Demand as a Function of Rental Rate

Mkt demand, $D(r)$ is:

$$D(r) = \int_a^b X(\theta, c+r)\, d\theta$$

where θ indexes buyers and $X(\theta; c+r)$ is the amount of good used.

Δ in demand from Δ in rental rate:

$$\frac{\partial D}{\partial r} = \int_a^b \frac{\partial X}{\partial r}\, d\theta$$

From FOC, $\theta \cdot u'(x) = c + r$ and so (since $\frac{\partial X}{\partial r} = \frac{1}{\theta \cdot u''(x)}$)

$$\frac{\partial D}{\partial r} = \int_a^b \frac{1}{\theta \cdot u''(x)}\, d\theta$$

and since $1 = \frac{\theta \cdot u'(x)}{c+r}$ (From FOC)

$$\frac{\partial D}{\partial r} = \int_a^b \frac{1}{c+r} \cdot \frac{u'(x)}{u''(x)}\, d\theta .$$

since $\frac{\partial X}{\partial \theta} = -\frac{u'(x)}{\theta \cdot u''(x)}$,

$$\frac{\partial D}{\partial r} = \frac{1}{c+r} \int_a^b \theta \cdot \frac{\partial X}{\partial \theta}\, d\theta$$

By integration by parts,

$$\frac{\partial D}{\partial r} = \frac{1}{c+r}\left[\theta \cdot X(\theta) \Big|_a^b - \int_a^b X(\theta)\, d\theta \right]$$

Since $\theta \cdot X(\theta) = 0$ when $\theta = a$

$$\frac{\partial D}{\partial r} = \frac{1}{c+r}\left[\underbrace{\bar{\theta}_B\, X(\theta_B)}_{Z} - D(r) \right]$$

(an ODE) we can easily solve

$$D(r) = \frac{rZ}{c+r} + \frac{C[1]}{c+r}$$

where $C[1]$ is a constant. at $D(0) = D_0$

$$D(0) = \frac{C[1]}{c} = D_0 \quad \text{so} \quad C[1] = D_0 \cdot c$$

$$\boxed{D(r) = \frac{r \cdot Z}{c+r} + \frac{c \cdot D_0}{c+r}}$$

D_0 is demand when rental rate is zero.

What's nice – utility function drops out – only observables matter.