# Chapter 3 Notes

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# 1 Class Notes

Proof by contradiction:

The aim is to show that a statement $r$ is true. The method is to show that an implication $\neg r \to F$ is true for some statement F which is false.

## 1.1 Example

Let A,B,C be sets and $A \setminus B \in C$. Show that if $x \in A \setminus C$ then $x \in B$.

*Proof.* By contradiction: Suppose $x \in A \setminus C$ and $x \notin B$, then $x \in A$ and $x \notin C$ and $x \notin B$. Thus $x \in A$ and $x \notin B$ and $x \notin C$. Therefore $x \in A \setminus B$ and $x \notin C$. Since $A \setminus B \in C$, $x \in C$ and $x \notin C$, which is a contradiction. □

## 1.2 Example

Let $x, y \in \mathbb{R}$. Show that if $x^2 + y = 13$ and $y \neq 4$ then $x \neq 3$

*Proof.* Indirectly.

Suppose $x = 3$, we will show that $x^2 + y \neq 13$ or $y = 4$. Suppose $x^2 + y = 13$. Then $3^2 + y = 13$, $y = 4$.

Contradiction.

Suppose $x^2 + y = 13$ and $y \neq 4$ and $x = 3$. Then $9 + y = 13$ and $y \neq 4$. Therefore $y = 4$ and $y \neq 4$, which is a contradiction. □

## 1.3 Example

Let $n \in \mathbb{Z}$. Show that if n is even then $n^2$ is even.

*Proof.* Suppose n is even. Then $n = 2k$ for some integer $k$. Then $n^2 = 4k^2 = 2(2k^2)$ and $2k^2 \in \mathbb{Z}$. Therefore $n^2$ is even. □

### 1.3.1 Additional

Show that if $n^2$ is even then n is even.

*Proof.* Suppose n is not even. Since $n \in \mathbb{Z}$, n is odd.

Then $n = 2k + 1$ for some $k \in \mathbb{Z}$ and $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ and $2k^2 + 2k \in \mathbb{Z}$ Thus $n^2$ is odd. □

## 1.4 Example

$\sqrt{2}$ is irrational

*Proof.* By contradiction. Suppose $\sqrt{2}$ is rational. Then $\sqrt{2} = \frac{p}{q}$ for some integers p,q where $q \neq 0$ and $\frac{p}{q}$ is reduced (p,q have no common factors).

Then $\sqrt{2}\dot{q} = p$ and so $2q^2 = p^2$ (*). From (*) $p^2$ is even. Then, by the previous fact, p is even. Thus $p = 2k$ for some integer k.

Thus in (*) we have $2q^2 = (2k)^2$; $2q^2 = 4k^2$; $q^2 = 2k^2$. Therefore $q^2$ is even, and so q is even. And so p,q are both even and $\frac{p}{q}$ is reduced which is a contradiction. □

## 1.5 Example

Let A,B be sets. Show that if $A \wedge B = A$ then $A \in B$ $(A \in B \equiv \forall x\, x \in A, x \in B)$

*Proof.* Let $x \in A$. Since $A = A \cap B, x \in A \cap B$. Thus $x \in A$ and $x \in B$. In particular $x \in B$. □

## 1.6 Example

Let $x \in \mathbb{R}$. Show that if $x > 0$ then there is $y \in \mathbb{R}$ $y(y+1) = x$. $\forall x(x > 0 \rightarrow \exists y y(y+1) = x)$

*Proof.* Let $x > 0$. Consider $y = \frac{-1+\sqrt{1+4x}}{2}$ Then $y \in \mathbb{R}$ because $x > 0$ and $y(y+1) = (\frac{-1+\sqrt{1+4x}}{2})(\frac{-1+\sqrt{1+4x}}{2} + 1) =$ □