MAT300 Spring 2021 Homework 6

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Problem 1

Let $S = \{(1,2),(2,1),(3,3),(1,3),(3,2)\}$ and $R = \{(1,2),(2,2),(3,1),(1,3),(3,3)\}$ be relations on $\{1,2,3\}$. Find:

(a) $S \circ R^{-1}$

Solution: $\{(2,2),(2,3),(2,1),(1,3),(1,2),(3,2),(3,3),(3,3),(3,2)\}$

(b) $R \circ (R \circ R)$

Solution: $\{(1,2),(2,2),(3,2),(3,1),(3,3),(1,2),(1,3),(1,1),(1,3),(3,2),(3,3)\}$

Problem 2

Let R, S be relations on A. Show:

(a) If $R \subseteq S$, then $R^{-1} \subseteq S^{-1}$

Proof. Let $R \subseteq S$ and suppose there is an arbitrary ordered pair $(x,y) \in R$ and $x, y \in A$. By definition of the subset, this means that there exists a $(x,y) \in S$ such that $(x,y) \in R = (x,y) \in S$. Inverting the ordered pairs would give $(y,x) \in R^{-1}$ and $(y,x) \in S^{-1}$ which shows $R^{-1} \subseteq S^{-1}$. $\square$

(b) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

Proof. 1) Suppose an arbitrary ordered pair $(x, y) \in (R \cup S)^{-1}$ and $x, y \in A$. Then $(y, x) \in R \cup S$ and so $(y, x) \in R \vee (y, x) \in S$. The inverse would be $(x, y) \in R^{-1} \vee (x, y) \in S^{-1}$ thus $(x, y) \in R^{-1} \cup S^{-1}$.

2) Suppose an arbitrary order pair $(x, y) \in R^{-1} \cup S^{-1}$ and so $(x, y) \in R^{-1} \vee (x, y) \in S^{-1}$ which can be rewritten as $(y, x) \in R \vee (y, x) \in S$. So that $(y, x) \in (R \cup S)$; inverting the ordered pair shows $(x, y) \in (R \cup S)^{-1}$. □

Problem 3

Let R be a relation on A. Show that if R is transitive and symmetric, then so is R^{-1}:

Proof. Suppose R is transitive and symmetric, then $\forall_{x \in A} \forall_{y \in A}((x, y) \in R \to (y, x) \in R)$ and $\forall_{x \in A} \forall_{y \in A} \forall_{\mathbb{Z} \in A}(((x, y) \in R \wedge (y, z) \in R) \to (x, z) \in R)$.

Then the inverse would be $\forall_{x \in A} \forall_{y \in A}((y, x) \in R^{-1} \to (x, y) \in R^{-1})$ and $\forall_{x \in A} \forall_{y \in A} \forall_{\mathbb{Z} \in A}(((y, x) \in R^{-1} \wedge (z, y) \in R) \to (z, x) \in R^{-1})$. So it is seen that R^{-1} is also transitive and symmetric. □

Problem 4

Let R be the following relation on $\mathbb{Z}$, aRb if $a - b \leq 10$. Check if R is reflexive, symmetric, antisymmetric, and transitive.

(a) R is reflexive.

Let $a \in \mathbb{Z}$. Then $a - a = 0 \leq 10$. So aRa.

(b) R is not symmetric.

Let $a, b \in \mathbb{Z}$. And suppose aRb then there exists $b - a > 10$. Let $(1, 12) = (a, b)$ then $1 - 12 = -11 \leq 10$ but $12 - 1 = 11 \not\leq 10$.

(c) R is not antisymmetric.

Let $a, b \in \mathbb{Z}$ and suppose $(aRb \wedge bRa)$ and consider $(a, b) = (1, 2)$ then indeed $1 - 2 \leq 10 \wedge 2 - 1 \leq 10$ but $1 \neq 2$.

(d) R is not transitive.

Let $a, b, c \in \mathbb{Z}$ and suppose $(aRb \wedge bRc)$. Consider $a = 2, b = 1, c = -9$, then indeed $2 - 1 \leq 10 \wedge 1 - (-9) \leq 10$ but $2 - (-9) = 11 \not\leq 10$.

Problem 5

Let R be a relation on $\mathbb{Z} \times \mathbb{Z}$ defined as $(a, b)R(c, d)$ when $2|(a^2 + c)$ and $3|(b - d)$. Check if R is reflexive, symmetric, antisymmetric, and transitive.

(a) R is reflexive

Let $(a, b) \in \mathbb{Z} \times \mathbb{Z}$. Then $b - b = 0 \cdot 3; 3|(b - b)$. And proving $2|(a^2 + a)$:

Case 1: a is even

Let $a = 2j$ where $j \in \mathbb{Z}$. Then $a^2 + a = 4j^2 + 2j = 2(2j^2 + j); 2|(a^2 + a)$.

Case 2: a is odd

Let $a = 2j + 1$ where $j \in \mathbb{Z}$. Then

$$\begin{aligned} a^2 + a &= (2j + 1)^2 + 2j + 1 \\ &= (4j^2 + 4j + 1) + 2j + 1 \\ &= 4j^2 + 6j + 2 \\ &= 2(2j^2 + 3j + 1) \end{aligned}$$

So $2|(a^2 + a)$ and $3|(b - b)$ as required.

(b) R is symmetric

Let $(a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z}$. And suppose $a^2 + c = 2k$ and $b - d = 3j$ where $k, j \in \mathbb{Z}$. Then $d - b = 3(-j)$ and in proving $2|(c^2 + a)$, let $a = \sqrt{2k - c}$:

Case 1: c is even

Let $c = 2i$ where $i \in \mathbb{Z}$. Then:

$$\begin{aligned} c^2 + a &= c^2 + \sqrt{2k - c} \\ &= c^4 + 2k - c \\ &= 16i^4 + 2k - 2i \\ &= 2(8i^4 + k - i) \end{aligned}$$

Case 2: c is odd

Let $c = 2i + 1$ where $i \in \mathbb{Z}$. Then:

$$\begin{aligned} c^2 + a &= c^2 + \sqrt{2k - c} \\ &= c^4 + 2k - c \\ &= 16i^4 + 32i^3 + 24i^2 + 8i + 1 + 2k - 2i - 1 \\ &= 16i^4 + 32i^3 + 24i^2 + 6i + 2k \\ &= 2(8i^4 + 16i^3 + 12i^2 + 3i + k) \end{aligned}$$

So $3|(d - b)$ and $2|(c^2 + a)$ as required.

(c) R is not antisymmetric

Let $(a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z}$. And consider $(a, b) = (1, 5)$ and $(c, d) = (3, 2)$, then it is indeed true that $(1, 5)R(1, 2)$ and $(1, 2)R(1, 5)$ but $(3, 2) \neq (1, 5)$.

(d) R is transitive

Let $(a, b), (c, d), (e, f) \in \mathbb{Z} \times \mathbb{Z}$. And suppose $(a, b)R(c, d)$ and $(c, d)R(e, f)$, then $2|(a^2 + c)$ and $3|(b - d)$ and $2|(c^2 + e)$ and $3|(d - f)$.

Then $b - d = 3j$ and $d - f = 3k$ for some $j, k \in \mathbb{Z}$. Thus $b - d + d - f = 3j + 3k$, then $b - f = 3(j + k)$, so $3|(b - f)$ as required.

Then $a^2 + c = 2g$ and $c^2 + e = 2i$ for some $g, i \in \mathbb{Z}$. Then $c = \sqrt{2i - e}$ and so $a^2 + \sqrt{2i - e} = 2g$, which is the same as $a^4 + 2i - e = 4g^2$. For $2|(a^2 + c)$, both a and c have to be both even or both odd, the same is true for $2|(c^2 + e)$. And since a^4 will be odd when a is odd and even when a is even, $a^4 + 2i - e = 4g^2$ can be written as $a^2 - e = 2(2g^2 - i)$ or $2|a^2 - e$ as required.

Problem 6

Let R be a relation on $\mathbb{Z}^+ \times \mathbb{Z}^+$ defined as follows $(x, y)R(z, w)$ if $x|z$ and $y - w \geq 0$.

(a) Prove that R is a partial order but not a total order.

Proof. R is reflexive.

Let $(x, y), (z, w) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. Then $x = 1 \cdot x$ so $x|x$ and $y - y = 0 \geq 0$. So $(x, y)R(x, y)$ as required.

R is transitive.

Let $(x, y), (z, w), (a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. And suppose $(x, y)R(z, w)$ and $(z, w)R(a, b)$. So that

$$\begin{aligned} &x|z \\ &y - w \geq 0 \\ &z|a \\ &w - b \geq 0 \end{aligned}$$

It is seen that $y - w + w - b \geq 0$. Which can be written as $y - b \geq 0$ which satisfies one half of the relation.

Consider $x|z = x \cdot k = z$ and $z|a = z \cdot j = a$ for some $k, j \in \mathbb{Z}^+$. Thus $a = x \cdot k \cdot j$ and so $x|a$.

So $(x, y)R(a, b)$ as required.

R is antisymmetric

Let $(x, y), (z, w) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ and suppose $(x, y)R(z, w)$ and $(z, w)R(x, y)$. Thus $x \cdot k = z$ and $z \cdot j = x$ for some $k, j \in \mathbb{Z}^+$ and $y - w \geq 0$ and $w - y \geq 0$.

Then $z = j \cdot k \cdot z$ so $z|z$ and $x = j \cdot k \cdot x$ so $x|x$ so $x = z$. And $y - w + w - y \geq 0$ and $w - y + y - w \geq 0$ so $y = w$.

Thus $(x, y) = (z, w)$ as required.

There exists an x and y such that $(x, y)\not R(z, w)$ and $(z, w)\not R(x, y)$

Consider $(x, y) = (4, 2)$ and $(z, w) = (3, 5)$ then it can be seen that $4 \cdot j \neq 3$ for some $j \in \mathbb{Z}^+$ and $2 - 5 \not\geq 0$. And $3 \cdot k \neq 4$ for some $k \in \mathbb{Z}^+$.

□

Problem 7

Give an example of a partially ordered set which contains exactly one minimal element x such that x is not the smallest element.

Let set $G = \{(x, y) \mid (x = 2 \wedge y = 1) \vee (x = 1 \wedge y \in \mathbb{N})\}$ and let relation $R = \{((x_1, y_1), (x_2, y_2)) \mid (x_1 > x_2 \wedge y_1 = y_2 = 1) \vee (x_1 = x_2 = 1 \wedge y_1 > y_2) \rightarrow (x_1, y_1) < (x_2, y_2)\}$.