Homework 2

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Problem 1:

(a) $(\forall_x\, x < 2) \to (\exists_x x > 2)$

The truth value for this statement is false since x can be a value less than 2.

(b) $(\forall_x\, x < 0) \longleftrightarrow (\exists_x x^2 + 2 < 0)$

The truth value for this statement is false because there doesn't exist an x where $\exists_x x^2 + 2 < 0$ would be true. Thus making $(\forall_x\, x < 0) \to (\exists_x x^2 + 2 < 0) \wedge (\exists_x x^2 + 2 < 0) \to (\forall_x\, x < 0)$ false.

(c) $\forall_x (x < 0 \to x^2 = 1)$

The truth value for this statement is false since x can be any value less than 0 and where $x^2 \neq 1$ such as $x = -2$

(d) $\exists_x (x < 0 \to x^2 = 1)$

The truth value for this statement is true since $x = -1$ would make this statement true.

Problem 2:

(a) $\forall_x \exists_y\, x + y = 5$

The truth value for this statement is true. Consider $y_x = 5 - x$, thus $y \in \mathbb{Z}$ and $x + y_x = x + 5 - x = 5$.

(b) $\exists_x \forall_y\, x + y = 5$

The truth value for this statement is false because for a certain x, not all y would make $x + y = 5$ true. Consider $y_x = 6 - x$, then $y_x \in \mathbb{Z}$ and $x + y_x = 6 \neq 5$.

(c) $\forall_x \exists_y\, (x + y = 5 \to x^2 + y^3 > 7)$

The truth value for this statement is true because the only way to make this statement false is to make $x^2 + y^3 > 7$ false and $x + y = 5$ true. But any values that would make $x^2 + y^3 > 7$ false would make $x + y = 5$ false thus making the conditional true.

(d) $\exists_x \forall_y\, (x + y = 5 \to xy \geq 0)$

The truth value for this statement is false. Consider $y = 5 - x$, then $y \in \mathbb{Z}$ and $x + y = x + 5 - x = 5$ and $xy = 5x - x^2 \geq 0$ which is false for $x > 5$. Thus this conditional can be false for some value of x and all values of y.

Problem 3:

See above.

Problem 4:

(a)

$\forall_x \exists_y (|x - y| < 1)$

(b)

(c)

(d)

$(x > 1) \wedge \neg \exists_y \exists_z (x = yz \wedge y < x \wedge z < x)$

Problem 5:

(a)

$\exists_{x \in \mathbb{Z}} \forall_{y \in \mathbb{Z}} (x^2 \geq y)$

(b)

$\exists_{x \in \mathbb{Z}} \forall_{y \in \mathbb{Z}} (x < y \wedge xy \geq 1)$

(c)

$\forall_{x \in \mathbb{R}} \exists_{v \in \mathbb{Z}} ((x + v = 1 \wedge v \leq x) \vee (x + v \neq 1 \wedge v > x))$

Problem 6:

The statement $\exists_x P(x)$ says there exists at least one value of x for which $P(x)$ is true. Thus the negation of $\exists_x P(x)$ says there doesn't exists a value of x for which $P(x)$ is true. Which can be also stated as for all values of x, $P(x)$ is false. Which is the same as $\forall_x (\neg P(x))$. Thus $\neg \exists_x P(x) \equiv \forall_x (\neg P(x))$.

Problem 7:

The statement $\exists_x (P(x) \rightarrow Q(x))$ says there exists at least one value of x such that $P(x)$ implies $Q(x)$. This statement is further defined as, there exists at least one value of x such that either $P(x)$ is false or $Q(x)$ is true. Since the conditional is defined as an or statement, the statement could be phrased so that it only relies on $Q(x)$ to make the conditional true. Thus $\exists_x (P(x) \rightarrow Q(x)) \equiv \forall_x P(x) \rightarrow \exists_x Q(x)$.

Problem 8:

The statement $\exists! x P(x)$ says there exists a unique x such that P(x) is true. This statement could be restated as there exists a x and no other x exists such that $P(x)$ is true.
This can be written as $\exists_x P(x) \wedge$ (no other x exists such that $P(x)$ is true). Taking the negation of the right hand side would give the statement, $\neg$(other x exists such that $P(x)$ is true). Other x can be replaced with y where x $\neq$ y.
This leaves us with $\exists_x P(x) \wedge \neg$ (y exists such that $P(y)$ and $x \neq y$). This can be further defined using quantifiers.
$\exists_x P(x) \wedge \neg(\exists_y P(y) \wedge x \neq y)$
$\exists_x P(x) \wedge (\forall_y \neg P(y) \vee x = y)$
We can apply that same process to $\exists! x Q(x)$. Leaving us with:
$(\exists_x P(x) \wedge (\forall_y \neg P(y) \vee x = y)) \rightarrow (\exists_x Q(x) \wedge (\forall_y \neg Q(y) \vee x = y))$