Hohmann Transfer to Lunar Orbit via Differing Propulsion Systems

John J. Murray[1]

The University of Alabama, Tuscaloosa, AL, 35487

In order to estimate the magnitude of delta-v required to transfer a spacecraft from Earth orbit to lunar orbit, two scenarios were considered. The first, examining a chemical-hybrid engine, was found to simplify to two finite burns, each imparting a part of the total required delta-v to complete the full maneuver. The second maneuver employs a constant, low thrusting engine for a longer time of flight and higher overall delta-v.

Nomenclature

a	=	length of semi-major axis
Δv	=	change in velocity as a result of thrust
$\mu_\oplus$	=	gravitational parameter (GM) of the Earth
P	=	period of motion

Methods and Equations

The velocity of an object at a particular point in its elliptical orbit is given by

$$v = \sqrt{2\mu\left(\frac{1}{r} - \frac{1}{2a}\right)} \tag{1}$$

Where μ is the gravitational constant of the central body (398,600 km^3/s^2 for the Earth[2]), r is the distance of the spacecraft from the center of the central body, and a is the semi-major axis of the ellipse. For circular orbits, this simplifies to a constant orbital speed

$$v = \sqrt{\frac{\mu}{r}} \tag{2}$$

The period of an elliptical orbit is found by

$$P = 2\pi\sqrt{\frac{a^3}{\mu}} \tag{3}$$

[1] M.S. Candidate, Department of Aerospace Engineering and Mechanics, The University of Alabama.

2 https://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html

The given problem specifies the initial Earth orbit to be circular, with a constant altitude of 12,800 km (for a total distance from Earth's center of 19,178 km when added to the planet's radius[2]). The Moon orbits Earth at a distance of 384,240 km, a value that serves as the basis for the spacecraft's target position.[3] The sources for the values used in this work have been added as footnotes where applicable.

Hohmann Transfer Orbit – Chemical-Hybrid System

The use of a chemical-hybrid motor to power a spacecraft from Earth orbit to lunar orbit involves three stages of interest: transfer from Earth orbit to a transfer trajectory (escape), flight along this transfer orbit, and finally transfer to lunar orbit (capture). This propulsion scheme allows for impulsive, finite, controlled burns that can be started and stopped as required.

Stage 1 – Circular Earth orbit and escape to transfer orbit

The escape maneuver is herein modeled as an impulse burn where the required velocity change is achieved instantaneously.

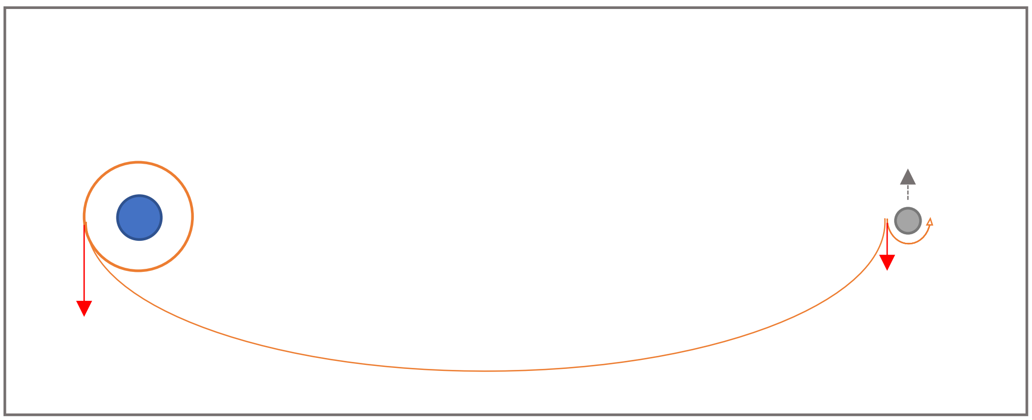

Figure 1 - Hohmann transfer orbit modeled in this scenario with two finite impulse burns (red)

The transfer orbit takes the shape of an ellipse with the Earth at one of its foci, the spacecraft's current position as its periapsis, and the position of the Moon's orbit on the opposite side of the Earth, minus the Moon's radius and the intended orbital height, as its apoapsis. It is assumed to be the case that the spacecraft is orbiting the Earth in the same plane as the Moon, and therefore no inclination change is required. Given the periapsis condition, no change in the direction of motion need occur—only a change in speed. This transfer orbit is defined as follows:

Element	**Transfer Orbit**	**Circular Earth Orbit**
r_p	19,178 km	19,178 km
r_a	382,562 km	19,178 km

3 https://solarsystem.nasa.gov/moons/earths-moon/overview/

Spacecraft velocity before the burn is the velocity of the circular orbit, which is found (using Equation 2) to be 4.56 km/s. The required spacecraft velocity at the same point on the transfer orbit is 6.30 km/s, found using Equation 1 with a being 200,870 km and r defined as r_p for the transfer orbit above. Therefore, the Δv required is the difference between the two velocities, or an increase of **1.74 km/s**.

Stage 2 – Transfer orbit and time of flight

The vehicle continues along its transfer orbit for a time equal to one-half the period of the full elliptical orbit. Given a semi-major axis of 200,870 km and the Earth as the central body, Equation 3 gives the time of flight as

$$TOF = \frac{P_t}{2} = \pi\sqrt{\frac{200{,}870^{\,3}}{\mu_\oplus}} \approx 448{,}000\ s \ \approx 124.4\ h \approx \mathbf{5.3}\ \boldsymbol{days}$$

Stage 3 – Capture into circular lunar orbit

The lunar capture maneuver is taken to occur at apoapsis of the transfer orbit. At this point, the velocity of the spacecraft relative to Earth is determined via Equation 1 to be

$$v = \sqrt{2\mu_\oplus\left(\frac{1}{r_a} - \frac{1}{2a_t}\right)} = \ 0.309\ km/s$$

The intended velocity of the spacecraft to complete the lunar capture is that of the Moon in its orbit around Earth, plus the required velocity to maintain a constant altitude orbit around the Moon. The velocity of the Moon in its orbit, relative to Earth and assuming a circular path, is found via Equation 2 to be 1.02 km/s in the same direction of the spacecraft's motion. The scenario is said to take the form of the Moon "catching up" to the spacecraft, which is ahead of it on the Earth-facing side. Therefore, it is convenient to place the spacecraft in a lunar orbit in the reverse direction of its transfer orbit, and claim some of the velocity difference to establish this orbit. A circular orbit of 100 km above the lunar surface, as stipulated by the problem, requires a velocity of—using Equation 2 again but with the lunar gravitational constant this time—1.63 km/s. Given that the spacecraft is moving relative to the Moon with a velocity of

$$0.309 – 1.02 = -0.711\ km/s$$

(the negative velocity indicating that the spacecraft is moving "backwards" relative to and towards the Moon) it is more fuel-efficient to increase the spacecraft velocity in the direction of this relative motion to achieve the 1.15 km/s value required. This requires a Δv in the direction opposite its motion relative to the Earth of **0.919 km/s**.

The total Δv required for the entire maneuver, then is the sum of the magnitudes of the two impulse burns, **2.66 km/s**.

Hohmann Transfer Orbit – Constant, low-thrust propulsion engine

Reducing the third-order equation of motion to first-order substitutions

Given the third-order equation

$$\frac{d}{d\tau}\left(\rho^3\left(\frac{d^2\rho}{d\tau^2} + \rho\right)^{\frac{1}{2}} = v\rho \qquad (4)$$

where τ and ρ are nondimensionalized parameters such that

$$\tau = \sqrt{\frac{\mu_{\oplus}}{r_0^3}}\, t\, ; \rho = \frac{r}{r_0}$$

making the substitution

$$A = \frac{d\rho}{d\tau} \tag{5}$$

we find

$$\frac{d}{d\tau}\left(\rho^3 \frac{dA}{d\tau} + \rho\right)^{\frac{1}{2}} = v\rho \tag{6}$$

Then, making another substitution

$$\frac{dA}{d\tau} = \frac{B^2 - \rho}{\rho^3} \tag{7}$$

gives

$$\frac{d}{d\tau}\left(\rho^3 \frac{B^2 - \rho}{\rho^3} + \rho\right)^{\frac{1}{2}} = v\rho \tag{8}$$

which in turn reduces to a final first-order substitution

$$\frac{d}{d\tau} B = v\rho \tag{9}$$

Solving the system of first-order equations and determining TOF and Δv

Equations 5, 7, and 9 are solved simultaneously through a MATLAB ordinary differential equation solver. The results are presented below, with the intended orbital height marked for reference. The spacecraft (v ~ 0.001) attains this height after approximately 4.75 million seconds of constant thrust, or **1316 hours (55 days)**.

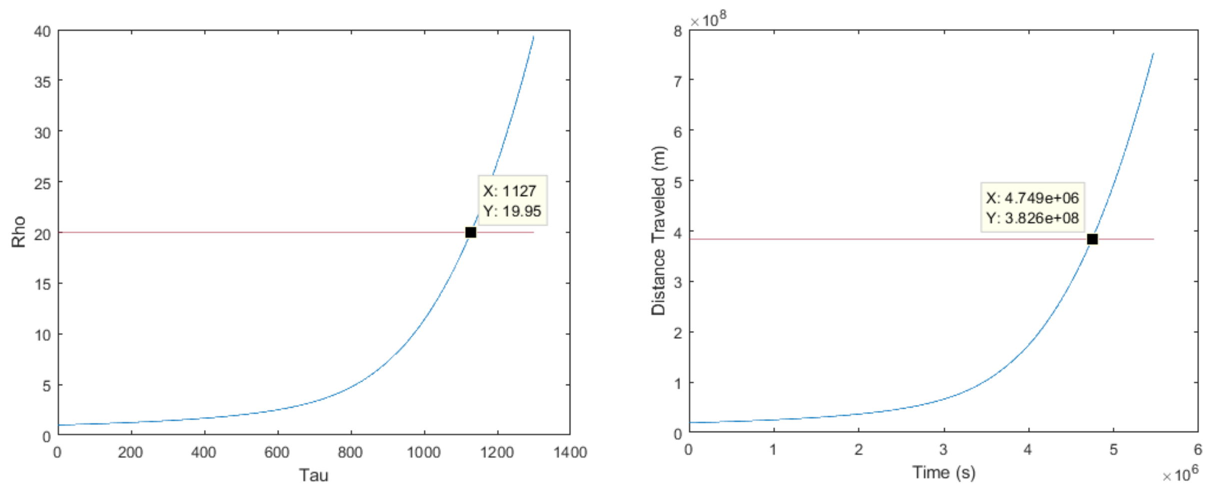

Figure 2 - Non-dimensional parameters and translation into standard units

The Δv over this time can be calculated by integrating the acceleration of the spacecraft over the time of flight

$$\Delta v = \int_{t_i}^{t_f} a_T dt = \int_0^{TOF} a_g dt = \int_0^{TOF} \frac{\mu_\oplus}{r^2} dt \tag{10}$$

where a_g is the changing acceleration due to gravity as the spacecraft moves farther from Earth, equal to **12.93 km/s.**

Conclusions

To summarize, under the chemical-hybrid engine scheme, the time of flight to achieve lunar orbit is found to be 124.4 hours, with a resultant delta-v of 2.66 km/s. Under the constant, low-thrust engine, the time of flight was found to be 1316 hours, with a resultant delta-v of 12.93 km/s.