An Introduction to Stochastic Calculus

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Probability Spaces

On a measurable space (Ω, F), the probability measure is $P : F \to [0,1]$. The following conditions apply:

a. $P(\emptyset) = 0, P(\Omega) = 1$

b. if $A_1, A_2, ... \in F$ and $(A_i)_{i=1}^{\infty}$ is disjoint, then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

A probability space contains (Ω, F, P) which the variables indicating:

- P is the exact probability measure
- Ω is the a space with all the possible outcomes
- F is the collection of possible events where each event is a subset of Ω

Stochastic Processes and Brownian Motion

A stochastic process is a parameterized collection of random variables $\{X_t\}$, defined on a probability space (Ω, F, P) with values in $\mathbb{R}^n$. We often have $t \in [0, \infty)$ for the case of continuous stochastic processes. This can be thought of a function of time, where the outcome at each time is a random variable.

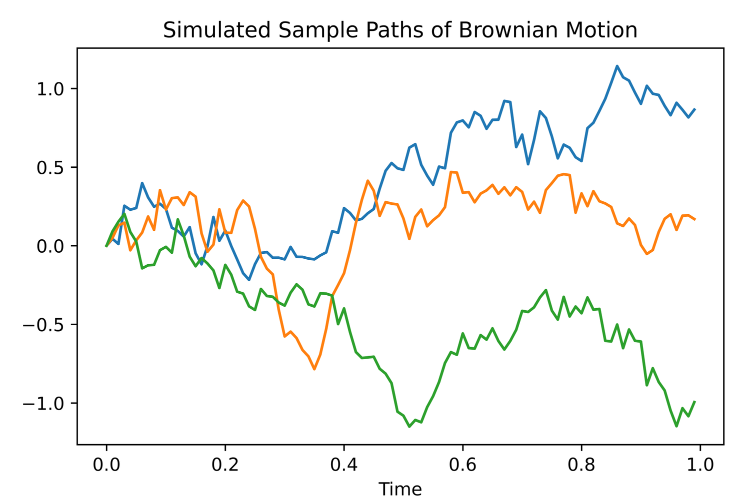
Brownian Motion

Brownian motion was observed by botanist Robert Brown while studying pollen grains, which moved in liquid in a jittery motion. This movement can be described mathematically by a 2 dimensional Brownian motion.

A sequence of random variables, B_t, for $t \geq 0$, is defined as a standard Brownian motion if:

1. $B_0 = 0$
2. B_t has continuous sample paths
3. For every t and s, with $s < t$, we have that $B_t - B_s$ is has a normal distribution with variance $t - s$ and mean 0.
4. The distribution of $B_t - B_s$ is independent of the behavior of B_r, for $r < s$.

The result of these properties is that Brownian Motion has independent, stationary increments with mean zero.



Quadratic Variation

Let $0 = t_0 < t_1 < t_2 < ... < t_n = T$ be a partition of a time interval $[0, T]$. For some stochastic process X_t, let

$$Q_n(T, X) = \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})^2$$

The quadratic variation of X_t on the interval is the limit of $Q_n(T, X)$ as n gets large (or as Δt gets small). For a Brownian motion $\{B_t\}_{t \geq 0}$ the quadratic variation is equal to T with probability 1, as the expected value of quadratic variation is T and the limit of the variance of $Q_n(T, B)$ approaches 0. One can show that the total variation of the path is infinite, with probability 1, and the paths $t \to B_t$ of Brownian motion are nowhere differentiable. The total variation of a process, X_t, on $[0, T]$, is defined as

$$\lim_{\Delta t \to 0} \sum_{i=1}^{n-1} |X_{t_{i+1}} - X_{t_i}|$$

Stochastic Integration and the Itô Integral

We will now define the Itô Integral of a stochastic process (under certain conditions that we omit for simplicity). It is possible to generalize the following definitions to multiple dimensions, but we will only focus on the one dimensional case. It is also important to note that this integral is a random variable.

Definition of the Itô Integral

A elementary function h has the form $h_t = \sum_i e_i I_{[t_i, t_{i+1})}(t)$, where I is the indicator function. Note that h_t is a piece wise continuous random process. We define the Itô integral of elementary functions as

$$\int_S^T h_t dB_t = \sum_{i=0}^{n-1} e_i (B_{t_{i+1}} - B_{t_i})$$

Then, for a more general process, X_t, we define

$$\int_S^T X_t dB_t = \lim_{n \to \infty} \int_S^T X_t^{(n)} dB_t$$

where $X_t^{(n)}$ is a elementary function such that

$$\mathbb{E}[\int_S^T (X_t - X_t^{(n)})^2 dt] \to 0 \text{ as } n \to 0$$

When computing the Itô integral, the e_i term for our elementary process $X_t^{(n)}$ becomes X_{t_i}, which is a left endpoint definition. Another seemingly reasonable choice would be to use to use $X_{t_{i+1}}$, which is the right endpoint. Under the Riemann-Stieltjes integral for a real valued function, this choice does not change the result of the integral. However, due to the large variations of the paths of B_t this choice results in different solutions to the integrals. The choice of a midpoint $\frac{t_i + t_{i+1}}{2}$ leads to the Stratonovich integral, which has different properties than the Itô integral. For example, the Itô integral is a martingale whereas the Stratonovich integral is not.

Computing the Itô integral of Brownian motion: As an example, we compute the value of $\int_0^T B_t dB_t$. Let $B_t^{(n)} = \sum_{i=0}^{n} B_{t_i} I_{[t_i, t_{i+1})}(t)$, then

$$\int_0^T B_t dB_t = \lim_{n \to \infty} \int_0^T B_t^{(n)} dB_t \quad (1)$$

$$= \lim_{n \to \infty} \sum_{i=0}^{n-1} B_{t_i} (B_{t_{i+1}} - B_{t_i}) \quad (2)$$

$$= \lim_{n \to \infty} (\frac{1}{2} B_T^2 - \frac{1}{2} \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2) \quad (3)$$

$$= \frac{1}{2} B_T^2 - \frac{1}{2} T \quad (4)$$

because the quadratic variation of Brownian motion is T almost surely. In line (3) we also use that:

$$B_T^2 = \sum_{i=0}^{n-1} (B_{t_{i+1}}^2 - B_{t_i}^2) = \sum_{i=0}^{n-1} ((B_{t_{i+1}} - B_{t_i})^2 - 2B_{t_i}(B_{t_{i+1}} - B_{t_i}))$$

Properties of the Itô Integral:

For f and g that are stochastic processes, and $0 \leq S < U < T$, we have

i. $\int_S^T f dB_t = \int_S^U f dB_t + \int_U^T f dB_t$

ii. $\int_S^T (cf + g) dB_t = c \int_S^T f dB_t + \int_S^T g dB_t$, where c is a constant

iii. $\mathbb{E}[\int_S^T f dB_t] = 0$

iv. $\int_S^T f dB_t$ is F_T-measurable

(Intuitively, a function is F_t-measurable if its value can be determined from the path of a Brownian Motion up to t. For example, B_{2t} is not F_t-measurable.)

Itô Processes and Stochastic Differential Equations

An Itô process X_t is a stochastic process that can be written as

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t b_s dB_s,$$

with special conditions on a_t and b_t which are random functions of time. We can write the above equality in a shorthand differential form:

$$dX_t = a_t dt + b_t dB_t$$

Itô's Lemma

Given a 1-dimensional Itô Process X_t and $f(t, x) : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ a twice continuously differentiable function, if $Z_t := f(t, X_t)$, then we have that:

$$dZ_t = \frac{\partial f}{\partial t}(t, X_t) dt + \frac{\partial f}{\partial x}(t, X_t) dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, X_t)(dX_t)^2$$
$$= \left(\frac{\partial f}{\partial t}(t, X_t) + a_t \frac{\partial f}{\partial x}(t, X_t) + \frac{1}{2} b_t^2 \frac{\partial^2 f}{\partial x^2}(t, X_t) \right) dt + b_t \frac{\partial f}{\partial x}(t, X_t) dB_t$$

Stochastic Differential Equations

A stochastic differential equation (SDE) describes a stochastic process which is equal to an Itô integral of a function of that process. A SDE has the form:

$$X_t = X_0 + \int_0^t a(X_s, s) ds + \int_0^t b(X_s, s) dB_s$$

This can be written in differential notation as:

$$dX_t = a(X_t, t) dt + b(X_t, t) dB_t; \ X_0 = x$$

Geometric Brownian Motion: A SDE which models asset prices in finance is

$$S_T = S_0 + \int_0^T r S_t dt + \int_0^T \sigma S_t dB_t$$

which is written in differential form as

$$dS_t = r S_t dt + \sigma S_t dB_t$$

This model, called geometric Brownian motion, describes a stochastic process which grows at a rate of r plus some random "noise". In finance, this r term represents an interest rate and σ represents the volatility of the asset. If we let $Y_t = Y_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma B_t}$ and apply Itô's Lemma, we will see that Y_t satisfies the SDE described above.

$$dY_t = (r - \frac{1}{2}\sigma^2) Y_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma B_t} dt + \sigma Y_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma B_t} dB_t \quad (5)$$

$$+ \frac{1}{2}\sigma^2 Y_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma B_t} (dB_t)^2) \quad (6)$$

$$= (r Y_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma B_t} dt + \sigma Y_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma B_t} dB_t) \quad (7)$$

$$= r Y_t dt + \sigma Y_t dB_t \quad (8)$$

where we use that $(dB_t)^2 = dt$. So, $Y_t = Y_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma B_t}$ solves the SDE.

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References

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