A Space-Limited Algorithm for Computing Collatz Sequences

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Abstract

This paper describes a space-bounded algorithm for computing Collatz sequences. It relies on performing calculation using a double-base numbers system [**?**], specifically $2,3$-integers. This representational scheme simplifies the Collatz operations on x of $3x+1$ and $x/2$ to left-shift plus 1 and down-shift. These operations do not destroy terms and will consume unbounded space if left unchecked. It is possible to create boundaries and fold terms back onto a constained playing field. For any positive integer, x, it is possible to perform Collatz iterations within an area $log_2 x + 2$ by $log_2 x$ bits using at most an additional $2 log_2 n$ bits of additional storage to hold the escaping bits before processing. Therefore, any Collatz sequence starting from a positive integer will eventually cycle.

1 Introduction

The Collatz conjecture is known by many names: the Ulam problem, the Syracuse problem, and the $3n+1$ conjecture. It is an unsolved problem in mathematics that has been around for over eighty years regarding sequences of number that follow a simple set of rules.

$$C(x) = \begin{cases} 3x+1 & \text{if } x \text{ is odd} \\ x/2 & \text{otherwise} \end{cases} \tag{1}$$

In this paper, a similar version will be used that recognizes that every $3x+1$ operation is always followed by a division since the value is even ($3(2x+1)+1 = 6x+4$).

$$C(x) = \begin{cases} (3x+1)/2 & \text{if } x \text{ is odd} \\ x/2 & \text{otherwise} \end{cases} \tag{2}$$

4	$2^4 3^0$	$2^4 3^1$	$2^4 3^2$	$2^4 3^3$	$2^4 3^4$
3	$2^3 3^0$	$2^3 3^1$	$2^3 3^2$	$2^3 3^3$	$2^3 3^4$
2	$2^2 3^0$	$2^2 3^1$	$2^2 3^2$	$2^2 3^3$	$2^2 3^4$
1	$2^1 3^0$	$2^1 3^1$	$2^1 3^2$	$2^1 3^3$	$2^1 3^4$
0	$2^0 3^0$	$2^0 3^1$	$2^0 3^2$	$2^0 3^3$	$2^0 3^4$
i/j	0	1	2	3	4

Table 1: Bit coefficients for the planar representation of $3,2$-integers.

```
1                         1
1                         0
1                         0
1                         0
1                         001
1                         0                         001
1                         0001                      10011
```

Base 2 | Minimal number of bits | Base 3

Figure 1: Several ways to represent 127 as $3,2$-integers.

It has been observed that for positive integer, iterating either of of the functions eventually produces cyclic behavior ($4 \to 2 \to 1 \to 4$ and $2 \to 1 \to 2$, respectively.

This paper describes a space-bounded algorithm for computing the Collatz sequences. By utilizing a double-base number system to carry out calculations, it is possible to construct boundaries that limit the number of bits used to represent sequence elements. The constraints are determined *a priori* from the length of the binary representation of the starting number. The technique involves returning any escaping bits back into the fixed region.

Number Representation A double-base number system using combations of powers of two relatively prime numbers. Here is formulation of 2,3-integers

$$\sum_{i=0}^{n}\sum_{j=0}^{n} x_{ij} 2^i 3^j \tag{3}$$

where x_{ij} are either zero or one. This number system is not unique, as it is possible to represent integers in multiple forms (e.g. 4 is also $1 + 3$). A cartesian-layout will be used for visualizations

Under this system, multiplication and division by either base is a shift. Thus, $3x$ is a right-shift ($x \rightarrow$) and $x/2$ is a down-shift ($x \downarrow$). In the binary system, division is still a shift, but multiplication by three requires a shift and add.

2	*1*	*2*	*1*	*2*
1	0	0	0	0
0	1	0	0	0
----	----	----	----	----
		11	0	1
			11	0
				011

Figure 2: A $2-1$ oscillation. The dashes separate the integer part above from the "fractional" part bellow. Sometimes integers can be represented as sums of fractional terms. The middle 2 is represented by $1/2+3/2$.

Now, the Collatz transformation becomes

$$C(x)=\begin{cases}(x\rightarrow +1)\downarrow & \text{if } x \text{ is odd}\\ x\downarrow & \text{otherwise}\end{cases} \tag{4}$$

Determining parity of 23-intgers is a bit more involved. The only terms that impact parity are those of the form $x_{0j}3^j$. It is necessary to count these terms modulo two in order to compute parity.

It should be clear that under this representation, the Collatz function never destroys a bit once it has been added to the value. By extending the notation to cover fractional values (where $i<0$), bit patterns propagate unchanged down and to the right across the representational plane. At any given time, the number of bits is the number of starting bits and the number of multiplies that have happend so far. In fact, the final 2-1 oscillation of the system is one of value and not representation.

2 Boundaries