MAT300 Spring 2021 Homework 7

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Problem 1

Check if the following relations R on $\mathbb{Z}$ are functions from $\mathbb{Z}$ to $\mathbb{Z}$.

(a) aRb if $a \leq b$

This relation is not a function. Suppose $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ and $a \leq b$ then $a \leq b - 1$. Since $a \leq b$ and $a \leq b - 1$, this is not a function.

(b) aRb if $a^2 = b^2$

This relation is not a function. Suppose $a = 1$ then $b = 1, -1$. There is more than one $b \in \mathbb{Z}$ such that aRb.

(c) aRb if $\{a\} \subseteq \{b\}$

This relation is a function. Suppose $\{a\} \subseteq \{b\}$, then $a = b$ since $a, b \in \mathbb{Z}$.

Problem 2

Let A, B be two disjoint sets and let $f : A \to A \cup B,\ g : B \to A \cup B$ be functions. Show that $f \cup g$ is a function from $A \cup B$ to $A \cup B$.

Proof. Suppose A, B are disjoint sets and let $f : A \to A \cup B,\ g : B \to A \cup B$ be functions. Consider $a \in A$, then there exists $x \in A \cup B$ such that $(a, x) \in f$. Consider $b \in B$, then there exists $y \in A \cup B$ such that $(b, y) \in g$. Consider $f \cup g$, then $(a, x), (b, y) \in f \cup g$ and $a \neq b$ since A, B are disjoint. And $a, b \in A \cup B$ and $x, y \in A \cup B$. Thus $f \cup g$ is a function from $A \cup B$ to $A \cup B$. $\square$

Problem 3

Which of the following functions are injective and which are surjective (justify).

(a) $f : \mathbb{N} \to \mathbb{N},\ f(n) = \lceil \frac{n+2}{3} \rceil$

This function is not injective. Consider $n = 2$ then $f(2) = 2$, now consider $n = 3$ then $f(3) = 2$ so $f(3) = f(2)$ so this function is not injective.

This function is not surjective since $\forall_{n\in\mathbb{N}}(f(n) > 0)$.

(b) $f : \mathbb{Z} \to \mathbb{N},\ f(n) = n^2$

This function is not injective. Consider $n = -1$ then $f(-1) = 1$, now consider $n = 1$ then $f(1) = 1$ so $f(-1) = f(1)$, thus this function is not injective.

This function is not surjective since $\forall_{n\in\mathbb{Z}}(f(n) \neq 2)$.

(c) $f : \mathbb{N} \to \mathbb{N},\ f(n) = |\lfloor \frac{n-1}{5} \rfloor - 1|$

This function is not injective. Suppose $f(n) = |\lfloor \frac{n-1}{5} \rfloor - 1|$ and consider $n = 0$, $f(0) = 1$. Now consider $n = 1$, $f(1) = 1$; and $1, 0 \in \mathbb{N}$. Since $f(0) = f(1)$ this function is not injective.

This function is surjective. Consider $b \in \mathbb{N}$ and $a = \pm 5b + 6$.

Case 1: $a = 5b + 6$.

$$\begin{aligned} f(a) &= |\lfloor \frac{a-1}{5} \rfloor - 1| \\ &= |\lfloor \frac{5b+6-1}{5} \rfloor - 1| \\ &= |\lfloor b+1 \rfloor - 1| \\ &= |b+1-1| && \text{(Since } b \in \mathbb{N}) \\ &= b \end{aligned}$$

Case 2: $a = -5b + 6$

$$\begin{aligned} f(a) &= |\lfloor \frac{a-1}{5} \rfloor - 1| \\ &= |\lfloor \frac{-5b+6-1}{5} \rfloor - 1| \\ &= |\lfloor -b+1 \rfloor - 1| \\ &= |-b+1-1| && \text{(Since } b \in \mathbb{N}) \\ &= b \end{aligned}$$

Problem 4

Give an example of a function from $\mathbb{Z}$ to $\mathbb{Z}$ which is injective but not surjective and an example of a function which is surjective but not injective.

Injective but not surjective.

$f(n) = 2n$

Surjective but not injective.

$f(n) = \lfloor \frac{n}{2} \rfloor$

Problem 5

Find a bijective function from $\mathbb{Z}$ to the set $\mathbb{Z} \setminus \{-1, 0, 1\}$ (Show that your example is indeed a bijection).

$$f(n) = \begin{cases} x + 2, & \text{if } n \geq 0 \\ -x - 1, & \text{if } n < 0 \end{cases}$$

This function is injective.

Case 1: $n \geq 0$. Let $x, y \in \mathbb{N}$ and suppose $f(x) = f(y)$. Then $x + 2 = y + 2$ so $x = y$.
Case 2: $n < 0$. Let $x, y \in \mathbb{N}$ and suppose $f(x) = f(y)$. Then $-x - 1 = -y - 1$ so $x = y$.

This function is surjective.

Case 1: $n \geq 0$. Let $y \in \mathbb{N}$ and suppose $x = y - 2$. Then $y - 2 + 2 = y$. Case 2: $n < 0$. Let $y \in \mathbb{N}$ and suppose $x = -y - 1$. Then $-(-y - 1) - 1 = y + 1 - 1 = y$.

Problem 6

Which of the following functions are injective and which are surjective (justify).

(a) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z},\ f((a, b)) = a^2 + b$

This function is not injective. Consider $a = 1, b = 0$ then $f((1, 0)) = 1$, now consider $a = 2, b = -3$ then $f((2, -3)) = 1$ and $(1, 0), (2, -3) \in \mathbb{Z} \times \mathbb{Z}$ so $f((2, -3)) = f((1, 0))$.

This function is surjective. Let $y \in \mathbb{Z}$ consider $x = (0, y)$. Then $f(x) = f((0, y)) = 0 + y = y$ and $(0, y) \in \mathbb{N} \times \mathbb{N}$.

(b) $f : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z},\ f(a) = (\lceil \frac{a}{2} \rceil,\ a)$

This function is injective. Suppose $f(x) = f(y)$ then $(\lceil \frac{x}{2} \rceil, x) = (\lceil \frac{x}{2} \rceil, y)$ Then $\lceil \frac{x}{2} \rceil = \lceil \frac{y}{2} \rceil$, so $x = y$. And looking at the second element of the pair, $x = y$.

This function is surjective. Let $y \in \mathbb{Z}$ and consider $x = y$ where $x \in \mathbb{Z}$. Then $f(x) = f(2y) = (\lceil \frac{y}{2} \rceil, y)$.

(c) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z},\ f((a, b)) = (a + b, 2a - b)$

This function is injective. Suppose $f(x, y) = f(c, d)$ then $(x+y, 2x-y) = (c+d, 2c-d)$. Then $x + y = c + d$ and $2x - y = 2c - d$. Adding these two functions gives $3x - y + y = 3c - d + d = 3x = 3c$ so $x = c$. Since $x = c$, $c + y = c + d = y = c - c + d$ so $y = d$. Thus $(x, y) = (c, d)$.

This function is not surjective. For $x, y \in \mathbb{Z}$ and suppose $x + y = 0$ then $2x - y = 2(-y) - y = -3y \neq 1$. So $(x + y, 2x - y) \neq (0, 1)$.

Problem 7

Let f, g be functions from A to A. Show that:

(a) If both f, g are injective then so is $f \circ g$.

Proof. Suppose f, g are injective. Let a_1 and a_2 be arbitrary elements of A and suppose that $(f \circ g)(a_1) = (f \circ g)(a_2)$. This means $f(g(a_1)) = f(g(a_2))$. Since f is injective, $g(a_1) = g(a_2)$ and similarly since g is injective, $a_1 = a_2$. Thus $f \circ g$ is injective. □

(b) If both f, g are surjective then so is $f \circ g$.

Proof. Suppose f, g are surjective and let a be an arbitrary element of A. Since f is surjective, there is some $b \in A$ such that $f(b) = a$. Similarly, since g is surjective, there is some $c \in A$ such that $f(c) = b$. Then $(f \circ g)(c) = f(g(c)) = f(b) = a$. Thus, $f \circ g$ is surjective. □