Homework 1

John J Li

2021 January 26

Problem 1:

(a) $((p \wedge q) \vee p) \wedge q$

$$
\begin{gathered}
((p \wedge q) \vee p) \wedge q \\
((p \vee p) \wedge (q \vee p)) \vee p \\
(\cancel{(p \vee p)} \wedge (q \vee p)) \vee p \\
q \wedge q \vee p \wedge q \\
\cancel{q \wedge q} \vee p \wedge q \\
p \wedge q
\end{gathered}
$$

Not a tautology because $p \wedge q$ can be false.

(b) $p \rightarrow (p \vee q)$

$$
\begin{gathered}
p \rightarrow (p \vee q) \\
\neg p \vee (p \vee q) \\
(\neg p \vee p) \vee (\neg p \vee q)
\end{gathered}
$$

This is a tautology because $\neg p \vee p$ is always true and so $(\neg p \vee p) \vee (\neg p \vee q)$ is always true as well.

(c) $(\neg p \vee q) \rightarrow (p \rightarrow q)$

$$
\begin{gathered}
\text{Let } A \equiv (p \rightarrow q) \\
(\neg p \vee q) \rightarrow (p \rightarrow q) \\
(p \rightarrow q) \rightarrow (p \rightarrow q) \\
A \rightarrow A
\end{gathered}
$$

This is a tautology because the statement is conditionally comparing the same things.

Problem 2:

(a) $x \in \mathbb{R}$ and if $x < 5$, then $x = 5$

$\boxed{\{x \in \mathbb{R} \mid x < 5 \rightarrow x = 5\} = \{x \geq 5\}}$

(b) $z \in \mathbb{R}$ and $2 \in \{x \in \mathbb{R} \mid x \cdot z < 2\}$

$\boxed{\{z \in \mathbb{R} \mid 2 \in \{x \in \mathbb{R} \mid x \cdot z < 2\}\} = \{z < 2\}}$

(c) $n \in \mathbb{Z}$ and $n \in \{2^i \mid i \in \mathbb{Z}^+\}$

$\boxed{\{0, 1, 2, 3...\}}$

Problem 3:

(a) $x \notin A \setminus (B \cup C)$

$$x \notin A \setminus (B \cup C)$$
$$x \notin A \wedge \neg(x \notin B \vee x \notin C)$$
$$x \notin A \wedge x \in B \wedge x \in C$$

$\boxed{x \notin A \wedge x \in B \wedge x \in C}$

(a) $x \in A \setminus (B \setminus C)$

$$x \in A \setminus (B \setminus C)$$
$$x \in A \wedge \neg(x \in B \wedge \neg(x \in C))$$
$$x \in A \wedge (x \notin B \vee x \in C)$$

$\boxed{x \in A \wedge (x \notin B \vee x \in C)}$

Problem 4:

(a) $\emptyset \subseteq \{\emptyset\}$

This statement is true.

(b) $(\{1, \emptyset\} \cap \{2, \emptyset\}) \subseteq \{1, 2\}$

This statement is false because $(\{1, \emptyset\} \cap \{2, \emptyset\}) \equiv \{\emptyset\}$ which is not a subset of $\{1, 2\}$.

Problem 5:

(a) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

$$
\begin{gathered}
A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C) \\
A \wedge \neg(B \vee C) = (A \wedge \neg B) \wedge (A \wedge \neg C) \\
\boxed{A \wedge \neg B \wedge \neg B = A \wedge \neg B \wedge \neg B}
\end{gathered}
$$

(b) $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$

$$
\begin{gathered}
(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B) \\
(A \wedge \neg B) \vee (B \wedge \neg A) = (A \vee B) \wedge \neg(A \wedge B) \\
(A \vee B) \wedge (A \vee \neg A) \wedge (\neg B \vee B) \wedge (\neg B \vee \neg A) = (A \vee B) \wedge (\neg A \vee \neg B) \\
(A \vee B) \wedge \cancel{(A \vee \neg A)} \wedge \cancel{(\neg B \vee B)} \wedge (\neg B \vee \neg A) = (A \vee B) \wedge (\neg A \vee \neg B) \\
\boxed{(A \vee B) \wedge (\neg A \vee \neg B) = (A \vee B) \wedge (\neg A \vee \neg B)}
\end{gathered}
$$

Problem 6:

$$
\begin{gathered}
(p \to q) \wedge (\neg p \to q) \equiv p \leftrightarrow q \\
(\neg p \vee q) \wedge (p \vee \neg q) \equiv (p \to q) \wedge (q \to p) \\
\boxed{(\neg p \vee q) \wedge (\neg q \vee p) \equiv (\neg p \vee q) \wedge (\neg q \vee p)}
\end{gathered}
$$

Problem 7:

(a) If $x + y < 2$ and $y + z = 2$, then $x^2 + y^2 + z^2 = 2$ or $x^2 \geq y^2$.

Converse:

$\boxed{\text{If } x^2 + y^2 + z^2 = 2 \text{ or } x^2 \geq y^2, \text{ then } x + y < 2 \text{ and } y + z = 2.}$

Contrapositive:

$\boxed{\text{If } x + y \leq 2 \text{ or } y + z \neq 2, \text{ then } x^2 + y^2 + z^2 \neq 2 \text{ and } x^2 < y^2}$

(b) If two integers x, y are both even, then at least one of $x^2 + y,\ y^2 + x$ is even.

Converse:

$\boxed{\text{If at least one of } x^2 + y,\ y^2 + x \text{ is true, then the two integers } x, y \text{ are both even.}}$

Contrapositive:

If two integers x, y are both odd, then both $x^2 + y,\ y^2 + x$ are odd.

Problem 8:

(a) $P(x) : x < 2 \rightarrow x > 2$

$\boxed{3}$

(b) $P(x) : x \in \mathbb{Z}^+ \rightarrow x = 0$

$\boxed{-1}$

(c) $P(x) : (x^2 + 1 = x) \rightleftarrows x < 2$

$\boxed{-1}$

Problem 9:

(a) X says "Both of us are knaves".

X is a knave while Y is a knight. This is because this statement is always false – knaves cannot tell the truth so we know X and Y are not both knaves. Therefore, X and Y could both be knights but since we know X is lying, we can conclude that Y is a knight.

(b) Y says "Either I am a knave or X is a knight"

X and Y can be both knights or Y is a knave and X is a knight.

(c) X says "I am a knave but Y isn't"

This statement is always false; we can automatically conclude that y is also a knave. So both X and Y are knaves – since knights can't say they are knaves.

Problem 10:

The number of words we use to define the smallest positive integer which cannot be described in fewer than sixteen words is 16. 16 is a subset of A since we can described 16 in fewer than sixteen words.