Quadratic Cost Sharing Model

A consumer that consumes x of the sharing good has utility :

```
u[x_] := 2 * α * x - x^2;
```

The parameter α determines how much the consumer values consuming the good. In the absence of rental costs, α is the amount consumed by that user, conditional upon purchasing. Note that a Cobb-Douglas style utility function does not "work" as everyone consumes at least a little of the good.
We also need a money-metric utility function so that we can accomodate rental income and rental expenses.

```
In[32]:= sol = Solve[u'[x] == 0, x]
         xstar = x /. sol[[1]];
Out[32]= {{x → α}}
```

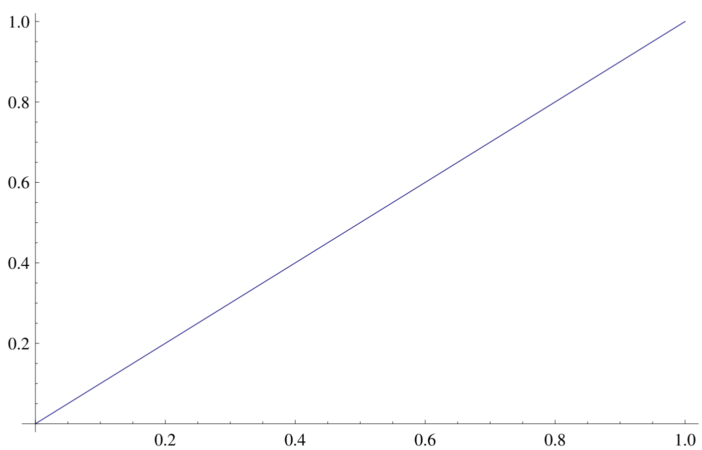
```
In[34]:= sol2 = Solve[xstar == 1, α]
Out[34]= {{α → 1}}
```

Indirect utility :

```
In[38]:= v[α_] = u[xstar] - p
```

Out[38]= $-p + \alpha^2$

```
Plot[xstar, {α, 0, 1}, AxesOrigin → {0, 0}]
```



Consumer purchase decision

A consumer must decide whether to purchase the good at a price p or go without.
Everyone with an α greater than the square root of the price purchases the good.

```
Solve[p == v[α], α]
```

$\{\{\alpha \to -\sqrt{p}\}, \{\alpha \to \sqrt{p}\}\}$

Four Consumer possibilities :

1) Everyone buys : $\alpha L > \sqrt{p}$ (think tootbrush)

2) High types buy but low types do not : $\alpha H > \sqrt{p} > \alpha L$ (think vacation homes)

3) No one buys : $\alpha H < \sqrt{p}$ (mainframe computers in the 1950 s)

When P2P renting is possible

Let us define the short-run as the period of time before anyone can revise their purchase decisions. The consumer's new optimization problem is:

Optimal usage in the presence of renting (for renters):

```
In[19]:= uR[x_] := 2 * α * x - x^2 - r * x;
solR = Collect[Solve[uR'[x] == 0, x], α]
```

Out[20]= $\left\{\left\{x \to -\frac{r}{2} + \alpha\right\}\right\}$

Note that for the renter to consume any of the good, 2 α > r.

Optimal usage in the presence of renting (for owners) :

```
In[39]:= uO[x_] := 2 * α * x - x^2 + (1 - x) * r;
solO = Collect[Solve[uO'[x] == 0, x], α]
```

Out[40]= $\left\{\left\{x \to -\frac{r}{2} + \alpha\right\}\right\}$

```
In[11]:= xstar2[r_, α_] = x /. solR[[1]]
```

Out[11]= $-\frac{r}{2} + \alpha$

Existence and Uniqueness of a P2P Rental Equilibrium

So long as there is not a glut (i.e., the comined consumption of the two types is greater than 1) a market clearing P2P rental equilibrium exists.

```
In[71]:= Solve[1 - xstar2[r, αH] == xstar2[r, αL], r]
```

Out[71]= $\{\{r \to -1 + \alpha H + \alpha L\}\}$

Indirect Utility for Both Types (assuming no product market changes)

For renters :

In[41]:=
```
vR[α_, r_] = uR[xstar2[r, α]] // FullSimplify
```

Out[41]= $\frac{1}{4}(r-2\alpha)^2$

Comparative Statics

In[45]:=
```
D[vR[α, r], α]
```

Out[45]= $-r+2\alpha$

A renter's surplus is increasing in their valuation parameter α.

In[49]:=
```
Resolve[ForAll[{α, r}, 2 α > r, D[vR[α, r], α] > 0]]
```

Out[49]= True

A renter's surplus is decreasing in the rental rate, r.

In[50]:=
```
D[vR[α, r], r]
```

In[52]:=
```
Resolve[ForAll[{α, r}, 2 * α > r, D[vR[α, r], r] < 0]]
```

Out[52]= True

Change in indirect utility for owners

In[53]:=
```
vO[α_, r_] = uO[xstar2[r, α]] - p // FullSimplify
```

Out[53]= $-p+r+\frac{r^2}{4}-r\alpha+\alpha^2$

A owner' s surplus is increasing in their valuation parameter.

In[64]:=
```
D[vR[α, r], α]
```

Out[64]= $-r+2\alpha$

A owner' s surplus is increasing in the rental rate.

In[62]:=
```
D[vO[α, r], r]
```

Out[62]= $1+\frac{r}{2}-\alpha$

In[63]:=
```
Resolve[ForAll[{α, r}, α > 0 && α < 1 && 2 * α > r && r > 0, D[vO[α, r], r] > 0]]
```

Out[63]= True

Change in Utility Verus the Status Quo (with no product market changes)

In[65]:=
```
conditions := α > 0 && α < 1 && 2 * α > r && r > 0
```

In[44]:=
```
vO[α, r] - v[α] // FullSimplify
```

Owner utility increases with the possibility of P2P rental

```
Resolve[ForAll[{α, r}, conditions, vO[α, r] - v[α] > 0]]
```

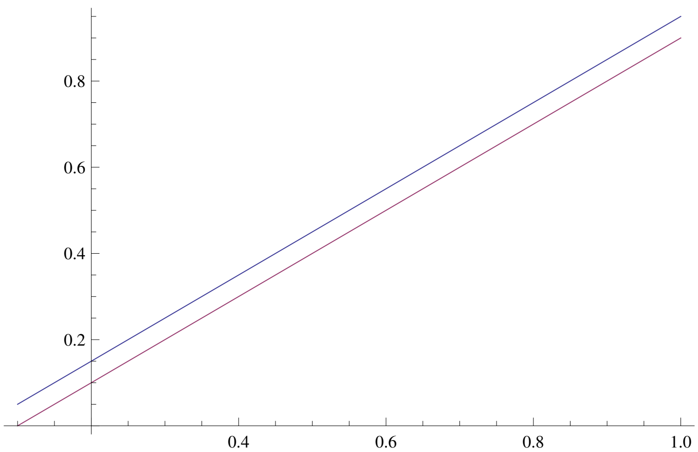
Out[67]=
```
True
```

A renter’s utility increases with the possibility of P2P rental

In[69]:=
```
Resolve[ForAll[{α, r}, conditions, vR[α, r] > 0]]
```

Out[69]=
```
True
```

```
Plot[{xstar2[0.1, α], xstar2[0.2, α]}, {α, .1, 1}]
```




```
Solve[(1 - xstar[r, αH]) == xstar[r, αL], r] // FullSimplify
```

$\{\{r \to -1 + \alpha H + \alpha L\}\}$

Equilibrium Rental Rate

$\{\{r \to -1 + \alpha H + \alpha L\}\}$

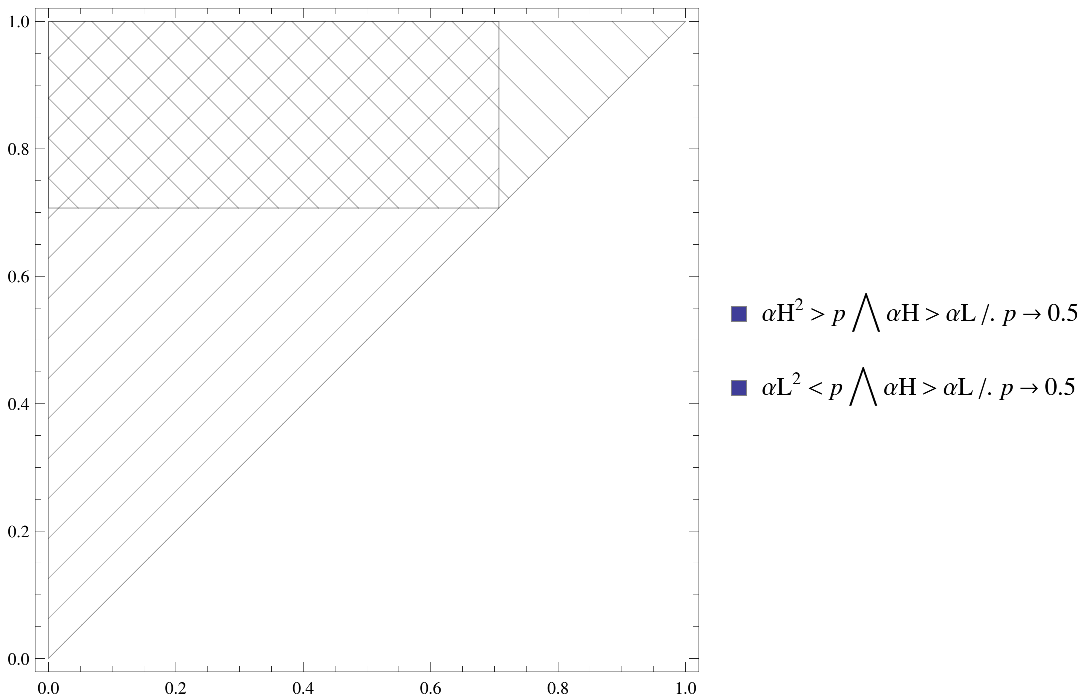
Comment : If either the high or low types value the good more, the rental rate increases. An increase in αH reduces supply, whereas an increase in αL reduces demand.

```
Solve[-1 + αH + αL == αH^2, αH] // FullSimplify
```

$$\left\{\left\{\alpha H \to \frac{1}{2}\left(1 - \sqrt{-3 + 4\,\alpha L}\right)\right\}, \left\{\alpha H \to \frac{1}{2}\left(1 + \sqrt{-3 + 4\,\alpha L}\right)\right\}\right\}$$

```
g1 = RegionPlot[αH^2 > p && αH > αL /. p → 0.5, {αL, 0, 1},
   {αH, 0, 1}, PlotLegends → "Expressions", MeshFunctions → {#1 + #2 &},
   Mesh → {Range[-2 Pi, 2 Pi, Pi / 50]}, PlotStyle → None];
g2 = RegionPlot[αL^2 < p && αH > αL /. p → 0.5, {αL, 0, 1},
   {αH, 0, 1}, PlotLegends → "Expressions",
   MeshFunctions → {#1 - #2 &}, Mesh → {Range[-Pi, Pi, Pi / 50]}, PlotStyle → None];
```

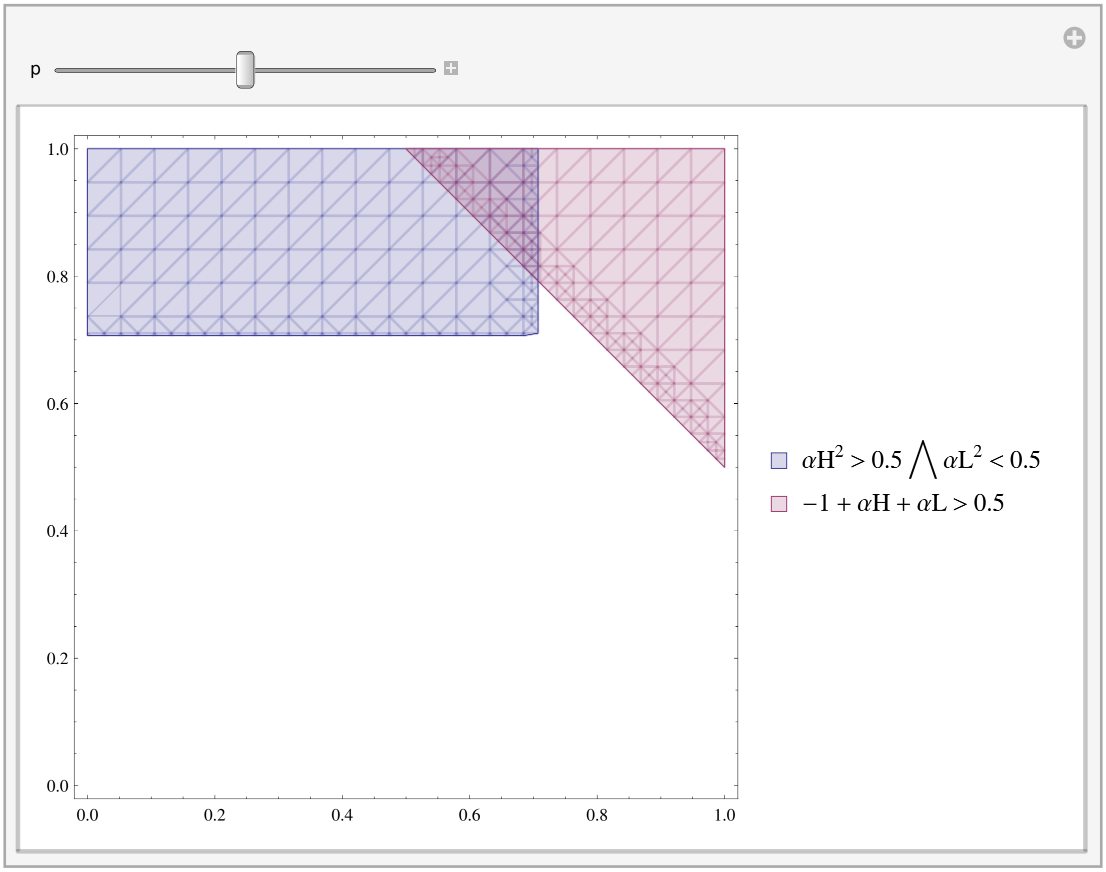
```
gCombo = Show[{g1, g2}]
```



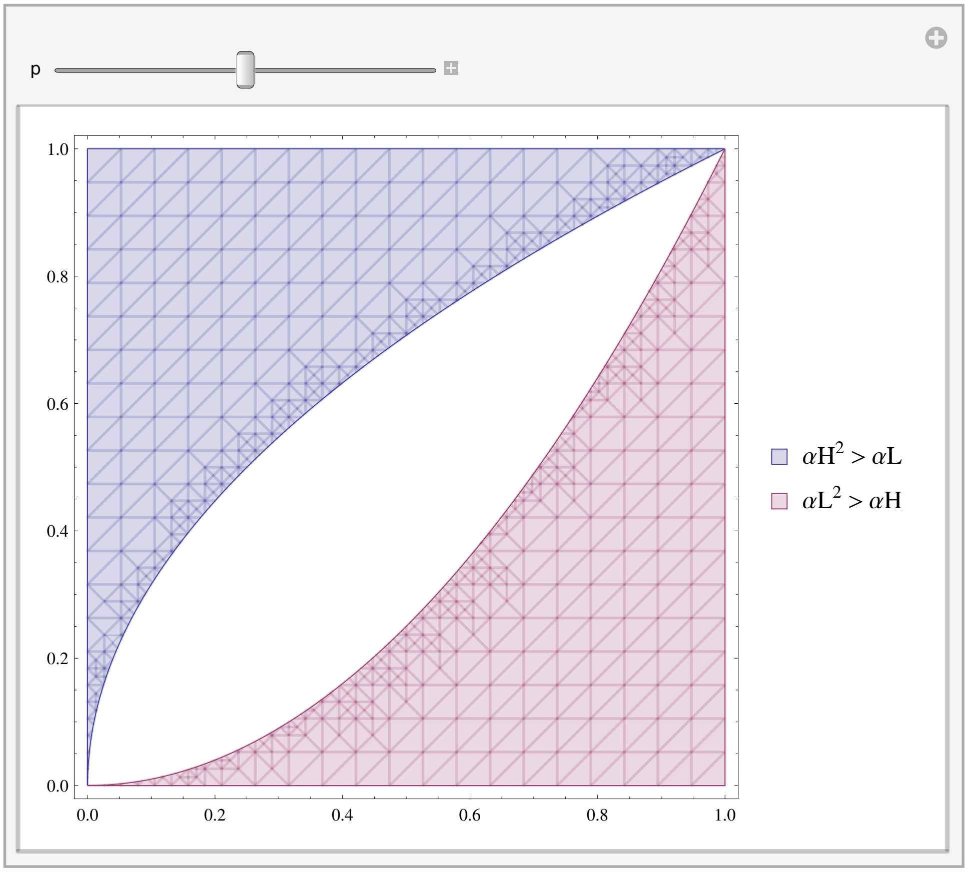

```
Export[ "../writeup/images/three_parts.pdf", gCombo]
```

```
../writeup/images/three_parts.pdf
```

```
Manipulate[RegionPlot[{αH^2 > p && αL^2 < p, -1 + αH + αL > p},
  {αL, 0, 1}, {αH, 0, 1}, PlotLegends → "Expressions"], {{p, 0.5}, 0, 1}]
```



```
Manipulate[RegionPlot[{αH^2 > αL, αL^2 > αH}, {αL, 0, 1},
  {αH, 0, 1}, PlotLegends → "Expressions"], {{p, 0.5}, 0, 1}]
```



Jevon' s Paradox

As a function of the product market price, what is the fraction of:
- all buy/no buy/high and lows buy

Product Market Demand

In[72]:=
```
d[p_] := If[p > αH^2, 0, If[p > αL^2, 1, 2]]
```

In[74]:=
```
Plot[d[p] /. {αH → 0.8, αL → 0.7}, {p, 0, 1}]
```

Out[74]=