],
... post_process = lambda x: x if ((is_prime(x[Integer(0)]) and is_
↪prime(x[Integer(1)])) and ((x[Integer(0)] - x[Integer(1)]) == Integer(2)))␣
↪else None)
>>> p = S.breadth_first_search_iterator()
>>> [next(p), next(p), next(p), next(p), next(p), next(p), next(p)]
[(5, 3), (7, 5), (13, 11), (19, 17), (31, 29), (43, 41), (61, 59)]
```

children(*x*)

Return the children of the element `x`.

The result can be a list, an iterable, an iterator, or even a generator.

EXAMPLES:

```
sage: from sage.sets.recursively_enumerated_set import␣
↪RecursivelyEnumeratedSet_forest
sage: I = RecursivelyEnumeratedSet_forest([(0,0)], lambda l: [(l[0]+1, l[1]),␣
↪(l[0], 1)] if l[1] == 0 else [(l[0], l[1]+1)])
sage: [i for i in I.children((0,0))]
[(1, 0), (0, 1)]
sage: [i for i in I.children((1,0))]
[(2, 0), (1, 1)]
sage: [i for i in I.children((1,1))]
[(1, 2)]
sage: [i for i in I.children((4,1))]
[(4, 2)]
sage: [i for i in I.children((4,0))]
[(5, 0), (4, 1)]
```

```
>>> from sage.all import *
>>> from sage.sets.recursively_enumerated_set import RecursivelyEnumeratedSet_
↪forest
>>> I = RecursivelyEnumeratedSet_forest([(Integer(0),Integer(0))], lambda l:␣
↪[(l[Integer(0)]+Integer(1), l[Integer(1)]), (l[Integer(0)], Integer(1))] if␣
↪l[Integer(1)] == Integer(0) else [(l[Integer(0)],␣
↪l[Integer(1)]+Integer(1))])
>>> [i for i in I.children((Integer(0),Integer(0)))]
[(1, 0), (0, 1)]
>>> [i for i in I.children((Integer(1),Integer(0)))]
[(2, 0), (1, 1)]
>>> [i for i in I.children((Integer(1),Integer(1)))]
[(1, 2)]
>>> [i for i in I.children((Integer(4),Integer(1)))]
[(4, 2)]
>>> [i for i in I.children((Integer(4),Integer(0)))]
```

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```
[(5, 0), (4, 1)]
```

depth_first_search_iterator()

Return a depth first search iterator over the elements of `self`.

EXAMPLES:

```
sage: from sage.sets.recursively_enumerated_set import␣
↪RecursivelyEnumeratedSet_forest
sage: f = RecursivelyEnumeratedSet_forest([[]],
....:                   lambda l: [l + [0], l + [1]] if len(l) < 3 else [])
sage: list(f.depth_first_search_iterator())
[[], [0], [0, 0], [0, 0, 0], [0, 0, 1], [0, 1], [0, 1, 0], [0, 1, 1],
     [1], [1, 0], [1, 0, 0], [1, 0, 1], [1, 1], [1, 1, 0], [1, 1, 1]]
```

```
>>> from sage.all import *
>>> from sage.sets.recursively_enumerated_set import RecursivelyEnumeratedSet_
↪forest
>>> f = RecursivelyEnumeratedSet_forest([[]],
...                   lambda l: [l + [Integer(0)], l + [Integer(1)]] if len(l)
↪< Integer(3) else [])
>>> list(f.depth_first_search_iterator())
[[], [0], [0, 0], [0, 0, 0], [0, 0, 1], [0, 1], [0, 1, 0], [0, 1, 1],
     [1], [1, 0], [1, 0, 0], [1, 0, 1], [1, 1], [1, 1, 0], [1, 1, 1]]
```

elements_of_depth_iterator(*depth=0*)

Return an iterator over the elements of `self` of given depth. An element of depth n can be obtained by applying the children function n times from a root.

EXAMPLES:

```
sage: from sage.sets.recursively_enumerated_set import␣
↪RecursivelyEnumeratedSet_forest
sage: S = RecursivelyEnumeratedSet_forest([(0,0)] ,
....:         lambda x : [(x[0], x[1]+1)] if x[1] != 0 else [(x[0]+1,0), (x[0],
↪1)],
....:         post_process = lambda x: x if ((is_prime(x[0]) and is_
↪prime(x[1]))
....:                                        and ((x[0] - x[1]) == 2)) else␣
↪None)
sage: p = S.elements_of_depth_iterator(8)
sage: next(p)
(5, 3)
sage: S = RecursivelyEnumeratedSet_forest(NN, lambda x : [],
....:                         lambda x: x^2 if x.is_prime() else None)
sage: p = S.elements_of_depth_iterator(0)
sage: [next(p), next(p), next(p), next(p), next(p)]
[4, 9, 25, 49, 121]
```

```
>>> from sage.all import *
>>> from sage.sets.recursively_enumerated_set import RecursivelyEnumeratedSet_
↪forest
```

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```
>>> S = RecursivelyEnumeratedSet_forest([(Integer(0),Integer(0))] ,
...        lambda x : [(x[Integer(0)], x[Integer(1)]+Integer(1))] if␣
↪x[Integer(1)] != Integer(0) else [(x[Integer(0)]+Integer(1),Integer(0)),␣
↪(x[Integer(0)],Integer(1))],
...        post_process = lambda x: x if ((is_prime(x[Integer(0)]) and is_
↪prime(x[Integer(1)]))
...                                       and ((x[Integer(0)] -␣
↪x[Integer(1)]) == Integer(2))) else None)
>>> p = S.elements_of_depth_iterator(Integer(8))
>>> next(p)
(5, 3)
>>> S = RecursivelyEnumeratedSet_forest(NN, lambda x : [],
...                        lambda x: x**Integer(2) if x.is_prime() else None)
>>> p = S.elements_of_depth_iterator(Integer(0))
>>> [next(p), next(p), next(p), next(p), next(p)]
[4, 9, 25, 49, 121]
```

map_reduce(*map_function=None*, *reduce_function=None*, *reduce_init=None*)

Apply a Map/Reduce algorithm on `self`.

INPUT:

- `map_function` – a function from the element of `self` to some set with a reduce operation (e.g.: a monoid). The default value is the constant function `1`.
- `reduce_function` – the reduce function (e.g.: the addition of a monoid); the default value is `+`
- `reduce_init` – the initialisation of the reduction (e.g.: the neutral element of the monoid); the default value is `0`

Note

the effect of the default values is to compute the cardinality of `self`.

EXAMPLES:

```
sage: seeds = [([i], i, i) for i in range(1, 10)]
sage: def succ(t):
....:     list, sum, last = t
....:     return [(list + [i], sum + i, i) for i in range(1, last)]
sage: F = RecursivelyEnumeratedSet(seeds, succ,
....:         structure='forest', enumeration='depth')

sage: # needs sage.symbolic
sage: y = var('y')
sage: def map_function(t):
....:     li, sum, _ = t
....:     return y ^ sum
sage: def reduce_function(x, y):
....:     return x + y
sage: F.map_reduce(map_function, reduce_function, 0)
y^45 + y^44 + y^43 + 2*y^42 + 2*y^41 + 3*y^40 + 4*y^39 + 5*y^38 + 6*y^37
+ 8*y^36 + 9*y^35 + 10*y^34 + 12*y^33 + 13*y^32 + 15*y^31 + 17*y^30
```

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```
+ 18*y^29 + 19*y^28 + 21*y^27 + 21*y^26 + 22*y^25 + 23*y^24 + 23*y^23
+ 23*y^22 + 23*y^21 + 22*y^20 + 21*y^19 + 21*y^18 + 19*y^17 + 18*y^16
+ 17*y^15 + 15*y^14 + 13*y^13 + 12*y^12 + 10*y^11 + 9*y^10 + 8*y^9 + 6*y^8
+ 5*y^7 + 4*y^6 + 3*y^5 + 2*y^4 + 2*y^3 + y^2 + y
```

```
>>> from sage.all import *
>>> seeds = [([i], i, i) for i in range(Integer(1), Integer(10))]
>>> def succ(t):
...     list, sum, last = t
...     return [(list + [i], sum + i, i) for i in range(Integer(1), last)]
>>> F = RecursivelyEnumeratedSet(seeds, succ,
...         structure='forest', enumeration='depth')

>>> # needs sage.symbolic
>>> y = var('y')
>>> def map_function(t):
...     li, sum, _ = t
...     return y ** sum
>>> def reduce_function(x, y):
...     return x + y
>>> F.map_reduce(map_function, reduce_function, Integer(0))
y^45 + y^44 + y^43 + 2*y^42 + 2*y^41 + 3*y^40 + 4*y^39 + 5*y^38 + 6*y^37
+ 8*y^36 + 9*y^35 + 10*y^34 + 12*y^33 + 13*y^32 + 15*y^31 + 17*y^30
+ 18*y^29 + 19*y^28 + 21*y^27 + 21*y^26 + 22*y^25 + 23*y^24 + 23*y^23
+ 23*y^22 + 23*y^21 + 22*y^20 + 21*y^19 + 21*y^18 + 19*y^17 + 18*y^16
+ 17*y^15 + 15*y^14 + 13*y^13 + 12*y^12 + 10*y^11 + 9*y^10 + 8*y^9 + 6*y^8
+ 5*y^7 + 4*y^6 + 3*y^5 + 2*y^4 + 2*y^3 + y^2 + y
```

Here is an example with the default values:

```
sage: F.map_reduce()
511
```

```
>>> from sage.all import *
>>> F.map_reduce()
511
```

See also

`sage.parallel.map_reduce`

roots()

Return an iterable over the roots of `self`.

EXAMPLES:

```
sage: from sage.sets.recursively_enumerated_set import␣
↪RecursivelyEnumeratedSet_forest
sage: I = RecursivelyEnumeratedSet_forest([(0,0)], lambda l: [(l[0]+1, l[1]),␣
↪(l[0], 1)] if l[1] == 0 else [(l[0], l[1]+1)])
sage: [i for i in I.roots()]
```

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```
[(0, 0)]
sage: I = RecursivelyEnumeratedSet_forest([(0,0),(1,1)], lambda l: [(l[0]+1,␣
↪l[1]), (l[0], 1)] if l[1] == 0 else [(l[0], l[1]+1)])
sage: [i for i in I.roots()]
[(0, 0), (1, 1)]
```

```
>>> from sage.all import *
>>> from sage.sets.recursively_enumerated_set import RecursivelyEnumeratedSet_
↪forest
>>> I = RecursivelyEnumeratedSet_forest([(Integer(0),Integer(0))], lambda l:␣
↪[(l[Integer(0)]+Integer(1), l[Integer(1)]), (l[Integer(0)], Integer(1))] if␣
↪l[Integer(1)] == Integer(0) else [(l[Integer(0)],␣
↪l[Integer(1)]+Integer(1))])
>>> [i for i in I.roots()]
[(0, 0)]
>>> I = RecursivelyEnumeratedSet_forest([(Integer(0),Integer(0)),(Integer(1),
↪Integer(1))], lambda l: [(l[Integer(0)]+Integer(1), l[Integer(1)]),␣
↪(l[Integer(0)], Integer(1))] if l[Integer(1)] == Integer(0) else␣
↪[(l[Integer(0)], l[Integer(1)]+Integer(1))])
>>> [i for i in I.roots()]
[(0, 0), (1, 1)]
```

class sage.sets.recursively_enumerated_set.**RecursivelyEnumeratedSet_generic**

Bases: Parent

A generic recursively enumerated set.

For more information, see *RecursivelyEnumeratedSet()*.

EXAMPLES:

```
sage: f = lambda a:[a+1]
```

```
>>> from sage.all import *
>>> f = lambda a:[a+Integer(1)]
```

Different structure for the sets:

```
sage: RecursivelyEnumeratedSet([0], f, structure=None)
A recursively enumerated set (breadth first search)
sage: RecursivelyEnumeratedSet([0], f, structure='graded')
A recursively enumerated set with a graded structure (breadth first search)
sage: RecursivelyEnumeratedSet([0], f, structure='symmetric')
A recursively enumerated set with a symmetric structure (breadth first search)
sage: RecursivelyEnumeratedSet([0], f, structure='forest')
An enumerated set with a forest structure
```

```
>>> from sage.all import *
>>> RecursivelyEnumeratedSet([Integer(0)], f, structure=None)
A recursively enumerated set (breadth first search)
>>> RecursivelyEnumeratedSet([Integer(0)], f, structure='graded')
A recursively enumerated set with a graded structure (breadth first search)
>>> RecursivelyEnumeratedSet([Integer(0)], f, structure='symmetric')
```

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```
A recursively enumerated set with a symmetric structure (breadth first search)
>>> RecursivelyEnumeratedSet([Integer(0)], f, structure='forest')
An enumerated set with a forest structure
```

Different default enumeration algorithms:

```
sage: RecursivelyEnumeratedSet([0], f, enumeration='breadth')
A recursively enumerated set (breadth first search)
sage: RecursivelyEnumeratedSet([0], f, enumeration='naive')
A recursively enumerated set (naive search)
sage: RecursivelyEnumeratedSet([0], f, enumeration='depth')
A recursively enumerated set (depth first search)
```

```
>>> from sage.all import *
>>> RecursivelyEnumeratedSet([Integer(0)], f, enumeration='breadth')
A recursively enumerated set (breadth first search)
>>> RecursivelyEnumeratedSet([Integer(0)], f, enumeration='naive')
A recursively enumerated set (naive search)
>>> RecursivelyEnumeratedSet([Integer(0)], f, enumeration='depth')
A recursively enumerated set (depth first search)
```

breadth_first_search_iterator(*max_depth=None*)

Iterate on the elements of `self` (breadth first).

This code remembers every element generated.

The elements are guaranteed to be enumerated in the order in which they are first visited (left-to-right traversal).

INPUT:

- `max_depth` – (default: `self._max_depth`) specifies the maximal depth to which elements are computed

EXAMPLES:

```
sage: f = lambda a: [a+3, a+5]
sage: C = RecursivelyEnumeratedSet([0], f)
sage: it = C.breadth_first_search_iterator()
sage: [next(it) for _ in range(10)]
[0, 3, 5, 6, 8, 10, 9, 11, 13, 15]
```

```
>>> from sage.all import *
>>> f = lambda a: [a+Integer(3), a+Integer(5)]
>>> C = RecursivelyEnumeratedSet([Integer(0)], f)
>>> it = C.breadth_first_search_iterator()
>>> [next(it) for _ in range(Integer(10))]
[0, 3, 5, 6, 8, 10, 9, 11, 13, 15]
```

depth_first_search_iterator()

Iterate on the elements of `self` (depth first).

This code remembers every element generated.

The elements are traversed right-to-left, so the last element returned by the successor function is visited first.

See Wikipedia article Depth-first_search.

EXAMPLES:

```
sage: f = lambda a: [a+3, a+5]
sage: C = RecursivelyEnumeratedSet([0], f)
sage: it = C.depth_first_search_iterator()
sage: [next(it) for _ in range(10)]
[0, 5, 10, 15, 20, 25, 30, 35, 40, 45]
```

```
>>> from sage.all import *
>>> f = lambda a: [a+Integer(3), a+Integer(5)]
>>> C = RecursivelyEnumeratedSet([Integer(0)], f)
>>> it = C.depth_first_search_iterator()
>>> [next(it) for _ in range(Integer(10))]
[0, 5, 10, 15, 20, 25, 30, 35, 40, 45]
```

elements_of_depth_iterator(*depth*)

Iterate over the elements of `self` of given depth.

An element of depth n can be obtained by applying the successor function n times to a seed.

INPUT:

- `depth` – integer

OUTPUT: an iterator

EXAMPLES:

```
sage: f = lambda a: [a-1, a+1]
sage: S = RecursivelyEnumeratedSet([5, 10], f, structure='symmetric')
sage: it = S.elements_of_depth_iterator(2)
sage: sorted(it)
[3, 7, 8, 12]
```

```
>>> from sage.all import *
>>> f = lambda a: [a-Integer(1), a+Integer(1)]
>>> S = RecursivelyEnumeratedSet([Integer(5), Integer(10)], f, structure=
↪'symmetric')
>>> it = S.elements_of_depth_iterator(Integer(2))
>>> sorted(it)
[3, 7, 8, 12]
```

graded_component(*depth*)

Return the graded component of given depth.

This method caches each lower graded component.

A graded component is a set of elements of the same depth where the depth of an element is its minimal distance to a root.

It is currently implemented only for graded or symmetric structure.

INPUT:

- `depth` – integer

OUTPUT: set

EXAMPLES:

```
sage: f = lambda a: [a+3, a+5]
sage: C = RecursivelyEnumeratedSet([0], f)
sage: C.graded_component(0)
Traceback (most recent call last):
...
NotImplementedError: graded_component_iterator method currently implemented␣
↪only for graded or symmetric structure
```

```
>>> from sage.all import *
>>> f = lambda a: [a+Integer(3), a+Integer(5)]
>>> C = RecursivelyEnumeratedSet([Integer(0)], f)
>>> C.graded_component(Integer(0))
Traceback (most recent call last):
...
NotImplementedError: graded_component_iterator method currently implemented␣
↪only for graded or symmetric structure
```

graded_component_iterator()

Iterate over the graded components of `self`.

A graded component is a set of elements of the same depth.

It is currently implemented only for graded or symmetric structure.

OUTPUT: an iterator of sets

EXAMPLES:

```
sage: f = lambda a: [a+3, a+5]
sage: C = RecursivelyEnumeratedSet([0], f)
sage: it = C.graded_component_iterator()     # todo: not implemented
```

```
>>> from sage.all import *
>>> f = lambda a: [a+Integer(3), a+Integer(5)]
>>> C = RecursivelyEnumeratedSet([Integer(0)], f)
>>> it = C.graded_component_iterator()     # todo: not implemented
```

naive_search_iterator()

Iterate on the elements of `self` (in no particular order).

This code remembers every element generated.

seeds()

Return an iterable over the seeds of `self`.

EXAMPLES:

```
sage: R = RecursivelyEnumeratedSet([1], lambda x: [x + 1, x - 1])
sage: R.seeds()
[1]
```

```
>>> from sage.all import *
>>> R = RecursivelyEnumeratedSet([Integer(1)], lambda x: [x + Integer(1), x -␣
↪Integer(1)])
>>> R.seeds()
[1]
```

successors

to_digraph(*max_depth=None*, *loops=True*, *multiedges=True*)

Return the directed graph of the recursively enumerated set.

INPUT:

- `max_depth` – (default: `self._max_depth`) specifies the maximal depth for which outgoing edges of elements are computed
- `loops` – boolean (default: `True`); option for the digraph
- `multiedges` – boolean (default: `True`); option of the digraph

OUTPUT: a directed graph

Warning

If the set is infinite, this will loop forever unless `max_depth` is finite.

EXAMPLES:

```
sage: child = lambda i: [(i+3) % 10, (i+8) % 10]
sage: R = RecursivelyEnumeratedSet([0], child)
sage: R.to_digraph()                                                    #␣
↪needs sage.graphs
Looped multi-digraph on 10 vertices
```

```
>>> from sage.all import *
>>> child = lambda i: [(i+Integer(3)) % Integer(10), (i+Integer(8)) %␣
↪Integer(10)]
>>> R = RecursivelyEnumeratedSet([Integer(0)], child)
>>> R.to_digraph()                                                      #␣
↪needs sage.graphs
Looped multi-digraph on 10 vertices
```

Digraph of a recursively enumerated set with a symmetric structure of infinite cardinality using `max_depth` argument:

```
sage: succ = lambda a: [(a[0]-1,a[1]), (a[0],a[1]-1), (a[0]+1,a[1]), (a[0],
↪a[1]+1)]
sage: seeds = [(0,0)]
sage: C = RecursivelyEnumeratedSet(seeds, succ, structure='symmetric')
sage: C.to_digraph(max_depth=3)                                         #␣
↪needs sage.graphs
Looped multi-digraph on 41 vertices
```

```
>>> from sage.all import *
>>> succ = lambda a: [(a[Integer(0)]-Integer(1),a[Integer(1)]),␣
↪(a[Integer(0)],a[Integer(1)]-Integer(1)), (a[Integer(0)]+Integer(1),
↪a[Integer(1)]), (a[Integer(0)],a[Integer(1)]+Integer(1))]
>>> seeds = [(Integer(0),Integer(0))]
>>> C = RecursivelyEnumeratedSet(seeds, succ, structure='symmetric')
>>> C.to_digraph(max_depth=Integer(3))                                    ␣
↪     # needs sage.graphs
Looped multi-digraph on 41 vertices
```

The `max_depth` argument can be given at the creation of the set:

```
sage: C = RecursivelyEnumeratedSet(seeds, succ, structure='symmetric',
....:                              max_depth=2)
sage: C.to_digraph()                                                    #␣
↪needs sage.graphs
Looped multi-digraph on 25 vertices
```

```
>>> from sage.all import *
>>> C = RecursivelyEnumeratedSet(seeds, succ, structure='symmetric',
...                              max_depth=Integer(2))
>>> C.to_digraph()                                                      #␣
↪needs sage.graphs
Looped multi-digraph on 25 vertices
```

Digraph of a recursively enumerated set with a graded structure:

```
sage: f = lambda a: [a+1, a+I]
sage: C = RecursivelyEnumeratedSet([0], f, structure='graded')
sage: C.to_digraph(max_depth=4)                                         #␣
↪needs sage.graphs
Looped multi-digraph on 21 vertices
```

```
>>> from sage.all import *
>>> f = lambda a: [a+Integer(1), a+I]
>>> C = RecursivelyEnumeratedSet([Integer(0)], f, structure='graded')
>>> C.to_digraph(max_depth=Integer(4))                                    ␣
↪     # needs sage.graphs
Looped multi-digraph on 21 vertices
```

class sage.sets.recursively_enumerated_set.**RecursivelyEnumeratedSet_graded**

Bases: *RecursivelyEnumeratedSet_generic*

Generic tool for constructing ideals of a graded relation.

INPUT:

- `seeds` – list (or iterable) of hashable objects
- `successors` – function (or callable) returning a list (or iterable)
- `enumeration` – `'depth'`, `'breadth'` or `None` (default: `None`)
- `max_depth` – integer (default: `float("inf")`)

EXAMPLES:

```
sage: f = lambda a: [(a[0]+1,a[1]), (a[0],a[1]+1)]
sage: C = RecursivelyEnumeratedSet([(0,0)], f, structure='graded', max_depth=3)
sage: C
A recursively enumerated set with a graded structure (breadth first
search) with max_depth=3
sage: list(C)
[(0, 0),
 (1, 0), (0, 1),
 (2, 0), (1, 1), (0, 2),
 (3, 0), (2, 1), (1, 2), (0, 3)]
```

```
>>> from sage.all import *
>>> f = lambda a: [(a[Integer(0)]+Integer(1),a[Integer(1)]), (a[Integer(0)],
↪a[Integer(1)]+Integer(1))]
>>> C = RecursivelyEnumeratedSet([(Integer(0),Integer(0))], f, structure='graded',
↪ max_depth=Integer(3))
>>> C
A recursively enumerated set with a graded structure (breadth first
search) with max_depth=3
>>> list(C)
[(0, 0),
 (1, 0), (0, 1),
 (2, 0), (1, 1), (0, 2),
 (3, 0), (2, 1), (1, 2), (0, 3)]
```

breadth_first_search_iterator(*max_depth=None*)

Iterate on the elements of `self` (breadth first).

This iterator makes use of the graded structure by remembering only the elements of the current depth.

The elements are guaranteed to be enumerated in the order in which they are first visited (left-to-right traversal).

INPUT:

- `max_depth` – (default: `self._max_depth`) specifies the maximal depth to which elements are computed

EXAMPLES:

```
sage: f = lambda a: [(a[0]+1,a[1]), (a[0],a[1]+1)]
sage: C = RecursivelyEnumeratedSet([(0,0)], f, structure='graded')
sage: list(C.breadth_first_search_iterator(max_depth=3))
[(0, 0),
 (1, 0), (0, 1),
 (2, 0), (1, 1), (0, 2),
 (3, 0), (2, 1), (1, 2), (0, 3)]
```

```
>>> from sage.all import *
>>> f = lambda a: [(a[Integer(0)]+Integer(1),a[Integer(1)]), (a[Integer(0)],
↪a[Integer(1)]+Integer(1))]
>>> C = RecursivelyEnumeratedSet([(Integer(0),Integer(0))], f, structure=
↪'graded')
>>> list(C.breadth_first_search_iterator(max_depth=Integer(3)))
[(0, 0),
```

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```
 (1, 0), (0, 1),
 (2, 0), (1, 1), (0, 2),
 (3, 0), (2, 1), (1, 2), (0, 3)]
```

graded_component(*depth*)

Return the graded component of given depth.

This method caches each lower graded component. See *graded_component_iterator()* to generate each graded component without caching the previous ones.

A graded component is a set of elements of the same depth where the depth of an element is its minimal distance to a root.

INPUT:

- `depth` – integer

OUTPUT: set

EXAMPLES:

```
sage: # needs sage.symbolic
sage: def f(a):
....:     return [a + 1, a + I]
sage: C = RecursivelyEnumeratedSet([0], f, structure='graded')
sage: for i in range(5): sorted(C.graded_component(i))
[0]
[I, 1]
[2*I, I + 1, 2]
[3*I, 2*I + 1, I + 2, 3]
[4*I, 3*I + 1, 2*I + 2, I + 3, 4]
```

```
>>> from sage.all import *
>>> # needs sage.symbolic
>>> def f(a):
...     return [a + Integer(1), a + I]
>>> C = RecursivelyEnumeratedSet([Integer(0)], f, structure='graded')
>>> for i in range(Integer(5)): sorted(C.graded_component(i))
[0]
[I, 1]
[2*I, I + 1, 2]
[3*I, 2*I + 1, I + 2, 3]
[4*I, 3*I + 1, 2*I + 2, I + 3, 4]
```

graded_component_iterator()

Iterate over the graded components of `self`.

A graded component is a set of elements of the same depth.

The algorithm remembers only the current graded component generated since the structure is graded.

OUTPUT: an iterator of sets

EXAMPLES:

```
sage: f = lambda a: [(a[0]+1,a[1]), (a[0],a[1]+1)]
sage: C = RecursivelyEnumeratedSet([(0,0)], f, structure='graded', max_
```

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```
↪depth=3)
sage: it = C.graded_component_iterator()
sage: for _ in range(4): sorted(next(it))
[(0, 0)]
[(0, 1), (1, 0)]
[(0, 2), (1, 1), (2, 0)]
[(0, 3), (1, 2), (2, 1), (3, 0)]
```

```
>>> from sage.all import *
>>> f = lambda a: [(a[Integer(0)]+Integer(1),a[Integer(1)]), (a[Integer(0)],
↪a[Integer(1)]+Integer(1))]
>>> C = RecursivelyEnumeratedSet([(Integer(0),Integer(0))], f, structure=
↪'graded', max_depth=Integer(3))
>>> it = C.graded_component_iterator()
>>> for _ in range(Integer(4)): sorted(next(it))
[(0, 0)]
[(0, 1), (1, 0)]
[(0, 2), (1, 1), (2, 0)]
[(0, 3), (1, 2), (2, 1), (3, 0)]
```

class sage.sets.recursively_enumerated_set.**RecursivelyEnumeratedSet_symmetric**

Bases: *RecursivelyEnumeratedSet_generic*

Generic tool for constructing ideals of a symmetric relation.

INPUT:

- `seeds` – list (or iterable) of hashable objects
- `successors` – function (or callable) returning a list (or iterable)
- `enumeration` – `'depth'`, `'breadth'` or `None` (default: `None`)
- `max_depth` – integer (default: `float("inf")`)

EXAMPLES:

```
sage: f = lambda a: [a-1,a+1]
sage: C = RecursivelyEnumeratedSet([0], f, structure='symmetric')
sage: C
A recursively enumerated set with a symmetric structure (breadth first search)
sage: it = iter(C)
sage: [next(it) for _ in range(7)]
[0, -1, 1, -2, 2, -3, 3]
```

```
>>> from sage.all import *
>>> f = lambda a: [a-Integer(1),a+Integer(1)]
>>> C = RecursivelyEnumeratedSet([Integer(0)], f, structure='symmetric')
>>> C
A recursively enumerated set with a symmetric structure (breadth first search)
>>> it = iter(C)
>>> [next(it) for _ in range(Integer(7))]
[0, -1, 1, -2, 2, -3, 3]
```

breadth_first_search_iterator(*max_depth=None*)

Iterate on the elements of `self` (breadth first).

This iterator makes use of the graded structure by remembering only the last two graded components since the structure is symmetric.

The elements are guaranteed to be enumerated in the order in which they are first visited (left-to-right traversal).

INPUT:

- `max_depth` – (default: `self._max_depth`) specifies the maximal depth to which elements are computed

EXAMPLES:

```
sage: f = lambda a: [(a[0]-1,a[1]), (a[0],a[1]-1), (a[0]+1,a[1]), (a[0],
↪a[1]+1)]
sage: C = RecursivelyEnumeratedSet([(0,0)], f, structure='symmetric')
sage: s = list(C.breadth_first_search_iterator(max_depth=2)); s
[(0, 0),
 (-1, 0), (0, -1), (1, 0), (0, 1),
 (-2, 0), (-1, -1), (-1, 1), (0, -2), (1, -1), (2, 0), (1, 1), (0, 2)]
```

```
>>> from sage.all import *
>>> f = lambda a: [(a[Integer(0)]-Integer(1),a[Integer(1)]), (a[Integer(0)],
↪a[Integer(1)]-Integer(1)), (a[Integer(0)]+Integer(1),a[Integer(1)]),␣
↪(a[Integer(0)],a[Integer(1)]+Integer(1))]
>>> C = RecursivelyEnumeratedSet([(Integer(0),Integer(0))], f, structure=
↪'symmetric')
>>> s = list(C.breadth_first_search_iterator(max_depth=Integer(2))); s
[(0, 0),
 (-1, 0), (0, -1), (1, 0), (0, 1),
 (-2, 0), (-1, -1), (-1, 1), (0, -2), (1, -1), (2, 0), (1, 1), (0, 2)]
```

This iterator is used by default for symmetric structure:

```
sage: it = iter(C)
sage: s == [next(it) for _ in range(13)]
True
```

```
>>> from sage.all import *
>>> it = iter(C)
>>> s == [next(it) for _ in range(Integer(13))]
True
```

graded_component(*depth*)

Return the graded component of given depth.

This method caches each lower graded component. See *graded_component_iterator()* to generate each graded component without caching the previous ones.

A graded component is a set of elements of the same depth where the depth of an element is its minimal distance to a root.

INPUT:

- `depth` – integer

OUTPUT: set

EXAMPLES:

```
sage: f = lambda a: [a-1,a+1]
sage: C = RecursivelyEnumeratedSet([10, 15], f, structure='symmetric')
sage: for i in range(5): sorted(C.graded_component(i))
[10, 15]
[9, 11, 14, 16]
[8, 12, 13, 17]
[7, 18]
[6, 19]
```

```
>>> from sage.all import *
>>> f = lambda a: [a-Integer(1),a+Integer(1)]
>>> C = RecursivelyEnumeratedSet([Integer(10), Integer(15)], f, structure=
↪'symmetric')
>>> for i in range(Integer(5)): sorted(C.graded_component(i))
[10, 15]
[9, 11, 14, 16]
[8, 12, 13, 17]
[7, 18]
[6, 19]
```

graded_component_iterator()

Iterate over the graded components of `self`.

A graded component is a set of elements of the same depth.

The enumeration remembers only the last two graded components generated since the structure is symmetric.

OUTPUT: an iterator of sets

EXAMPLES:

```
sage: f = lambda a: [a-1, a+1]
sage: S = RecursivelyEnumeratedSet([10], f, structure='symmetric')
sage: it = S.graded_component_iterator()
sage: [sorted(next(it)) for _ in range(5)]
[[10], [9, 11], [8, 12], [7, 13], [6, 14]]
```

```
>>> from sage.all import *
>>> f = lambda a: [a-Integer(1), a+Integer(1)]
>>> S = RecursivelyEnumeratedSet([Integer(10)], f, structure='symmetric')
>>> it = S.graded_component_iterator()
>>> [sorted(next(it)) for _ in range(Integer(5))]
[[10], [9, 11], [8, 12], [7, 13], [6, 14]]
```

Starting with two generators:

```
sage: f = lambda a: [a-1, a+1]
sage: S = RecursivelyEnumeratedSet([5, 10], f, structure='symmetric')
sage: it = S.graded_component_iterator()
sage: [sorted(next(it)) for _ in range(5)]
[[5, 10], [4, 6, 9, 11], [3, 7, 8, 12], [2, 13], [1, 14]]
```

```
>>> from sage.all import *
>>> f = lambda a: [a-Integer(1), a+Integer(1)]
>>> S = RecursivelyEnumeratedSet([Integer(5), Integer(10)], f, structure=
↪'symmetric')
>>> it = S.graded_component_iterator()
>>> [sorted(next(it)) for _ in range(Integer(5))]
[[5, 10], [4, 6, 9, 11], [3, 7, 8, 12], [2, 13], [1, 14]]
```

Gaussian integers:

```
sage: # needs sage.symbolic
sage: def f(a):
....:     return [a + 1, a + I]
sage: S = RecursivelyEnumeratedSet([0], f, structure='symmetric')
sage: it = S.graded_component_iterator()
sage: [sorted(next(it)) for _ in range(7)]
[[0],
 [I, 1],
 [2*I, I + 1, 2],
 [3*I, 2*I + 1, I + 2, 3],
 [4*I, 3*I + 1, 2*I + 2, I + 3, 4],
 [5*I, 4*I + 1, 3*I + 2, 2*I + 3, I + 4, 5],
 [6*I, 5*I + 1, 4*I + 2, 3*I + 3, 2*I + 4, I + 5, 6]]
```

```
>>> from sage.all import *
>>> # needs sage.symbolic
>>> def f(a):
...     return [a + Integer(1), a + I]
>>> S = RecursivelyEnumeratedSet([Integer(0)], f, structure='symmetric')
>>> it = S.graded_component_iterator()
>>> [sorted(next(it)) for _ in range(Integer(7))]
[[0],
 [I, 1],
 [2*I, I + 1, 2],
 [3*I, 2*I + 1, I + 2, 3],
 [4*I, 3*I + 1, 2*I + 2, I + 3, 4],
 [5*I, 4*I + 1, 3*I + 2, 2*I + 3, I + 4, 5],
 [6*I, 5*I + 1, 4*I + 2, 3*I + 3, 2*I + 4, I + 5, 6]]
```

sage.sets.recursively_enumerated_set.**search_forest_iterator**(*roots*, *children*, *algorithm='depth'*)

Return an iterator on the nodes of the forest having the given roots, and where `children(x)` returns the children of the node `x` of the forest. Note that every node of the tree is returned, not simply the leaves.

INPUT:

- `roots` – list (or iterable)
- `children` – a function returning a list (or iterable)
- `algorithm` – `'depth'` or `'breadth'` (default: `'depth'`)

EXAMPLES:

We construct the prefix tree of binary sequences of length at most three, and enumerate its nodes:

```
sage: from sage.sets.recursively_enumerated_set import search_forest_iterator
sage: list(search_forest_iterator([[]], lambda l: [l + [0], l + [1]]
....:                                   if len(l) < 3 else []))
[[], [0], [0, 0], [0, 0, 0], [0, 0, 1], [0, 1], [0, 1, 0],
 [0, 1, 1], [1], [1, 0], [1, 0, 0], [1, 0, 1], [1, 1], [1, 1, 0], [1, 1, 1]]
```

```
>>> from sage.all import *
>>> from sage.sets.recursively_enumerated_set import search_forest_iterator
>>> list(search_forest_iterator([[]], lambda l: [l + [Integer(0)], l +␣
↪[Integer(1)]]
...                                   if len(l) < Integer(3) else []))
[[], [0], [0, 0], [0, 0, 0], [0, 0, 1], [0, 1], [0, 1, 0],
 [0, 1, 1], [1], [1, 0], [1, 0, 0], [1, 0, 1], [1, 1], [1, 1, 0], [1, 1, 1]]
```

By default, the nodes are iterated through by depth first search. We can instead use a breadth first search (increasing depth):

```
sage: list(search_forest_iterator([[]], lambda l: [l + [0], l + [1]]
....:                                   if len(l) < 3 else [],
....:                              algorithm='breadth'))
[[],
 [0], [1],
 [0, 0], [0, 1], [1, 0], [1, 1],
 [0, 0, 0], [0, 0, 1], [0, 1, 0], [0, 1, 1],
 [1, 0, 0], [1, 0, 1], [1, 1, 0], [1, 1, 1]]
```

```
>>> from sage.all import *
>>> list(search_forest_iterator([[]], lambda l: [l + [Integer(0)], l +␣
↪[Integer(1)]]
...                                   if len(l) < Integer(3) else [],
...                              algorithm='breadth'))
[[],
 [0], [1],
 [0, 0], [0, 1], [1, 0], [1, 1],
 [0, 0, 0], [0, 0, 1], [0, 1, 0], [0, 1, 1],
 [1, 0, 0], [1, 0, 1], [1, 1, 0], [1, 1, 1]]
```

This allows for iterating through trees of infinite depth:

```
sage: it = search_forest_iterator([[]], lambda l: [l + [0], l + [1]],
....:                              algorithm='breadth')
sage: [ next(it) for i in range(16) ]
[[],
 [0], [1], [0, 0], [0, 1], [1, 0], [1, 1],
 [0, 0, 0], [0, 0, 1], [0, 1, 0], [0, 1, 1],
 [1, 0, 0], [1, 0, 1], [1, 1, 0], [1, 1, 1],
 [0, 0, 0, 0]]
```

```
>>> from sage.all import *
>>> it = search_forest_iterator([[]], lambda l: [l + [Integer(0)], l +␣
↪[Integer(1)]],
...                              algorithm='breadth')
```

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```
>>> [ next(it) for i in range(Integer(16)) ]
[[],
 [0], [1], [0, 0], [0, 1], [1, 0], [1, 1],
 [0, 0, 0], [0, 0, 1], [0, 1, 0], [0, 1, 1],
 [1, 0, 0], [1, 0, 1], [1, 1, 0], [1, 1, 1],
 [0, 0, 0, 0]]
```

Here is an iterator through the prefix tree of sequences of letters in $0, 1, 2$ without repetitions, sorted by length; the leaves are therefore permutations:

```
sage: list(search_forest_iterator([[]], lambda l: [l + [i] for i in range(3) if i␣
↪not in l],
....:                             algorithm='breadth'))
[[],
 [0], [1], [2],
 [0, 1], [0, 2], [1, 0], [1, 2], [2, 0], [2, 1],
 [0, 1, 2], [0, 2, 1], [1, 0, 2], [1, 2, 0], [2, 0, 1], [2, 1, 0]]
```

```
>>> from sage.all import *
>>> list(search_forest_iterator([[]], lambda l: [l + [i] for i in␣
↪range(Integer(3)) if i not in l],
...                             algorithm='breadth'))
[[],
 [0], [1], [2],
 [0, 1], [0, 2], [1, 0], [1, 2], [2, 0], [2, 1],
 [0, 1, 2], [0, 2, 1], [1, 0, 2], [1, 2, 0], [2, 0, 1], [2, 1, 0]]
```

1.9 Subsets of a Universe Defined by Predicates

class sage.sets.condition_set.**ConditionSet**(*universe*, **predicates*, *names=None*, *category=None*)

Bases: Set_generic, *Set_base*, *Set_boolean_operators*, *Set_add_sub_operators*, UniqueRepresentation

Set of elements of a universe that satisfy given predicates.

INPUT:

- `universe` – set
- `*predicates` – callables
- `vars` or `names` – (default: inferred from `predicates` if any predicate is an element of a CallableSymbolicExpressionRing_class) variables or names of variables
- `category` – (default: inferred from `universe`) a category

EXAMPLES:

```
sage: Evens = ConditionSet(ZZ, is_even); Evens
{ x ∈ Integer Ring : <function is_even at 0x...>(x) }
sage: 2 in Evens
True
sage: 3 in Evens
False
```

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```
sage: 2.0 in Evens
True

sage: Odds = ConditionSet(ZZ, is_odd); Odds
{ x ∈ Integer Ring : <function is_odd at 0x...>(x) }
sage: EvensAndOdds = Evens | Odds; EvensAndOdds
Set-theoretic union of
 { x ∈ Integer Ring : <function is_even at 0x...>(x) } and
 { x ∈ Integer Ring : <function is_odd at 0x...>(x) }
sage: 5 in EvensAndOdds
True
sage: 7/2 in EvensAndOdds
False

sage: var('y')                                                              #␣
↪needs sage.symbolic
y
sage: SmallOdds = ConditionSet(ZZ, is_odd, abs(y) <= 11, vars=[y]); SmallOdds   #␣
↪needs sage.symbolic
{ y ∈ Integer Ring : abs(y) <= 11, <function is_odd at 0x...>(y) }

sage: # needs sage.geometry.polyhedron
sage: P = polytopes.cube(); P
A 3-dimensional polyhedron in ZZ^3 defined as the convex hull of 8 vertices
sage: P.rename('P')
sage: P_inter_B = ConditionSet(P, lambda x: x.norm() < 1.2); P_inter_B
{ x ∈ P : <function <lambda> at 0x...>(x) }
sage: vector([1, 0, 0]) in P_inter_B
True
sage: vector([1, 1, 1]) in P_inter_B                                        #␣
↪needs sage.symbolic
False

sage: # needs sage.symbolic
sage: predicate(x, y, z) = sqrt(x^2 + y^2 + z^2) < 1.2; predicate
(x, y, z) |--> sqrt(x^2 + y^2 + z^2) < 1.20000000000000
sage: P_inter_B_again = ConditionSet(P, predicate); P_inter_B_again          #␣
↪needs sage.geometry.polyhedron
{ (x, y, z) ∈ P : sqrt(x^2 + y^2 + z^2) < 1.20000000000000 }
sage: vector([1, 0, 0]) in P_inter_B_again                                  #␣
↪needs sage.geometry.polyhedron
True
sage: vector([1, 1, 1]) in P_inter_B_again                                  #␣
↪needs sage.geometry.polyhedron
False
```

```
>>> from sage.all import *
>>> Evens = ConditionSet(ZZ, is_even); Evens
{ x ∈ Integer Ring : <function is_even at 0x...>(x) }
>>> Integer(2) in Evens
True
>>> Integer(3) in Evens
```

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```
False
>>> RealNumber('2.0') in Evens
True

>>> Odds = ConditionSet(ZZ, is_odd); Odds
{ x ∈ Integer Ring : <function is_odd at 0x...>(x) }
>>> EvensAndOdds = Evens | Odds; EvensAndOdds
Set-theoretic union of
 { x ∈ Integer Ring : <function is_even at 0x...>(x) } and
 { x ∈ Integer Ring : <function is_odd at 0x...>(x) }
>>> Integer(5) in EvensAndOdds
True
>>> Integer(7)/Integer(2) in EvensAndOdds
False

>>> var('y')                                                                #␣
↪needs sage.symbolic
y
>>> SmallOdds = ConditionSet(ZZ, is_odd, abs(y) <= Integer(11), vars=[y]);␣
↪SmallOdds   # needs sage.symbolic
{ y ∈ Integer Ring : abs(y) <= 11, <function is_odd at 0x...>(y) }

>>> # needs sage.geometry.polyhedron
>>> P = polytopes.cube(); P
A 3-dimensional polyhedron in ZZ^3 defined as the convex hull of 8 vertices
>>> P.rename('P')
>>> P_inter_B = ConditionSet(P, lambda x: x.norm() < RealNumber('1.2')); P_inter_B
{ x ∈ P : <function <lambda> at 0x...>(x) }
>>> vector([Integer(1), Integer(0), Integer(0)]) in P_inter_B
True
>>> vector([Integer(1), Integer(1), Integer(1)]) in P_inter_B                ␣
↪                       # needs sage.symbolic
False

>>> # needs sage.symbolic
>>> __tmp__=var("x,y,z"); predicate = symbolic_expression(sqrt(x**Integer(2) +␣
↪y**Integer(2) + z**Integer(2)) < RealNumber('1.2')).function(x,y,z); predicate
(x, y, z) |--> sqrt(x^2 + y^2 + z^2) < 1.20000000000000
>>> P_inter_B_again = ConditionSet(P, predicate); P_inter_B_again            #␣
↪needs sage.geometry.polyhedron
{ (x, y, z) ∈ P : sqrt(x^2 + y^2 + z^2) < 1.20000000000000 }
>>> vector([Integer(1), Integer(0), Integer(0)]) in P_inter_B_again          ␣
↪                       # needs sage.geometry.polyhedron
True
>>> vector([Integer(1), Integer(1), Integer(1)]) in P_inter_B_again          ␣
↪                       # needs sage.geometry.polyhedron
False
```

Iterating over subsets determined by predicates:

```
sage: Odds = ConditionSet(ZZ, is_odd); Odds
{ x ∈ Integer Ring : <function is_odd at 0x...>(x) }
```

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```
sage: list(Odds.iterator_range(stop=6))
[1, -1, 3, -3, 5, -5]

sage: R = IntegerModRing(8)
sage: R_primes = ConditionSet(R, is_prime); R_primes
{ x ∈ Ring of integers modulo 8 : <function is_prime at 0x...>(x) }
sage: R_primes.is_finite()
True
sage: list(R_primes)
[2, 6]
```

```
>>> from sage.all import *
>>> Odds = ConditionSet(ZZ, is_odd); Odds
{ x ∈ Integer Ring : <function is_odd at 0x...>(x) }
>>> list(Odds.iterator_range(stop=Integer(6)))
[1, -1, 3, -3, 5, -5]

>>> R = IntegerModRing(Integer(8))
>>> R_primes = ConditionSet(R, is_prime); R_primes
{ x ∈ Ring of integers modulo 8 : <function is_prime at 0x...>(x) }
>>> R_primes.is_finite()
True
>>> list(R_primes)
[2, 6]
```

Using `ConditionSet` without predicates provides a way of attaching variable names to a set:

```
sage: Z3 = ConditionSet(ZZ^3, vars=['x', 'y', 'z']); Z3                    #␣
↪needs sage.modules
{ (x, y, z) ∈ Ambient free module of rank 3
              over the principal ideal domain Integer Ring }
sage: Z3.variable_names()                                                  #␣
↪needs sage.modules
('x', 'y', 'z')
sage: Z3.arguments()                                                       #␣
↪needs sage.modules sage.symbolic
(x, y, z)

sage: Q4.<a, b, c, d> = ConditionSet(QQ^4); Q4                             #␣
↪needs sage.modules sage.symbolic
{ (a, b, c, d) ∈ Vector space of dimension 4 over Rational Field }
sage: Q4.variable_names()                                                  #␣
↪needs sage.modules sage.symbolic
('a', 'b', 'c', 'd')
sage: Q4.arguments()                                                       #␣
↪needs sage.modules sage.symbolic
(a, b, c, d)
```

```
>>> from sage.all import *
>>> Z3 = ConditionSet(ZZ**Integer(3), vars=['x', 'y', 'z']); Z3           ␣
↪      # needs sage.modules
{ (x, y, z) ∈ Ambient free module of rank 3
```

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```
            over the principal ideal domain Integer Ring }
>>> Z3.variable_names()                                                    #␣
↪needs sage.modules
('x', 'y', 'z')
>>> Z3.arguments()                                                         #␣
↪needs sage.modules sage.symbolic
(x, y, z)

>>> Q4 = ConditionSet(QQ**Integer(4), names=('a', 'b', 'c', 'd',)); (a, b, c, d,)␣
↪= Q4._first_ngens(4); Q4                                  # needs sage.modules␣
↪sage.symbolic
{ (a, b, c, d) ∈ Vector space of dimension 4 over Rational Field }
>>> Q4.variable_names()                                                    #␣
↪needs sage.modules sage.symbolic
('a', 'b', 'c', 'd')
>>> Q4.arguments()                                                         #␣
↪needs sage.modules sage.symbolic
(a, b, c, d)
```

ambient()

Return the universe of `self`.

EXAMPLES:

```
sage: Evens = ConditionSet(ZZ, is_even); Evens
{ x ∈ Integer Ring : <function is_even at 0x...>(x) }
sage: Evens.ambient()
Integer Ring
```

```
>>> from sage.all import *
>>> Evens = ConditionSet(ZZ, is_even); Evens
{ x ∈ Integer Ring : <function is_even at 0x...>(x) }
>>> Evens.ambient()
Integer Ring
```

arguments()

Return the variables of `self` as elements of the symbolic ring.

EXAMPLES:

```
sage: Odds = ConditionSet(ZZ, is_odd); Odds
{ x ∈ Integer Ring : <function is_odd at 0x...>(x) }
sage: args = Odds.arguments(); args                                       #␣
↪needs sage.symbolic
(x,)
sage: args[0].parent()                                                    #␣
↪needs sage.symbolic
Symbolic Ring
```

```
>>> from sage.all import *
>>> Odds = ConditionSet(ZZ, is_odd); Odds
{ x ∈ Integer Ring : <function is_odd at 0x...>(x) }
```

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```
>>> args = Odds.arguments(); args                                              #␣
↪needs sage.symbolic
(x,)
>>> args[Integer(0)].parent()                                                  ␣
↪     # needs sage.symbolic
Symbolic Ring
```

intersection(*X*)

Return the intersection of `self` and `X`.

EXAMPLES:

```
sage: # needs sage.modules sage.symbolic
sage: in_small_oblong(x, y) = x^2 + 3 * y^2 <= 42
sage: SmallOblongUniverse = ConditionSet(QQ^2, in_small_oblong)
sage: SmallOblongUniverse
{ (x, y) ∈ Vector space of dimension 2
                    over Rational Field : x^2 + 3*y^2 <= 42 }
sage: parity_check(x, y) = abs(sin(pi/2*(x + y))) < 1/1000
sage: EvenUniverse = ConditionSet(ZZ^2, parity_check); EvenUniverse
{ (x, y) ∈ Ambient free module of rank 2 over the principal ideal
           domain Integer Ring : abs(sin(1/2*pi*x + 1/2*pi*y)) < (1/1000) }
sage: SmallOblongUniverse & EvenUniverse
{ (x, y) ∈ Free module of degree 2 and rank 2 over Integer Ring
Echelon basis matrix:
[1 0]
[0 1] : x^2 + 3*y^2 <= 42, abs(sin(1/2*pi*x + 1/2*pi*y)) < (1/1000) }
```

```
>>> from sage.all import *
>>> # needs sage.modules sage.symbolic
>>> __tmp__=var("x,y"); in_small_oblong = symbolic_expression(x**Integer(2) +␣
↪Integer(3) * y**Integer(2) <= Integer(42)).function(x,y)
>>> SmallOblongUniverse = ConditionSet(QQ**Integer(2), in_small_oblong)
>>> SmallOblongUniverse
{ (x, y) ∈ Vector space of dimension 2
                    over Rational Field : x^2 + 3*y^2 <= 42 }
>>> __tmp__=var("x,y"); parity_check = symbolic_expression(abs(sin(pi/
↪Integer(2)*(x + y))) < Integer(1)/Integer(1000)).function(x,y)
>>> EvenUniverse = ConditionSet(ZZ**Integer(2), parity_check); EvenUniverse
{ (x, y) ∈ Ambient free module of rank 2 over the principal ideal
           domain Integer Ring : abs(sin(1/2*pi*x + 1/2*pi*y)) < (1/1000) }
>>> SmallOblongUniverse & EvenUniverse
{ (x, y) ∈ Free module of degree 2 and rank 2 over Integer Ring
Echelon basis matrix:
[1 0]
[0 1] : x^2 + 3*y^2 <= 42, abs(sin(1/2*pi*x + 1/2*pi*y)) < (1/1000) }
```

Combining two `ConditionSet``s with different formal variables works correctly. The formal variables of the intersection are taken from ``self`:

```
sage: # needs sage.modules sage.symbolic
sage: SmallMirrorUniverse = ConditionSet(QQ^2, in_small_oblong,
```

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```
....:                                     vars=(y, x))
sage: SmallMirrorUniverse
{ (y, x) ∈ Vector space of dimension 2
                   over Rational Field : 3*x^2 + y^2 <= 42 }
sage: SmallOblongUniverse & SmallMirrorUniverse
{ (x, y) ∈ Vector space of dimension 2
                   over Rational Field : x^2 + 3*y^2 <= 42 }
sage: SmallMirrorUniverse & SmallOblongUniverse
{ (y, x) ∈ Vector space of dimension 2
                   over Rational Field : 3*x^2 + y^2 <= 42 }
```

```
>>> from sage.all import *
>>> # needs sage.modules sage.symbolic
>>> SmallMirrorUniverse = ConditionSet(QQ**Integer(2), in_small_oblong,
...                                    vars=(y, x))
>>> SmallMirrorUniverse
{ (y, x) ∈ Vector space of dimension 2
                   over Rational Field : 3*x^2 + y^2 <= 42 }
>>> SmallOblongUniverse & SmallMirrorUniverse
{ (x, y) ∈ Vector space of dimension 2
                   over Rational Field : x^2 + 3*y^2 <= 42 }
>>> SmallMirrorUniverse & SmallOblongUniverse
{ (y, x) ∈ Vector space of dimension 2
                   over Rational Field : 3*x^2 + y^2 <= 42 }
```

1.10 Maps between finite sets

This module implements parents modeling the set of all maps between two finite sets. At the user level, any such parent should be constructed using the factory class *FiniteSetMaps* which properly selects which of its subclasses to use.

AUTHORS:

- Florent Hivert

class sage.sets.finite_set_maps.**FiniteSetEndoMaps_N**(*n*, *action*, *category=None*)

Bases: *FiniteSetMaps_MN*

The sets of all maps from $\{1, 2, \ldots, n\}$ to itself.

Users should use the factory class *FiniteSetMaps* to create instances of this class.

INPUT:

- `n` – integer
- `category` – the category in which the sets of maps is constructed. It must be a sub-category of `Monoids().Finite()` and `EnumeratedSets().Finite()` which is the default value.

Element

alias of *FiniteSetEndoMap_N*

an_element()

Return a map in `self`.

EXAMPLES:

```
sage: M = FiniteSetMaps(4)
sage: M.an_element()
[3, 2, 1, 0]
```

```
>>> from sage.all import *
>>> M = FiniteSetMaps(Integer(4))
>>> M.an_element()
[3, 2, 1, 0]
```

one()

EXAMPLES:

```
sage: M = FiniteSetMaps(4)
sage: M.one()
[0, 1, 2, 3]
```

```
>>> from sage.all import *
>>> M = FiniteSetMaps(Integer(4))
>>> M.one()
[0, 1, 2, 3]
```

class sage.sets.finite_set_maps.**FiniteSetEndoMaps_Set**(*domain*, *action*, *category=None*)

Bases: *FiniteSetMaps_Set*, *FiniteSetEndoMaps_N*

The sets of all maps from a set to itself.

Users should use the factory class *FiniteSetMaps* to create instances of this class.

INPUT:

- `domain` – an object in the category `FiniteSets()`
- `category` – the category in which the sets of maps is constructed. It must be a sub-category of `Monoids().Finite()` and `EnumeratedSets().Finite()` which is the default value.

Element

alias of *FiniteSetEndoMap_Set*

class sage.sets.finite_set_maps.**FiniteSetMaps**

Bases: `UniqueRepresentation`, `Parent`

Maps between finite sets.

Constructs the set of all maps between two sets. The sets can be given using any of the three following ways:

1. an object in the category `Sets()`.
2. a finite iterable. In this case, an object of the class *FiniteEnumeratedSet* is constructed from the iterable.
3. an integer `n` designing the set $\{0, 1, \ldots, n-1\}$. In this case an object of the class *IntegerRange* is constructed.

INPUT:

- `domain` – set, finite iterable, or integer
- `codomain` – set, finite iterable, integer, or `None` (default). In this last case, the maps are endo-maps of the domain.

- `action` – `'left'` (default) or `'right'`. The side where the maps act on the domain. This is used in particular to define the meaning of the product (composition) of two maps.
- `category` – the category in which the sets of maps is constructed. By default, this is `FiniteMonoids()` if the domain and codomain coincide, and `FiniteEnumeratedSets()` otherwise.

OUTPUT:

an instance of a subclass of *FiniteSetMaps* modeling the set of all maps between `domain` and `codomain`.

EXAMPLES:

We construct the set `M` of all maps from $\{a, b\}$ to $\{3, 4, 5\}$:

```
sage: M = FiniteSetMaps(["a", "b"], [3, 4, 5]); M
Maps from {'a', 'b'} to {3, 4, 5}
sage: M.cardinality()
9
sage: M.domain()
{'a', 'b'}
sage: M.codomain()
{3, 4, 5}
sage: for f in M: print(f)
map: a -> 3, b -> 3
map: a -> 3, b -> 4
map: a -> 3, b -> 5
map: a -> 4, b -> 3
map: a -> 4, b -> 4
map: a -> 4, b -> 5
map: a -> 5, b -> 3
map: a -> 5, b -> 4
map: a -> 5, b -> 5
```

```
>>> from sage.all import *
>>> M = FiniteSetMaps(["a", "b"], [Integer(3), Integer(4), Integer(5)]); M
Maps from {'a', 'b'} to {3, 4, 5}
>>> M.cardinality()
9
>>> M.domain()
{'a', 'b'}
>>> M.codomain()
{3, 4, 5}
>>> for f in M: print(f)
map: a -> 3, b -> 3
map: a -> 3, b -> 4
map: a -> 3, b -> 5
map: a -> 4, b -> 3
map: a -> 4, b -> 4
map: a -> 4, b -> 5
map: a -> 5, b -> 3
map: a -> 5, b -> 4
map: a -> 5, b -> 5
```

Elements can be constructed from functions and dictionaries:

```
sage: M(lambda c: ord(c)-94)
map: a -> 3, b -> 4

sage: M.from_dict({'a':3, 'b':5})
map: a -> 3, b -> 5
```

```
>>> from sage.all import *
>>> M(lambda c: ord(c)-Integer(94))
map: a -> 3, b -> 4

>>> M.from_dict({'a':Integer(3), 'b':Integer(5)})
map: a -> 3, b -> 5
```

If the domain is equal to the codomain, then maps can be composed:

```
sage: M = FiniteSetMaps([1, 2, 3])
sage: f = M.from_dict({1:2, 2:1, 3:3}); f
map: 1 -> 2, 2 -> 1, 3 -> 3
sage: g = M.from_dict({1:2, 2:3, 3:1}); g
map: 1 -> 2, 2 -> 3, 3 -> 1

sage: f * g
map: 1 -> 1, 2 -> 3, 3 -> 2
```

```
>>> from sage.all import *
>>> M = FiniteSetMaps([Integer(1), Integer(2), Integer(3)])
>>> f = M.from_dict({Integer(1):Integer(2), Integer(2):Integer(1),␣
↪Integer(3):Integer(3)}); f
map: 1 -> 2, 2 -> 1, 3 -> 3
>>> g = M.from_dict({Integer(1):Integer(2), Integer(2):Integer(3),␣
↪Integer(3):Integer(1)}); g
map: 1 -> 2, 2 -> 3, 3 -> 1

>>> f * g
map: 1 -> 1, 2 -> 3, 3 -> 2
```

This makes M into a monoid:

```
sage: M.category()
Category of finite enumerated monoids
sage: M.one()
map: 1 -> 1, 2 -> 2, 3 -> 3
```

```
>>> from sage.all import *
>>> M.category()
Category of finite enumerated monoids
>>> M.one()
map: 1 -> 1, 2 -> 2, 3 -> 3
```

By default, composition is from right to left, which corresponds to an action on the left. If one specifies `action` to right, then the composition is from left to right:

```
sage: M = FiniteSetMaps([1, 2, 3], action = 'right')
sage: f = M.from_dict({1:2, 2:1, 3:3})
sage: g = M.from_dict({1:2, 2:3, 3:1})
sage: f * g
map: 1 -> 3, 2 -> 2, 3 -> 1
```

```
>>> from sage.all import *
>>> M = FiniteSetMaps([Integer(1), Integer(2), Integer(3)], action = 'right')
>>> f = M.from_dict({Integer(1):Integer(2), Integer(2):Integer(1),␣
↪Integer(3):Integer(3)})
>>> g = M.from_dict({Integer(1):Integer(2), Integer(2):Integer(3),␣
↪Integer(3):Integer(1)})
>>> f * g
map: 1 -> 3, 2 -> 2, 3 -> 1
```

If the domains and codomains are both of the form $\{0, \dots\}$, then one can use the shortcut:

```
sage: M = FiniteSetMaps(2,3); M
Maps from {0, 1} to {0, 1, 2}
sage: M.cardinality()
9
```

```
>>> from sage.all import *
>>> M = FiniteSetMaps(Integer(2),Integer(3)); M
Maps from {0, 1} to {0, 1, 2}
>>> M.cardinality()
9
```

For a compact notation, the elements are then printed as lists $[f(i), i = 0, \dots]$:

```
sage: list(M)
[[0, 0], [0, 1], [0, 2], [1, 0], [1, 1], [1, 2], [2, 0], [2, 1], [2, 2]]
```

```
>>> from sage.all import *
>>> list(M)
[[0, 0], [0, 1], [0, 2], [1, 0], [1, 1], [1, 2], [2, 0], [2, 1], [2, 2]]
```

cardinality()

The cardinality of `self`.

EXAMPLES:

```
sage: FiniteSetMaps(4, 3).cardinality()
81
```

```
>>> from sage.all import *
>>> FiniteSetMaps(Integer(4), Integer(3)).cardinality()
81
```

class sage.sets.finite_set_maps.**FiniteSetMaps_MN**(*m*, *n*, *category=None*)

Bases: *FiniteSetMaps*

The set of all maps from $\{1, 2, \dots, m\}$ to $\{1, 2, \dots, n\}$.

Users should use the factory class *FiniteSetMaps* to create instances of this class.

INPUT:

- m, n – integers
- category – the category in which the sets of maps is constructed. It must be a sub-category of EnumeratedSets().Finite() which is the default value.

Element

alias of *FiniteSetMap_MN*

an_element()

Return a map in self.

EXAMPLES:

```
sage: M = FiniteSetMaps(4, 2)
sage: M.an_element()
[0, 0, 0, 0]

sage: M = FiniteSetMaps(0, 0)
sage: M.an_element()
[]
```

```
>>> from sage.all import *
>>> M = FiniteSetMaps(Integer(4), Integer(2))
>>> M.an_element()
[0, 0, 0, 0]

>>> M = FiniteSetMaps(Integer(0), Integer(0))
>>> M.an_element()
[]
```

An exception EmptySetError is raised if this set is empty, that is if the codomain is empty and the domain is not.

```
sage: M = FiniteSetMaps(4, 0)
sage: M.cardinality()
0
sage: M.an_element()
Traceback (most recent call last):
...
EmptySetError
```

```
>>> from sage.all import *
>>> M = FiniteSetMaps(Integer(4), Integer(0))
>>> M.cardinality()
0
>>> M.an_element()
Traceback (most recent call last):
...
EmptySetError
```

codomain()

The codomain of self.

EXAMPLES:

```
sage: FiniteSetMaps(3,2).codomain()
{0, 1}
```

```
>>> from sage.all import *
>>> FiniteSetMaps(Integer(3),Integer(2)).codomain()
{0, 1}
```

domain()

The domain of `self`.

EXAMPLES:

```
sage: FiniteSetMaps(3,2).domain()
{0, 1, 2}
```

```
>>> from sage.all import *
>>> FiniteSetMaps(Integer(3),Integer(2)).domain()
{0, 1, 2}
```

class sage.sets.finite_set_maps.**FiniteSetMaps_Set**(*domain*, *codomain*, *category=None*)

Bases: *FiniteSetMaps_MN*

The sets of all maps between two sets.

Users should use the factory class *FiniteSetMaps* to create instances of this class.

INPUT:

- `domain` – an object in the category `FiniteSets()`
- `codomain` – an object in the category `FiniteSets()`
- `category` – the category in which the sets of maps is constructed. It must be a sub-category of `EnumeratedSets().Finite()` which is the default value.

Element

alias of *FiniteSetMap_Set*

codomain()

The codomain of `self`.

EXAMPLES:

```
sage: FiniteSetMaps(["a", "b"], [3, 4, 5]).codomain()
{3, 4, 5}
```

```
>>> from sage.all import *
>>> FiniteSetMaps(["a", "b"], [Integer(3), Integer(4), Integer(5)]).codomain()
{3, 4, 5}
```

domain()

The domain of `self`.

EXAMPLES:

```
sage: FiniteSetMaps(["a", "b"], [3, 4, 5]).domain()
{'a', 'b'}
```

```
>>> from sage.all import *
>>> FiniteSetMaps(["a", "b"], [Integer(3), Integer(4), Integer(5)]).domain()
{'a', 'b'}
```

from_dict(*d*)

Create a map from a dictionary.

EXAMPLES:

```
sage: M = FiniteSetMaps(["a", "b"], [3, 4, 5])
sage: M.from_dict({"a": 4, "b": 3})
map: a -> 4, b -> 3
```

```
>>> from sage.all import *
>>> M = FiniteSetMaps(["a", "b"], [Integer(3), Integer(4), Integer(5)])
>>> M.from_dict({"a": Integer(4), "b": Integer(3)})
map: a -> 4, b -> 3
```

1.11 Data structures for maps between finite sets

This module implements several fast Cython data structures for maps between two finite set. Those classes are not intended to be used directly. Instead, such a map should be constructed via its parent, using the class *FiniteSetMaps*.

EXAMPLES:

To create a map between two sets, one first creates the set of such maps:

```
sage: M = FiniteSetMaps(["a", "b"], [3, 4, 5])
```

```
>>> from sage.all import *
>>> M = FiniteSetMaps(["a", "b"], [Integer(3), Integer(4), Integer(5)])
```

The map can then be constructed either from a function:

```
sage: f1 = M(lambda c: ord(c)-94); f1
map: a -> 3, b -> 4
```

```
>>> from sage.all import *
>>> f1 = M(lambda c: ord(c)-Integer(94)); f1
map: a -> 3, b -> 4
```

or from a dictionary:

```
sage: f2 = M.from_dict({'a':3, 'b':4}); f2
map: a -> 3, b -> 4
```

```
>>> from sage.all import *
>>> f2 = M.from_dict({'a':Integer(3), 'b':Integer(4)}); f2
map: a -> 3, b -> 4
```

The two created maps are equal:

```
sage: f1 == f2
True
```

```
>>> from sage.all import *
>>> f1 == f2
True
```

Internally, maps are represented as the list of the ranks of the images `f(x)` in the co-domain, in the order of the domain:

```
sage: list(f2)
[0, 1]
```

```
>>> from sage.all import *
>>> list(f2)
[0, 1]
```

A third fast way to create a map it to use such a list. it should be kept for internal use:

```
sage: f3 = M._from_list_([0, 1]); f3
map: a -> 3, b -> 4
sage: f1 == f3
True
```

```
>>> from sage.all import *
>>> f3 = M._from_list_([Integer(0), Integer(1)]); f3
map: a -> 3, b -> 4
>>> f1 == f3
True
```

AUTHORS:

- Florent Hivert

class sage.sets.finite_set_map_cy.**FiniteSetEndoMap_N**

Bases: *FiniteSetMap_MN*

Map from `range(n)` to itself.

See also

FiniteSetMap_MN for assumptions on the parent

class sage.sets.finite_set_map_cy.**FiniteSetEndoMap_Set**

Bases: *FiniteSetMap_Set*

Map from a set to itself.

See also

FiniteSetMap_Set for assumptions on the parent

class sage.sets.finite_set_map_cy.**FiniteSetMap_MN**

Bases: ClonableIntArray

Data structure for maps from `range(m)` to `range(n)`.

We assume that the parent given as argument is such that:

- `m` – stored in `self.parent()._m`
- `n` – stored in `self.parent()._n`
- the domain is in `self.parent().domain()`
- the codomain is in `self.parent().codomain()`

check()

Perform checks on `self`.

Check that `self` is a proper function and then calls `parent.check_element(self)` where `parent` is the parent of `self`.

codomain()

Return the codomain of `self`.

EXAMPLES:

```
sage: FiniteSetMaps(4, 3)([1, 0, 2, 1]).codomain()
{0, 1, 2}
```

```
>>> from sage.all import *
>>> FiniteSetMaps(Integer(4), Integer(3))([Integer(1), Integer(0), Integer(2),
↪ Integer(1)]).codomain()
{0, 1, 2}
```

domain()

Return the domain of `self`.

EXAMPLES:

```
sage: FiniteSetMaps(4, 3)([1, 0, 2, 1]).domain()
{0, 1, 2, 3}
```

```
>>> from sage.all import *
>>> FiniteSetMaps(Integer(4), Integer(3))([Integer(1), Integer(0), Integer(2),
↪ Integer(1)]).domain()
{0, 1, 2, 3}
```

fibers()

Return the fibers of `self`.

OUTPUT:

a dictionary `d` such that `d[y]` is the set of all `x` in `domain` such that `f(x) = y`

EXAMPLES:

```
sage: FiniteSetMaps(4, 3)([1, 0, 2, 1]).fibers()
{0: {1}, 1: {0, 3}, 2: {2}}
sage: F = FiniteSetMaps(["a", "b", "c"])
```

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```
sage: F.from_dict({"a": "b", "b": "a", "c": "b"}).fibers() == {'a': {'b'}, 'b
↪': {'a', 'c'}}
True
```

```
>>> from sage.all import *
>>> FiniteSetMaps(Integer(4), Integer(3))([Integer(1), Integer(0), Integer(2),
↪ Integer(1)]).fibers()
{0: {1}, 1: {0, 3}, 2: {2}}
>>> F = FiniteSetMaps(["a", "b", "c"])
>>> F.from_dict({"a": "b", "b": "a", "c": "b"}).fibers() == {'a': {'b'}, 'b':
↪{'a', 'c'}}
True
```

getimage(*i*)

Return the image of `i` by `self`.

INPUT:

- `i` – any object

Note

if you need speed, please use instead `_getimage()`

EXAMPLES:

```
sage: fs = FiniteSetMaps(4, 3)([1, 0, 2, 1])
sage: fs.getimage(0), fs.getimage(1), fs.getimage(2), fs.getimage(3)
(1, 0, 2, 1)
```

```
>>> from sage.all import *
>>> fs = FiniteSetMaps(Integer(4), Integer(3))([Integer(1), Integer(0),␣
↪Integer(2), Integer(1)])
>>> fs.getimage(Integer(0)), fs.getimage(Integer(1)), fs.getimage(Integer(2)),
↪ fs.getimage(Integer(3))
(1, 0, 2, 1)
```

image_set()

Return the image set of `self`.

EXAMPLES:

```
sage: FiniteSetMaps(4, 3)([1, 0, 2, 1]).image_set()
{0, 1, 2}
sage: FiniteSetMaps(4, 3)([1, 0, 0, 1]).image_set()
{0, 1}
```

```
>>> from sage.all import *
>>> FiniteSetMaps(Integer(4), Integer(3))([Integer(1), Integer(0), Integer(2),
↪ Integer(1)]).image_set()
{0, 1, 2}
>>> FiniteSetMaps(Integer(4), Integer(3))([Integer(1), Integer(0), Integer(0),
```

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```
↪ Integer(1)]).image_set()
{0, 1}
```

items()

The items of `self`.

Return the list of the ordered pairs `(x, self(x))`

EXAMPLES:

```
sage: FiniteSetMaps(4, 3)([1, 0, 2, 1]).items()
[(0, 1), (1, 0), (2, 2), (3, 1)]
```

```
>>> from sage.all import *
>>> FiniteSetMaps(Integer(4), Integer(3))([Integer(1), Integer(0), Integer(2),
↪ Integer(1)]).items()
[(0, 1), (1, 0), (2, 2), (3, 1)]
```

setimage(i, j)

Set the image of `i` as `j` in `self`.

Warning

`self` must be mutable; otherwise an exception is raised.

INPUT:

- `i`, `j` – two `object`'s

OUTPUT: none

Note

if you need speed, please use instead `_setimage()`

EXAMPLES:

```
sage: fs = FiniteSetMaps(4, 3)([1, 0, 2, 1])
sage: fs2 = copy(fs)
sage: fs2.setimage(2, 1)
sage: fs2
[1, 0, 1, 1]
sage: with fs.clone() as fs3:
....:     fs3.setimage(0, 2)
....:     fs3.setimage(1, 2)
sage: fs3
[2, 2, 2, 1]
```

```
>>> from sage.all import *
>>> fs = FiniteSetMaps(Integer(4), Integer(3))([Integer(1), Integer(0),␣
↪Integer(2), Integer(1)])
```

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```
>>> fs2 = copy(fs)
>>> fs2.setimage(Integer(2), Integer(1))
>>> fs2
[1, 0, 1, 1]
>>> with fs.clone() as fs3:
...     fs3.setimage(Integer(0), Integer(2))
...     fs3.setimage(Integer(1), Integer(2))
>>> fs3
[2, 2, 2, 1]
```

class sage.sets.finite_set_map_cy.**FiniteSetMap_Set**

Bases: *FiniteSetMap_MN*

Data structure for maps.

We assume that the parent given as argument is such that:

- the domain is in `parent.domain()`
- the codomain is in `parent.codomain()`
- `parent._m` contains the cardinality of the domain
- `parent._n` contains the cardinality of the codomain
- `parent._unrank_domain` and `parent._rank_domain` is a pair of reciprocal rank and unrank functions between the domain and `range(parent._m)`.
- `parent._unrank_codomain` and `parent._rank_codomain` is a pair of reciprocal rank and unrank functions between the codomain and `range(parent._n)`.

classmethod from_dict(*t*, *parent*, *d*)

Create a `FiniteSetMap` from a dictionary.

Warning

no check is performed !

classmethod from_list(*t*, *parent*, *lst*)

Create a `FiniteSetMap` from a list.

Warning

no check is performed !

getimage(*i*)

Return the image of `i` by `self`.

INPUT:

- `i` – integer

EXAMPLES:

```
sage: F = FiniteSetMaps(["a", "b", "c", "d"], ["u", "v", "w"])
sage: fs = F._from_list_([1, 0, 2, 1])
sage: list(map(fs.getimage, ["a", "b", "c", "d"]))
['v', 'u', 'w', 'v']
```

```
>>> from sage.all import *
>>> F = FiniteSetMaps(["a", "b", "c", "d"], ["u", "v", "w"])
>>> fs = F._from_list_([Integer(1), Integer(0), Integer(2), Integer(1)])
>>> list(map(fs.getimage, ["a", "b", "c", "d"]))
['v', 'u', 'w', 'v']
```

image_set()

Return the image set of `self`.

EXAMPLES:

```
sage: F = FiniteSetMaps(["a", "b", "c"])
sage: sorted(F.from_dict({"a": "b", "b": "a", "c": "b"}).image_set())
['a', 'b']
sage: F = FiniteSetMaps(["a", "b", "c"])
sage: F(lambda x: "c").image_set()
{'c'}
```

```
>>> from sage.all import *
>>> F = FiniteSetMaps(["a", "b", "c"])
>>> sorted(F.from_dict({"a": "b", "b": "a", "c": "b"}).image_set())
['a', 'b']
>>> F = FiniteSetMaps(["a", "b", "c"])
>>> F(lambda x: "c").image_set()
{'c'}
```

items()

The items of `self`.

Return the list of the couple `(x, self(x))`

EXAMPLES:

```
sage: F = FiniteSetMaps(["a", "b", "c"])
sage: F.from_dict({"a": "b", "b": "a", "c": "b"}).items()
[('a', 'b'), ('b', 'a'), ('c', 'b')]
```

```
>>> from sage.all import *
>>> F = FiniteSetMaps(["a", "b", "c"])
>>> F.from_dict({"a": "b", "b": "a", "c": "b"}).items()
[('a', 'b'), ('b', 'a'), ('c', 'b')]
```

setimage(i, j)

Set the image of `i` as `j` in `self`.

> **Warning**
>
> `self` must be mutable otherwise an exception is raised.

INPUT:

- `i`, `j` – two `object`'s

OUTPUT: none

EXAMPLES:

```
sage: F = FiniteSetMaps(["a", "b", "c", "d"], ["u", "v", "w"])
sage: fs = F(lambda x: "v")
sage: fs2 = copy(fs)
sage: fs2.setimage("a", "w")
sage: fs2
map: a -> w, b -> v, c -> v, d -> v
sage: with fs.clone() as fs3:
....:     fs3.setimage("a", "u")
....:     fs3.setimage("c", "w")
sage: fs3
map: a -> u, b -> v, c -> w, d -> v
```

```
>>> from sage.all import *
>>> F = FiniteSetMaps(["a", "b", "c", "d"], ["u", "v", "w"])
>>> fs = F(lambda x: "v")
>>> fs2 = copy(fs)
>>> fs2.setimage("a", "w")
>>> fs2
map: a -> w, b -> v, c -> v, d -> v
>>> with fs.clone() as fs3:
...     fs3.setimage("a", "u")
...     fs3.setimage("c", "w")
>>> fs3
map: a -> u, b -> v, c -> w, d -> v
```

`sage.sets.finite_set_map_cy.`**`FiniteSetMap_Set_from_dict`**(*t*, *parent*, *d*)

Create a `FiniteSetMap` from a dictionary.

Warning

no check is performed !

`sage.sets.finite_set_map_cy.`**`FiniteSetMap_Set_from_list`**(*t*, *parent*, *lst*)

Create a `FiniteSetMap` from a list.

Warning

no check is performed !

`sage.sets.finite_set_map_cy.`**`fibers`**(*f*, *domain*)

Return the fibers of the function `f` on the finite set `domain`.

INPUT:

- `f` – a function or callable

- `domain` – a finite iterable

OUTPUT:

- a dictionary `d` such that `d[y]` is the set of all `x` in `domain` such that `f(x) = y`

EXAMPLES:

```
sage: from sage.sets.finite_set_map_cy import fibers, fibers_args
sage: fibers(lambda x: 1, [])
{}
sage: fibers(lambda x: x^2, [-1, 2, -3, 1, 3, 4])
{1: {1, -1}, 4: {2}, 9: {3, -3}, 16: {4}}
sage: fibers(lambda x: 1,   [-1, 2, -3, 1, 3, 4])
{1: {1, 2, 3, 4, -3, -1}}
sage: fibers(lambda x: 1, [1,1,1])
{1: {1}}
```

```
>>> from sage.all import *
>>> from sage.sets.finite_set_map_cy import fibers, fibers_args
>>> fibers(lambda x: Integer(1), [])
{}
>>> fibers(lambda x: x**Integer(2), [-Integer(1), Integer(2), -Integer(3),␣
↪Integer(1), Integer(3), Integer(4)])
{1: {1, -1}, 4: {2}, 9: {3, -3}, 16: {4}}
>>> fibers(lambda x: Integer(1),   [-Integer(1), Integer(2), -Integer(3),␣
↪Integer(1), Integer(3), Integer(4)])
{1: {1, 2, 3, 4, -3, -1}}
>>> fibers(lambda x: Integer(1), [Integer(1),Integer(1),Integer(1)])
{1: {1}}
```

See also

fibers_args() if one needs to pass extra arguments to `f`.

`sage.sets.finite_set_map_cy.`**`fibers_args`**(*f*, *domain*, **args*, ***opts*)

Return the fibers of the function `f` on the finite set `domain`.

It is the same as *fibers()* except that one can pass extra argument for `f` (with a small overhead)

EXAMPLES:

```
sage: from sage.sets.finite_set_map_cy import fibers_args
sage: fibers_args(operator.pow, [-1, 2, -3, 1, 3, 4], 2)
{1: {1, -1}, 4: {2}, 9: {3, -3}, 16: {4}}
```

```
>>> from sage.all import *
>>> from sage.sets.finite_set_map_cy import fibers_args
>>> fibers_args(operator.pow, [-Integer(1), Integer(2), -Integer(3), Integer(1),␣
↪Integer(3), Integer(4)], Integer(2))
{1: {1, -1}, 4: {2}, 9: {3, -3}, 16: {4}}
```

1.12 Totally Ordered Finite Sets

AUTHORS:

- Stepan Starosta (2012): Initial version

class sage.sets.totally_ordered_finite_set.**TotallyOrderedFiniteSet**(*elements*, *facade=True*)

Bases: *FiniteEnumeratedSet*

Totally ordered finite set.

This is a finite enumerated set assuming that the elements are ordered based upon their rank (i.e. their position in the set).

INPUT:

- `elements` – list of elements in the set
- `facade` – boolean (default: `True`); if `True`, a facade is used. It should be set to `False` if the elements do not inherit from `Element` or if you want a funny order. See examples for more details.

See also

FiniteEnumeratedSet

EXAMPLES:

```
sage: S = TotallyOrderedFiniteSet([1,2,3])
sage: S
{1, 2, 3}
sage: S.cardinality()
3
```

```
>>> from sage.all import *
>>> S = TotallyOrderedFiniteSet([Integer(1),Integer(2),Integer(3)])
>>> S
{1, 2, 3}
>>> S.cardinality()
3
```

By default, totally ordered finite set behaves as a facade:

```
sage: S(1).parent()
Integer Ring
```

```
>>> from sage.all import *
>>> S(Integer(1)).parent()
Integer Ring
```

It makes comparison fails when it is not the standard order:

```
sage: T1 = TotallyOrderedFiniteSet([3,2,5,1])
sage: T1(3) < T1(1)
False
sage: T2 = TotallyOrderedFiniteSet([3, x])                                  #␣
↪needs sage.symbolic
```

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```
sage: T2(3) < T2(x)                                                             #␣
↪needs sage.symbolic
3 < x
```

```
>>> from sage.all import *
>>> T1 = TotallyOrderedFiniteSet([Integer(3),Integer(2),Integer(5),Integer(1)])
>>> T1(Integer(3)) < T1(Integer(1))
False
>>> T2 = TotallyOrderedFiniteSet([Integer(3), x])                               ␣
↪     # needs sage.symbolic
>>> T2(Integer(3)) < T2(x)                                                      ␣
↪     # needs sage.symbolic
3 < x
```

To make the above example work, you should set the argument facade to `False` in the constructor. In that case, the elements of the set have a dedicated class:

```
sage: A = TotallyOrderedFiniteSet([3,2,0,'a',7,(0,0),1], facade=False)
sage: A
{3, 2, 0, 'a', 7, (0, 0), 1}
sage: x = A.an_element()
sage: x
3
sage: x.parent()
{3, 2, 0, 'a', 7, (0, 0), 1}
sage: A(3) < A(2)
True
sage: A('a') < A(7)
True
sage: A(3) > A(2)
False
sage: A(1) < A(3)
False
sage: A(3) == A(3)
True
```

```
>>> from sage.all import *
>>> A = TotallyOrderedFiniteSet([Integer(3),Integer(2),Integer(0),'a',Integer(7),
↪(Integer(0),Integer(0)),Integer(1)], facade=False)
>>> A
{3, 2, 0, 'a', 7, (0, 0), 1}
>>> x = A.an_element()
>>> x
3
>>> x.parent()
{3, 2, 0, 'a', 7, (0, 0), 1}
>>> A(Integer(3)) < A(Integer(2))
True
>>> A('a') < A(Integer(7))
True
>>> A(Integer(3)) > A(Integer(2))
False
```

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```
>>> A(Integer(1)) < A(Integer(3))
False
>>> A(Integer(3)) == A(Integer(3))
True
```

But then, the equality comparison is always False with elements outside of the set:

```
sage: A(1) == 1
False
sage: 1 == A(1)
False
sage: 'a' == A('a')
False
sage: A('a') == 'a'
False
```

```
>>> from sage.all import *
>>> A(Integer(1)) == Integer(1)
False
>>> Integer(1) == A(Integer(1))
False
>>> 'a' == A('a')
False
>>> A('a') == 'a'
False
```

Since Issue #16280, totally ordered sets support elements that do not inherit from `sage.structure.element.Element`, whether they are facade or not:

```
sage: S = TotallyOrderedFiniteSet(['a','b'])
sage: S('a')
'a'
sage: S = TotallyOrderedFiniteSet(['a','b'], facade = False)
sage: S('a')
'a'
```

```
>>> from sage.all import *
>>> S = TotallyOrderedFiniteSet(['a','b'])
>>> S('a')
'a'
>>> S = TotallyOrderedFiniteSet(['a','b'], facade = False)
>>> S('a')
'a'
```

Multiple elements are automatically deleted:

```
sage: TotallyOrderedFiniteSet([1,1,2,1,2,2,5,4])
{1, 2, 5, 4}
```

```
>>> from sage.all import *
>>> TotallyOrderedFiniteSet([Integer(1),Integer(1),Integer(2),Integer(1),
```

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```
↪Integer(2),Integer(2),Integer(5),Integer(4)])
{1, 2, 5, 4}
```

Element

alias of *TotallyOrderedFiniteSetElement*

le(*x*, *y*)

Return `True` if $x \leq y$ for the order of `self`.

EXAMPLES:

```
sage: T = TotallyOrderedFiniteSet([1,3,2], facade=False)
sage: T1, T3, T2 = T.list()
sage: T.le(T1,T3)
True
sage: T.le(T3,T2)
True
```

```
>>> from sage.all import *
>>> T = TotallyOrderedFiniteSet([Integer(1),Integer(3),Integer(2)],␣
↪facade=False)
>>> T1, T3, T2 = T.list()
>>> T.le(T1,T3)
True
>>> T.le(T3,T2)
True
```

class sage.sets.totally_ordered_finite_set.**TotallyOrderedFiniteSetElement**(*parent*, *data*)

Bases: `Element`

Element of a finite totally ordered set.

EXAMPLES:

```
sage: S = TotallyOrderedFiniteSet([2,7], facade=False)
sage: x = S(2)
sage: print(x)
2
sage: x.parent()
{2, 7}
```

```
>>> from sage.all import *
>>> S = TotallyOrderedFiniteSet([Integer(2),Integer(7)], facade=False)
>>> x = S(Integer(2))
>>> print(x)
2
>>> x.parent()
{2, 7}
```

1.13 Set of all objects of a given Python class

sage.sets.pythonclass.**Set_PythonType**(*typ*)

Return the (unique) Parent that represents the set of Python objects of a specified type.

EXAMPLES:

```
sage: from sage.sets.pythonclass import Set_PythonType
sage: Set_PythonType(list)
Set of Python objects of class 'list'
sage: Set_PythonType(list) is Set_PythonType(list)
True
sage: S = Set_PythonType(tuple)
sage: S([1,2,3])
(1, 2, 3)
```

```
>>> from sage.all import *
>>> from sage.sets.pythonclass import Set_PythonType
>>> Set_PythonType(list)
Set of Python objects of class 'list'
>>> Set_PythonType(list) is Set_PythonType(list)
True
>>> S = Set_PythonType(tuple)
>>> S([Integer(1),Integer(2),Integer(3)])
(1, 2, 3)
```

S is a parent which models the set of all lists:

```
sage: S.category()
Category of infinite sets
```

```
>>> from sage.all import *
>>> S.category()
Category of infinite sets
```

class sage.sets.pythonclass.**Set_PythonType_class**

Bases: Set_generic

The set of Python objects of a given class.

The elements of this set are not instances of Element; they are instances of the given class.

INPUT:

- typ – a Python (new-style) class

EXAMPLES:

```
sage: from sage.sets.pythonclass import Set_PythonType
sage: S = Set_PythonType(int); S
Set of Python objects of class 'int'
sage: int('1') in S
True
sage: Integer('1') in S
False
```

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```
sage: Set_PythonType(2)
Traceback (most recent call last):
...
TypeError: must be initialized with a class, not 2
```

```
>>> from sage.all import *
>>> from sage.sets.pythonclass import Set_PythonType
>>> S = Set_PythonType(int); S
Set of Python objects of class 'int'
>>> int('1') in S
True
>>> Integer('1') in S
False

>>> Set_PythonType(Integer(2))
Traceback (most recent call last):
...
TypeError: must be initialized with a class, not 2
```

cardinality()

EXAMPLES:

```
sage: from sage.sets.pythonclass import Set_PythonType
sage: S = Set_PythonType(bool)
sage: S.cardinality()
2
sage: S = Set_PythonType(int)
sage: S.cardinality()
+Infinity
```

```
>>> from sage.all import *
>>> from sage.sets.pythonclass import Set_PythonType
>>> S = Set_PythonType(bool)
>>> S.cardinality()
2
>>> S = Set_PythonType(int)
>>> S.cardinality()
+Infinity
```

object()

EXAMPLES:

```
sage: from sage.sets.pythonclass import Set_PythonType
sage: Set_PythonType(tuple).object()
<... 'tuple'>
```

```
>>> from sage.all import *
>>> from sage.sets.pythonclass import Set_PythonType
>>> Set_PythonType(tuple).object()
<... 'tuple'>
```

CHAPTER

TWO

SETS OF NUMBERS

2.1 Integer Range

AUTHORS:

- Nicolas Borie (2010-03): First release.
- Florent Hivert (2010-03): Added a class factory + cardinality method.
- Vincent Delecroix (2012-02): add methods rank/unrank, make it compliant with Python int.

class sage.sets.integer_range.**IntegerRange**

Bases: UniqueRepresentation, Parent

The class of Integer ranges.

Returns an enumerated set containing an arithmetic progression of integers.

INPUT:

- begin – integer, Infinity or -Infinity
- end – integer, Infinity or -Infinity
- step – a nonzero integer (default: 1)
- middle_point – integer inside the set (default: None)

OUTPUT:

A parent in the category FiniteEnumeratedSets() or InfiniteEnumeratedSets() depending on the arguments defining self.

IntegerRange(i, j) returns the set of $\{i, i+1, i+2, \ldots, j-1\}$. start (!) defaults to 0. When step is given, it specifies the increment. The default increment is 1. IntegerRange allows begin and end to be infinite.

IntegerRange is designed to have similar interface Python range. However, whereas range accept and returns Python int, IntegerRange deals with Integer.

If middle_point is given, then the elements are generated starting from it, in a alternating way: $\{m, m+1, m-2, m+2, m-2\ldots\}$.

EXAMPLES:

```
sage: list(IntegerRange(5))
[0, 1, 2, 3, 4]
sage: list(IntegerRange(2,5))
[2, 3, 4]
sage: I = IntegerRange(2,100,5); I
```

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```
{2, 7, ..., 97}
sage: list(I)
[2, 7, 12, 17, 22, 27, 32, 37, 42, 47, 52, 57, 62, 67, 72, 77, 82, 87, 92, 97]
sage: I.category()
Category of facade finite enumerated sets
sage: I[1].parent()
Integer Ring
```

```
>>> from sage.all import *
>>> list(IntegerRange(Integer(5)))
[0, 1, 2, 3, 4]
>>> list(IntegerRange(Integer(2),Integer(5)))
[2, 3, 4]
>>> I = IntegerRange(Integer(2),Integer(100),Integer(5)); I
{2, 7, ..., 97}
>>> list(I)
[2, 7, 12, 17, 22, 27, 32, 37, 42, 47, 52, 57, 62, 67, 72, 77, 82, 87, 92, 97]
>>> I.category()
Category of facade finite enumerated sets
>>> I[Integer(1)].parent()
Integer Ring
```

When `begin` and `end` are both finite, `IntegerRange(begin, end, step)` is the set whose list of elements is equivalent to the python construction `range(begin, end, step)`:

```
sage: list(IntegerRange(4,105,3)) == list(range(4,105,3))
True
sage: list(IntegerRange(-54,13,12)) == list(range(-54,13,12))
True
```

```
>>> from sage.all import *
>>> list(IntegerRange(Integer(4),Integer(105),Integer(3))) ==␣
↪list(range(Integer(4),Integer(105),Integer(3)))
True
>>> list(IntegerRange(-Integer(54),Integer(13),Integer(12))) == list(range(-
↪Integer(54),Integer(13),Integer(12)))
True
```

Except for the type of the numbers:

```
sage: type(IntegerRange(-54,13,12)[0]), type(list(range(-54,13,12))[0])
(<... 'sage.rings.integer.Integer'>, <... 'int'>)
```

```
>>> from sage.all import *
>>> type(IntegerRange(-Integer(54),Integer(13),Integer(12))[Integer(0)]),␣
↪type(list(range(-Integer(54),Integer(13),Integer(12)))[Integer(0)])
(<... 'sage.rings.integer.Integer'>, <... 'int'>)
```

When `begin` is finite and `end` is +Infinity, `self` is the infinite arithmetic progression starting from the `begin` by step `step`:

```
sage: I = IntegerRange(54,Infinity,3); I
{54, 57, ...}
sage: I.category()
Category of facade infinite enumerated sets
sage: p = iter(I)
sage: (next(p), next(p), next(p), next(p), next(p), next(p))
(54, 57, 60, 63, 66, 69)

sage: I = IntegerRange(54,-Infinity,-3); I
{54, 51, ...}
sage: I.category()
Category of facade infinite enumerated sets
sage: p = iter(I)
sage: (next(p), next(p), next(p), next(p), next(p), next(p))
(54, 51, 48, 45, 42, 39)
```

```
>>> from sage.all import *
>>> I = IntegerRange(Integer(54),Infinity,Integer(3)); I
{54, 57, ...}
>>> I.category()
Category of facade infinite enumerated sets
>>> p = iter(I)
>>> (next(p), next(p), next(p), next(p), next(p), next(p))
(54, 57, 60, 63, 66, 69)

>>> I = IntegerRange(Integer(54),-Infinity,-Integer(3)); I
{54, 51, ...}
>>> I.category()
Category of facade infinite enumerated sets
>>> p = iter(I)
>>> (next(p), next(p), next(p), next(p), next(p), next(p))
(54, 51, 48, 45, 42, 39)
```

When `begin` and `end` are both infinite, you will have to specify the extra argument `middle_point`. `self` is then defined by a point and a progression/regression setting by `step`. The enumeration is done this way: (let us call m the `middle_point`) $\{m, m+step, m-step, m+2step, m-2step, m+3step, \dots\}$:

```
sage: I = IntegerRange(-Infinity,Infinity,37,-12); I
Integer progression containing -12 with increment 37 and bounded with -Infinity␣
↪and +Infinity
sage: I.category()
Category of facade infinite enumerated sets
sage: -12 in I
True
sage: -15 in I
False
sage: p = iter(I)
sage: (next(p), next(p), next(p), next(p), next(p), next(p), next(p), next(p))
(-12, 25, -49, 62, -86, 99, -123, 136)
```

```
>>> from sage.all import *
>>> I = IntegerRange(-Infinity,Infinity,Integer(37),-Integer(12)); I
```

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```
Integer progression containing -12 with increment 37 and bounded with -Infinity␣
↪and +Infinity
>>> I.category()
Category of facade infinite enumerated sets
>>> -Integer(12) in I
True
>>> -Integer(15) in I
False
>>> p = iter(I)
>>> (next(p), next(p), next(p), next(p), next(p), next(p), next(p), next(p))
(-12, 25, -49, 62, -86, 99, -123, 136)
```

It is also possible to use the argument `middle_point` for other cases, finite or infinite. The set will be the same as if you didn't give this extra argument but the enumeration will begin with this `middle_point`:

```
sage: I = IntegerRange(123,-12,-14); I
{123, 109, ..., -3}
sage: list(I)
[123, 109, 95, 81, 67, 53, 39, 25, 11, -3]
sage: J = IntegerRange(123,-12,-14,25); J
Integer progression containing 25 with increment -14 and bounded with 123 and -12
sage: list(J)
[25, 11, 39, -3, 53, 67, 81, 95, 109, 123]
```

```
>>> from sage.all import *
>>> I = IntegerRange(Integer(123),-Integer(12),-Integer(14)); I
{123, 109, ..., -3}
>>> list(I)
[123, 109, 95, 81, 67, 53, 39, 25, 11, -3]
>>> J = IntegerRange(Integer(123),-Integer(12),-Integer(14),Integer(25)); J
Integer progression containing 25 with increment -14 and bounded with 123 and -12
>>> list(J)
[25, 11, 39, -3, 53, 67, 81, 95, 109, 123]
```

Remember that, like for range, if you define a non empty set, `begin` is supposed to be included and `end` is supposed to be excluded. In the same way, when you define a set with a `middle_point`, the `begin` bound will be supposed to be included and the `end` bound supposed to be excluded:

```
sage: I = IntegerRange(-100,100,10,0)
sage: J = list(range(-100,100,10))
sage: 100 in I
False
sage: 100 in J
False
sage: -100 in I
True
sage: -100 in J
True
sage: list(I)
[0, 10, -10, 20, -20, 30, -30, 40, -40, 50, -50, 60, -60, 70, -70, 80, -80, 90, -
↪90, -100]
```

```
>>> from sage.all import *
>>> I = IntegerRange(-Integer(100),Integer(100),Integer(10),Integer(0))
>>> J = list(range(-Integer(100),Integer(100),Integer(10)))
>>> Integer(100) in I
False
>>> Integer(100) in J
False
>>> -Integer(100) in I
True
>>> -Integer(100) in J
True
>>> list(I)
[0, 10, -10, 20, -20, 30, -30, 40, -40, 50, -50, 60, -60, 70, -70, 80, -80, 90, -
↪90, -100]
```

Note

The input is normalized so that:

```
sage: IntegerRange(1, 6, 2) is IntegerRange(1, 7, 2)
True
sage: IntegerRange(1, 8, 3) is IntegerRange(1, 10, 3)
True
```

```
>>> from sage.all import *
>>> IntegerRange(Integer(1), Integer(6), Integer(2)) is␣
↪IntegerRange(Integer(1), Integer(7), Integer(2))
True
>>> IntegerRange(Integer(1), Integer(8), Integer(3)) is␣
↪IntegerRange(Integer(1), Integer(10), Integer(3))
True
```

element_class

alias of `Integer`

class sage.sets.integer_range.**IntegerRangeEmpty**(*elements*)

Bases: *IntegerRange*, *FiniteEnumeratedSet*

A singleton class for empty integer ranges.

See *IntegerRange* for more details.

class sage.sets.integer_range.**IntegerRangeFinite**(*begin*, *end*, *step=1*)

Bases: *IntegerRange*

The class of finite enumerated sets of integers defined by finite arithmetic progressions

See *IntegerRange* for more details.

cardinality()

Return the cardinality of `self`.

EXAMPLES:

```
sage: IntegerRange(123,12,-4).cardinality()
28
sage: IntegerRange(-57,12,8).cardinality()
9
sage: IntegerRange(123,12,4).cardinality()
0
```

```
>>> from sage.all import *
>>> IntegerRange(Integer(123),Integer(12),-Integer(4)).cardinality()
28
>>> IntegerRange(-Integer(57),Integer(12),Integer(8)).cardinality()
9
>>> IntegerRange(Integer(123),Integer(12),Integer(4)).cardinality()
0
```

rank(x)

EXAMPLES:

```
sage: I = IntegerRange(-57,36,8)
sage: I.rank(23)
10
sage: I.unrank(10)
23
sage: I.rank(22)
Traceback (most recent call last):
...
IndexError: 22 not in self
sage: I.rank(87)
Traceback (most recent call last):
...
IndexError: 87 not in self
```

```
>>> from sage.all import *
>>> I = IntegerRange(-Integer(57),Integer(36),Integer(8))
>>> I.rank(Integer(23))
10
>>> I.unrank(Integer(10))
23
>>> I.rank(Integer(22))
Traceback (most recent call last):
...
IndexError: 22 not in self
>>> I.rank(Integer(87))
Traceback (most recent call last):
...
IndexError: 87 not in self
```

unrank(i)

Return the i-th element of this integer range.

EXAMPLES:

```
sage: I = IntegerRange(1,13,5)
sage: I[0], I[1], I[2]
(1, 6, 11)
sage: I[3]
Traceback (most recent call last):
...
IndexError: out of range
sage: I[-1]
11
sage: I[-4]
Traceback (most recent call last):
...
IndexError: out of range

sage: I = IntegerRange(13,1,-1)
sage: l = I.list()
sage: [I[i] for i in range(I.cardinality())] == l
True
sage: l.reverse()
sage: [I[i] for i in range(-1,-I.cardinality()-1,-1)] == l
True
```

```
>>> from sage.all import *
>>> I = IntegerRange(Integer(1),Integer(13),Integer(5))
>>> I[Integer(0)], I[Integer(1)], I[Integer(2)]
(1, 6, 11)
>>> I[Integer(3)]
Traceback (most recent call last):
...
IndexError: out of range
>>> I[-Integer(1)]
11
>>> I[-Integer(4)]
Traceback (most recent call last):
...
IndexError: out of range

>>> I = IntegerRange(Integer(13),Integer(1),-Integer(1))
>>> l = I.list()
>>> [I[i] for i in range(I.cardinality())] == l
True
>>> l.reverse()
>>> [I[i] for i in range(-Integer(1),-I.cardinality()-Integer(1),-
↪Integer(1))] == l
True
```

class sage.sets.integer_range.**IntegerRangeFromMiddle**(*begin*, *end*, *step=1*, *middle_point=1*)

Bases: *IntegerRange*

The class of finite or infinite enumerated sets defined with an inside point, a progression and two limits.

See *IntegerRange* for more details.

next(*elt*)

Return the next element of `elt` in `self`.

EXAMPLES:

```
sage: from sage.sets.integer_range import IntegerRangeFromMiddle
sage: I = IntegerRangeFromMiddle(-100,100,10,0)
sage: (I.next(0), I.next(10), I.next(-10), I.next(20), I.next(-100))
(10, -10, 20, -20, None)
sage: I = IntegerRangeFromMiddle(-Infinity,Infinity,10,0)
sage: (I.next(0), I.next(10), I.next(-10), I.next(20), I.next(-100))
(10, -10, 20, -20, 110)
sage: I.next(1)
Traceback (most recent call last):
...
LookupError: 1 not in Integer progression containing 0 with increment 10 and␣
↪bounded with -Infinity and +Infinity
```

```
>>> from sage.all import *
>>> from sage.sets.integer_range import IntegerRangeFromMiddle
>>> I = IntegerRangeFromMiddle(-Integer(100),Integer(100),Integer(10),
↪Integer(0))
>>> (I.next(Integer(0)), I.next(Integer(10)), I.next(-Integer(10)), I.
↪next(Integer(20)), I.next(-Integer(100)))
(10, -10, 20, -20, None)
>>> I = IntegerRangeFromMiddle(-Infinity,Infinity,Integer(10),Integer(0))
>>> (I.next(Integer(0)), I.next(Integer(10)), I.next(-Integer(10)), I.
↪next(Integer(20)), I.next(-Integer(100)))
(10, -10, 20, -20, 110)
>>> I.next(Integer(1))
Traceback (most recent call last):
...
LookupError: 1 not in Integer progression containing 0 with increment 10 and␣
↪bounded with -Infinity and +Infinity
```

class sage.sets.integer_range.**IntegerRangeInfinite**(*begin*, *step=1*)

Bases: *IntegerRange*

The class of infinite enumerated sets of integers defined by infinite arithmetic progressions.

See *IntegerRange* for more details.

rank(*x*)

EXAMPLES:

```
sage: I = IntegerRange(-57,Infinity,8)
sage: I.rank(23)
10
sage: I.unrank(10)
23
sage: I.rank(22)
Traceback (most recent call last):
...
IndexError: 22 not in self
```

```
>>> from sage.all import *
>>> I = IntegerRange(-Integer(57),Infinity,Integer(8))
>>> I.rank(Integer(23))
10
>>> I.unrank(Integer(10))
23
>>> I.rank(Integer(22))
Traceback (most recent call last):
...
IndexError: 22 not in self
```

unrank(*i*)

Return the `i`-th element of `self`.

EXAMPLES:

```
sage: I = IntegerRange(-8,Infinity,3)
sage: I.unrank(1)
-5
```

```
>>> from sage.all import *
>>> I = IntegerRange(-Integer(8),Infinity,Integer(3))
>>> I.unrank(Integer(1))
-5
```

2.2 Positive Integers

class sage.sets.positive_integers.**PositiveIntegers**

Bases: *IntegerRangeInfinite*

The enumerated set of positive integers. To fix the ideas, we mean $\{1, 2, 3, 4, 5, \dots\}$.

This class implements the set of positive integers, as an enumerated set (see `InfiniteEnumeratedSets`).

This set is an integer range set. The construction is therefore done by IntegerRange (see *IntegerRange*).

EXAMPLES:

```
sage: PP = PositiveIntegers()
sage: PP
Positive integers
sage: PP.cardinality()
+Infinity
sage: TestSuite(PP).run()
sage: PP.list()
Traceback (most recent call last):
...
NotImplementedError: cannot list an infinite set
sage: it = iter(PP)
sage: (next(it), next(it), next(it), next(it), next(it))
(1, 2, 3, 4, 5)
sage: PP.first()
1
```

```
>>> from sage.all import *
>>> PP = PositiveIntegers()
>>> PP
Positive integers
>>> PP.cardinality()
+Infinity
>>> TestSuite(PP).run()
>>> PP.list()
Traceback (most recent call last):
...
NotImplementedError: cannot list an infinite set
>>> it = iter(PP)
>>> (next(it), next(it), next(it), next(it), next(it))
(1, 2, 3, 4, 5)
>>> PP.first()
1
```

an_element()

Return an element of `self`.

EXAMPLES:

```
sage: PositiveIntegers().an_element()
42
```

```
>>> from sage.all import *
>>> PositiveIntegers().an_element()
42
```

2.3 Non Negative Integers

class sage.sets.non_negative_integers.**NonNegativeIntegers**(*category=None*)

Bases: `UniqueRepresentation`, `Parent`

The enumerated set of nonnegative integers.

This class implements the set of nonnegative integers, as an enumerated set (see `InfiniteEnumeratedSets`).

EXAMPLES:

```
sage: NN = NonNegativeIntegers()
sage: NN
Non negative integers
sage: NN.cardinality()
+Infinity
sage: TestSuite(NN).run()
sage: NN.list()
Traceback (most recent call last):
...
NotImplementedError: cannot list an infinite set
sage: NN.element_class
<... 'sage.rings.integer.Integer'>
sage: it = iter(NN)
```

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```
sage: [next(it), next(it), next(it), next(it), next(it)]
[0, 1, 2, 3, 4]
sage: NN.first()
0
```

```
>>> from sage.all import *
>>> NN = NonNegativeIntegers()
>>> NN
Non negative integers
>>> NN.cardinality()
+Infinity
>>> TestSuite(NN).run()
>>> NN.list()
Traceback (most recent call last):
...
NotImplementedError: cannot list an infinite set
>>> NN.element_class
<... 'sage.rings.integer.Integer'>
>>> it = iter(NN)
>>> [next(it), next(it), next(it), next(it), next(it)]
[0, 1, 2, 3, 4]
>>> NN.first()
0
```

Currently, this is just a "facade" parent; namely its elements are plain Sage `Integers` with `Integer Ring` as parent:

```
sage: x = NN(15); type(x)
<... 'sage.rings.integer.Integer'>
sage: x.parent()
Integer Ring
sage: x+3
18
```

```
>>> from sage.all import *
>>> x = NN(Integer(15)); type(x)
<... 'sage.rings.integer.Integer'>
>>> x.parent()
Integer Ring
>>> x+Integer(3)
18
```

In a later version, there will be an option to specify whether the elements should have `Integer Ring` or `Non negative integers` as parent:

```
sage: NN = NonNegativeIntegers(facade = False)  # todo: not implemented
sage: x = NN(5)                                 # todo: not implemented
sage: x.parent()                                # todo: not implemented
Non negative integers
```

```
>>> from sage.all import *
>>> NN = NonNegativeIntegers(facade = False) # todo: not implemented
```

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```
>>> x = NN(Integer(5))                                    # todo: not implemented
>>> x.parent()                                        # todo: not implemented
Non negative integers
```

This runs generic sanity checks on `NN`:

```
sage: TestSuite(NN).run()
```

```
>>> from sage.all import *
>>> TestSuite(NN).run()
```

TODO: do not use `NN` any more in the doctests for `NonNegativeIntegers`.

Element

alias of `Integer`

an_element()

EXAMPLES:

```
sage: NonNegativeIntegers().an_element()
42
```

```
>>> from sage.all import *
>>> NonNegativeIntegers().an_element()
42
```

from_integer

alias of `Integer`

next(*o*)

EXAMPLES:

```
sage: NN = NonNegativeIntegers()
sage: NN.next(3)
4
```

```
>>> from sage.all import *
>>> NN = NonNegativeIntegers()
>>> NN.next(Integer(3))
4
```

some_elements()

EXAMPLES:

```
sage: NonNegativeIntegers().some_elements()
[0, 1, 3, 42]
```

```
>>> from sage.all import *
>>> NonNegativeIntegers().some_elements()
[0, 1, 3, 42]
```

unrank(*rnk*)

EXAMPLES:

```
sage: NN = NonNegativeIntegers()
sage: NN.unrank(100)
100
```

```
>>> from sage.all import *
>>> NN = NonNegativeIntegers()
>>> NN.unrank(Integer(100))
100
```

2.4 The set of prime numbers

AUTHORS:

- William Stein (2005): original version
- Florent Hivert (2009-11): adapted to the category framework.

class sage.sets.primes.**Primes**(*proof*)

Bases: Set_generic, UniqueRepresentation

The set of prime numbers.

EXAMPLES:

```
sage: P = Primes(); P
Set of all prime numbers: 2, 3, 5, 7, ...
```

```
>>> from sage.all import *
>>> P = Primes(); P
Set of all prime numbers: 2, 3, 5, 7, ...
```

We show various operations on the set of prime numbers:

```
sage: P.cardinality()
+Infinity
sage: R = Primes()
sage: P == R
True
sage: 5 in P
True
sage: 100 in P
False

sage: len(P)
Traceback (most recent call last):
...
NotImplementedError: infinite set
```

```
>>> from sage.all import *
>>> P.cardinality()
+Infinity
```

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```
>>> R = Primes()
>>> P == R
True
>>> Integer(5) in P
True
>>> Integer(100) in P
False

>>> len(P)
Traceback (most recent call last):
...
NotImplementedError: infinite set
```

first()

Return the first prime number.

EXAMPLES:

```
sage: P = Primes()
sage: P.first()
2
```

```
>>> from sage.all import *
>>> P = Primes()
>>> P.first()
2
```

next(*pr*)

Return the next prime number.

EXAMPLES:

```
sage: P = Primes()
sage: P.next(5)                                                             #␣
↪needs sage.libs.pari
7
```

```
>>> from sage.all import *
>>> P = Primes()
>>> P.next(Integer(5))                                                      ␣
↪     # needs sage.libs.pari
7
```

unrank(*n*)

Return the n-th prime number.

EXAMPLES:

```
sage: P = Primes()
sage: P.unrank(0)                                                           #␣
↪needs sage.libs.pari
2
sage: P.unrank(5)                                                           #␣
```

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```
↪needs sage.libs.pari
13
sage: P.unrank(42)                                                          #␣
↪needs sage.libs.pari
191
```

```
>>> from sage.all import *
>>> P = Primes()
>>> P.unrank(Integer(0))                                                    ␣
↪     # needs sage.libs.pari
2
>>> P.unrank(Integer(5))                                                    ␣
↪     # needs sage.libs.pari
13
>>> P.unrank(Integer(42))                                                   ␣
↪     # needs sage.libs.pari
191
```

2.5 Subsets of the Real Line

This module contains subsets of the real line that can be constructed as the union of a finite set of open and closed intervals.

EXAMPLES:

```
sage: RealSet(0,1)
(0, 1)
sage: RealSet((0,1), [2,3])
(0, 1) ∪ [2, 3]
sage: RealSet((1,3), (0,2))
(0, 3)
sage: RealSet(-oo, oo)
(-oo, +oo)
```

```
>>> from sage.all import *
>>> RealSet(Integer(0),Integer(1))
(0, 1)
>>> RealSet((Integer(0),Integer(1)), [Integer(2),Integer(3)])
(0, 1) ∪ [2, 3]
>>> RealSet((Integer(1),Integer(3)), (Integer(0),Integer(2)))
(0, 3)
>>> RealSet(-oo, oo)
(-oo, +oo)
```

Brackets must be balanced in Python, so the naive notation for half-open intervals does not work:

```
sage: RealSet([0,1))
Traceback (most recent call last):
...
SyntaxError: ...
```

```
>>> from sage.all import *
>>> RealSet([Integer(0),Integer(1)))
Traceback (most recent call last):
...
SyntaxError: ...
```

Instead, you can use the following construction functions:

```
sage: RealSet.open_closed(0,1)
(0, 1]
sage: RealSet.closed_open(0,1)
[0, 1)
sage: RealSet.point(1/2)
{1/2}
sage: RealSet.unbounded_below_open(0)
(-oo, 0)
sage: RealSet.unbounded_below_closed(0)
(-oo, 0]
sage: RealSet.unbounded_above_open(1)
(1, +oo)
sage: RealSet.unbounded_above_closed(1)
[1, +oo)
```

```
>>> from sage.all import *
>>> RealSet.open_closed(Integer(0),Integer(1))
(0, 1]
>>> RealSet.closed_open(Integer(0),Integer(1))
[0, 1)
>>> RealSet.point(Integer(1)/Integer(2))
{1/2}
>>> RealSet.unbounded_below_open(Integer(0))
(-oo, 0)
>>> RealSet.unbounded_below_closed(Integer(0))
(-oo, 0]
>>> RealSet.unbounded_above_open(Integer(1))
(1, +oo)
>>> RealSet.unbounded_above_closed(Integer(1))
[1, +oo)
```

The lower and upper endpoints will be sorted if necessary:

```
sage: RealSet.interval(1, 0, lower_closed=True, upper_closed=False)
[0, 1)
```

```
>>> from sage.all import *
>>> RealSet.interval(Integer(1), Integer(0), lower_closed=True, upper_closed=False)
[0, 1)
```

Relations containing symbols and numeric values or constants:

```
sage: # needs sage.symbolic
sage: RealSet(x != 0)
(-oo, 0) ∪ (0, +oo)
```

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```
sage: RealSet(x == pi)
{pi}
sage: RealSet(x < 1/2)
(-oo, 1/2)
sage: RealSet(1/2 < x)
(1/2, +oo)
sage: RealSet(1.5 <= x)
[1.50000000000000, +oo)
```

```
>>> from sage.all import *
>>> # needs sage.symbolic
>>> RealSet(x != Integer(0))
(-oo, 0) ∪ (0, +oo)
>>> RealSet(x == pi)
{pi}
>>> RealSet(x < Integer(1)/Integer(2))
(-oo, 1/2)
>>> RealSet(Integer(1)/Integer(2) < x)
(1/2, +oo)
>>> RealSet(RealNumber('1.5') <= x)
[1.50000000000000, +oo)
```

Note that multiple arguments are combined as union:

```
sage: RealSet(x >= 0, x < 1)                                                #␣
↪needs sage.symbolic
(-oo, +oo)
sage: RealSet(x >= 0, x > 1)                                                #␣
↪needs sage.symbolic
[0, +oo)
sage: RealSet(x >= 0, x > -1)                                               #␣
↪needs sage.symbolic
(-1, +oo)
```

```
>>> from sage.all import *
>>> RealSet(x >= Integer(0), x < Integer(1))                                 ␣
↪              # needs sage.symbolic
(-oo, +oo)
>>> RealSet(x >= Integer(0), x > Integer(1))                                 ␣
↪              # needs sage.symbolic
[0, +oo)
>>> RealSet(x >= Integer(0), x > -Integer(1))                                ␣
↪              # needs sage.symbolic
(-1, +oo)
```

AUTHORS:

- Laurent Claessens (2010-12-10): Interval and ContinuousSet, posted to sage-devel at http://www.mail-archive.com/sage-support@googlegroups.com/msg21326.html.
- Ares Ribo (2011-10-24): Extended the previous work defining the class RealSet.
- Jordi Saludes (2011-12-10): Documentation and file reorganization.

- Volker Braun (2013-06-22): Rewrite
- Yueqi Li, Yuan Zhou (2022-07-31): Rewrite RealSet. Adapt faster operations by scan-line (merging) techniques from the code by Matthias Köppe et al., at https://github.com/mkoeppe/cutgeneratingfunctionology/blob/master/cutgeneratingfunctionology/igp/intervals.py

class sage.sets.real_set.**InternalRealInterval**(*lower*, *lower_closed*, *upper*, *upper_closed*, *check=True*)

Bases: UniqueRepresentation, Parent

A real interval.

You are not supposed to create *InternalRealInterval* objects yourself. Always use *RealSet* instead.

INPUT:

- lower – real or minus infinity; the lower bound of the interval
- lower_closed – boolean; whether the interval is closed at the lower bound
- upper – real or (plus) infinity; the upper bound of the interval
- upper_closed – boolean; whether the interval is closed at the upper bound
- check – boolean; whether to check the other arguments for validity

boundary_points()

Generate the boundary points of self.

EXAMPLES:

```
sage: list(RealSet.open_closed(-oo, 1)[0].boundary_points())
[1]
sage: list(RealSet.open(1, 2)[0].boundary_points())
[1, 2]
```

```
>>> from sage.all import *
>>> list(RealSet.open_closed(-oo, Integer(1))[Integer(0)].boundary_points())
[1]
>>> list(RealSet.open(Integer(1), Integer(2))[Integer(0)].boundary_points())
[1, 2]
```

closure()

Return the closure.

OUTPUT: the closure as a new *InternalRealInterval*

EXAMPLES:

```
sage: RealSet.open(0,1)[0].closure()
[0, 1]
sage: RealSet.open(-oo,1)[0].closure()
(-oo, 1]
sage: RealSet.open(0, oo)[0].closure()
[0, +oo)
```

```
>>> from sage.all import *
>>> RealSet.open(Integer(0),Integer(1))[Integer(0)].closure()
[0, 1]
>>> RealSet.open(-oo,Integer(1))[Integer(0)].closure()
```

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```
(-oo, 1]
>>> RealSet.open(Integer(0), oo)[Integer(0)].closure()
[0, +oo)
```

contains(*x*)

Return whether x is contained in the interval.

INPUT:

- `x` – a real number

OUTPUT: boolean

EXAMPLES:

```
sage: i = RealSet.open_closed(0,2)[0]; i
(0, 2]
sage: i.contains(0)
False
sage: i.contains(1)
True
sage: i.contains(2)
True
```

```
>>> from sage.all import *
>>> i = RealSet.open_closed(Integer(0),Integer(2))[Integer(0)]; i
(0, 2]
>>> i.contains(Integer(0))
False
>>> i.contains(Integer(1))
True
>>> i.contains(Integer(2))
True
```

convex_hull(*other*)

Return the convex hull of the two intervals.

OUTPUT: the convex hull as a new *InternalRealInterval*

EXAMPLES:

```
sage: I1 = RealSet.open(0, 1)[0];  I1
(0, 1)
sage: I2 = RealSet.closed(1, 2)[0];  I2
[1, 2]
sage: I1.convex_hull(I2)
(0, 2]
sage: I2.convex_hull(I1)
(0, 2]
sage: I1.convex_hull(I2.interior())
(0, 2)
sage: I1.closure().convex_hull(I2.interior())
[0, 2)
sage: I1.closure().convex_hull(I2)
[0, 2]
```

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```
sage: I3 = RealSet.closed(1/2, 3/2)[0]; I3
[1/2, 3/2]
sage: I1.convex_hull(I3)
(0, 3/2]
```

```
>>> from sage.all import *
>>> I1 = RealSet.open(Integer(0), Integer(1))[Integer(0)];  I1
(0, 1)
>>> I2 = RealSet.closed(Integer(1), Integer(2))[Integer(0)];  I2
[1, 2]
>>> I1.convex_hull(I2)
(0, 2]
>>> I2.convex_hull(I1)
(0, 2]
>>> I1.convex_hull(I2.interior())
(0, 2)
>>> I1.closure().convex_hull(I2.interior())
[0, 2)
>>> I1.closure().convex_hull(I2)
[0, 2]
>>> I3 = RealSet.closed(Integer(1)/Integer(2), Integer(3)/
↪Integer(2))[Integer(0)]; I3
[1/2, 3/2]
>>> I1.convex_hull(I3)
(0, 3/2]
```

element_class

alias of `LazyFieldElement`

interior()

Return the interior.

OUTPUT: the interior as a new *InternalRealInterval*

EXAMPLES:

```
sage: RealSet.closed(0, 1)[0].interior()
(0, 1)
sage: RealSet.open_closed(-oo, 1)[0].interior()
(-oo, 1)
sage: RealSet.closed_open(0, oo)[0].interior()
(0, +oo)
```

```
>>> from sage.all import *
>>> RealSet.closed(Integer(0), Integer(1))[Integer(0)].interior()
(0, 1)
>>> RealSet.open_closed(-oo, Integer(1))[Integer(0)].interior()
(-oo, 1)
>>> RealSet.closed_open(Integer(0), oo)[Integer(0)].interior()
(0, +oo)
```

intersection(*other*)

Return the intersection of the two intervals.

INPUT:

- `other` – a *InternalRealInterval*

OUTPUT: the intersection as a new *InternalRealInterval*

EXAMPLES:

```
sage: I1 = RealSet.open(0, 2)[0];  I1
(0, 2)
sage: I2 = RealSet.closed(1, 3)[0];  I2
[1, 3]
sage: I1.intersection(I2)
[1, 2)
sage: I2.intersection(I1)
[1, 2)
sage: I1.closure().intersection(I2.interior())
(1, 2]
sage: I2.interior().intersection(I1.closure())
(1, 2]

sage: I3 = RealSet.closed(10, 11)[0];  I3
[10, 11]
sage: I1.intersection(I3)
(0, 0)
sage: I3.intersection(I1)
(0, 0)
```

```
>>> from sage.all import *
>>> I1 = RealSet.open(Integer(0), Integer(2))[Integer(0)];  I1
(0, 2)
>>> I2 = RealSet.closed(Integer(1), Integer(3))[Integer(0)];  I2
[1, 3]
>>> I1.intersection(I2)
[1, 2)
>>> I2.intersection(I1)
[1, 2)
>>> I1.closure().intersection(I2.interior())
(1, 2]
>>> I2.interior().intersection(I1.closure())
(1, 2]

>>> I3 = RealSet.closed(Integer(10), Integer(11))[Integer(0)];  I3
[10, 11]
>>> I1.intersection(I3)
(0, 0)
>>> I3.intersection(I1)
(0, 0)
```

is_connected(*other*)

Test whether two intervals are connected.

OUTPUT:

boolean; whether the set-theoretic union of the two intervals has a single connected component.

EXAMPLES:

```
sage: I1 = RealSet.open(0, 1)[0];  I1
(0, 1)
sage: I2 = RealSet.closed(1, 2)[0];  I2
[1, 2]
sage: I1.is_connected(I2)
True
sage: I1.is_connected(I2.interior())
False
sage: I1.closure().is_connected(I2.interior())
True
sage: I2.is_connected(I1)
True
sage: I2.interior().is_connected(I1)
False
sage: I2.closure().is_connected(I1.interior())
True
sage: I3 = RealSet.closed(1/2, 3/2)[0]; I3
[1/2, 3/2]
sage: I1.is_connected(I3)
True
sage: I3.is_connected(I1)
True
```

```
>>> from sage.all import *
>>> I1 = RealSet.open(Integer(0), Integer(1))[Integer(0)];  I1
(0, 1)
>>> I2 = RealSet.closed(Integer(1), Integer(2))[Integer(0)];  I2
[1, 2]
>>> I1.is_connected(I2)
True
>>> I1.is_connected(I2.interior())
False
>>> I1.closure().is_connected(I2.interior())
True
>>> I2.is_connected(I1)
True
>>> I2.interior().is_connected(I1)
False
>>> I2.closure().is_connected(I1.interior())
True
>>> I3 = RealSet.closed(Integer(1)/Integer(2), Integer(3)/
↪Integer(2))[Integer(0)]; I3
[1/2, 3/2]
>>> I1.is_connected(I3)
True
>>> I3.is_connected(I1)
True
```

is_empty()

Return whether the interval is empty.

The normalized form of *RealSet* has all intervals non-empty, so this method usually returns `False`.

OUTPUT: boolean

EXAMPLES:

```
sage: I = RealSet(0, 1)[0]
sage: I.is_empty()
False
```

```
>>> from sage.all import *
>>> I = RealSet(Integer(0), Integer(1))[Integer(0)]
>>> I.is_empty()
False
```

is_point()

Return whether the interval consists of a single point.

OUTPUT: boolean

EXAMPLES:

```
sage: I = RealSet(0, 1)[0]
sage: I.is_point()
False
```

```
>>> from sage.all import *
>>> I = RealSet(Integer(0), Integer(1))[Integer(0)]
>>> I.is_point()
False
```

lower()

Return the lower bound.

OUTPUT: the lower bound as it was originally specified

EXAMPLES:

```
sage: I = RealSet(0, 1)[0]
sage: I.lower()
0
sage: I.upper()
1
```

```
>>> from sage.all import *
>>> I = RealSet(Integer(0), Integer(1))[Integer(0)]
>>> I.lower()
0
>>> I.upper()
1
```

lower_closed()

Return whether the interval is open at the lower bound.

OUTPUT: boolean

EXAMPLES:

```
sage: I = RealSet.open_closed(0, 1)[0];  I
(0, 1]
sage: I.lower_closed()
False
sage: I.lower_open()
True
sage: I.upper_closed()
True
sage: I.upper_open()
False
```

```
>>> from sage.all import *
>>> I = RealSet.open_closed(Integer(0), Integer(1))[Integer(0)];  I
(0, 1]
>>> I.lower_closed()
False
>>> I.lower_open()
True
>>> I.upper_closed()
True
>>> I.upper_open()
False
```

lower_open()

Return whether the interval is closed at the upper bound.

OUTPUT: boolean

EXAMPLES:

```
sage: I = RealSet.open_closed(0, 1)[0];  I
(0, 1]
sage: I.lower_closed()
False
sage: I.lower_open()
True
sage: I.upper_closed()
True
sage: I.upper_open()
False
```

```
>>> from sage.all import *
>>> I = RealSet.open_closed(Integer(0), Integer(1))[Integer(0)];  I
(0, 1]
>>> I.lower_closed()
False
>>> I.lower_open()
True
>>> I.upper_closed()
True
>>> I.upper_open()
False
```

upper()

Return the upper bound.

OUTPUT: the upper bound as it was originally specified

EXAMPLES:

```
sage: I = RealSet(0, 1)[0]
sage: I.lower()
0
sage: I.upper()
1
```

```
>>> from sage.all import *
>>> I = RealSet(Integer(0), Integer(1))[Integer(0)]
>>> I.lower()
0
>>> I.upper()
1
```

upper_closed()

Return whether the interval is closed at the lower bound.

OUTPUT: boolean

EXAMPLES:

```
sage: I = RealSet.open_closed(0, 1)[0];  I
(0, 1]
sage: I.lower_closed()
False
sage: I.lower_open()
True
sage: I.upper_closed()
True
sage: I.upper_open()
False
```

```
>>> from sage.all import *
>>> I = RealSet.open_closed(Integer(0), Integer(1))[Integer(0)];  I
(0, 1]
>>> I.lower_closed()
False
>>> I.lower_open()
True
>>> I.upper_closed()
True
>>> I.upper_open()
False
```

upper_open()

Return whether the interval is closed at the upper bound.

OUTPUT: boolean

EXAMPLES:

```
sage: I = RealSet.open_closed(0, 1)[0];  I
(0, 1]
sage: I.lower_closed()
False
sage: I.lower_open()
True
sage: I.upper_closed()
True
sage: I.upper_open()
False
```

```
>>> from sage.all import *
>>> I = RealSet.open_closed(Integer(0), Integer(1))[Integer(0)];  I
(0, 1]
>>> I.lower_closed()
False
>>> I.lower_open()
True
>>> I.upper_closed()
True
>>> I.upper_open()
False
```

class sage.sets.real_set.**RealSet**(**intervals*, *normalized=True*)

Bases: UniqueRepresentation, Parent, *Set_base*, *Set_boolean_operators*, *Set_add_sub_operators*

A subset of the real line, a finite union of intervals.

INPUT:

- `*args` – arguments defining a real set. Possibilities are either:
  - two extended real numbers `a, b`, to construct the open interval (a, b), or
  - a list/tuple/iterable of (not necessarily disjoint) intervals or real sets, whose union is taken. The individual intervals can be specified by either
    - a tuple `(a, b)` of two extended real numbers (constructing an open interval),
    - a list `[a, b]` of two real numbers (constructing a closed interval),
    - an *InternalRealInterval*,
    - an OpenInterval.
- `structure` – (default: `None`) if `None`, construct the real set as an instance of *RealSet*; if `'differentiable'`, construct it as a subset of an instance of RealLine, representing the differentiable manifold **R**.
- `ambient` – (default: `None`) an instance of RealLine; construct a subset of it. Using this keyword implies `structure='differentiable'`.
- `names` or `coordinate` – coordinate symbol for the canonical chart; see RealLine. Using these keywords implies `structure='differentiable'`.
- `name`, `latex_name`, `start_index` – see RealLine
- `normalized` – (default: `None`) if `True`, the input is already normalized, i.e., `*args` are the connected components (type *InternalRealInterval*) of the real set in ascending order; no other keyword is provided.

There are also specialized constructors for various types of intervals:

Constructor	Interval
RealSet.open()	(a, b)
RealSet.closed()	$[a, b]$
RealSet.point()	$\{a\}$
RealSet.open_closed()	$(a, b]$
RealSet.closed_open()	$[a, b)$
RealSet.unbounded_below_closed()	$(-\infty, b]$
RealSet.unbounded_below_open()	$(-\infty, b)$
RealSet.unbounded_above_closed()	$[a, +\infty)$
RealSet.unbounded_above_open()	$(a, +\infty)$
RealSet.real_line()	$(-\infty, +\infty)$
RealSet.interval()	any

EXAMPLES:

```
sage: RealSet(0, 1)     # open set from two numbers
(0, 1)
sage: RealSet(1, 0)     # the two numbers will be sorted
(0, 1)
sage: s1 = RealSet((1,2)); s1     # tuple of two numbers = open set
(1, 2)
sage: s2 = RealSet([3,4]); s2     # list of two numbers = closed set
[3, 4]
sage: i1, i2 = s1[0], s2[0]
sage: RealSet(i2, i1)             # union of intervals
(1, 2) ∪ [3, 4]
sage: RealSet((-oo, 0), x > 6, i1, RealSet.point(5),                          #␣
↪needs sage.symbolic
....:          RealSet.closed_open(4, 3))
(-oo, 0) ∪ (1, 2) ∪ [3, 4) ∪ {5} ∪ (6, +oo)
```

```
>>> from sage.all import *
>>> RealSet(Integer(0), Integer(1))     # open set from two numbers
(0, 1)
>>> RealSet(Integer(1), Integer(0))     # the two numbers will be sorted
(0, 1)
>>> s1 = RealSet((Integer(1),Integer(2))); s1     # tuple of two numbers = open set
(1, 2)
>>> s2 = RealSet([Integer(3),Integer(4)]); s2     # list of two numbers = closed␣
↪set
[3, 4]
>>> i1, i2 = s1[Integer(0)], s2[Integer(0)]
>>> RealSet(i2, i1)             # union of intervals
(1, 2) ∪ [3, 4]
>>> RealSet((-oo, Integer(0)), x > Integer(6), i1, RealSet.point(Integer(5)),    ␣
↪                        # needs sage.symbolic
...          RealSet.closed_open(Integer(4), Integer(3)))
(-oo, 0) ∪ (1, 2) ∪ [3, 4) ∪ {5} ∪ (6, +oo)
```

Initialization from manifold objects:

```
sage: # needs sage.symbolic
sage: R = manifolds.RealLine(); R
Real number line ℝ
sage: RealSet(R)
(-oo, +oo)
sage: I02 = manifolds.OpenInterval(0, 2); I
I
sage: RealSet(I02)
(0, 2)
sage: I01_of_R = manifolds.OpenInterval(0, 1, ambient_interval=R); I01_of_R
Real interval (0, 1)
sage: RealSet(I01_of_R)
(0, 1)
sage: RealSet(I01_of_R.closure())
[0, 1]
sage: I01_of_I02 = manifolds.OpenInterval(0, 1,
....:                                     ambient_interval=I02); I01_of_I02
Real interval (0, 1)
sage: RealSet(I01_of_I02)
(0, 1)
sage: RealSet(I01_of_I02.closure())
(0, 1]
```

```
>>> from sage.all import *
>>> # needs sage.symbolic
>>> R = manifolds.RealLine(); R
Real number line ℝ
>>> RealSet(R)
(-oo, +oo)
>>> I02 = manifolds.OpenInterval(Integer(0), Integer(2)); I
I
>>> RealSet(I02)
(0, 2)
>>> I01_of_R = manifolds.OpenInterval(Integer(0), Integer(1), ambient_interval=R);
↪ I01_of_R
Real interval (0, 1)
>>> RealSet(I01_of_R)
(0, 1)
>>> RealSet(I01_of_R.closure())
[0, 1]
>>> I01_of_I02 = manifolds.OpenInterval(Integer(0), Integer(1),
...                                     ambient_interval=I02); I01_of_I02
Real interval (0, 1)
>>> RealSet(I01_of_I02)
(0, 1)
>>> RealSet(I01_of_I02.closure())
(0, 1]
```

Real sets belong to a subcategory of topological spaces:

```
sage: RealSet().category()
Join of
 Category of finite sets and
```

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```
 Category of subobjects of sets and
 Category of connected topological spaces
sage: RealSet.point(1).category()
Join of
 Category of finite sets and
 Category of subobjects of sets and
 Category of connected topological spaces
sage: RealSet([1, 2]).category()
Join of
 Category of infinite sets and
 Category of compact topological spaces and
 Category of subobjects of sets and
 Category of connected topological spaces
sage: RealSet((1, 2), (3, 4)).category()
Join of
 Category of infinite sets and
 Category of subobjects of sets and
 Category of topological spaces
```

```
>>> from sage.all import *
>>> RealSet().category()
Join of
 Category of finite sets and
 Category of subobjects of sets and
 Category of connected topological spaces
>>> RealSet.point(Integer(1)).category()
Join of
 Category of finite sets and
 Category of subobjects of sets and
 Category of connected topological spaces
>>> RealSet([Integer(1), Integer(2)]).category()
Join of
 Category of infinite sets and
 Category of compact topological spaces and
 Category of subobjects of sets and
 Category of connected topological spaces
>>> RealSet((Integer(1), Integer(2)), (Integer(3), Integer(4))).category()
Join of
 Category of infinite sets and
 Category of subobjects of sets and
 Category of topological spaces
```

Constructing real sets as manifolds or manifold subsets by passing `structure='differentiable'`:

```
sage: # needs sage.symbolic
sage: RealSet(-oo, oo, structure='differentiable')
Real number line ℝ
sage: RealSet([0, 1], structure='differentiable')
Subset [0, 1] of the Real number line ℝ
sage: _.category()
Category of subobjects of sets
sage: RealSet.open_closed(0, 5, structure='differentiable')
```

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```
Subset (0, 5] of the Real number line ℝ
```

```
>>> from sage.all import *
>>> # needs sage.symbolic
>>> RealSet(-oo, oo, structure='differentiable')
Real number line ℝ
>>> RealSet([Integer(0), Integer(1)], structure='differentiable')
Subset [0, 1] of the Real number line ℝ
>>> _.category()
Category of subobjects of sets
>>> RealSet.open_closed(Integer(0), Integer(5), structure='differentiable')
Subset (0, 5] of the Real number line ℝ
```

This is implied when a coordinate name is given using the keywords `coordinate` or `names`:

```
sage: RealSet(0, 1, coordinate='λ')                                             #␣
↪needs sage.symbolic
Open subset (0, 1) of the Real number line ℝ
sage: _.category()                                                              #␣
↪needs sage.symbolic
Join of
 Category of smooth manifolds over Real Field with 53 bits of precision and
 Category of connected manifolds over Real Field with 53 bits of precision and
 Category of subobjects of sets
```

```
>>> from sage.all import *
>>> RealSet(Integer(0), Integer(1), coordinate='λ')                              ␣
↪              # needs sage.symbolic
Open subset (0, 1) of the Real number line ℝ
>>> _.category()                                                               #␣
↪needs sage.symbolic
Join of
 Category of smooth manifolds over Real Field with 53 bits of precision and
 Category of connected manifolds over Real Field with 53 bits of precision and
 Category of subobjects of sets
```

It is also implied by assigning a coordinate name using generator notation:

```
sage: R_xi.<ξ> = RealSet.real_line(); R_xi                                      #␣
↪needs sage.symbolic
Real number line ℝ
sage: R_xi.canonical_chart()                                                    #␣
↪needs sage.symbolic
Chart (ℝ, (ξ,))
```

```
>>> from sage.all import *
>>> R_xi = RealSet.real_line(names=('ξ',)); (ξ,) = R_xi._first_ngens(1); R_xi    ␣
↪                                  # needs sage.symbolic
Real number line ℝ
>>> R_xi.canonical_chart()                                                     #␣
↪needs sage.symbolic
Chart (ℝ, (ξ,))
```

With the keyword `ambient`, we can construct a subset of a previously constructed manifold:

```
sage: # needs sage.symbolic
sage: P_xi = RealSet(0, oo, ambient=R_xi); P_xi
Open subset (0, +oo) of the Real number line ℝ
sage: P_xi.default_chart()
Chart ((0, +oo), (ξ,))
sage: B_xi = RealSet(0, 1, ambient=P_xi); B_xi
Open subset (0, 1) of the Real number line ℝ
sage: B_xi.default_chart()
Chart ((0, 1), (ξ,))
sage: R_xi.subset_family()
Set {(0, +oo), (0, 1), ℝ} of open subsets of the Real number line ℝ
sage: F = RealSet.point(0).union(RealSet.point(1)).union(RealSet.point(2)); F
{0} ∪ {1} ∪ {2}
sage: F_tau = RealSet(F, names='τ'); F_tau
Subset {0} ∪ {1} ∪ {2} of the Real number line ℝ
sage: F_tau.manifold().canonical_chart()
Chart (ℝ, (τ,))
```

```
>>> from sage.all import *
>>> # needs sage.symbolic
>>> P_xi = RealSet(Integer(0), oo, ambient=R_xi); P_xi
Open subset (0, +oo) of the Real number line ℝ
>>> P_xi.default_chart()
Chart ((0, +oo), (ξ,))
>>> B_xi = RealSet(Integer(0), Integer(1), ambient=P_xi); B_xi
Open subset (0, 1) of the Real number line ℝ
>>> B_xi.default_chart()
Chart ((0, 1), (ξ,))
>>> R_xi.subset_family()
Set {(0, +oo), (0, 1), ℝ} of open subsets of the Real number line ℝ
>>> F = RealSet.point(Integer(0)).union(RealSet.point(Integer(1))).union(RealSet.
↪point(Integer(2))); F
{0} ∪ {1} ∪ {2}
>>> F_tau = RealSet(F, names='τ'); F_tau
Subset {0} ∪ {1} ∪ {2} of the Real number line ℝ
>>> F_tau.manifold().canonical_chart()
Chart (ℝ, (τ,))
```

ambient()

Return the ambient space (the real line).

EXAMPLES:

```
sage: s = RealSet(RealSet.open_closed(0,1), RealSet.closed_open(2,3))
sage: s.ambient()
(-oo, +oo)
```

```
>>> from sage.all import *
>>> s = RealSet(RealSet.open_closed(Integer(0),Integer(1)), RealSet.closed_
↪open(Integer(2),Integer(3)))
>>> s.ambient()
(-oo, +oo)
```

static are_pairwise_disjoint(**real_set_collection*)

Test whether the real sets are pairwise disjoint.

INPUT:

- `*real_set_collection` – list/tuple/iterable of *RealSet* or data that defines one

OUTPUT: boolean

See also

is_disjoint()

EXAMPLES:

```
sage: s1 = RealSet((0, 1), (2, 3))
sage: s2 = RealSet((1, 2))
sage: s3 = RealSet.point(3)
sage: RealSet.are_pairwise_disjoint(s1, s2, s3)
True
sage: RealSet.are_pairwise_disjoint(s1, s2, s3, [10,10])
True
sage: RealSet.are_pairwise_disjoint(s1, s2, s3, [-1, 1/2])
False
```

```
>>> from sage.all import *
>>> s1 = RealSet((Integer(0), Integer(1)), (Integer(2), Integer(3)))
>>> s2 = RealSet((Integer(1), Integer(2)))
>>> s3 = RealSet.point(Integer(3))
>>> RealSet.are_pairwise_disjoint(s1, s2, s3)
True
>>> RealSet.are_pairwise_disjoint(s1, s2, s3, [Integer(10),Integer(10)])
True
>>> RealSet.are_pairwise_disjoint(s1, s2, s3, [-Integer(1), Integer(1)/
↪Integer(2)])
False
```

boundary()

Return the topological boundary of `self` as a new *RealSet*.

EXAMPLES:

```
sage: RealSet(-oo, oo).boundary()
{}
sage: RealSet().boundary()
{}
sage: RealSet.point(2).boundary()
{2}
sage: RealSet([1, 2], (3, 4)).boundary()
{1} ∪ {2} ∪ {3} ∪ {4}
sage: RealSet((1, 2), (2, 3)).boundary()
{1} ∪ {2} ∪ {3}
```

```
>>> from sage.all import *
>>> RealSet(-oo, oo).boundary()
{}
>>> RealSet().boundary()
{}
>>> RealSet.point(Integer(2)).boundary()
{2}
>>> RealSet([Integer(1), Integer(2)], (Integer(3), Integer(4))).boundary()
{1} ∪ {2} ∪ {3} ∪ {4}
>>> RealSet((Integer(1), Integer(2)), (Integer(2), Integer(3))).boundary()
{1} ∪ {2} ∪ {3}
```

cardinality()

Return the cardinality of the subset of the real line.

OUTPUT:

Integer or infinity; the size of a discrete set is the number of points; the size of a real interval is Infinity.

EXAMPLES:

```
sage: RealSet([0, 0], [1, 1], [3, 3]).cardinality()
3
sage: RealSet(0,3).cardinality()
+Infinity
```

```
>>> from sage.all import *
>>> RealSet([Integer(0), Integer(0)], [Integer(1), Integer(1)], [Integer(3),␣
↪Integer(3)]).cardinality()
3
>>> RealSet(Integer(0),Integer(3)).cardinality()
+Infinity
```

static closed(*lower*, *upper*, ***kwds*)

Construct a closed interval.

INPUT:

- `lower`, `upper` – two real numbers or infinity; they will be sorted if necessary
- `**kwds` – see *RealSet*

OUTPUT: a new *RealSet*

EXAMPLES:

```
sage: RealSet.closed(1, 0)
[0, 1]
```

```
>>> from sage.all import *
>>> RealSet.closed(Integer(1), Integer(0))
[0, 1]
```

static closed_open(*lower*, *upper*, ***kwds*)

Construct a half-open interval.

INPUT:

- `lower`, `upper` – two real numbers or infinity; they will be sorted if necessary
- `**kwds` – see *RealSet*

OUTPUT:

A new *RealSet* that is closed at the lower bound and open at the upper bound.

EXAMPLES:

```
sage: RealSet.closed_open(1, 0)
[0, 1)
```

```
>>> from sage.all import *
>>> RealSet.closed_open(Integer(1), Integer(0))
[0, 1)
```

closure()

Return the topological closure of `self` as a new *RealSet*.

EXAMPLES:

```
sage: RealSet(-oo, oo).closure()
(-oo, +oo)
sage: RealSet((1, 2), (2, 3)).closure()
[1, 3]
sage: RealSet().closure()
{}
```

```
>>> from sage.all import *
>>> RealSet(-oo, oo).closure()
(-oo, +oo)
>>> RealSet((Integer(1), Integer(2)), (Integer(2), Integer(3))).closure()
[1, 3]
>>> RealSet().closure()
{}
```

complement()

Return the complement.

OUTPUT: the set-theoretic complement as a new *RealSet*

EXAMPLES:

```
sage: RealSet(0,1).complement()
(-oo, 0] ∪ [1, +oo)

sage: s1 = RealSet(0,2) + RealSet.unbounded_above_closed(10);  s1
(0, 2) ∪ [10, +oo)
sage: s1.complement()
(-oo, 0] ∪ [2, 10)

sage: s2 = RealSet(1,3) + RealSet.unbounded_below_closed(-10);  s2
(-oo, -10] ∪ (1, 3)
sage: s2.complement()
(-10, 1] ∪ [3, +oo)
```

```
>>> from sage.all import *
>>> RealSet(Integer(0),Integer(1)).complement()
(-oo, 0] ∪ [1, +oo)

>>> s1 = RealSet(Integer(0),Integer(2)) + RealSet.unbounded_above_
↪closed(Integer(10));  s1
(0, 2) ∪ [10, +oo)
>>> s1.complement()
(-oo, 0] ∪ [2, 10)

>>> s2 = RealSet(Integer(1),Integer(3)) + RealSet.unbounded_below_closed(-
↪Integer(10));  s2
(-oo, -10] ∪ (1, 3)
>>> s2.complement()
(-10, 1] ∪ [3, +oo)
```

contains(x)

Return whether x is contained in the set.

INPUT:

- `x` – a real number

OUTPUT: boolean

EXAMPLES:

```
sage: s = RealSet(0,2) + RealSet.unbounded_above_closed(10);  s
(0, 2) ∪ [10, +oo)
sage: s.contains(1)
True
sage: s.contains(0)
False
sage: s.contains(10.0)
True
sage: 10 in s    # syntactic sugar
True
sage: s.contains(+oo)
False
sage: RealSet().contains(1)
False
```

```
>>> from sage.all import *
>>> s = RealSet(Integer(0),Integer(2)) + RealSet.unbounded_above_
↪closed(Integer(10));  s
(0, 2) ∪ [10, +oo)
>>> s.contains(Integer(1))
True
>>> s.contains(Integer(0))
False
>>> s.contains(RealNumber('10.0'))
True
>>> Integer(10) in s    # syntactic sugar
True
```

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```
>>> s.contains(+oo)
False
>>> RealSet().contains(Integer(1))
False
```

static convex_hull(**real_set_collection*)

Return the convex hull of real sets.

INPUT:

- `*real_set_collection` – list/tuple/iterable of *RealSet* or data that defines one

OUTPUT: the convex hull as a new *RealSet*

EXAMPLES:

```
sage: s1 = RealSet(0,2) + RealSet.unbounded_above_closed(10);  s1 # unbounded␣
↪set
(0, 2) ∪ [10, +oo)
sage: s2 = RealSet(1,3) + RealSet.unbounded_below_closed(-10);  s2
(-oo, -10] ∪ (1, 3)
sage: s3 = RealSet((0,2), RealSet.point(8)); s3
(0, 2) ∪ {8}
sage: s4 = RealSet(); s4  # empty set
{}
sage: RealSet.convex_hull(s1)
(0, +oo)
sage: RealSet.convex_hull(s2)
(-oo, 3)
sage: RealSet.convex_hull(s3)
(0, 8]
sage: RealSet.convex_hull(s4)
{}
sage: RealSet.convex_hull(s1, s2)
(-oo, +oo)
sage: RealSet.convex_hull(s2, s3)
(-oo, 8]
sage: RealSet.convex_hull(s2, s3, s4)
(-oo, 8]
```

```
>>> from sage.all import *
>>> s1 = RealSet(Integer(0),Integer(2)) + RealSet.unbounded_above_
↪closed(Integer(10));  s1 # unbounded set
(0, 2) ∪ [10, +oo)
>>> s2 = RealSet(Integer(1),Integer(3)) + RealSet.unbounded_below_closed(-
↪Integer(10));  s2
(-oo, -10] ∪ (1, 3)
>>> s3 = RealSet((Integer(0),Integer(2)), RealSet.point(Integer(8))); s3
(0, 2) ∪ {8}
>>> s4 = RealSet(); s4  # empty set
{}
>>> RealSet.convex_hull(s1)
(0, +oo)
>>> RealSet.convex_hull(s2)
```

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```
(-oo, 3)
>>> RealSet.convex_hull(s3)
(0, 8]
>>> RealSet.convex_hull(s4)
{}
>>> RealSet.convex_hull(s1, s2)
(-oo, +oo)
>>> RealSet.convex_hull(s2, s3)
(-oo, 8]
>>> RealSet.convex_hull(s2, s3, s4)
(-oo, 8]
```

difference(**other*)

Return `self` with `other` subtracted.

INPUT:

- `other` – a *RealSet* or data that defines one

OUTPUT:

The set-theoretic difference of `self` with `other` removed as a new *RealSet*.

EXAMPLES:

```
sage: s1 = RealSet(0,2) + RealSet.unbounded_above_closed(10);  s1
(0, 2) ∪ [10, +oo)
sage: s2 = RealSet(1,3) + RealSet.unbounded_below_closed(-10);  s2
(-oo, -10] ∪ (1, 3)
sage: s1.difference(s2)
(0, 1] ∪ [10, +oo)
sage: s1 - s2    # syntactic sugar
(0, 1] ∪ [10, +oo)
sage: s2.difference(s1)
(-oo, -10] ∪ [2, 3)
sage: s2 - s1    # syntactic sugar
(-oo, -10] ∪ [2, 3)
sage: s1.difference(1,11)
(0, 1] ∪ [11, +oo)
```

```
>>> from sage.all import *
>>> s1 = RealSet(Integer(0),Integer(2)) + RealSet.unbounded_above_
↪closed(Integer(10));  s1
(0, 2) ∪ [10, +oo)
>>> s2 = RealSet(Integer(1),Integer(3)) + RealSet.unbounded_below_closed(-
↪Integer(10));  s2
(-oo, -10] ∪ (1, 3)
>>> s1.difference(s2)
(0, 1] ∪ [10, +oo)
>>> s1 - s2    # syntactic sugar
(0, 1] ∪ [10, +oo)
>>> s2.difference(s1)
(-oo, -10] ∪ [2, 3)
>>> s2 - s1    # syntactic sugar
```

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```
(-oo, -10] ∪ [2, 3)
>>> s1.difference(Integer(1),Integer(11))
(0, 1] ∪ [11, +oo)
```

get_interval(*i*)

Return the `i`-th connected component.

Note that the intervals representing the real set are always normalized, i.e., they are sorted, disjoint and not connected.

INPUT:

- `i` – integer

OUTPUT: the i-th connected component as a *InternalRealInterval*

EXAMPLES:

```
sage: s = RealSet(RealSet.open_closed(0,1), RealSet.closed_open(2,3))
sage: s.get_interval(0)
(0, 1]
sage: s[0]    # shorthand
(0, 1]
sage: s.get_interval(1)
[2, 3)
sage: s[0] == s.get_interval(0)
True
```

```
>>> from sage.all import *
>>> s = RealSet(RealSet.open_closed(Integer(0),Integer(1)), RealSet.closed_
↪open(Integer(2),Integer(3)))
>>> s.get_interval(Integer(0))
(0, 1]
>>> s[Integer(0)]    # shorthand
(0, 1]
>>> s.get_interval(Integer(1))
[2, 3)
>>> s[Integer(0)] == s.get_interval(Integer(0))
True
```

inf()

Return the infimum.

OUTPUT: a real number or infinity

EXAMPLES:

```
sage: s1 = RealSet(0,2) + RealSet.unbounded_above_closed(10);  s1
(0, 2) ∪ [10, +oo)
sage: s1.inf()
0

sage: s2 = RealSet(1,3) + RealSet.unbounded_below_closed(-10);  s2
(-oo, -10] ∪ (1, 3)
sage: s2.inf()
-Infinity
```

```
>>> from sage.all import *
>>> s1 = RealSet(Integer(0),Integer(2)) + RealSet.unbounded_above_
↪closed(Integer(10));  s1
(0, 2) ∪ [10, +oo)
>>> s1.inf()
0

>>> s2 = RealSet(Integer(1),Integer(3)) + RealSet.unbounded_below_closed(-
↪Integer(10));  s2
(-oo, -10] ∪ (1, 3)
>>> s2.inf()
-Infinity
```

interior()

Return the topological interior of `self` as a new *RealSet*.

EXAMPLES:

```
sage: RealSet(-oo, oo).interior()
(-oo, +oo)
sage: RealSet().interior()
{}
sage: RealSet.point(2).interior()
{}
sage: RealSet([1, 2], (3, 4)).interior()
(1, 2) ∪ (3, 4)
```

```
>>> from sage.all import *
>>> RealSet(-oo, oo).interior()
(-oo, +oo)
>>> RealSet().interior()
{}
>>> RealSet.point(Integer(2)).interior()
{}
>>> RealSet([Integer(1), Integer(2)], (Integer(3), Integer(4))).interior()
(1, 2) ∪ (3, 4)
```

intersection(**real_set_collection*)

Return the intersection of real sets.

INPUT:

- `*real_set_collection` – list/tuple/iterable of *RealSet* or data that defines one

OUTPUT: the set-theoretic intersection as a new *RealSet*

EXAMPLES:

```
sage: s1 = RealSet(0,2) + RealSet.unbounded_above_closed(10);  s1
(0, 2) ∪ [10, +oo)
sage: s2 = RealSet(1,3) + RealSet.unbounded_below_closed(-10);  s2
(-oo, -10] ∪ (1, 3)
sage: s1.intersection(s2)
(1, 2)
sage: s1 & s2    # syntactic sugar
```

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```
(1, 2)
sage: s3 = RealSet((0, 1), (2, 3));  s3
(0, 1) ∪ (2, 3)
sage: s4 = RealSet([0, 1], [2, 3]);  s4
[0, 1] ∪ [2, 3]
sage: s3.intersection(s4)
(0, 1) ∪ (2, 3)
sage: s3.intersection([1, 2])
{}
sage: s4.intersection([1, 2])
{1} ∪ {2}
sage: s4.intersection(1, 2)
{}
sage: s5 = RealSet.closed_open(1, 10);  s5
[1, 10)
sage: s5.intersection(-oo, +oo)
[1, 10)
sage: s5.intersection(x != 2, (-oo, 3), RealSet.real_line()[0])             #␣
↪needs sage.symbolic
[1, 2) ∪ (2, 3)
```

```
>>> from sage.all import *
>>> s1 = RealSet(Integer(0),Integer(2)) + RealSet.unbounded_above_
↪closed(Integer(10));  s1
(0, 2) ∪ [10, +oo)
>>> s2 = RealSet(Integer(1),Integer(3)) + RealSet.unbounded_below_closed(-
↪Integer(10));  s2
(-oo, -10] ∪ (1, 3)
>>> s1.intersection(s2)
(1, 2)
>>> s1 & s2    # syntactic sugar
(1, 2)
>>> s3 = RealSet((Integer(0), Integer(1)), (Integer(2), Integer(3)));  s3
(0, 1) ∪ (2, 3)
>>> s4 = RealSet([Integer(0), Integer(1)], [Integer(2), Integer(3)]);  s4
[0, 1] ∪ [2, 3]
>>> s3.intersection(s4)
(0, 1) ∪ (2, 3)
>>> s3.intersection([Integer(1), Integer(2)])
{}
>>> s4.intersection([Integer(1), Integer(2)])
{1} ∪ {2}
>>> s4.intersection(Integer(1), Integer(2))
{}
>>> s5 = RealSet.closed_open(Integer(1), Integer(10));  s5
[1, 10)
>>> s5.intersection(-oo, +oo)
[1, 10)
>>> s5.intersection(x != Integer(2), (-oo, Integer(3)), RealSet.real_
↪line()[Integer(0)])             # needs sage.symbolic
[1, 2) ∪ (2, 3)
```

static interval(*lower*, *upper*, *lower_closed*, *upper_closed*, ***kwds*)

Construct an interval.

INPUT:

- `lower`, `upper` – two real numbers or infinity; they will be sorted if necessary
- `lower_closed`, `upper_closed` – boolean; whether the interval is closed at the lower and upper bound of the interval, respectively
- `**kwds` – see *RealSet*

OUTPUT: a new *RealSet*

EXAMPLES:

```
sage: RealSet.interval(1, 0, lower_closed=True, upper_closed=False)
[0, 1)
```

```
>>> from sage.all import *
>>> RealSet.interval(Integer(1), Integer(0), lower_closed=True, upper_
↪closed=False)
[0, 1)
```

is_closed()

Return whether `self` is a closed set.

EXAMPLES:

```
sage: RealSet().is_closed()
True
sage: RealSet.point(1).is_closed()
True
sage: RealSet([1, 2]).is_closed()
True
sage: RealSet([1, 2], (3, 4)).is_closed()
False
sage: RealSet(-oo, +oo).is_closed()
True
```

```
>>> from sage.all import *
>>> RealSet().is_closed()
True
>>> RealSet.point(Integer(1)).is_closed()
True
>>> RealSet([Integer(1), Integer(2)]).is_closed()
True
>>> RealSet([Integer(1), Integer(2)], (Integer(3), Integer(4))).is_closed()
False
>>> RealSet(-oo, +oo).is_closed()
True
```

is_connected()

Return whether `self` is a connected set.

OUTPUT: boolean

EXAMPLES:

```
sage: s1 = RealSet((1, 2), (2, 4));  s1
(1, 2) ∪ (2, 4)
sage: s1.is_connected()
False
sage: s2 = RealSet((1, 2), (2, 4), RealSet.point(2));  s2
(1, 4)
sage: s2.is_connected()
True
sage: s3 = RealSet(1,3) + RealSet.unbounded_below_closed(-10);  s3
(-oo, -10] ∪ (1, 3)
sage: s3.is_connected()
False
sage: RealSet(x != 0).is_connected()                                    #␣
↪needs sage.symbolic
False
sage: RealSet(-oo, oo).is_connected()
True
sage: RealSet().is_connected()
False
```

```
>>> from sage.all import *
>>> s1 = RealSet((Integer(1), Integer(2)), (Integer(2), Integer(4)));  s1
(1, 2) ∪ (2, 4)
>>> s1.is_connected()
False
>>> s2 = RealSet((Integer(1), Integer(2)), (Integer(2), Integer(4)), RealSet.
↪point(Integer(2)));  s2
(1, 4)
>>> s2.is_connected()
True
>>> s3 = RealSet(Integer(1),Integer(3)) + RealSet.unbounded_below_closed(-
↪Integer(10));  s3
(-oo, -10] ∪ (1, 3)
>>> s3.is_connected()
False
>>> RealSet(x != Integer(0)).is_connected()                             ␣
↪     # needs sage.symbolic
False
>>> RealSet(-oo, oo).is_connected()
True
>>> RealSet().is_connected()
False
```

is_disjoint(**other*)

Test whether the two sets are disjoint.

INPUT:

- `other` – a *RealSet* or data defining one

OUTPUT: boolean

See also

are_pairwise_disjoint()

EXAMPLES:

```
sage: s = RealSet((0, 1), (2, 3));  s
(0, 1) ∪ (2, 3)
sage: s.is_disjoint(RealSet([1, 2]))
True
sage: s.is_disjoint([3/2, 5/2])
False
sage: s.is_disjoint(RealSet())
True
sage: s.is_disjoint(RealSet().real_line())
False
```

```
>>> from sage.all import *
>>> s = RealSet((Integer(0), Integer(1)), (Integer(2), Integer(3)));  s
(0, 1) ∪ (2, 3)
>>> s.is_disjoint(RealSet([Integer(1), Integer(2)]))
True
>>> s.is_disjoint([Integer(3)/Integer(2), Integer(5)/Integer(2)])
False
>>> s.is_disjoint(RealSet())
True
>>> s.is_disjoint(RealSet().real_line())
False
```

is_disjoint_from(**args*, ***kwds*)

Deprecated: Use *is_disjoint()* instead. See Issue #31927 for details.

is_empty()

Return whether the set is empty.

EXAMPLES:

```
sage: RealSet(0, 1).is_empty()
False
sage: RealSet(0, 0).is_empty()
True
sage: RealSet.interval(1, 1, lower_closed=False, upper_closed=True).is_empty()
True
sage: RealSet.interval(1, -1, lower_closed=False, upper_closed=True).is_
↪empty()
False
```

```
>>> from sage.all import *
>>> RealSet(Integer(0), Integer(1)).is_empty()
False
>>> RealSet(Integer(0), Integer(0)).is_empty()
True
>>> RealSet.interval(Integer(1), Integer(1), lower_closed=False, upper_
```

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```
↪closed=True).is_empty()
True
>>> RealSet.interval(Integer(1), -Integer(1), lower_closed=False, upper_
↪closed=True).is_empty()
False
```

is_included_in(**args*, ***kwds*)

Deprecated: Use *is_subset()* instead. See Issue #31927 for details.

is_open()

Return whether `self` is an open set.

EXAMPLES:

```
sage: RealSet().is_open()
True
sage: RealSet.point(1).is_open()
False
sage: RealSet((1, 2)).is_open()
True
sage: RealSet([1, 2], (3, 4)).is_open()
False
sage: RealSet(-oo, +oo).is_open()
True
```

```
>>> from sage.all import *
>>> RealSet().is_open()
True
>>> RealSet.point(Integer(1)).is_open()
False
>>> RealSet((Integer(1), Integer(2))).is_open()
True
>>> RealSet([Integer(1), Integer(2)], (Integer(3), Integer(4))).is_open()
False
>>> RealSet(-oo, +oo).is_open()
True
```

is_subset(**other*)

Return whether `self` is a subset of `other`.

INPUT:

- `*other` – a *RealSet* or something that defines one

OUTPUT: boolean

EXAMPLES:

```
sage: I = RealSet((1,2))
sage: J = RealSet((1,3))
sage: K = RealSet((2,3))
sage: I.is_subset(J)
True
sage: J.is_subset(K)
False
```

```
>>> from sage.all import *
>>> I = RealSet((Integer(1),Integer(2)))
>>> J = RealSet((Integer(1),Integer(3)))
>>> K = RealSet((Integer(2),Integer(3)))
>>> I.is_subset(J)
True
>>> J.is_subset(K)
False
```

is_universe()

Return whether the set is the ambient space (the real line).

EXAMPLES:

```
sage: RealSet().ambient().is_universe()
True
```

```
>>> from sage.all import *
>>> RealSet().ambient().is_universe()
True
```

lift(*x*)

Lift `x` to the ambient space for `self`.

This version of the method just returns `x`.

EXAMPLES:

```
sage: s = RealSet(0, 2); s
(0, 2)
sage: s.lift(1)
1
```

```
>>> from sage.all import *
>>> s = RealSet(Integer(0), Integer(2)); s
(0, 2)
>>> s.lift(Integer(1))
1
```

n_components()

Return the number of connected components.

See also *get_interval()*.

EXAMPLES:

```
sage: s = RealSet(RealSet.open_closed(0,1), RealSet.closed_open(2,3))
sage: s.n_components()
2
```

```
>>> from sage.all import *
>>> s = RealSet(RealSet.open_closed(Integer(0),Integer(1)), RealSet.closed_
↪open(Integer(2),Integer(3)))
>>> s.n_components()
2
```

normalize(*intervals*)

Bring a collection of intervals into canonical form.

INPUT:

- `intervals` – list/tuple/iterable of intervals

OUTPUT: a tuple of intervals such that

- they are sorted in ascending order (by lower bound)
- there is a gap between each interval
- all intervals are non-empty

EXAMPLES:

```
sage: i1 = RealSet((0, 1))[0]
sage: i2 = RealSet([1, 2])[0]
sage: i3 = RealSet((2, 3))[0]
sage: RealSet.normalize([i1, i2, i3])
((0, 3),)
```

```
>>> from sage.all import *
>>> i1 = RealSet((Integer(0), Integer(1)))[Integer(0)]
>>> i2 = RealSet([Integer(1), Integer(2)])[Integer(0)]
>>> i3 = RealSet((Integer(2), Integer(3)))[Integer(0)]
>>> RealSet.normalize([i1, i2, i3])
((0, 3),)
```

static open(*lower*, *upper*, ***kwds*)

Construct an open interval.

INPUT:

- `lower`, `upper` – two real numbers or infinity; they will be sorted if necessary
- `**kwds` – see *RealSet*

OUTPUT: a new *RealSet*

EXAMPLES:

```
sage: RealSet.open(1, 0)
(0, 1)
```

```
>>> from sage.all import *
>>> RealSet.open(Integer(1), Integer(0))
(0, 1)
```

static open_closed(*lower*, *upper*, ***kwds*)

Construct a half-open interval.

INPUT:

- `lower`, `upper` – two real numbers or infinity; they will be sorted if necessary
- `**kwds` – see *RealSet*

OUTPUT:

A new *RealSet* that is open at the lower bound and closed at the upper bound.

EXAMPLES:

```
sage: RealSet.open_closed(1, 0)
(0, 1]
```

```
>>> from sage.all import *
>>> RealSet.open_closed(Integer(1), Integer(0))
(0, 1]
```

static point(*p*, ***kwds*)

Construct an interval containing a single point.

INPUT:

- `p` – a real number
- `**kwds` – see *RealSet*

OUTPUT: a new *RealSet*

EXAMPLES:

```
sage: RealSet.open(1, 0)
(0, 1)
```

```
>>> from sage.all import *
>>> RealSet.open(Integer(1), Integer(0))
(0, 1)
```

static real_line(***kwds*)

Construct the real line.

INPUT:

- `**kwds` – see *RealSet*

EXAMPLES:

```
sage: RealSet.real_line()
(-oo, +oo)
```

```
>>> from sage.all import *
>>> RealSet.real_line()
(-oo, +oo)
```

retract(*x*)

Retract `x` to `self`.

It raises an error if `x` does not lie in the set `self`.

EXAMPLES:

```
sage: s = RealSet(0, 2); s
(0, 2)
sage: s.retract(1)
1
sage: s.retract(2)
```

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```
Traceback (most recent call last):
...
ValueError: 2 is not an element of (0, 2)
```

```
>>> from sage.all import *
>>> s = RealSet(Integer(0), Integer(2)); s
(0, 2)
>>> s.retract(Integer(1))
1
>>> s.retract(Integer(2))
Traceback (most recent call last):
...
ValueError: 2 is not an element of (0, 2)
```

sup()

Return the supremum.

OUTPUT: a real number or infinity

EXAMPLES:

```
sage: s1 = RealSet(0,2) + RealSet.unbounded_above_closed(10);  s1
(0, 2) ∪ [10, +oo)
sage: s1.sup()
+Infinity

sage: s2 = RealSet(1,3) + RealSet.unbounded_below_closed(-10);  s2
(-oo, -10] ∪ (1, 3)
sage: s2.sup()
3
```

```
>>> from sage.all import *
>>> s1 = RealSet(Integer(0),Integer(2)) + RealSet.unbounded_above_
↪closed(Integer(10));  s1
(0, 2) ∪ [10, +oo)
>>> s1.sup()
+Infinity

>>> s2 = RealSet(Integer(1),Integer(3)) + RealSet.unbounded_below_closed(-
↪Integer(10));  s2
(-oo, -10] ∪ (1, 3)
>>> s2.sup()
3
```

symmetric_difference(**other*)

Return the symmetric difference of `self` and `other`.

INPUT:

- `other` – a *RealSet* or data that defines one

OUTPUT:

The set-theoretic symmetric difference of `self` and `other` as a new *RealSet*.

EXAMPLES:

```
sage: s1 = RealSet(0,2); s1
(0, 2)
sage: s2 = RealSet.unbounded_above_open(1); s2
(1, +oo)
sage: s1.symmetric_difference(s2)
(0, 1] ∪ [2, +oo)
```

```
>>> from sage.all import *
>>> s1 = RealSet(Integer(0),Integer(2)); s1
(0, 2)
>>> s2 = RealSet.unbounded_above_open(Integer(1)); s2
(1, +oo)
>>> s1.symmetric_difference(s2)
(0, 1] ∪ [2, +oo)
```

static unbounded_above_closed(*bound*, ***kwds*)

Construct a semi-infinite interval.

INPUT:

- `bound` – a real number
- `**kwds` – see *RealSet*

OUTPUT:

A new *RealSet* from the bound (including) to plus infinity.

EXAMPLES:

```
sage: RealSet.unbounded_above_closed(1)
[1, +oo)
```

```
>>> from sage.all import *
>>> RealSet.unbounded_above_closed(Integer(1))
[1, +oo)
```

static unbounded_above_open(*bound*, ***kwds*)

Construct a semi-infinite interval.

INPUT:

- `bound` – a real number
- `**kwds` – see *RealSet*

OUTPUT:

A new *RealSet* from the bound (excluding) to plus infinity.

EXAMPLES:

```
sage: RealSet.unbounded_above_open(1)
(1, +oo)
```

```
>>> from sage.all import *
>>> RealSet.unbounded_above_open(Integer(1))
(1, +oo)
```

static unbounded_below_closed(*bound*, ***kwds*)

Construct a semi-infinite interval.

INPUT:

- `bound` – a real number

OUTPUT:

A new *RealSet* from minus infinity to the bound (including).

- `**kwds` – see *RealSet*

EXAMPLES:

```
sage: RealSet.unbounded_below_closed(1)
(-oo, 1]
```

```
>>> from sage.all import *
>>> RealSet.unbounded_below_closed(Integer(1))
(-oo, 1]
```

static unbounded_below_open(*bound*, ***kwds*)

Construct a semi-infinite interval.

INPUT:

- `bound` – a real number

OUTPUT:

A new *RealSet* from minus infinity to the bound (excluding).

- `**kwds` – see *RealSet*

EXAMPLES:

```
sage: RealSet.unbounded_below_open(1)
(-oo, 1)
```

```
>>> from sage.all import *
>>> RealSet.unbounded_below_open(Integer(1))
(-oo, 1)
```

union(**real_set_collection*)

Return the union of real sets.

INPUT:

- `*real_set_collection` – list/tuple/iterable of *RealSet* or data that defines one

OUTPUT: the set-theoretic union as a new *RealSet*

EXAMPLES:

```
sage: s1 = RealSet(0,2)
sage: s2 = RealSet(1,3)
sage: s1.union(s2)
(0, 3)
sage: s1.union(1,3)
```

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```
(0, 3)
sage: s1 | s2    # syntactic sugar
(0, 3)
sage: s1 + s2    # syntactic sugar
(0, 3)
sage: RealSet().union(RealSet.real_line())
(-oo, +oo)
sage: s = RealSet().union([1, 2], (2, 3)); s
[1, 3)
sage: RealSet().union((-oo, 0), x > 6, s[0],                                    #␣
↪needs sage.symbolic
....:                 RealSet.point(5.0), RealSet.closed_open(2, 4))
(-oo, 0) ∪ [1, 4) ∪ {5} ∪ (6, +oo)
```

```
>>> from sage.all import *
>>> s1 = RealSet(Integer(0),Integer(2))
>>> s2 = RealSet(Integer(1),Integer(3))
>>> s1.union(s2)
(0, 3)
>>> s1.union(Integer(1),Integer(3))
(0, 3)
>>> s1 | s2    # syntactic sugar
(0, 3)
>>> s1 + s2    # syntactic sugar
(0, 3)
>>> RealSet().union(RealSet.real_line())
(-oo, +oo)
>>> s = RealSet().union([Integer(1), Integer(2)], (Integer(2), Integer(3))); s
[1, 3)
>>> RealSet().union((-oo, Integer(0)), x > Integer(6), s[Integer(0)],        ␣
↪                       # needs sage.symbolic
...                 RealSet.point(RealNumber('5.0')), RealSet.closed_
↪open(Integer(2), Integer(4)))
(-oo, 0) ∪ [1, 4) ∪ {5} ∪ (6, +oo)
```

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