Homework 4

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Problem 1:

Prove that for every integer n, n^3 is odd if and only if n is odd.

Proof. 1) n^3 is odd $\to$ n is odd

Indirectly. Suppose n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$. Therefore $n^3 = (2k)^3 = 8k^3 = 2(4k^3)$ and $4k^3 \in \mathbb{Z}$. Thus n^3 is even.

2) n is odd $\to$ n^3 is odd

Suppose n is odd. Then $n = 2k+1$ for some $k \in \mathbb{Z}$. Therefore $n^3 = (2k+1)^2 = 8k^3 + 6k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1$ and $4k^3 + 6k^2 + 3k \in \mathbb{Z}$. Thus n^3 is odd. $\square$

Problem 2:

Prove that for every two integers m, n, $m + n$ is even if and only if $3m^2 + n^3$ is even.

Proof. **1)** $m + n$ is even $\to$ $3m^2 + n^3$ is even.

(Case 1) Suppose m is even and n is even.

Then it could be said that $m = 2k$ and $n = 2j$ for some $k, j \in \mathbb{Z}$. Then $3(2k)^2 + (2j)^3$ and then $12k + 8j$ which can be written: $2(6k+4j)$ which is even.

(Case 2) Suppose m is odd and n is odd.

Then it could be said that $m = 2k + 1$ and $n = 2j + 1$ for some $k, j \in \mathbb{Z}$. Then $3(2k+1)^2 + (2j+1)^3$ and then $12k^2 + 12k + 8j^3 + 12j^2 + 6j + 4$ which can be written: $2(6k^2 + 6k + 4j^3 + 6j^2 + 3j + 2)$ which is even.

2) $3m^2 + n^3$ is even $\to$ $m + n$ is even.

Indirectly. Suppose $m + n$ is odd.

(Case 1) Suppose m is odd and n is even

Then it could be said that $m = 2k + 1$ and $n = 2j$ for some $k, j \in \mathbb{Z}$. Then $3(2k+1)^2 + (2j)^3$ and then $12k^2 + 12k + 8j^3 + 3$ which can be written: $2(6k^2 + 6k + 4j^3 + 1) + 1$ which is odd.

(Case 2) Suppose m is even and n is odd

Then it could be said that $m = 2k$ and $n = 2j + 1$ for some $k, j \in \mathbb{Z}$. Then $3(2k)^2 + (2j+1)^3$ and then $12k^2 + 12k + 8j^3 + 12j^2 + 6j + 1$ which can be written:

$2(6k^2 + 6k + 4j^3 + 6j^2 + 3j) + 1$ which is odd. $\square$

Problem 3:

Prove that for every $x \in \mathbb{R}$, $|x - 4| > 4$, then $x^2 > 8x$.

Proof. Indirectly. Suppose $x^2 \leq 8x$. This implies $0 \leq x^2 \leq 8x$ and by dividing by x, the equation becomes: $0 \leq x \leq 8$ so that adding -4 to all sides gives $-4 \leq x - 4 \leq 4$ which is the same as $|x - 4| \leq 4$. $\square$

Problem 4:

(a) Prove that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$

Proof. Suppose $x \in \mathcal{P}(A) \cup \mathcal{P}(B)$ then $x \subseteq A$ or $x \subseteq B$. Which can be written as $x \subseteq A \cup B$ which implies $x \in \mathcal{P}(A \cup B)$. Thus, since $x \in \mathcal{P}(A) \cup \mathcal{P}(B)$ and $x \in \mathcal{P}(A \cup B)$, $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. $\square$

(b) Prove that $P(A \cup B) = P(A) \cup P(B)$ if and only if either $A \subseteq B$ or $B \subseteq A$.

Proof. Indirectly. Suppose $A \nsubseteq B$ and $B \nsubseteq A$. There there exists $a \in A \setminus B$ and $b \in B \setminus A$.

Let $x = \{a, b\}$.

- Then $x \subseteq A \cup B$ and so $x \in \mathcal{P}(A \cup B)$.
- Then $x \notin \mathcal{P}(A)$ because $b \in B \setminus A$.
- Then $x \notin \mathcal{P}(B)$ because $a \in A \setminus B$.
- Thus $x \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

Thus $\mathcal{P}(A \cup B) \neq \mathcal{P}(A) \cup \mathcal{P}(B)$. $\square$

Problem 5:

Let A, B be sets. Prove that $A \setminus (A \setminus B) = A \cap B$.

Proof. Suppose $y, x \in A$ and $z, x \in B$ but $y \notin B$ and $z \notin A$. Then $x \in (A \setminus (A \setminus B))$ but $y, z \notin (A \setminus (A \setminus B))$. So it can be said that for every element in A that is not in B will not be in $A \setminus (A \setminus B)$ and vice versa for B, but for every element in both A and B, it will also be in $A \setminus (A \setminus B)$ which fullfils the definition of $A \cap B$. Thus $A \setminus (A \setminus B) = A \cap B$. $\square$

Problem 6:

Prove that for any integers $a,\ b,\ c$ if $a|b$ or $a|c$ then $a|(c \cdot b)$.

Proof. 1) Suppose $a|b \rightarrow a|(c \cdot b)$.

Then $a(i) = b$ for some $i \in \mathbb{Z}$. Then suppose each side is multiplied by some integer c such that $a(i)c = (b \cdot c)$. Since $i, c \in \mathbb{Z}$, it can be written: $a|(b \cdot c)$. So if $a|b$ then $a|(c \cdot b)$.

2) Suppose $a|c \rightarrow a|(c \cdot b)$.

Then $a(j) = c$ for some $j \in \mathbb{Z}$. Then suppose each side is multiplied by some integer b such that $a(j)b = (b \cdot c)$. Since $j, c \in \mathbb{Z}$, it can be written: $a|(b \cdot c)$. So if $a|c$ then $a|(c \cdot b)$.

3) Suppose $a|b$ and $a|c \rightarrow a|(c \cdot b)$

Then $a(i) = b$ and $a(j) = c$ for some $i, j \in \mathbb{Z}$. Then $b \cdot c = a(i) \cdot a(j)$ and so $b \cdot c = a^2ij$. Since i, j are integers, $a^2|(b \cdot c)$ which implies $a|(b \cdot c)$. So if $a|b$ or $a|c$ then $a|(c \cdot b)$. $\square$

Problem 7:

Prove that for every integer n, $15|n$ if and only if $3|n$ and $5|n$.

Proof. 1) $15|n \rightarrow 3|n$ and $5|n$

Suppose $15|n$ then $15(k) = n$ for some $k \in \mathbb{Z}$. And since $15 = 3 \cdot 5$, then $3(5)k = n$, since $5k$ is an integer it can be subsitute for $j \in \mathbb{Z}$. Then $3j = n$ which implies $3|n$. Subsequently, $3k$ can be subsituted for $i \in \mathbb{Z}$ since $3k$ is an integer, thus giving $5i = n$ which implies $5|n$. So if $15|n$ then $3|n$ and $5|n$.

2) $3|n$ and $5|n \rightarrow 15|n$

Suppose $3|n$ and $5|n$ then $3j = n$ and $5i = n$ for some $j, i \in \mathbb{Z}$. Multiply $3j = n$ and $5i = n$ together to get $15ij = n^2$. Since $j, i \in \mathbb{Z}$ they can be subsituted for $k \in \mathbb{Z}$ to get $15k = n^2$ which is $15|n^2$ which implies $15|n$. And so if $3|n$ and $5|n$ then $15|n$. $\square$

Problem 8:

Show that $\sqrt{3}$ is irrational.

Proof. By contradiction.

Suppose $\sqrt{3}$ is rational. Then $\sqrt{3} = \frac{p}{q}$ where p and q are integers, share no common factors, and $q \neq 0$. Then $3 = \frac{p^2}{q^2}$ and so $3q^2 = p^2$. So $3|p^2$ which implies $3|p$ and $3k = p$ where $k \in \mathbb{Z}$ Thus $3q^2 = (3k)^2$ then $q^2 = 3k$ so $3|q^2$ which implies $3|q$. And so $q = 3j$ for some $j \in \mathbb{Z}$.

This contradicts our statement that p and q share no common factors. So, there are no integers p and q such that $\sqrt{3}$ is rational by contradiction. Thus, $\sqrt{3}$ must be irrational. □

Problem 9:

Let U be a nonempty set. Show that there is a unique set $A \in \mathcal{P}(U)$ such that for every set $B \in \mathcal{P}(U)$, $A \cap B = B$.

Proof. Existence: Since both $A, B \in \mathcal{P}(U)$, it can be stated that $\forall B\ (U \cap B = B)$, so U has the required property.

Uniqueness: Suppose there are arbitrary sets $C, D \in \mathcal{P}(U)$ and $\forall B\ (C \cap B = B)$ and $\forall B\ (D \cap B = B)$. Applying the first assumption to D it is seen that $C \cap D = D$, and applying the second to C, we get $D \cap C = C$. Clearly, $C \cap D = D \cap C$, so $C = D$. □

Problem 10:

Prove that if $\lim_{x \to a} f(x) = C$ and $\lim_{x \to a} g(x) = D$, then $\lim_{x \to a}(f(x) + g(x)) = C + D$.

Proof. Suppose $\lim_{x \to a} f(x) = C$ and $\lim_{x \to a} g(x) = D$.

Let $\epsilon_1 > 0$ and $\delta_1 > 0$. Let $x \in \mathbb{R}$ and if $0 < |x - a| < \delta_1$ then $|f(x) - C| < \epsilon_1$. Let $\epsilon_2 > 0$ and $\delta_2 > 0$. Let $x \in \mathbb{R}$ and if $0 < |x - a| < \delta_2$ then $|g(x) - D| < \epsilon_2$. Consider ϵ_1 and ϵ_2 to be $\frac{\epsilon}{2}$ and δ to be smaller than δ_1 and δ_2 so that if $0 < |x - a| < \delta$ then $|f(x) - C| < \frac{\epsilon}{2}$ and if $0 < |x - a| < \delta$ then $|g(x) - D| < \frac{\epsilon}{2}$.

After adding both inequalities together, if $0 < |x - a| < \delta$ then $|f(x) - C| + |g(x) - D| < 2 \cdot \frac{\epsilon}{2} = \epsilon$.

Because $|f(x) - C|$ and $|g(x) - D| \geq |f(x) - C + g(x) - D|$, it can be written: if $0 < |x - a| < \delta$ then $|f(x) - C + g(x) - D| \leq |f(x) - C| + |g(x) - D| < \epsilon$, and so if $0 < |x - a| < \delta$ then $|f(x) - C + g(x) - D| < \epsilon$.

Then if $0 < |x - a| < \delta$ then $|(f(x) + g(x)) - (D + C)| < \epsilon$ which is the definition of $\lim_{x \to a}(f(x) + g(x)) = C + D$. So if $\lim_{x \to a} f(x) = C$ and $\lim_{x \to a} g(x) = D$, then $\lim_{x \to a}(f(x) + g(x)) = C + D$. □