MAT300 Spring 2021 Homework 5

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Problem 1

Show that for every $n \in \mathbb{N}, \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Solution:

Proof. Using mathematical induction.

Base case: Setting $n = 1$, it is seen that $1^2 = \frac{1((1)+1)(2(1)+1)}{6} = 1$ as required.

Induction step: Let n be an arbitrary natural number and suppose that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$. Then,

$$\begin{aligned} 1^2 + 2^2 + ... + (n+1)^2 &= (1^2 + 2^2 + ... + n^2) + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \\ &= \frac{2n^3 + 9n^2 + 13n + 6}{6} \\ &= \frac{(n+1)(n+2)(2n+3)}{6} \end{aligned}$$

□

Problem 2

Show that for every $n \in \mathbb{N}, \sum_{i=0}^{n}(-\frac{1}{2})^i = \frac{2^{n+1}+(-1)^n}{3 \cdot 2^n}$.

Solution:

Proof. Using mathematical induction.

Base case: Setting $n = 0$, it is seen that $(-\frac{1}{2})^0 = \frac{2^1+(-1)^0}{3\cdot 2^0} = 1$ as required.

Induction step: Let n be an arbitrary natural number and suppose $\sum_{i=0}^{n}(-\frac{1}{2})^i = \frac{2^{n+1}+(-1)^n}{3\cdot 2^n}$. Then,

$$\begin{aligned}
\left(-\frac{1}{2}\right)^0 + \left(-\frac{1}{2}\right)^1 + \dots + \left(-\frac{1}{2}\right)^{n+1} &= \left(\left(-\frac{1}{2}\right)^0 + \left(-\frac{1}{2}\right)^1 + \dots + \left(-\frac{1}{2}\right)^n\right) + \left(-\frac{1}{2}\right)^{n+1} \\
&= \frac{2^{n+1}+(-1)^n}{3\cdot 2^n} + \left(-\frac{1}{2}\right)^{n+1} \\
&= \frac{(2)^{n+1}\left(2^{n+1}+(-1)^n\right)}{3(2)^n(2)^{n+1}} + \frac{3(2)^n(-1)^{n+1}}{3(2)^n(2)^{n+1}} \\
&= \frac{(2)^n 2^{n+2} + 2^{n+1}(-1)^n - 3(2)^n(-1)^n}{3(2)^n 2^{n+1}} \\
&= \frac{(2)^n 2^{n+2} + (2(2^n) - 3(2)^n)(-1)^n}{3(2)^n 2^{n+1}} \\
&= \frac{(2)^n 2^{n+2} + 2^n(2-3)(-1)^n}{3(2)^n 2^{n+1}} \\
&= \frac{2^{n+2}+(-1)^{n+1}}{3(2)^{n+1}}
\end{aligned}$$

□

Problem 3

Show that for every $n \in \mathbb{N}, 9|(4^n + 6n - 1)$.

Proof. Using mathematical induction.

Base case: If $n = 0$, then $4^n + 6n - 1 = 0 = 9 \cdot 0$, so $9|(4^n + 6n - 1)$.

Induction step: Let n be an arbitrary natural number and suppose $9|(4^n + 6n - 1)$. Then $9k = 4^n + 6n - 1$ for some integer k. Thus,

$$\begin{aligned}
4^{n+1} + 6(n+1) - 1 &= 4(4)^n + 6n + 5 \\
&= 4(4)^n + 24n - 4 - 18n + 9 \\
&= 4(4^n + 6n - 1) - 18n + 9 \\
&= 4(9k) - 18n + 9 \\
&= 9(4k - 2n + 1)
\end{aligned}$$

Therefore $9|(4^{n+1} + 6(n+1) - 1)$, as required. □

Problem 4

Show that for all non-negative integers $n, 3|(2^{2n} - 1)$.

Proof. Using mathematical induction.

Base case: If $n = 0$, then $2^0 - 1 = 0 = 3 \cdot 0$, so $3|(2^{2n} - 1)$.

Induction step: Let n be an non-negative integer and suppose $3|(2^{2n} - 1)$. Then $3k = 2^{2n} - 1$ for some integer k. Thus,

$$\begin{aligned} 2^{2n+2} - 1 &= 4(2)^{2n} - 1 \\ &= 4(2)^{2n} - 4 + 3 \\ &= 4(2^{2n} - 1) + 3 \\ &= 4(3k) + 3 \\ &= 3(4k + 1) \end{aligned}$$

Therefore $3|(2^{2n+2} - 1)$, as required. □

Problem 5

For all non-negative integers $n, 101|(10^{2n} - (-1)^n)$.

Proof. Using mathematical induction.

Base case: If $n = 0$, then $10^0 - (-1)^0 = 0 = 101 \cdot 0$, so $101|(10^{2n} - (-1)^n)$.

Induction step: Let n be an non-negative integer and suppose $101|(10^{2n} - (-1)^n)$. Then $101k = 10^{2n} - (-1)^n$ for some integer k. Thus,

$$\begin{aligned} 10^{2n+2} - (-1)^{n+1} &= 100(10)^{2n} + (-1)^n \\ &= 100(10)^{2n} - 100(-1)^n + 101(-1)^n \\ &= 100(10^{2n} - (-1)^n) + 101(-1)^n \\ &= 100(101k) + 101(-1)^n \\ &= 101(100k + (-1)^n) \end{aligned}$$

Therefore $101|(10^{2n+2} - (-1)^{n+1})$, as required. □

Problem 6

For all non-negative integers $n, n^2 \leq 3^n$.

Proof. Using mathematical induction.

Base case: When $n = 0$, it is seen that $n^2 = 0 \leq 1 = 3^n$.

Induction step: Let n be an arbitrary non-negative integer and suppose $n^2 \leq 3^n$ Then,

$$\begin{aligned}
(n+1)^2 &= n^2 + 2n + 1 \\
&\leq n^2 + n^2 + n^2 + 1 \\
&= 3(n)^2 + 1 \\
&\leq 3(3)^n && \text{(inductive hypothesis)} \\
&= 3^{n+1}.
\end{aligned}$$

□

Problem 7

Show that for $n \in \mathbb{Z}^+$ if $n \geq 5$ then $(n+1)^2 \leq n!$.

Proof. Using mathematical induction.

Base case: When $n = 5$, it is seen that $(5+1)^2 = 36 \leq 120 = 5!$.

Induction step: Let $n \geq 5$ be arbitrary, and suppose $(n+1)^2 \leq n!$. Then,

$$\begin{aligned}
(n+2)^2 &= (n+1)^2 + 2n + 3 \\
&\leq n! + 2n + 3 && \text{(inductive hypothesis)} \\
&\leq (n+1)n! && \text{(since } n \geq 5\text{)} \\
&= (n+1)!
\end{aligned}$$

□

Problem 8

Show that for every $n \in \mathbb{N}$ and $h > -1, 1 + nh \leq (1+h)^n$.

Proof. Using mathematical induction.

Base case: When $n = 0$ and $h = 0$, it is seen that $1 + (0)(0) = 1 \leq 1 = (1 + (0))^0$.

Induction step: Let n be an arbitrary natural number, and suppose $1 + nh \leq (1+h)^n$. Then proving

$1 + (n+1)h \leq (1+h)^{n+1}$,

$$\begin{aligned}
1 + (n+1)h &\leq 1 + (n+1)h + nh^2 && \text{(since } h > -1\text{)} \\
&= (1+nh)(1+h) \\
&\leq (1+h)^n(1+h) && \text{(inductive hypothesis)} \\
&= (1+h)^{n+1}
\end{aligned}$$

□

Problem 9

Use mathematical induction to show that for $n \in \mathbb{Z}^+$, the number of 3-element subsets of the set $\{1, ..., n\}$ is $n(n-1)(n-2)/6$.

Proof. Since $n = 1, 2$ yields no possible 3-element subsets, $n = 3$ will be the base case. Let $n = 3$, then the number of 3-element subsets is $1 = 1 = 3(3-1)(3-2)/6$.

Let $\{1, ..., n\}$ be an arbitrary set, and suppose $\frac{n(n-1)(n-2)}{6}$. Then proving the number of 3-element subsets of the set with an additional element requires two cases, where the first case is when $n+1$ is an element of the 3-element subset and the second case being when $n+1$ is not an element of the 3-element subset.

Case 1: Suppose $n+1$ is an element of the 3-element subset. Thus there exists two other elements in $\{1, ..., n\}$.

Case 2: Suppose $n+1$ is not an element of the 3-element subset. Then there must exists 3 elements in $\{1, ..., n\}$ which makes up the 3-element subset. □