Analysis of Spacecraft Attitude and Inertia Properties in a Docking Scenario

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A docking scenario between the International Space Station and the SpaceX Dragon transport vehicle is modeled. The two spacecraft are analyzed separately at first, and their inertial properties and motion are determined. A rotation of the SpaceX Dragon docking vehicle is then considered to align the crafts. Upon completion of the docking scenario, the two spacecraft are considered as one combined body, and new inertial properties and equations of motion are calculated for the combined system. At various points of interest throughout the analysis the effects of Earth's gravity are taken into account. A numerical simulation of a particular scenario is performed and results are presented and discussed.

Nomenclature

G	=	Universal gravitational constant
ISS	=	International Space Station
$M_{\oplus}$	=	Mass of Earth
SXD	=	SpaceX Dragon Capsule

I. Introduction

TO meet the mission needs of the aerospace industry, spacecraft must often be designed to physically interact with other bodies, even other spacecraft. This allows for greater flexibility in mission planning, given that a spacecraft such as the International Space Station may be constantly resupplied and the crew exchanged through regular visits of a transport vehicle that can mate and de-mate from the station repeatedly. This functionality extends the mission life for spacecraft and eliminates the high cost of launching all supplies and crew with the craft itself. It can be, however, difficult to accurately position two vehicles for successful docking while in orbit, due to a number of factors that arise when dealing with space operations. For one, all spacecraft in orbit maintain a high velocity to preserve their orbit; space vehicles have a great deal of momentum, and bringing multiple such craft in close proximity to each other brings with it the risk of uncontrolled collision. Another hurdle in the successful docking of two spacecraft is the issue of controlling a vehicle remotely, often across great distances, with a great degree of precision. To accomplish a successful docking, a pilot needs accurate information about the relative motion between the two vehicles, and the controller of the receiving spacecraft needs to be able to anticipate the potential effects of the added mass on the stability of the combined system. In this analysis, investigation is performed into these quantities.

II. Discussion of Docking Scenario

Prior to docking, the receiving spacecraft and the docking spacecraft each have a unique inertia matrix and angular momentum properties. They also each have a unique orientation in space which may be defined by any of a number of sets of kinematic variables. To properly align the spacecraft for the scenario, the docking vehicle must undergo a rotation, represented by simple rotations through any number of its principal axes. This implies some angular momentum of the vehicle, which may be transferred to the receiving spacecraft upon docking if it is not completely eliminated. After docking, the two bodies are treated as one system with its own unique inertia matrix and angular momentum properties, with the condition that the total angular momentum of the system must be conserved.

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III. Spacecraft Attitude Dynamics – Concepts & Equations

A. Inertia Properties of Spacecraft in Orbit

The inertia matrix of a given body is represented as

$$I = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \tag{1}$$

which translates to an inertia dyad around the body's center of mass given by

$$\bar{\bar{I}}^{B}_{B^*} = I_1\hat{a}_1\hat{a}_1 + I_2\hat{a}_2\hat{a}_2 + I_3\hat{a}_3\hat{a}_3 \tag{2}$$

B. Equations of Motion for Spacecraft in Orbit

The rotation sequence used in this analysis will be the Body 3-1-3 sequence. This being the case, the angular velocity components of a rotating body relative to the inertial reference frame may be expressed in the body frame as

$$\bar{\omega} = \omega_i \cdot \hat{b}_i \tag{3}$$

where

$$\omega_1 = \dot{\theta}_1 \sin(\theta_2)\sin(\theta_3) + \dot{\theta}_2\cos(\theta_3) \tag{4}$$

$$\omega_2 = \dot{\theta}_1 \sin(\theta_2)\cos(\theta_3) - \dot{\theta}_2 \sin(\theta_3) \tag{5}$$

$$\omega_3 = \dot{\theta}_1\cos(\theta_2) + \dot{\theta}_3 \tag{6}$$

and θ_1, θ_2, and θ_3 represent the precession, nutation, and spin angles, respectively.

The kinematic variables used in this analysis will be the quaternions

$$\bar{\varepsilon} = \hat{\lambda}\sin\left(\frac{\Theta}{2}\right);\ \varepsilon_4 = \cos\left(\frac{\Theta}{2}\right)$$

where $\hat{\lambda}$ is the axis of rotation and Θ is the angle through which the body rotates about $\hat{\lambda}$.

The kinematic differential equations of motion of a body, then, are

$$\dot{\varepsilon}_1 = \frac{1}{2}(\omega_1\varepsilon_4 - \omega_2\varepsilon_3 + \omega_3\varepsilon_2) \tag{7}$$

$$\dot{\varepsilon}_2 = \frac{1}{2}(\omega_1\varepsilon_3 + \omega_2\varepsilon_4 - \omega_3\varepsilon_1) \tag{8}$$

$$\dot{\varepsilon}_3 = \frac{1}{2}(-\omega_1\varepsilon_2 + \omega_2\varepsilon_1 + \omega_3\varepsilon_4) \tag{9}$$

$$\dot{\varepsilon}_4 = -\frac{1}{2}(\omega_1\varepsilon_1 + \omega_2\varepsilon_2 + \omega_3\varepsilon_3) \tag{10}$$

C. Approximate Gravity Forces on a Spacecraft in Orbit

Given that this scenario involves spacecraft in orbit around the Earth such that their distance from the Earth's center is many times greater than their own length (R >> r), the force of gravity on each spacecraft may be given as the second-order approximation

$$\bar{F_g} = -\frac{GM_{\oplus}m}{R^2}\hat{a}_1 - \frac{3}{2}\left(\frac{GM_{\oplus}}{R^4}\right)\left[tr(\bar{\bar{I}}) - 5\hat{a}_1 \cdot \bar{\bar{I}} \cdot \hat{a}_1\right]\hat{a}_1 - \frac{3}{R^4}GM_{\oplus}\bar{\bar{I}} \cdot \hat{a}_1 \tag{11}$$

and the center of gravity of the spacecraft is located at a distance from the center of mass of the Earth

$$R_{cg} = \left(\frac{GmM_{\oplus}}{|\bar{F}|}\right)^{-\frac{1}{2}} \tag{12}$$

IV. Analytical Modeling of Docking Scenario

A. Defining Spacecraft Initial States

It is assumed in this scenario that the spacecraft are very near to each other, as this analysis will not cover any propulsion maneuvers needed to close the relative distance between them but will rather focus on the attitude dynamics of the problem.

The receiving spacecraft is said to have an inertia matrix[2]

$$I_{ISS} = \begin{bmatrix} I_{ISS,1} & 0 & 0 \\ 0 & I_{ISS,2} & 0 \\ 0 & 0 & I_{ISS,3} \end{bmatrix} = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 200 \end{bmatrix} \times 10^6 \; kg \cdot m^2$$

and the docking vehicle has an inertia matrix

$$I_{SXD} = \begin{bmatrix} I_{SXD,1} & 0 & 0 \\ 0 & I_{SXD,2} & 0 \\ 0 & 0 & I_{SXD,3} \end{bmatrix} = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 25 \end{bmatrix} \times 10^3 \; kg \cdot m^2$$

These values define each craft as an axisymmetric body, which simplifies the analysis greatly.

At the onset of the scenario, both spacecraft are said to maintain an attitude hold, so that their angular velocities are 0 in all directions and their precession, nutation, and spin angles are held constant. For the purposes of this analysis, however, the body-fixed dextral orthogonal triads of the receiving and docking spacecrafts are not aligned, so that a rotation is necessary to align the docking vehicle. The rotation matrix relating the body-fixed frames of the receiving and the docking spacecrafts, using the Body 3-1-3 sequence, is

$$^{ISS}C^{SXD} = \begin{bmatrix} -S_1C_2S_3 + C_3C_1 & -S_1C_2C_3 - S_3C_1 & S_1S_2 \\ C_1C_2S_3 + C_3S_1 & C_1C_2C_3 - S_3S_1 & -C_1S_2 \\ S_2S_3 & S_2C_3 & C_2 \end{bmatrix}$$

We now find the force of gravity on the receiving vehicle before docking. The mass of the ISS is 419,725 kg and is said to have an orbital height of 400 km for this scenario.[3] Additionally, we assume that the body-fixed unit vector $\mathbf{b}_1$ is parallel to the inertial frame vector $\mathbf{a}_1$ extending from the center of Earth through the center of mass of the ISS, so that the force of gravity is directed along this line and is not "pulled off" due to the orientation of the spacecraft. Using the approximation given by Equation 11, we find that the force of gravity on the ISS is about 9.155 x $10^{-9}GM_{\oplus}$

[2] Shi, J. and Ulrich, S., "Spacecraft Adaptive Attitude Control with Application to Space Station Free-Flyer Robotic Capture," *AIAA Guidance, Navigation, and Control Conference*, AIAA, Kissimmee, FL, 2015.
[3] Source: NASA

kg/km^{-2}. Using Equation 12, we find analytically that the center of gravity of the ISS is located a negligible distance of ~2.64 x 10^{-5} m towards Earth along the line $\mathbf{a_1}$ from the center of mass.

B. Rotation of Docking Vehicle to Align with Receiving Spacecraft

For the purposes of this analysis, the desired alignment of the docking vehicle is such that its body-fixed dextral orthogonal triad unit vectors are aligned with (parallel to) the body-frame unit vectors of the receiving spacecraft.

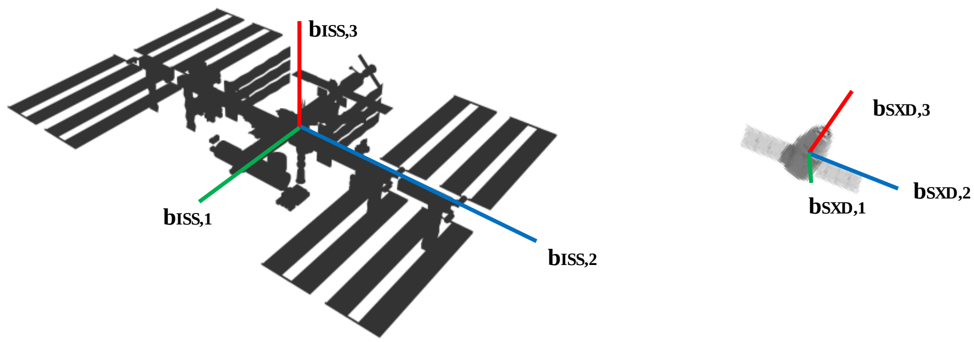

Figure 1. Body-fixed dextral orthogonal triads are not aligned.
Image credit: NASA, SpaceX

The Dragon capsule has a precession angle θ_1, a nutation angle θ_2, and a spin angle θ_3, that differentiate its body-fixed axes from those of the ISS. In order to align the two vehicles, the Dragon capsule will need to rotate through these angles, necessitating an applied torque to effect some angular velocity for some defined time. This angular velocity and its components are represented in Equations 3-6, and relate the motion of the docking spacecraft to the receiving spacecraft. Adapted for this scenario, we may rewrite these equations as

$$^{ISS}\overline{\omega}^{SXD} = \omega_i \cdot \hat{b}_{SXD,i} \tag{13}$$

where

$$\omega_1 = \dot{\theta}_1 \sin(\theta_2)\sin(\theta_3) + \dot{\theta}_2 \cos(\theta_3) \tag{14}$$

$$\omega_2 = \dot{\theta}_1 \sin(\theta_2)\cos(\theta_3) - \dot{\theta}_2 \sin(\theta_3) \tag{15}$$

$$\omega_3 = \dot{\theta}_1 \cos(\theta_2) + \dot{\theta}_3 \tag{16}$$

During this rotation, the Dragon capsule will have an angular momentum equal to

$$^{ISS}\overline{H}^{SXD} = \bar{\bar{I}}^{SXD}_{SXD*} \cdot \; ^{ISS}\overline{\omega}^{SXD} = I_{ij}\omega_k \hat{b}_i \delta_{jk} \tag{17}$$

where $\bar{\bar{I}}$ is the inertia dyad of the Dragon capsule, I is the inertia matrix (above written as I_{SXD}), and δ_{jk} is the Kronecker delta function for indices j, k.

C. Inertia Properties & Equations of Motion for Combined System

Once the two spacecraft are joined, we must rewrite the inertia equations and equations of motion to describe the new combined system.

For the purposes of this scenario and simplification of the analysis, the Dragon capsule is said to have docked to the ISS such that its center of mass is aligned along the (now shared) body-fixed vector $\mathbf{b_3}$ with the center of mass of the ISS. The vector $\mathbf{b}_{SXD,1}$ is parallel to $\mathbf{b}_{ISS,1}$ and the vector $\mathbf{b}_{SXD,2}$ is parallel to $\mathbf{b}_{ISS,2}$. Figure 2 illustrates this arrangement.

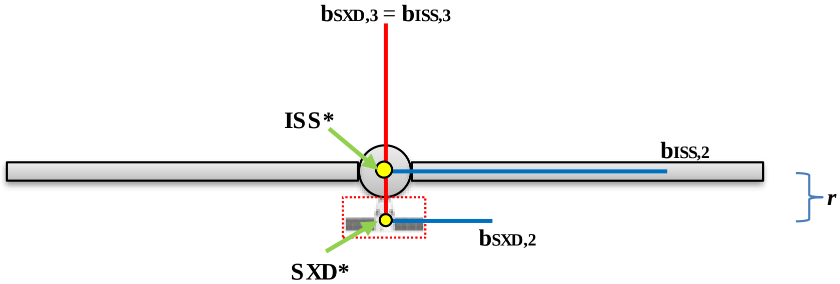

Figure 2. System after docking completion. The Dragon capsule's body-fixed vector triad is aligned with the ISS body-fixed vector triad. The respective centers of mass are indicated for illustrative purposes. Image credit: SpaceX

A new center of mass for the combined system must be located, a task made easier by the chosen docking location. Taking ISS* as the origin, denoting the distance between the two centers of mass as *r*, and mass-averaging, we find the location of the new center of mass is

$$y_{cm} = \frac{m_{SXD} r}{m_{SXD} + m_{ISS}} = 0.0141r \tag{18}$$

towards the Dragon capsule along $\mathbf{b}_3$.

To move the analysis forward, it is taken to be the case that this distance is negligible (even more so when taking into account the mass of the docking spacecraft compared to the mass of the receiving spacecraft) and that the center of mass of the combined system is essentially coincident with the center of mass of the ISS. This assumption allows for the simple addition of the two inertia dyads to find the resultant inertia dyad of the combined system *S* about its center of mass *S** (it is assumed that the body-fixed reference frame continues to align with the inertial reference frame)

$$\bar{\bar{I}}^{S}_{S^*} = (I_{ISS,1} + I_{SXD,1})\hat{n}_1\hat{n}_1 + (I_{ISS,2} + I_{SXD,2})\hat{n}_2\hat{n}_2 + (I_{ISS,3} + I_{SXD,3})\hat{n}_3\hat{n}_3 \tag{19}$$

It was taken to be the case that the ISS was in an attitude hold during the docking. The Dragon, however, had to rotate to position itself for docking, and it may still retain some rotational motion at the point of contact. Indeed, the margins of error for docking to the ISS, given the data[4] for the European cargo transfer vehicle ATV, are 5.0° of relative attitude error, and angular velocity errors of 0.40°/s (0.007 rad/s) for spin and 0.15°/s (0.0026 rad/s) in the transverse direction.

Thus we may investigate a transfer of angular momentum from the docking vehicle to the combined system. Given the absence of torques external to the two-spacecraft system, angular momentum must be conserved such that

$$\bar{H}_f^{\,SXD} + \bar{H}_f^{\,ISS} = \bar{H}_i^{\,SXD} + \bar{H}_i^{\,ISS} \tag{20}$$

which becomes

$$\bar{\bar{I}}^{S}_{S^*} \cdot \; {}^{B}\bar{\omega}^{B'} = \bar{I}^{SXD}_{SXD*} \cdot \; {}^{B}\bar{\omega}^{SXD} \tag{21}$$

where frame B is the equivalent to the initial body-fixed reference frame of the ISS and frame B' is the frame defined by the rotation of the combined system. We have $\bar{\bar{I}}^{S}_{S^*}$, $\bar{I}^{SXD}_{SXD*}$, and ${}^{ISS}\bar{\omega}^{SXD}$, so we may solve for the resultant angular

[4]Bourdon, J., Ganet-Schoeller, M., Delpy, P., and Ankersen, F., "Position Control Design and Validation Applied to ATV During Docking to ISS," *IFAC Automatic Control in Aerospace*, Saint Petersburg, Russia, 2004.

velocity of the combined system ${}^{B}\overline{\omega}^{B'}$, which the receiving spacecraft's attitude control system will have to account for and correct if it is to keep the station in proper orientation.

V. Further Analysis on System

A. Gravitational Moment on Combined System

It may be of interest to now analyze the behavior of the combined system as it continues its orbit around the Earth. We may define two new reference frames, in addition to the body-fixed frame $\hat{b}$ and the inertial frame $\hat{n}$: the orbit-fixed frame $\hat{a}$ and the non-physical frame $\hat{c}$. The orbit-fixed frame is useful in that it allows for the definition of the angular rate of the spacecraft around the Earth, given by

$$\Omega = \sqrt{\frac{\mu}{R}} \ ; \ \mu = GM_{\oplus} \tag{22}$$

The non-physical frame $\hat{c}$ provides a useful intermediate, a working frame to relate the body-fixed frame $\hat{b}$ to the orbit-fixed reference frame $\hat{a}$. It is at this point that another assumption will be taken: the system has a nonzero spin rate, defined as s, imparted by the docking vehicle. As $\hat{c}_3$ is equivalent to $\hat{b}_3$, the angular velocity ${}^{C}\overline{\omega}^{B} = s\hat{c}_3$

As for defining the system's inertial properties in the $\hat{c}$ frame,

$$I_k = \hat{c}_k \cdot \overline{\overline{I}}{}^{S}_{S^*} \cdot \hat{c}_k \tag{23}$$

Because the inertia dyad of the combined system reveals that it is still axisymmetric, we may express it as

$$I = \hat{c}_1 \cdot \overline{\overline{I}}{}^{S}_{S^*} \cdot \hat{c}_1 = \hat{c}_2 \cdot \overline{\overline{I}}{}^{S}_{S^*} \cdot \hat{c}_2$$

$$J = \hat{c}_3 \cdot \overline{\overline{I}}{}^{S}_{S^*} \cdot \hat{c}_3$$

We now must find the differential equations of motion for the combined system in orbit around Earth. The gravitational moment on the system is

$$\overline{M}^{B^*} = \frac{{}^{N}d\,{}^{N}\overline{H}{}^{S}_{S^*}}{dt} \tag{24}$$

Expressing $\frac{{}^{N}d\,{}^{N}\overline{H}{}^{B}_{B^*}}{dt}$ in $\hat{c}$ frame,

$$\frac{{}^{N}d\,{}^{N}\overline{H}{}^{S}_{S^*}}{dt} = \frac{{}^{C}d\,{}^{N}\overline{H}{}^{S}_{S^*}}{dt} + {}^{N}\overline{\omega}^{C} \times \overline{H} \tag{25}$$

where

$$\overline{H} = {}^{N}\overline{H}{}^{S}_{S^*} = \overline{\overline{I}}{}^{S}_{S^*} \cdot {}^{N}\overline{\omega}^{B} \tag{26}$$

and

$${}^{N}\overline{\omega}^{B} = \omega_1\hat{c}_1 + \omega_2\hat{c}_2 + \omega_3\hat{c}_3 = {}^{N}\overline{\omega}^{C} + {}^{C}\overline{\omega}^{B} \tag{27}$$

Solving these equations to express the derivative of angular momentum in the $\hat{c}$ frame,

$$\frac{{}^N d\, {}^N\bar{H}^{S/S^*}}{dt} = [I(\dot{\omega}_1 + s\omega_2) + (J - I)\omega_2\omega_3]\hat{c}_1 + [I(\dot{\omega}_2 - s\omega_1) - (J - I)\omega_1\omega_2]\hat{c}_2 + J\dot{\omega}_3\hat{c}_3 \tag{28}$$

Given the assumption

$$\bar{M}^{B^*} = \frac{3\mu}{R^3}\hat{a}_1 \times \bar{\bar{I}} \cdot \hat{a}_1 = 3\Omega^2\ \hat{a}_1 \times \bar{\bar{I}} \cdot \hat{a}_1 \tag{29}$$

Equation 24 expands to the dynamical differential equations

$$\dot{\omega}_1 = -s\omega_2 + \left(1 - \frac{J}{I}\right)[\omega_2\omega_3 - 12\Omega^2(\varepsilon_1\varepsilon_2 - \varepsilon_3\varepsilon_4)(\varepsilon_3\varepsilon_1 + \varepsilon_2\varepsilon_4)] \tag{30}$$

$$\dot{\omega}_2 = s\omega_1 - \left(1 - \frac{J}{I}\right)[\omega_1\omega_3 - 6\Omega^2(\varepsilon_3\varepsilon_1 + \varepsilon_2\varepsilon_4)(1 - 2\varepsilon_2^2 - 2\varepsilon_3^2)] \tag{31}$$

$$\underset{\text{(due to axisymmetry)}}{\dot{\omega}_3 = 0} \tag{32}$$

Due to the axisymmetric nature of the spacecraft about the spin axis $\hat{b}_3$ and the assumption that it is in a circular orbit around the Earth, the kinematical differential equations can be found to be

$$\dot{\varepsilon}_1 = \frac{1}{2}[\varepsilon_2(\omega_3 - s + \Omega) - \varepsilon_3\omega_2 + \varepsilon_4\omega_1] \tag{33}$$

$$\dot{\varepsilon}_2 = \frac{1}{2}[\varepsilon_3\omega_1 + \varepsilon_4\omega_2 - \varepsilon_1(\omega_3 - s + \Omega)] \tag{34}$$

$$\dot{\varepsilon}_3 = \frac{1}{2}[\varepsilon_4(\omega_3 - s - \Omega) + \varepsilon_1\omega_2 - \varepsilon_2\omega_1] \tag{35}$$

$$\dot{\varepsilon}_4 = \frac{1}{2}[-\varepsilon_1\omega_1 - \varepsilon_2\omega_2 - \varepsilon_3(\omega_3 - s - \Omega)] \tag{36}$$

B. Stability Analysis of Combined System

Of interest in any scenario regarding the motion of a spacecraft is the stability of the vehicle. Given a particular case of motion with selected initial conditions, we may conduct an analytical study of the stability of this given system of spacecraft.

To begin, we assume that for time t, the intermediate frame $\hat{c}_i$ is equivalent to the orbit-fixed frame $\hat{a}_i$, and that the system is aligned such that its angular velocity can be given as ${}^N\bar{\omega}^B = \omega_{3_0}\hat{a}_3$. We then exploit our ability to choose the spin rate of the system and set $s = \omega_{3_0} - \Omega$. Our dynamic and kinematic equations of motion thus become

$$\omega_1 = \omega_2 = 0\ ;\ \omega_3 = \omega_{3_0}$$
$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0\ ;\ \varepsilon_4 = 1$$

We now introduce a perturbation to this system to test its stability. This perturbation can be modeled as

$$\tilde{\varepsilon}, \tilde{\varepsilon}_4, \tilde{\omega}$$

so that it may be simply component-added to the defined nominal conditions above, and the system's equations of motion become

$$\bar{\varepsilon}_i = \tilde{\varepsilon}_i \text{ for i = 1, 2, 3}\ ;\ \varepsilon_4 = 1 + \tilde{\varepsilon}_4$$

$$\omega_j = \widetilde{\omega}_j \text{ for j = 1, 2 ; } \omega_3 = \omega_{3_0} + \widetilde{\omega}_3$$

Substituting these new nominal conditions into equations 30-36, and neglecting higher-order perturbation terms, we find new differential equations of motion for the perturbed system.

$$\dot{\tilde{\varepsilon}}_1 = \Omega\tilde{\varepsilon}_2 + \frac{\widetilde{\omega}_1}{2} \tag{37}$$

$$\dot{\tilde{\varepsilon}}_2 = \frac{\widetilde{\omega}_2}{2} - \Omega\tilde{\varepsilon}_1 \tag{38}$$

$$\dot{\tilde{\varepsilon}}_3 = \dot{\tilde{\varepsilon}}_4 = 0 \tag{39}$$

$$\dot{\widetilde{\omega}}_1 = -Q\Omega\widetilde{\omega}_2 \tag{40}$$

$$\dot{\widetilde{\omega}}_2 = Q\Omega\widetilde{\omega}_2 - 6x\Omega^2\tilde{\varepsilon}_2 \tag{41}$$

where

$$x = \frac{J}{I} - 1\,, y = \frac{\omega_{3_0}}{\Omega} - 1$$

and

$$Q = y + x(1 + y).$$

Thus the nonlinear system given in equations 30-36 has now been linearized, which allows for determination of the stability of the system through investigation of its eigenvalues. Two of these are known to be 0, given Equation 39. To obtain the others, we rewrite the remaining components of the system as a matrix multiplication $\dot{z} = zA$, where

$$z = [\tilde{\varepsilon}_1\ \ \tilde{\varepsilon}_2\ \ \widetilde{\omega}_1\ \ \widetilde{\omega}_2]$$

$$A = \begin{bmatrix} 0 & -\Omega & 0 & 0 \\ \Omega & 0 & 0 & -6 \\ \frac{1}{2} & 0 & 0 & Q\Omega \\ 0 & \frac{1}{2} & -Q\Omega & 0 \end{bmatrix}$$

Taking the determinant of $A - \lambda U$, where U is the unity matrix, we find the characteristic equation for the system to be

$$\lambda^4 + (3x + Q^2 + 1)\Omega^2\lambda^2 + (3xQ + Q^2)\Omega^4 = 0 \tag{42}$$

whose roots λ represent the eigenvalues of the linear system.

VI. Simulation of Results

A. Docking at Maximum Spin Error

Here we will investigate the docking scenario in which the Dragon capsule maintains a spin rate at the maximum allowable 0.007 rad/s. Using Equation 17, we find the angular momentum of the capsule relative to the ISS (in attitude hold relative to the capsule in anticipation of the docking) to be

$$^{ISS}\overline{H}^{SXD} = 175\frac{kg}{m^2 \cdot s}\hat{b}_3$$

where the third body-fixed axis is mutually shared between the two spacecraft.

Using Equation 21, we find that to conserve angular momentum, the combined system now experiences an angular velocity of 8.75 x 10^{-7} rad/s about its third body-fixed axis.

We now define the orientation and rotation of the combined system with regards to the orbit-fixed frame $\hat{a}_i$, using the quaternion

$$\bar{\varepsilon} = \bar{0}\ ;\ \varepsilon_4 = 1$$

and the angular velocity as given above

$$\bar{\omega} = 8.75 \text{ x } 10^{-7}\ \hat{a}_3 \text{ rad/s}$$

The system, being in orbit around the Earth, is said to have an orbital angular velocity $\Omega = 0.05\frac{rad}{s}$, and is subject to a gravitational moment. This calls for the use of the differential kinematic and dynamic equations of motion (Equations 30-36) to describe the system's motion in the inertial reference frame. The spin rate, s, is also defined as $0.05\ \frac{rad}{s}$. Solving the differential equations of motion simultaneously, we find the following solution for the kinematic variables:

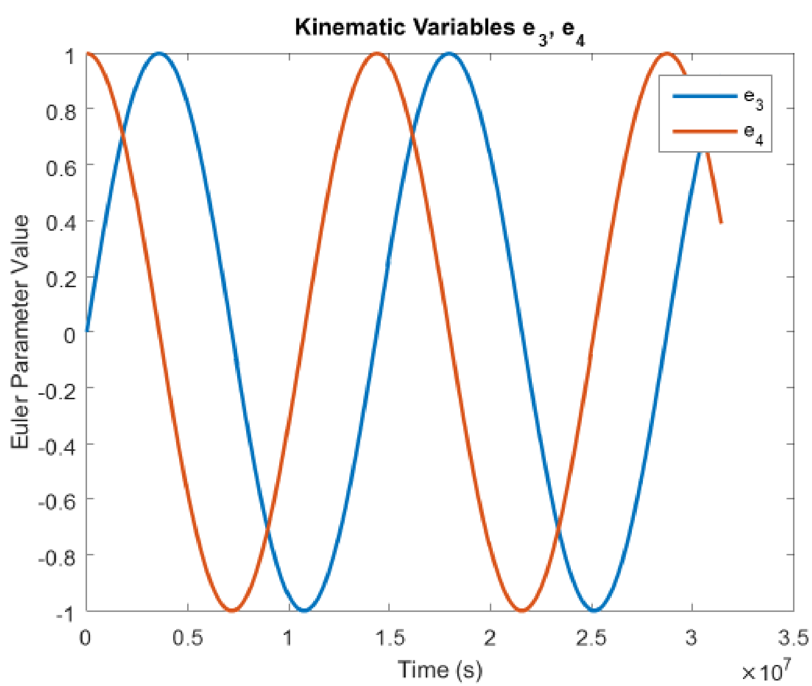

Figure 3. Variation of Euler parameters of combined system with time.

It is expected, given the initial conditions and upon inspection of the equations of motion, that the Euler parameters ε_3 and ε_4 are the only components of the quaternion to change with time, and that is what is observed. Given the constant spin rate of the combined system, it is also expected that these Euler parameters exhibit periodic motion with a constant frequency, and again, this is indeed what is viewed in Figure 3. The period of this motion is on the order of months, not surprising given the miniscule angular velocity imparted on the system by the Dragon capsule.

It is necessary to check the accuracy of the differential equation solver, in this case MATLAB's *ode45* function. This is accomplished by investigating the constraint equation for Euler parameters,

$$k = \sqrt{\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2} = 1$$

over the same time frame. Figure 4 shows the error from this expected value increasing over time.

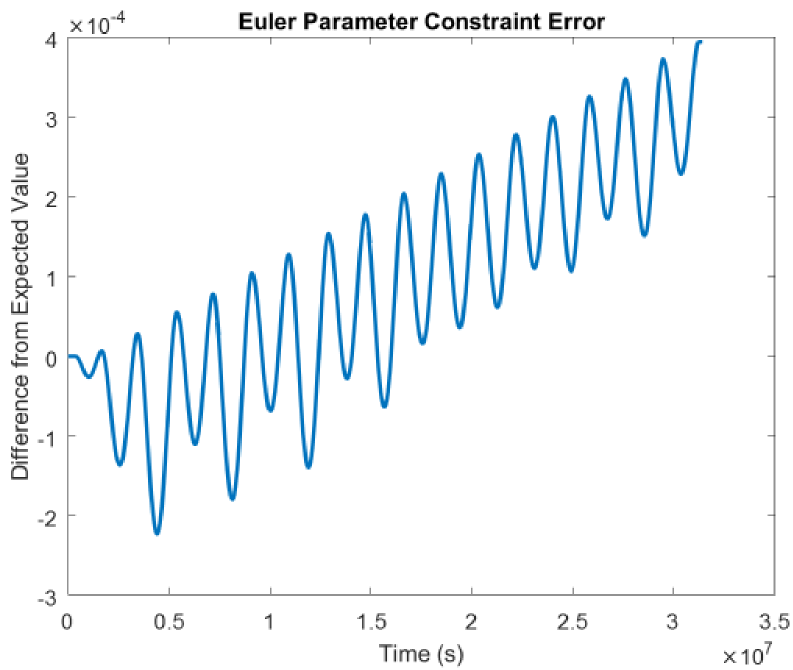

Figure 4. Euler parameter constraint error over time.

When we vary the spin rate of this system, this may wash out any effect of the added angular velocity, as can be seen in Figure 5 below. The red line is the condition depicted above, where the spin rate cancels out the orbital angular velocity and all that remains is the miniscule transferred angular velocity from the docking maneuver, which cannot be seen on a timescale as short as a few minutes.

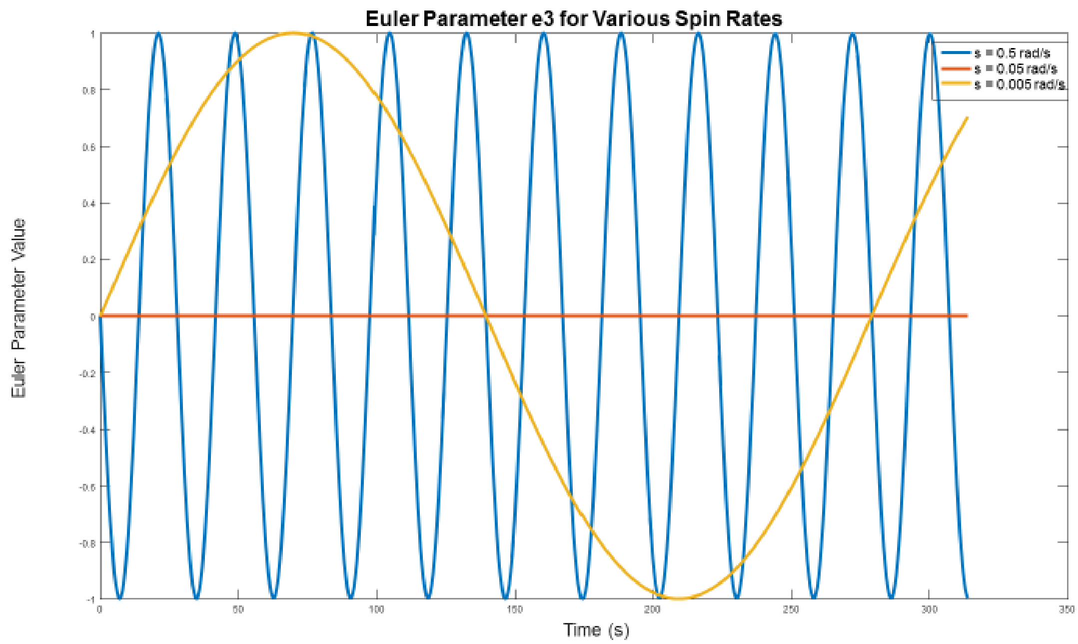

Figure 5. Euler parameter ε_3 over time for various spin rates. The effects of the added angular velocity are quickly overshadowed by any discrepancy between orbit angular velocity and spin rate.

B. Stability Analysis

For stability analysis, the system spin rate is locked at $s = \omega_{3_0} - \Omega$, and the analysis contained in Section VB performed. Two perturbations are added: one of 0.5 rad/s to the system's angular velocity ω_1, and one described by $\tilde{\varepsilon}_1 = 1$. Solving for the roots of the characteristic equation yielded the results contained in Table 1 below. The relationship of ω_{3_0} to Ω was varied, and eigenvalues for a particular solution of each case were determined, to investigate the stability of the system. Additionally, it is of value to investigate the hypothetical scenario in which a docking spacecraft were to affect the inertia matrix of the combined system in such a way that the system had a lower moment of inertia around the axis of symmetry (i.e., became more rod-like). Stability was determined for the four cases below by finding the eigenvalue roots of the system of equations as given by the conditions in the first column.

Table 1. Roots of the characteristic equation.

Conditions	Particular λ_1	Particular λ_2	Particular λ_3	Particular λ_4	Stability
s < 0	0 + 0.116*i*	0 - 0.116*i*	-0.031 + 0*i*	0.031 + 0*i*	Unstable
s > 0	0 + 0.169*i*	0 - 0.169*i*	0 + 0.063*i*	0 - 0.063*i*	Undetermined
s < 0 , J < I	-0.041 + 0.048*i*	-0.041- 0.048i	0.041+ 0.048*i*	0.041- 0.048*i*	Unstable
s > 0 , J < I	-0.035 + 0*i*	0.035 + 0*i*	0 + 0.001i	0 + 0.001i	Unstable

Simulating the first case, we simultaneously solve the differential equations of motion (Equations 37 – 41) to verify that indeed, the system is unstable, as our dynamic variables can be seen to increase without limit in Figure 6. In only a matter of seconds, if undetected and uncorrected, a perturbation may lead to an irreconcilable loss of control of the system.

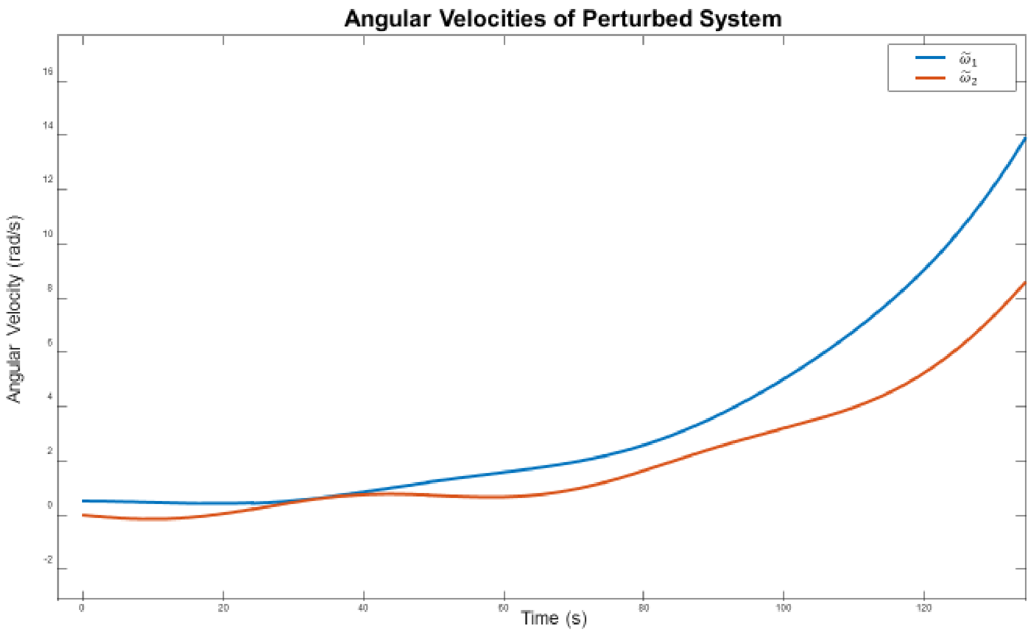

Figure 6. Angular velocities exhibiting uncontrolled increase due to perturbation.

VII. Conclusions

A docking scenario between the SpaceX Dragon capsule and the International Space Station was modeled from an inertia properties and an angular momentum standpoint. The effect of the Dragon capsule on the ISS, given the mass difference between the two vehicles of nearly three orders of magnitude, is nearly negligible in the short term. In the long term, however, the docking scenario may have a measurable effect on the nominal orientation of the ISS that must be accounted for to maintain mission success. These effects are quickly overshadowed, however, by nearly any measurable added spin rate. Great care must be taken, in the case of adding angular velocity, that the system remains stable, for an uncontrolled velocity change in the presence of gravity torque from the Earth may quickly lead to an unrecoverable, out-of-control spacecraft.

VIII. Appendix – Relevant Code

A. Function to solve differential equations of motion (Equations 30-36):

```
function ep_solutions = eom_solve(t, eom_params)
% bring in EPs
e1 = eom_params(1);
e2 = eom_params(2);
e3 = eom_params(3);
e4 = eom_params(4);
w1 = eom_params(5);
w2 = eom_params(6);
w3 = eom_params(7);
load shape.mat

w1_dot = -s*w2+(1-(J/I))*(w2*w3-12*(omega^2)*(e1*e2-e3*e4)*(e3*e1+e2*e4));
w2_dot = s*w1+(1-(J/I))*(w1*w3-2*(omega^2)*(e3*e1+e2*e4)*(1-2*e2^2-2*e3^2));
w3_dot = 0;

e1_dot = 0.5*(w1*e4 - w2*e3 + (w3-s+omega)*e2);
e2_dot = 0.5*(w1*e3 + w2*e4 - (w3-s+omega)*e1);
e3_dot = 0.5*(-w1*e2 + w2*e1 + (w3-s+omega)*e4);
e4_dot = -0.5*(w1*e1 + w2*e2 + (w3-s+omega)*e3);

ep_solutions = [e1_dot; e2_dot; e3_dot; e4_dot; w1_dot; w2_dot; w3_dot];
end
```

B. Function to solve differential equations of motion after perturbation (Equations 37-41):

```
function perturb_solutions = perturb_solve(t, perturb_params)

e1 = perturb_params(1);
e2 = perturb_params(2);
w1 = perturb_params(3);
w2 = perturb_params(4);
load shape.mat
e1_dot = omega*e2 + w1/2;
e2_dot = w2/2 - omega*e1;
w1_dot = -Q*omega*w2;
w2_dot = Q*omega*w1 - 6*x*omega^2*e2;

perturb_solutions = [e1_dot; e2_dot; w1_dot; w2_dot];
end
```

C. Setting up and solving characteristic equation (Equation 42):

```
I = I_sys(1,1);
J = I_sys(3,3);
omega = 0.05;
x = J/I - 1;
y = w3_0/omega - 1;
Q = y+x*(1+y);
p = [1 0 (3*x+Q^2+1)*omega^2 0 (3*x*Q+Q^2)*omega^4];
lambda = roots(p);
```