0 Introduction

0.1 Finite Sums

0.11 Progressions

0.12 Sums of powers of natural numbers

0.13 Sums of reciprocals of natural numbers

0.14 Sums of products of reciprocals of natural numbers

0.15 Sums of the binomial coefficients

0.2 Numerical Series and Infinite Products

0.21 The convergence of numerical series

0.22 Convergence tests

0.23–0.24 Examples of numerical series

0.25 Infinite products

0.26 Examples of infinite products

0.3 Functional Series

0.30 Definitions and theorems

0.31 Power series

0.32 Fourier series

0.33 Asymptotic series

0.4 Certain Formulas from Differential Calculus

0.41 Differentiation of a definite integral with respect to a parameter

0.42 The nth derivative of a product (Leibniz's rule)

0.43 The nth derivative of a composite function

0.44 Integration by substitution

1 Elementary Functions

1.1 Power of Binomials

1.11 Power series

1.12 Series of rational fractions

1.2 The Exponential Function

1.21 Series representation

1.22 Functional relations

1.23 Series of exponentials

1.3–1.4 Trigonometric and Hyperbolic Functions

1.30 Introduction

1.31 The basic functional relations

1.32 The representation of powers of trigonometric and hyperbolic functions in terms of functions of multiples of the argument (angle)

1.33 The representation of trigonometric and hyperbolic functions of multiples of the argument (angle) in terms of powers of these functions

1.34 Certain sums of trigonometric and hyperbolic functions

1.35 Sums of powers of trigonometric functions of multiple angles

1.36 Sums of products of trigonometric functions of multiple angles

1.37 Sums of tangents of multiple angles

1.38 Sums leading to hyperbolic tangents and cotangents

1.39 The representation of cosines and sines of multiples of the angle as finite products

1.41 The expansion of trigonometric and hyperbolic functions in power series

1.42 Expansion in series of simple fractions

1.43 Representation in the form of an infinite product

1.44–1.45 Trigonometric (Fourier) series

1.46 Series of products of exponential and trigonometric functions

1.47 Series of hyperbolic functions

1.48 Lobachevskiy's "Angle of Parallelism" $\Pi(x)$

1.49 The hyperbolic amplitude (the Gudermannian) $\operatorname{gd} x$

1.5 The Logarithm

1.51 Series representation

1.52 Series of logarithms (cf. 1.431)

1.6 The Inverse Trigonometric and Hyperbolic Functions

1.61 The domain of definition

1.62–1.63 Functional relations

1.64 Series representations

2 Indefinite Integrals of Elementary Functions

2.0 Introduction

2.00 General remarks

2.01 The basic integrals

2.02 General formulas

2.1 Rational Functions

2.10 General integration rules

2.11–2.13 Forms containing the binomial $a + bx^k$

2.14 Forms containing the binomial $1 \pm x^n$

2.15 Forms containing pairs of binomials: $a + bx$ and $\alpha + \beta x$

2.16 Forms containing the trinomial $a + bx^k + cx^{2k}$

2.17 Forms containing the quadratic trinomial $a+bx+cx^2$ and powers of x

2.18 Forms containing the quadratic trinomial $a + bx + cx^2$ and the binomial $\alpha + \beta x$

2.2 Algebraic Functions

2.20 Introduction

2.21 Forms containing the binomial $a + bx^k$ and $\sqrt{x}$

2.22–2.23 Forms containing $\sqrt[n]{(a + bx)^k}$

2.24 Forms containing $\sqrt{a + bx}$ and the binomial $\alpha + \beta x$

2.25 Forms containing $\sqrt{a + bx + cx^2}$

2.26 Forms containing $\sqrt{a + bx + cx^2}$ and integral powers of x

2.27 Forms containing $\sqrt{a + cx^2}$ and integral powers of x

2.28 Forms containing $\sqrt{a + bx + cx^2}$ and first- and second-degree polynomials

2.29 Integrals that can be reduced to elliptic or pseudo-elliptic integrals

2.3 The Exponential Function

2.31 Forms containing e^{ax}

2.32 The exponential combined with rational functions of x

2.4 Hyperbolic Functions

2.41–2.43 Powers of $\sinh x$, $\cosh x$, $\tanh x$, and $\coth x$

2.44–2.45 Rational functions of hyperbolic functions

2.46 Algebraic functions of hyperbolic functions

2.47 Combinations of hyperbolic functions and powers

2.48 Combinations of hyperbolic functions, exponentials, and powers

2.5–2.6 Trigonometric Functions

2.50 Introduction

2.51–2.52 Powers of trigonometric functions

2.53–2.54 Sines and cosines of multiple angles and of linear and more complicated functions of the argument

2.55–2.56 Rational functions of the sine and cosine

2.57 Integrals containing $\sqrt{a \pm b \sin x}$ or $\sqrt{a \pm b \cos x}$

2.58–2.62 Integrals reducible to elliptic and pseudo-elliptic integrals

2.63–2.65 Products of trigonometric functions and powers

2.66 Combinations of trigonometric functions and exponentials

2.67 Combinations of trigonometric and hyperbolic functions

2.7 Logarithms and Inverse-Hyperbolic Functions

2.71 The logarithm

2.72–2.73 Combinations of logarithms and algebraic functions

2.74 Inverse hyperbolic functions

2.8 Inverse Trigonometric Functions

2.81 Arcsines and arccosines

2.82 The arcsecant, the arccosecant, the arctangent, and the arccotangent

2.83 Combinations of arcsine or arccosine and algebraic functions

2.84 Combinations of the arcsecant and arccosecant with powers of x

2.85 Combinations of the arctangent and arccotangent with algebraic functions

3–4 Definite Integrals of Elementary Functions

3.0 Introduction

3.01 Theorems of a general nature

3.02 Change of variable in a definite integral

3.03 General formulas

3.04 Improper integrals

3.05 The principal values of improper integrals

3.1–3.2 Power and Algebraic Functions

3.11 Rational functions

3.12 Products of rational functions and expressions that can be reduced to square roots of first- and second-degree polynomials

3.13–3.17 Expressions that can be reduced to square roots of third- and fourth-degree polynomials and their products with rational functions

3.18 Expressions that can be reduced to fourth roots of second-degree polynomials and their products with rational functions

3.19–3.23 Combinations of powers of x and powers of binomials of the form $(\alpha + \beta x)$

3.24–3.27 Powers of x, of binomials of the form $\alpha + \beta x^p$ and of polynomials in x

3.3–3.4 Exponential Functions

3.31 Exponential functions

3.32–3.34 Exponentials of more complicated arguments

3.35 Combinations of exponentials and rational functions

3.36–3.37 Combinations of exponentials and algebraic functions

3.38–3.39 Combinations of exponentials and arbitrary powers

3.41–3.44 Combinations of rational functions of powers and exponentials

3.45 Combinations of powers and algebraic functions of exponentials

3.46–3.48 Combinations of exponentials of more complicated arguments and powers

3.5 Hyperbolic Functions

3.51 Hyperbolic functions

3.52–3.53 Combinations of hyperbolic functions and algebraic functions

3.54 Combinations of hyperbolic functions and exponentials

3.55–3.56 Combinations of hyperbolic functions, exponentials, and powers

3.6–4.1 Trigonometric Functions

3.61 Rational functions of sines and cosines and trigonometric functions of multiple angles

3.62 Powers of trigonometric functions

3.63 Powers of trigonometric functions and trigonometric functions of linear functions

3.64–3.65 Powers and rational functions of trigonometric functions

3.66 Forms containing powers of linear functions of trigonometric functions

3.67 Square roots of expressions containing trigonometric functions

3.68 Various forms of powers of trigonometric functions

3.69–3.71 Trigonometric functions of more complicated arguments

3.72–3.74 Combinations of trigonometric and rational functions

3.75 Combinations of trigonometric and algebraic functions

3.76–3.77 Combinations of trigonometric functions and powers

3.78–3.81 Rational functions of x and of trigonometric functions

3.82–3.83 Powers of trigonometric functions combined with other powers

3.84 Integrals containing $\sqrt{1-k^2\sin^2 x}$, $\sqrt{1-k^2\cos^2 x}$, and similar expressions

3.85–3.88 Trigonometric functions of more complicated arguments combined with powers

3.89–3.91 Trigonometric functions and exponentials

3.92 Trigonometric functions of more complicated arguments combined with exponentials

3.93 Trigonometric and exponential functions of trigonometric functions

3.94–3.97 Combinations involving trigonometric functions, exponentials, and powers

3.98–3.99 Combinations of trigonometric and hyperbolic functions

4.11–4.12 Combinations involving trigonometric and hyperbolic functions and powers

4.13 Combinations of trigonometric and hyperbolic functions and exponentials

4.14 Combinations of trigonometric and hyperbolic functions, exponentials, and powers

4.2–4.4 Logarithmic Functions

4.21 Logarithmic functions

4.22 Logarithms of more complicated arguments

4.23 Combinations of logarithms and rational functions

4.24 Combinations of logarithms and algebraic functions

4.25 Combinations of logarithms and powers

4.26–4.27 Combinations involving powers of the logarithm and other powers

4.28 Combinations of rational functions of $\ln x$ and powers

4.29–4.32 Combinations of logarithmic functions of more complicated arguments and powers

4.33–4.34 Combinations of logarithms and exponentials

4.35–4.36 Combinations of logarithms, exponentials, and powers

5.1 Elliptic Integrals and Functions

5.11 Complete elliptic integrals

5.12 Elliptic integrals

5.13 Jacobian elliptic functions

5.14 Weierstrass elliptic functions

5.2 The Exponential Integral Function

5.21 The exponential integral function

5.22 Combinations of the exponential integral function and powers

5.23 Combinations of the exponential integral and the exponential

5.3 The Sine Integral and the Cosine Integral

5.4 The Probability Integral and Fresnel Integrals

5.5 Bessel Functions

6–7 Definite Integrals of Special Functions

6.1 Elliptic Integrals and Functions

6.11 Forms containing $F(x, k)$

6.12 Forms containing $E(x, k)$

6.13 Integration of elliptic integrals with respect to the modulus

6.14–6.15 Complete elliptic integrals

6.16 The theta function

6.17 Generalized elliptic integrals

6.2–6.3 The Exponential Integral Function and Functions Generated by It

6.21 The logarithm integral

6.22–6.23 The exponential integral function

6.24–6.26 The sine integral and cosine integral functions

6.27 The hyperbolic sine integral and hyperbolic cosine integral functions

6.28–6.31 The probability integral

6.32 Fresnel integrals

6.4 The Gamma Function and Functions Generated by It

The gamma function $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$ and the family of special functions it generates pervade both pure mathematics and mathematical physics. Gradshteyn & Ryzhik sections 6.41–6.47 catalogue the integral identities; the annotations below describe the problems to which those identities apply.

6.41 The gamma function

Physics applications.

1. **Dimensional regularisation in quantum field theory.**One-loop Feynman integrals in $d = 4 - 2\varepsilon$ dimensions evaluate to ratios of gamma functions; for instance the scalar tadpole gives

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2+m^2)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n-d/2)}{\Gamma(n)} \left(\frac{1}{m^2}\right)^{n-d/2}.$$

   Ultraviolet divergences appear as poles of $\Gamma(\varepsilon)$ at $\varepsilon = 0$ and are absorbed by renormalisation counterterms [tV72; BG72].

2. **Volume of the n-sphere and solid angles.**The volume of the unit n-ball and the surface area of S^{n-1} are

$$V_n = \frac{\pi^{n/2}}{\Gamma(n/2+1)}, \qquad S_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

   These arise every time a d-dimensional integral is converted to polar coordinates—scattering cross-sections, the Stefan–Boltzmann law, and phase-space volumes in particle physics.

3. **Black-body radiation and Bose–Einstein integrals.** The Stefan–Boltzmann constant derives from $\int_0^\infty x^3(e^x-1)^{-1}\,dx = \Gamma(4)\,\zeta(4) = \pi^4/15$. More generally, $\int_0^\infty x^{s-1}(e^x-1)^{-1}\,dx = \Gamma(s)\,\zeta(s)$ controls the energy density of the cosmic microwave background and the Debye model of phonon specific heat.

4. **Coulomb phase shifts.** In charged-particle scattering the Coulomb phase shift is $\sigma_\ell = \arg\Gamma(\ell+1+i\eta)$, where η is the Sommerfeld parameter. The identity $|\Gamma(i\eta)|^2 = \pi/[\eta\sinh(\pi\eta)]$ governs the Gamow penetration factor in nuclear alpha-decay theory and thermonuclear reaction rates in stellar interiors.

5. **The Veneziano amplitude and the birth of string theory.** Veneziano's 1968 meson scattering amplitude $B(s,t) = \Gamma(s)\Gamma(t)/\Gamma(s+t)$, with s,t linear in Mandelstam variables, reproduces crossing symmetry and Regge behaviour [Ven68]. The gamma-function poles at non-positive integers correspond to the infinite tower of string resonances.

6. **Selberg integral and random matrix theory.** The partition function of the log-gas [MD63] is the Selberg integral [Sel44], a product of gamma functions that governs GUE/GOE/GSE eigenvalue distributions and the Calogero–Sutherland integrable system.

Mathematics applications.

1. **Functional equation of the Riemann zeta function.** The completed zeta function $\xi(s) = \pi^{-s/2}\Gamma(s/2)\,\zeta(s)$ satisfies $\xi(s) = \xi(1-s)$. The gamma factor encodes the archimedean place in the Euler product over primes; the proof uses the Mellin transform of the Jacobi theta function.

2. **Weierstrass product and entire function theory.** $1/\Gamma(z) = z\,e^{\gamma z}\prod_{n=1}^\infty(1+z/n)\,e^{-z/n}$ is the prototype for the Hadamard factorisation theorem and underlies the theory of zeta-regularised determinants.

3. **Interpolation of the factorial.** By the Bohr–Mollerup theorem, Γ is the unique log-convex extension of $n!$ to real and complex arguments. The binomial coefficient $\binom{\alpha}{k} = \Gamma(\alpha+1)/[\Gamma(k+1)\Gamma(\alpha-k+1)]$ for non-integer α is essential in fractional calculus and generalised hypergeometric series.

4. **Spectral zeta-regularised determinants.** For a positive self-adjoint operator A (e.g. the Laplacian on a compact Riemannian manifold), $\det'(A) = \exp(-\zeta_A'(0))$ is computed via the Mellin transform $\lambda^{-s} = \Gamma(s)^{-1}\int_0^\infty t^{s-1}e^{-\lambda t}\,dt$. This is central to one-loop quantum gravity and the Ray–Singer analytic torsion.

6.42 Combinations of the gamma function, the exponential, and powers

Physics applications.

1. **Schwinger proper-time parametrisation.**The identity

$$\frac{1}{(k^2+m^2)^n} = \frac{1}{\Gamma(n)} \int_0^\infty \alpha^{n-1}\, e^{-\alpha(k^2+m^2)}\, d\alpha$$

   converts momentum-space Feynman propagators into Gaussian integrals over proper-time parameters [Sch51]. Multi-loop calculations in QED and QCD chain multiple such parametrisations, producing integrands of products $\alpha_i^{n_i-1}$ times exponentials—exactly the class of integrals in G&R 6.42.

2. **Mellin–Barnes integrals for scattering amplitudes.**Feynman integrals are frequently represented as Mellin–Barnes contour integrals

$$I = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(a+s)\,\Gamma(b-s)}{\Gamma(c+s)}\, z^{-s}\, ds,$$

   i.e. products and ratios of gamma functions multiplied by exponentials and powers. The "method of brackets" [GM10] systematises such evaluations, extending Ramanujan's Master Theorem.

3. **Hawking radiation.**The Bogoliubov coefficients near a black-hole horizon involve $|\Gamma(i\omega/\kappa)|^2 = \pi/[\omega \sinh(\pi\omega/\kappa)]$, producing the thermal Hawking spectrum at temperature $T_H = \hbar\kappa/(2\pi k_B)$ [Haw75].

4. **Maxwell–Boltzmann moment integrals.**The n-th moment of the Maxwell speed distribution is $\langle v^n \rangle \propto (k_B T/m)^{n/2}\,\Gamma\big(\frac{n+3}{2}\big)$. These moments yield transport coefficients—viscosity, thermal conductivity—and appear in stellar structure equations.

5. **Statistical mechanics partition functions.**The Gibbs factor $N! = \Gamma(N+1)$ corrects for particle indistinguishability, and the density of states $g(\varepsilon) \propto \varepsilon^{d/2-1}/\Gamma(d/2)$ is a gamma-exponential-power combination that shapes the thermodynamics of ideal Bose and Fermi gases.

Mathematics applications.

1. **Ramanujan's Master Theorem.**If $f(x) = \sum_{k=0}^\infty \varphi(k)(-x)^k/k!$, then $\int_0^\infty x^{s-1} f(x)\, dx = \Gamma(s)\,\varphi(-s)$. This result (rigorised by Hardy [Har20]) is the prototype for the integrals in G&R 6.42 and underpins the modern method of brackets.

2. **Mellin transform theory.**The Mellin transform of e^{-x} is $\Gamma(s)$ itself. More generally, Mellin transforms of functions built from exponentials and powers produce gamma-function combinations. Mellin inversion and the Parseval-type identity (used in analytic number theory, e.g. Perron's formula) rely on the analytic properties of $\Gamma(s)$.

3. **Watson's lemma and asymptotic expansions.** Watson's lemma gives the large-$|z|$ asymptotic expansion of $\int_0^\infty t^{\lambda-1} e^{-zt} \phi(t)\, dt$: each term contributes $\Gamma(\lambda+n)/z^{\lambda+n}$, making the gamma function the organising structure for all Laplace-type asymptotic series, including Stirling's series.

6.43 Combinations of the gamma function and trigonometric functions

Physics applications.

1. **Euler's reflection formula and quantum scattering.** The identity $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ is the prototypical gamma-trigonometric combination. In charged-particle scattering, $|\Gamma(i\eta)|^2 = \pi/[\eta \sinh(\pi\eta)]$ gives the Gamow penetration factor for thermonuclear reactions in stellar interiors.

2. **Regge poles and partial-wave amplitudes.** In Regge theory the partial-wave amplitude, continued to complex angular momentum ℓ, takes the form $\beta(t)\,\Gamma(1-\alpha(t))/\sin(\pi\alpha(t))\,(-s)^{\alpha(t)}$ —a product of gamma and trigonometric functions of the Regge trajectory $\alpha(t)$ [Reg59]. This structure is inherited by the Veneziano amplitude and modern string amplitudes.

3. **Gutzwiller trace formula.** In semiclassical quantum mechanics, the density of energy levels in quantum billiards is expressed as a sum over classical periodic orbits involving gamma-trigonometric combinations, via the functional equation of spectral L-functions [Gut90].

4. **Fourier transforms of power laws and Lévy distributions.** The Fourier transform of $|x|^{-\alpha}$ involves $\Gamma((d-\alpha)/2)/\Gamma(\alpha/2)$, with intermediate steps yielding $\Gamma(s)\cos(\pi s/2)$. These appear in the theory of Lévy stable distributions, fractional diffusion equations, and turbulence theory (Kolmogorov spectrum).

Mathematics applications.

1. **Euler's sine product and entire function theory.** $\sin(\pi z)/(\pi z) = \prod_{n=1}^{\infty}(1 - z^2/n^2)$, combined with the Weierstrass product for $\Gamma(z)$, yields the reflection formula. This circle of ideas is foundational for the theory of entire functions of finite order and the Hadamard factorisation theorem.

2. **Dirichlet L-function functional equations.** The completed L-function involves gamma factors $\Gamma((s+a)/2)\,\pi^{-(s+a)/2}$, which pair with $\cos(\pi s/2)$ or $\sin(\pi s/2)$ through the duplication and reflection formulas. This structure extends to automorphic L-functions in the Langlands programme.

3. **Ramanujan's integral identities.** The identity $\int_0^\infty x^{s-1}/(1+x)\, dx = \pi/\sin(\pi s) = \Gamma(s)\Gamma(1-s)$ is the simplest of many gamma-trigonometric evaluations in Ramanujan's notebooks, rigorised by Hardy [Har20].

6.44 The logarithm of the gamma function*

Physics applications.

1. **Stirling's approximation and the thermodynamic limit.** The expansion $\ln\Gamma(z) \sim z\ln z - z - \frac{1}{2}\ln z + \frac{1}{2}\ln(2\pi) + \sum_{k=1}^{\infty} B_{2k}/[2k(2k-1)\,z^{2k-1}]$ (with Bernoulli numbers B_{2k}) is the workhorse of statistical mechanics: every computation of entropy, free energy, or chemical potential for N particles passes through $\ln N! \approx N\ln N - N$. More refined forms appear in finite-size scaling, nucleation theory, and the Sackur–Tetrode equation for ideal-gas entropy.

2. **Entropy of the Gamma distribution and Bayesian inference.** The differential entropy of a Gamma(α, θ) random variable is $H = \alpha + \ln\theta + \ln\Gamma(\alpha) + (1-\alpha)\,\psi(\alpha)$. This expression appears in variational inference (ELBO computations), Bayesian model comparison, and the maximum-entropy characterisation of the gamma distribution.

3. **Free energy of random matrix ensembles.** The large-n expansion of $\ln Z_n(\beta)$ (from the Selberg integral partition function) using the Stirling expansion of $\ln\Gamma$ yields the topological expansion of random matrix theory, with coefficients related to intersection numbers on moduli spaces of Riemann surfaces.

4. **One-loop effective actions in QFT.** The one-loop effective action $\Gamma^{(1)} = -\frac{1}{2}\ln\det(-\nabla^2 + m^2) = -\frac{1}{2}\zeta_A'(0)$ involves $\ln\Gamma$ through the spectral zeta function. The Barnes G-function (built from $\int \ln\Gamma$) appears in functional determinants on spheres and in conformal field theory.

Mathematics applications.

1. **Raabe's formula.** $\int_0^1 \ln\Gamma(x)\,dx = \frac{1}{2}\ln(2\pi)$ [Raa43], a fundamental identity connected to $\zeta'(0) = -\frac{1}{2}\ln(2\pi)$. Kummer's Fourier series for $\ln\Gamma(x)$ on $(0,1)$ expresses it in terms of $\ln\sin(\pi x)$ and a cosine series with coefficients involving $\ln k$.

2. **The Barnes G-function and multiple gamma functions.** $G(z+1) = \Gamma(z)\,G(z)$; its logarithm is built from iterated integrals of $\ln\Gamma$. Applications include: determinants of Laplacians on S^n [Var88; OPS88], the Glaisher–Kinkelin constant $A = e^{1/12-\zeta'(-1)}$, and exact Casimir energies on curved manifolds.

3. **The Riemann–Siegel theta function.** $\vartheta(t) = \arg\Gamma(\frac{1}{4} + \frac{it}{2}) - \frac{t}{2}\ln\pi$ governs the phase of $\zeta(\frac{1}{2} + it)$ on the critical line. Computing high zeros of $\zeta(s)$ requires accurate evaluation of $\ln\Gamma$ at complex arguments via the Stirling series.

6.45 The incomplete gamma function

The lower and upper incomplete gamma functions are

$$\gamma(s,x) = \int_0^x t^{s-1}e^{-t}\,dt, \qquad \Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}\,dt,$$

so that $\gamma(s,x)+\Gamma(s,x) = \Gamma(s)$. The regularised forms are $P(s,x) = \gamma(s,x)/\Gamma(s)$ and $Q(s,x) = \Gamma(s,x)/\Gamma(s)$.

Physics applications.

1. **Chi-squared distribution and experimental physics.** The CDF of the χ^2 distribution with k degrees of freedom is $F(x;k) = P(k/2,\, x/2) = \gamma(k/2,\, x/2)/\Gamma(k/2)$. Every goodness-of-fit p-value in experimental particle physics invokes the regularised incomplete gamma function.

2. **Poisson process cumulative probabilities.** $P(X \le k) = Q(k+1,\lambda) = \Gamma(k+1,\lambda)/\Gamma(k+1)$, connecting the incomplete gamma function to counting statistics in nuclear and particle physics detectors, radioactive decay counting, and queuing theory.

3. **The error function and Gaussian integrals.** $\mathrm{erf}(x) = \gamma(\frac{1}{2}, x^2)/\sqrt{\pi}$ is a special case. It appears in quantum-mechanical tunnelling probabilities, Gaussian noise analysis in signal processing, and diffusion in random media.

4. **Heat conduction and diffusion.** The fundamental solution of the heat equation in a semi-infinite rod with specific boundary conditions involves the incomplete gamma function. Generalised forms appear in thermal models of laser-heated biological tissue and in the n-dimensional Debye function for specific heat of solids.

5. **Nakagami fading in wireless communications.** The outage probability over a Nakagami-m fading channel is $P_{\mathrm{out}} = P(m,\, m\gamma_{\mathrm{th}}/\bar{\gamma})$, the regularised incomplete gamma function [AG99]. This is the standard analytical framework for outage analysis in 4G/5G systems.

6. **Radiative transfer and the exponential integral.** The exponential integral $E_n(x) = x^{n-1}\Gamma(1-n,x)$ appears in the Chandrasekhar equations for stellar atmospheres [Cha60], neutron transport theory, and electromagnetic wave attenuation in lossy media.

Mathematics applications.

1. **Generalised exponential integral and Mittag-Leffler function.** The incomplete gamma function is the building block for $E_p(z) = z^{p-1}\Gamma(1-p,z)$ at complex p. The three-parameter Mittag-Leffler function, central to fractional calculus and anomalous diffusion, can be expressed through incomplete gamma functions in certain parameter ranges.

2. **Uniform asymptotic expansions.** Temme [Tem79; Tem96] developed uniform asymptotic expansions of $Q(a, x)$ for large a valid uniformly in x/a, bridging the transition region around $x = a$. These expansions form the basis for high-precision numerical computation of chi-squared quantiles in standard mathematical libraries.

3. **Analytic combinatorics.** In the Flajolet–Sedgewick framework [FS09], the saddle-point method applied to generating functions involving e^z naturally produces incomplete gamma integrals. The number of permutations and partitions with restricted cycle structure often reduces to such integrals after contour deformation.

6.46–6.47 The function $\psi(x)$

The digamma function $\psi(x) = \Gamma'(x)/\Gamma(x) = d \ln \Gamma(x)/dx$ and the higher polygamma functions $\psi^{(n)}(x) = d^{n+1} \ln \Gamma(x)/dx^{n+1}$.

Physics applications.

1. **Renormalisation constants in QFT.** In dimensional regularisation, $\Gamma(\varepsilon) = 1/\varepsilon - \gamma_E + O(\varepsilon)$, where the Euler–Mascheroni constant is $\gamma_E = -\psi(1)$. More generally, expanding $\Gamma(n+\varepsilon)$ about integer n yields $\psi(n)$ and $\psi^{(k)}(n)$ in the finite parts of renormalised Green functions. These digamma values appear explicitly in the running of the fine-structure constant $\alpha(\mu)$ through the one-loop photon self-energy.

2. **Feynman diagram evaluation.** Feynman parameter integrals evaluate to linear combinations of $\psi(p/q)$ at rational arguments, which by Gauss's digamma theorem reduce to elementary functions [Cof05]. Polygamma values $\psi^{(n)}(1/2)$, $\psi^{(n)}(1/3)$, etc. appear at two-loop and three-loop order in the Standard Model.

3. **The Casimir effect and zeta regularisation.** The derivative $\partial_a \zeta'(0, a) = \psi(a)$ connects the digamma function to the Hurwitz zeta function. The Epstein zeta function for rectangular cavities involves polygamma functions in its Laurent expansion.

4. **Harmonic sums in QCD.** At positive integers, $\psi(n + 1) = H_n - \gamma_E$, where $H_n = \sum_{k=1}^{n} 1/k$ is the n-th harmonic number. The nested harmonic sums $S_{a_1,a_2,\ldots}(n)$ that appear in DGLAP splitting functions and anomalous dimensions at higher loop orders are expressible in terms of polygamma functions and multiple polylogarithms.

5. **Fisher information and information geometry.** For a Gamma(α, θ) distribution, the Fisher information has $I_{\alpha\alpha} = \psi^{(1)}(\alpha)$ (the trigamma function). The natural gradient in parameter estimation [Ama98] uses the inverse Fisher information metric, making the trigamma function central to efficient optimisation of gamma-family models in machine learning and Bayesian statistics.

6. **Maximum likelihood estimation for the Gamma distribution.** The MLE equation for the shape parameter α requires solving the transcendental equation $\psi(\hat{\alpha}) - \ln\hat{\alpha} = \overline{\ln x} - \ln\bar{x}$. This appears throughout survival analysis, hydrology, queuing theory, and insurance mathematics.

Mathematics applications.

1. **Summation of rational series.** Any convergent series $\sum P(n)/Q(n)$ with $\deg Q > \deg P + 1$ evaluates as a finite linear combination of ψ and $\psi^{(n)}$ at the roots of Q, via partial fractions and the identity $\psi(z+1) - \psi(z) = 1/z$.

2. **The Hurwitz zeta function.** The identity $\psi^{(m)}(z) = (-1)^{m+1}\, m!\, \zeta(m+1, z)$ for $m \geq 1$ connects G&R 6.46–6.47 to the entire theory of Hurwitz and Lerch zeta functions. At rational arguments, Gauss's digamma theorem and the Hurwitz formula give closed-form evaluations involving $\ln(2\pi)$, $\pi \cot$, and $\pi \csc$ terms.

3. **Bernoulli numbers and asymptotic expansions.** $\psi(z) \sim \ln z - 1/(2z) - \sum_{k=1}^{\infty} B_{2k}/(2k\, z^{2k})$ for large $|z|$. These expansions are essential for numerical computation and govern the large-order behaviour of perturbation series in quantum mechanics and QFT.

4. **Gauss's digamma theorem and arithmetic.** Special values $\psi(p/q)$ at rational arguments are connected to class numbers of imaginary quadratic fields through the Chowla–Selberg formula, linking the integrals of G&R 6.46–6.47 to deep algebraic number theory.

6.5–6.7 Bessel Functions

6.51 Bessel functions

6.52 Bessel functions combined with x and x^2

6.53–6.54 Combinations of Bessel functions and rational functions

6.55 Combinations of Bessel functions and algebraic functions

6.56–6.58 Combinations of Bessel functions and powers

6.59 Combinations of powers and Bessel functions of more complicated arguments

6.61 Combinations of Bessel functions and exponentials

6.62–6.63 Combinations of Bessel functions, exponentials, and powers

6.64 Combinations of Bessel functions of more complicated arguments, exponentials, and powers

6.65 Combinations of Bessel and exponential functions of more complicated arguments and powers

6.66 Combinations of Bessel, hyperbolic, and exponential functions

6.67–6.68 Combinations of Bessel and trigonometric functions

6.69–6.74 Combinations of Bessel and trigonometric functions and powers

6.75 Combinations of Bessel, trigonometric, and exponential functions and powers

6.76 Combinations of Bessel, trigonometric, and hyperbolic functions

6.77 Combinations of Bessel functions and the logarithm, or arctangent

6.78 Combinations of Bessel and other special functions

6.79 Integration of Bessel functions with respect to the order

6.8 Functions Generated by Bessel Functions

6.81 Struve functions

6.82 Combinations of Struve functions, exponentials, and powers

6.83 Combinations of Struve and trigonometric functions

6.84–6.85 Combinations of Struve and Bessel functions

6.86 Lommel functions

6.87 Thomson functions

6.9 Mathieu Functions

6.91 Mathieu functions

6.92 Combinations of Mathieu, hyperbolic, and trigonometric functions

6.93 Combinations of Mathieu and Bessel functions

6.94 Relationships between eigenfunctions of the Helmholtz equation in different coordinate systems

7.1–7.2 Associated Legendre Functions

7.11 Associated Legendre functions

7.12–7.13 Combinations of associated Legendre functions and powers

7.14 Combinations of associated Legendre functions, exponentials, and powers

7.15 Combinations of associated Legendre and hyperbolic functions

7.16 Combinations of associated Legendre functions, powers, and trigonometric functions

7.17 A combination of an associated Legendre function and the probability integral

7.18 Combinations of associated Legendre and Bessel functions

7.19 Combinations of associated Legendre functions and functions generated by Bessel functions

7.21 Integration of associated Legendre functions with respect to the order

7.22 Combinations of Legendre polynomials, rational functions, and algebraic functions

7.23 Combinations of Legendre polynomials and powers

7.24 Combinations of Legendre polynomials and other elementary functions

7.25 Combinations of Legendre polynomials and Bessel functions

7.3–7.4 Orthogonal Polynomials

7.31 Combinations of Gegenbauer polynomials $C_n^\nu(x)$ and powers

7.32 Combinations of Gegenbauer polynomials $C_n^\nu(x)$ and elementary functions

7.325* Complete System of Orthogonal Step Functions

7.33 Combinations of the polynomials $C_n^\nu(x)$ and Bessel functions; Integration of Gegenbauer functions with respect to the index

7.34 Combinations of Chebyshev polynomials and powers

7.35 Combinations of Chebyshev polynomials and elementary functions

7.36 Combinations of Chebyshev polynomials and Bessel functions

7.37–7.38 Hermite polynomials

7.39 Jacobi polynomials

7.41–7.42 Laguerre polynomials

7.5 Hypergeometric Functions

7.51 Combinations of hypergeometric functions and powers

7.52 Combinations of hypergeometric functions and exponentials

7.53 Hypergeometric and trigonometric functions

7.54 Combinations of hypergeometric and Bessel functions

7.6 Confluent Hypergeometric Functions

7.61 Combinations of confluent hypergeometric functions and powers

7.62–7.63 Combinations of confluent hypergeometric functions and exponentials

7.64 Combinations of confluent hypergeometric and trigonometric functions

7.65 Combinations of confluent hypergeometric functions and Bessel functions

7.66 Combinations of confluent hypergeometric functions, Bessel functions, and powers

7.67 Combinations of confluent hypergeometric functions, Bessel functions, exponentials, and powers

7.68 Combinations of confluent hypergeometric functions and other special functions

7.69 Integration of confluent hypergeometric functions with respect to the index

7.7 Parabolic Cylinder Functions

7.71 Parabolic cylinder functions

7.72 Combinations of parabolic cylinder functions, powers, and exponentials

7.73 Combinations of parabolic cylinder and hyperbolic functions

7.74 Combinations of parabolic cylinder and trigonometric functions

7.75 Combinations of parabolic cylinder and Bessel functions

7.76 Combinations of parabolic cylinder functions and confluent hypergeometric functions

7.77 Integration of a parabolic cylinder function with respect to the index

7.8 Meijer's and MacRobert's Functions (G and E)

7.81 Combinations of the functions G and E and the elementary functions

7.82 Combinations of the functions G and E and Bessel functions

7.83 Combinations of the functions G and E and other special functions

8–9 Special Functions

8.1 Elliptic Integrals and Functions

8.11 Elliptic integrals

8.12 Functional relations between elliptic integrals

8.13 Elliptic functions

8.14 Jacobian elliptic functions

8.15 Properties of Jacobian elliptic functions and functional relationships between them

8.16 The Weierstrass function $\wp(u)$

8.17 The functions $\zeta(u)$ and $\sigma(u)$

8.18–8.19 Theta functions

8.2 The Exponential Integral Function and Functions Generated by It

8.21 The exponential integral function $\mathrm{Ei}(x)$

8.22 The hyperbolic sine integral $\mathrm{shi}\, x$ and the hyperbolic cosine integral $\mathrm{chi}\, x$

8.23 The sine integral and the cosine integral: $\mathrm{si}\, x$ and $\mathrm{ci}\, x$

8.24 The logarithm integral $\mathrm{li}(x)$

8.25 The probability integral $\Phi(x)$, the Fresnel integrals $S(x)$ and $C(x)$, the error function $\mathrm{erf}(x)$, and the complementary error function $\mathrm{erfc}(x)$

8.26 Lobachevskiy's function $L(x)$

8.3 Euler's Integrals of the First and Second Kinds

8.31 The gamma function (Euler's integral of the second kind): $\Gamma(z)$

8.32 Representation of the gamma function as series and products

8.33 Functional relations involving the gamma function

8.34 The logarithm of the gamma function

8.35 The incomplete gamma function

8.36 The psi function $\psi(x)$

8.37 The function $\beta(x)$

8.38 The beta function (Euler's integral of the first kind): $\mathrm{B}(x, y)$

8.39 The incomplete beta function $\mathrm{B}_x(p, q)$

8.4–8.5 Bessel Functions and Functions Associated with Them

8.40 Definitions

8.41 Integral representations of the functions $J_\nu(z)$ and $N_\nu(z)$

8.42 Integral representations of the functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$

8.43 Integral representations of the functions $I_\nu(z)$ and $K_\nu(z)$

8.44 Series representation

8.45 Asymptotic expansions of Bessel functions

8.46 Bessel functions of order equal to an integer plus one-half

8.47–8.48 Functional relations

8.49 Differential equations leading to Bessel functions

8.51–8.52 Series of Bessel functions

8.53 Expansion in products of Bessel functions

8.54 The zeros of Bessel functions

8.55 Struve functions

8.56 Thomson functions and their generalizations

8.57 Lommel functions

8.58 Anger and Weber functions $J_\nu(z)$ and $\mathbf{E}_\nu(z)$

8.59 Neumann's and Schläfli's polynomials: $O_n(z)$ and $S_n(z)$

8.6 Mathieu Functions

8.60 Mathieu's equation

8.61 Periodic Mathieu functions

8.62 Recursion relations for the coefficients $A_{2r}^{(2n)}$, $A_{2r+1}^{(2n+1)}$, $B_{2r+1}^{(2n+1)}$, $B_{2r+2}^{(2n+2)}$

8.63 Mathieu functions with a purely imaginary argument

8.64 Non-periodic solutions of Mathieu's equation

8.65 Mathieu functions for negative q

8.66 Representation of Mathieu functions as series of Bessel functions

8.67 The general theory

8.7–8.8 Associated Legendre Functions

8.70 Introduction

8.71 Integral representations

8.72 Asymptotic series for large values of $|\nu|$

8.73–8.74 Functional relations

8.75 Special cases and particular values

8.76 Derivatives with respect to the order

8.77 Series representation

8.78 The zeros of associated Legendre functions

8.79 Series of associated Legendre functions

8.81 Associated Legendre functions with integer indices

8.82–8.83 Legendre functions

8.84 Conical functions

8.85 Toroidal functions

8.9 Orthogonal Polynomials

8.90 Introduction

8.91 Legendre polynomials

8.919 Series of products of Legendre and Chebyshev polynomials

8.92 Series of Legendre polynomials

8.93 Gegenbauer polynomials $C_n^\lambda(t)$

8.94 The Chebyshev polynomials $T_n(x)$ and $U_n(x)$

8.95 The Hermite polynomials $H_n(x)$

8.96 Jacobi's polynomials

8.97 The Laguerre polynomials

9.1 Hypergeometric Functions

9.10 Definition

9.11 Integral representations

9.12 Representation of elementary functions in terms of a hypergeometric functions

9.13 Transformation formulas and the analytic continuation of functions defined by hypergeometric series

9.14 A generalized hypergeometric series

9.15 The hypergeometric differential equation

9.16 Riemann's differential equation

9.17 Representing the solutions to certain second-order differential equations using a Riemann scheme

9.18 Hypergeometric functions of two variables

9.19 A hypergeometric function of several variables

9.2 Confluent Hypergeometric Functions

9.20 Introduction

9.21 The functions $\Phi(\alpha, \gamma; z)$ and $\Psi(\alpha, \gamma; z)$

9.22–9.23 The Whittaker functions $M_{\lambda,\mu}(z)$ and $W_{\lambda,\mu}(z)$

9.24–9.25 Parabolic cylinder functions $D_p(z)$

9.26 Confluent hypergeometric series of two variables

9.3 Meijer's G-Function

9.30 Definition

9.31 Functional relations

9.32 A differential equation for the G-function

9.33 Series of G-functions

9.34 Connections with other special functions

9.4 MacRobert's E-Function

9.41 Representation by means of multiple integrals

9.42 Functional relations

9.5 Riemann's Zeta Functions $\zeta(z, q)$ and $\zeta(z)$, and the Functions $\Phi(z, s, v)$ and $\xi(s)$

9.51 Definition and integral representations

9.52 Representation as a series or as an infinite product

9.53 Functional relations

9.54 Singular points and zeros

9.55 The Lerch function $\Phi(z, s, v)$

9.56 The function $\xi(s)$

9.6 Bernoulli Numbers and Polynomials, Euler Numbers

9.61 Bernoulli numbers

9.62 Bernoulli polynomials

9.63 Euler numbers

9.64 The functions $\nu(x)$, $\nu(x, \alpha)$, $\mu(x, \beta)$, $\mu(x, \beta, \alpha)$, and $\lambda(x, y)$

9.65 Euler polynomials

9.7 Constants

9.71 Bernoulli numbers

9.72 Euler numbers

9.73 Euler's and Catalan's constants

9.74 Stirling numbers

10 Vector Field Theory

10.1–10.8 Vectors, Vector Operators, and Integral Theorems

10.11 Products of vectors

10.12 Properties of scalar product

10.13 Properties of vector product

10.14 Differentiation of vectors

10.21 Operators grad, div, and curl

10.31 Properties of the operator ∇

10.41 Solenoidal fields

10.51–10.61 Orthogonal curvilinear coordinates

10.71–10.72 Vector integral theorems

10.81 Integral rate of change theorems

11 Algebraic Inequalities

11.1–11.3 General Algebraic Inequalities

11.11 Algebraic inequalities involving real numbers

11.21 Algebraic inequalities involving complex numbers

11.31 Inequalities for sets of complex numbers

12 Integral Inequalities

12.11 Mean Value Theorems

12.111 First mean value theorem

12.112 Second mean value theorem

12.113 First mean value theorem for infinite integrals

12.114 Second mean value theorem for infinite integrals

12.21 Differentiation of Definite Integral Containing a Parameter

12.211 Differentiation when limits are finite

12.212 Differentiation when a limit is infinite

12.31 Integral Inequalities

12.311 Cauchy-Schwarz-Buniakowsky inequality for integrals

12.312 Hölder's inequality for integrals

12.313 Minkowski's inequality for integrals

12.314 Chebyshev's inequality for integrals

12.315 Young's inequality for integrals

12.316 Steffensen's inequality for integrals

12.317 Gram's inequality for integrals

12.318 Ostrowski's inequality for integrals

12.41 Convexity and Jensen's Inequality

12.411 Jensen's inequality

12.412 Carleman's inequality for integrals

12.51 Fourier Series and Related Inequalities

12.511 Riemann-Lebesgue lemma

12.512 Dirichlet lemma

12.513 Parseval's theorem for trigonometric Fourier series

12.514 Integral representation of the $n^{\mathbf{th}}$ partial sum

12.515 Generalized Fourier series

12.516 Bessel's inequality for generalized Fourier series

12.517 Parseval's theorem for generalized Fourier series

13 Matrices and Related Results

13.11–13.12 Special Matrices

13.111 Diagonal matrix

13.112 Identity matrix and null matrix

13.113 Reducible and irreducible matrices

13.114 Equivalent matrices

13.115 Transpose of a matrix

13.116 Adjoint matrix

13.117 Inverse matrix

13.118 Trace of a matrix

13.119 Symmetric matrix

13.120 Skew-symmetric matrix

13.121 Triangular matrices

13.122 Orthogonal matrices

13.123 Hermitian transpose of a matrix

13.124 Hermitian matrix

13.125 Unitary matrix

13.126 Eigenvalues and eigenvectors

13.127 Nilpotent matrix

13.128 Idempotent matrix

13.129 Positive definite

13.130 Non-negative definite

13.131 Diagonally dominant

13.21 Quadratic Forms

13.211 Sylvester's law of inertia

13.212 Rank

13.213 Signature

13.214 Positive definite and semidefinite quadratic form

13.215 Basic theorems on quadratic forms

13.31 Differentiation of Matrices

13.41 The Matrix Exponential

13.411 Basic properties

14 Determinants

14.11 Expansion of Second- and Third-Order Determinants

14.12 Basic Properties

14.13 Minors and Cofactors of a Determinant

14.14 Principal Minors

14.15* Laplace Expansion of a Determinant

14.16 Jacobi's Theorem

14.17 Hadamard's Theorem

14.18 Hadamard's Inequality

14.21 Cramer's Rule

14.31 Some Special Determinants

14.311 Vandermonde's determinant (alternant)

14.312 Circulants

14.313 Jacobian determinant

14.314 Hessian determinants

14.315 Wronskian determinants

14.316 Properties

14.317 Gram-Kowalewski theorem on linear dependence

15 Norms

15.1–15.9 Vector Norms

15.11 General Properties

15.21 Principal Vector Norms

15.211 The norm $\|\mathbf{x}\|_1$

15.212 The norm $\|\mathbf{x}\|_2$ (Euclidean or L_2 norm)

15.213 The norm $\|\mathbf{x}\|_\infty$

15.31 Matrix Norms

15.311 General properties

15.312 Induced norms

15.313 Natural norm of unit matrix

15.41 Principal Natural Norms

15.411 Maximum absolute column sum norm

15.412 Spectral norm

15.413 Maximum absolute row sum norm

15.51 Spectral Radius of a Square Matrix

15.511 Inequalities concerning matrix norms and the spectral radius

15.512 Deductions from Gerschgorin's theorem (see 15.814)

15.61 Inequalities Involving Eigenvalues of Matrices

15.611 Cayley-Hamilton theorem

15.612 Corollaries

15.71 Inequalities for the Characteristic Polynomial

15.711 Named and unnamed inequalities

15.712 Parodi's theorem

15.713 Corollary of Brauer's theorem

15.714 Ballieu's theorem

15.715 Routh-Hurwitz theorem

15.81–15.82 Named Theorems on Eigenvalues

15.811 Schur's inequalities

15.812 Sturmian separation theorem

15.813 Poincaré's separation theorem

15.814 Gerschgorin's theorem

15.815 Brauer's theorem

15.816 Perron's theorem

15.817 Frobenius theorem

15.818 Perron–Frobenius theorem

15.819 Wielandt's theorem

15.820 Ostrowski's theorem

15.821 First theorem due to Lyapunov

15.822 Second theorem due to Lyapunov

15.823 Hermitian matrices and diophantine relations involving circular functions of rational angles due to Calogero and Perelomov

15.91 Variational Principles

15.911 Rayleigh quotient

15.912 Basic theorems

16 Ordinary Differential Equations

16.1–16.9 Results Relating to the Solution of Ordinary Differential Equations

16.11 First-Order Equations

16.111 Solution of a first-order equation

16.112 Cauchy problem

16.113 Approximate solution to an equation

16.114 Lipschitz continuity of a function

16.21 Fundamental Inequalities and Related Results

16.211 Gronwall's lemma

16.212 Comparison of approximate solutions of a differential equation

16.31 First-Order Systems

16.311 Solution of a system of equations

16.312 Cauchy problem for a system

16.313 Approximate solution to a system

16.314 Lipschitz continuity of a vector

16.315 Comparison of approximate solutions of a system

16.316 First-order linear differential equation

16.317 Linear systems of differential equations

16.41 Some Special Types of Elementary Differential Equations

16.411 Variables separable

16.412 Exact differential equations

16.413 Conditions for an exact equation

16.414 Homogeneous differential equations

16.51 Second-Order Equations

16.511 Adjoint and self-adjoint equations

16.512 Abel's identity

16.513 Lagrange identity

16.514 The Riccati equation

16.515 Solutions of the Riccati equation

16.516 Solution of a second-order linear differential equation

16.61–16.62 Oscillation and Non-Oscillation Theorems for Second-Order Equations

16.611 First basic comparison theorem

16.622 Second basic comparison theorem

16.623 Interlacing of zeros

16.624 Sturm separation theorem

16.625 Sturm comparison theorem

16.626 Szegö's comparison theorem

16.627 Picone's identity

16.628 Sturm-Picone theorem

16.629 Oscillation on the half line

16.71 Two Related Comparison Theorems

16.711 Theorem 1

16.712 Theorem 2

16.81–16.82 Non-Oscillatory Solutions

16.811 Kneser's non-oscillation theorem

16.822 Comparison theorem for non-oscillation

16.823 Necessary and sufficient conditions for non-oscillation

16.91 Some Growth Estimates for Solutions of Second-Order Equations

16.911 Strictly increasing and decreasing solutions

16.912 General result on dominant and subdominant solutions

16.913 Estimate of dominant solution

16.914 A theorem due to Lyapunov

16.92 Boundedness Theorems

16.921 All solutions of the equation

16.922 If all solutions of the equation

16.923 If $a(x) \to \infty$ monotonically as $x \to \infty$, then all solutions of

16.924 Consider the equation

16.93 Growth of maxima of $|y|$

17 Fourier, Laplace, and Mellin Transforms

17.1–17.4 Integral Transforms

17.11 Laplace transform

17.12 Basic properties of the Laplace transform

17.13 Table of Laplace transform pairs

17.21 Fourier transform

17.22 Basic properties of the Fourier transform

17.23 Table of Fourier transform pairs

17.24 Table of Fourier transform pairs for spherically symmetric functions

17.31 Fourier sine and cosine transforms

17.32 Basic properties of the Fourier sine and cosine transforms

17.33 Table of Fourier sine transforms

17.34 Table of Fourier cosine transforms

17.35 Relationships between transforms

17.41 Mellin transform

17.42 Basic properties of the Mellin transform

17.43 Table of Mellin transforms

18 The z-Transform

18.1–18.3 Definition, Bilateral, and Unilateral z-Transforms

18.1 Definitions

18.2 Bilateral z-transform

18.3 Unilateral z-transform

Bibliography

[tV72] Gerard 't Hooft and Martinus J. G. Veltman. "Regularization and renormalization of gauge fields". In: *Nuclear Physics B* 44.1 (1972), pp. 189–213. DOI: 10.1016/0550-3213(72)90279-9 (cit. on p. 12).

[AG99] Mohamed-Slim Alouini and Andrea J. Goldsmith. "A unified approach for calculating error rates of linearly modulated signals over generalized fading channels". In: *IEEE Transactions on Communications* 47.9 (1999), pp. 1324–1334. DOI: 10.1109/26.789668 (cit. on p. 17).

[Ama98] Shun-ichi Amari. "Natural gradient works efficiently in learning". In: *Neural Computation* 10.2 (1998), pp. 251–276. DOI: 10.1162/089976698300017746 (cit. on p. 18).

[BG72] Carlos G. Bollini and Juan José Giambiagi. "Dimensional renormalization: the number of dimensions as a regularizing parameter". In: *Il Nuovo Cimento B* 12.1 (1972), pp. 20–36. DOI: 10.1007/BF02895558 (cit. on p. 12).

[Cha60] Subrahmanyan Chandrasekhar. *Radiative Transfer*. Reprint of the 1950 Oxford edition. New York: Dover, 1960 (cit. on p. 17).

[Cof05] Mark W. Coffey. "On one-dimensional digamma and polygamma series related to the evaluation of Feynman diagrams". In: *Journal of Computational and Applied Mathematics* 183.1 (2005), pp. 84–100. DOI: 10.1016/j.cam.2005.01.003 (cit. on p. 18).

[FS09] Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics*. Cambridge University Press, 2009. DOI: 10.1017/CBO9780511801655 (cit. on p. 18).

[GM10] Ivan Gonzalez and Victor H. Moll. "Definite integrals by the method of brackets. Part 1". In: *Advances in Applied Mathematics* 45.1 (2010), pp. 50–73. DOI: 10.1016/j.aam.2009.11.003 (cit. on p. 14).

[Gut90] Martin C. Gutzwiller. *Chaos in Classical and Quantum Mechanics*. New York: Springer-Verlag, 1990. DOI: 10.1007/978-1-4612-0983-6 (cit. on p. 15).

[Har20] Godfrey Harold Hardy. "Some formulae in the theory of Bessel functions". In: *Proceedings of the London Mathematical Society* s2-23.1 (1920). Contains the rigorous proof of Ramanujan's Master Theorem, pp. 61–63 (cit. on pp. 14, 15).

[Haw75] Stephen W. Hawking. "Particle creation by black holes". In: *Communications in Mathematical Physics* 43.3 (1975), pp. 199–220. DOI: 10.1007/BF02345020 (cit. on p. 14).

[MD63] Madan Lal Mehta and Freeman J. Dyson. "Statistical theory of the energy levels of complex systems. V". In: *Journal of Mathematical Physics* 4.5 (1963), pp. 713–719. DOI: 10.1063/1.1704009 (cit. on p. 13).

[OPS88] Brad Osgood, Ralph Phillips, and Peter Sarnak. "Extremals of determinants of Laplacians". In: *Journal of Functional Analysis* 80.1 (1988), pp. 148–211. DOI: 10.1016/0022-1236(88)90070-5 (cit. on p. 16).

[Raa43] Joseph Ludwig Raabe. "Angenäherte Bestimmung der Factorenfolge $1 \cdot 2 \cdot 3 \cdots n = \Gamma(1+n) = \int x^n e^{-x}\,dx$, wenn n eine sehr grosse Zahl ist". In: *Journal für die reine und angewandte Mathematik* 25 (1843), pp. 146–159 (cit. on p. 16).

[Reg59] Tullio Regge. "Introduction to complex orbital momenta". In: *Il Nuovo Cimento* 14.5 (1959), pp. 951–976. DOI: 10.1007/BF02728177 (cit. on p. 15).

[Sch51] Julian Schwinger. "On gauge invariance and vacuum polarization". In: *Physical Review* 82.5 (1951), pp. 664–679. DOI: 10.1103/PhysRev.82.664 (cit. on p. 14).

[Sel44] Atle Selberg. "Remarks on a multiple integral". In: *Norsk Matematisk Tidsskrift* 26 (1944), pp. 71–78 (cit. on p. 13).

[Tem79] Nico M. Temme. "The asymptotic expansion of the incomplete gamma functions". In: *SIAM Journal on Mathematical Analysis* 10.4 (1979), pp. 757–766. DOI: 10.1137/0510071 (cit. on p. 18).

[Tem96] Nico M. Temme. *Special Functions: An Introduction to the Classical Functions of Mathematical Physics*. New York: Wiley, 1996 (cit. on p. 18).

[Var88] Ilan Vardi. "Determinants of Laplacians and multiple gamma functions". In: *SIAM Journal on Mathematical Analysis* 19.2 (1988), pp. 493–507. DOI: 10.1137/0519035 (cit. on p. 16).

[Ven68] Gabriele Veneziano. "Construction of a crossing-symmetric, Regge-behaved amplitude for linearly rising trajectories". In: *Il Nuovo Cimento A* 57.1 (1968), pp. 190–197. DOI: 10.1007/BF02824451 (cit. on p. 13).

Index