<u>Linear Algebra</u>

Vectors

The form that vectors take doesn't really matter. It can be anything, so long as there is some notion of adding and scaling vectors that follow these rules:

1. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
2. $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
3. There is a vector $\mathbf{0}$ such that $\mathbf{0} + \vec{v} = \vec{v}$ for all $\vec{v}$
4. For every vector $\vec{v}$ there is a vector $-\vec{v}$ so that $\vec{v} + (-\vec{v}) = 0$
5. $a(b\vec{v}) = (ab)\vec{v}$
6. $1\vec{v} = \vec{v}$
7. $a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w}$
8. $(a + b)\vec{v} = a\vec{v} + b\vec{v}$

It doesn't matter whether you think about vectors as fundamentally being arrows in space that happen to have a nice numerical representation, or fundamentally as lists of numbers that happen to have a nice geometric interpretation. The usefulness of linear algebra has less to do with either one of these views then it does with the ability to translate back and forth between them. It gives the data analyst a nice way to conceptualize many lists of numbers in a visual way that can seriously clarify patterns in data and give a global view of what certain operations do. And on the flip side, it gives people like physicists and computer graphics programmers a language to describe space and the manipulation of space using numbers that can be crunched and run through a computer.

Basis Vectors and Span

In the xy-coordinate system, there are two very special vectors. The one pointing to the right with length one, $\hat{\imath}$, and the one pointing straight up with length one, $\hat{\jmath}$. One way to think about vector coordinates is to think of each coordinate as a scalar. Think of the x-coordinate of our vector as a scalar that scales $\hat{\imath}$ and the y-coordinate as a scalar that scales $\hat{\jmath}$. In this sense, the vector that these coordinates describe is the sum of two scaled vectors.

For example, the vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is the vector that is the sum of $3\hat{\imath}$ and $4\hat{\jmath}$.

$\hat{\imath}$ and $\hat{\jmath}$ are the basis of a coordinate system. What this means basically, is that when you think about coordinates and their scalars, the basis vectors are what those scalars scale.

The span of $\vec{v}$ and $\vec{w}$ is the set of all their linear combinations. The span of most pairs of 2d vectors is all vectors of 2d space, but when they line up, their span is all vectors whose tip sits on a certain line. Linear algebra revolves around vector addition and scalar multiplication. The span of two vectors is basically a way of asking what are all the possible vectors you can reach using only these two fundamental operations, vector addition and scalar multiplication. The span of two 3d vectors is the collection of all possible linear combinations of those two vectors. This will be a flat sheet. When you have multiple vectors and you can remove one without reducing the span, the vectors are linearly dependent. In other words, one of the vectors can be expressed as a linear combination of the others since it is already in the span of the others. If each vector really does add another dimension to the span, they are said to be linearly independent.

The technical definition of a basis is the set of linearly independent vectors that span the full space.

Linear Transformations and Matrices

Transformation is a fancy word for function. It takes in input and spits out output. In linear algebra, we like to think about transformations that take in a vector and spits out another vector.

A transformation is linear if it has two properties:
1. all lines must remain lines, without getting curved
2. the origin must remain fixed in place

To understand a transformation, you only need to record where the two basis vectors, $\hat{\imath}$ and $\hat{\jmath}$, each land. This is because a transformation is just a linear combination of where $\hat{\imath}$ and $\hat{\jmath}$ land. A 2x2 matrix can be thought of as transforming space, where the columns are the two special vectors where $\hat{\imath}$ and $\hat{\jmath}$ each land.

Matrix Multiplication

Multiplying two matrices represents applying one transformation after another.

Determinant

The determinant of a linear transformation measures how much areas/volumes change during the transformation.

Column Space and Null Space

A linear system of equations has some kind of linear transformation associated with it. When that transformation has an inverse, you can use that inverse to solve the system.

When the output of a transformation is a line, meaning its one-dimensional, we say the transformation has a rank of 1. If all the vectors land on some two-dimensional plane, we say the transformation has a rank of 2. The rank is the number of dimensions in the output of a transformation. In the case of 2x2 matrices, rank 2 is the best it can be; it means the basis vectors continue to span the full 2 dimensions of space and the determinant is non-zero. In the case of 3x3 matrices, rank 2 means the vectors have collapsed, but not as much they would have collapsed for a rank 1 situation. If a 3x3 transformation has a non-zero determinant and its output fills all of 3d space, it has a rank of 3.

The set of all possible outputs for a matrix is the column space of the matrix. The columns of a matrix tell where the basis vectors land. And the span of those transformed basis vector gives all possible outputs. In other words, the column space is the span of the columns of the matrix. A more precise definition of rank is the number of definitions in the column space. When this rank is as high as it can be, it equals the number of columns, we call this full rank.

The null space or the kernel of a matrix is the set of vectors that land on the origin after a transformation. Its the space of all vectors that become null in the sense that they land on the zero vector.

Nonsquare Matrices

Nonsquare matrices are transformations between dimensions.

Dot Products and Duality

For the dot product between two vectors, v and w, imagine projecting w onto the line that passes through the origin and the tip of v. The dot product is multiplying the length of this projection by the length of v. This shows whether or not the two vectors tend to point in the same direction.

Any time you have a 2D to 1D linear transformation, whose output space is the number line, no matter how it is defined, there is going to be some unique vector v corresponding to that transformation, in the sense of applying that transformation is the same thing as taking the dot product with that vector. This is an example of duality - situations where you have a natural but surprising correspondence between two types of mathematical thing. The dual of a vector is the linear transformation it encodes. The dual of a linear transformation from some space to 1 dimension is a certain vector in that space.

Cross Products

The cross product of v and w can be thought of as the area of the parallelogram formed by v and w. Note, the cross product is larger when the vectors are perpendicular than when they are closer together.

More specifically, the cross product of v and w is a new vector with a length equal to the area of the parallelogram and pointing in the direction perpendicular to the parallelogram according to the right-hand rule.

Eigenvectors and Eigenvalues

The eigenvectors of a transformation are the vectors that stay on their own span after a transformation. Each eigenvector has an eigenvalue which is the factor by which the eigenvector is stretched or squished by the transformation.

Abstract Vector Spaces

A vector space is a set that is closed under finite vector addition and scalar multiplication. For example: arrows, lists of numbers, functions, etc.