Kempe recoloring of planar graphs

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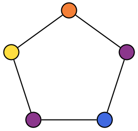
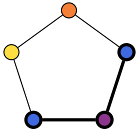
Joint work with Quentin Deschamps, Carl Feghali,
František Kardoš and Théo Pierron

Recoloring with Kempe changes

Kempe chain (1879)

Maximal bichromatic connected component in G





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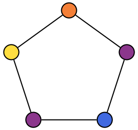

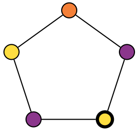

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Reconfiguration graph

- $V(R^k(G)) = k$-colorings of G.
- α and β adjacent if $\alpha \xleftrightarrow[\text{Kempe}]{} \beta$

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Reconfiguration graph

- $V(R^k(G)) = k$-colorings of G.
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Usual questions

- Given two k-colorings α and β, $\alpha \leftrightsquigarrow \beta$?
- Is $R^k(G)$ connected ?
- What is $\mathrm{diam}(R^k(G))$?
- Does the corresponding Markov chain mix well ?

Fundamental lemma

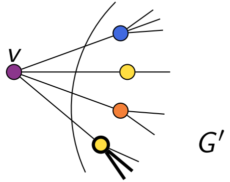
Las Vergnas, Meyniel 1981

All k-colorings of a d-degenerate graph are equivalent, for $k > d$

By induction: Let v with $\deg(v) \leq d$, consider
$G' = G \setminus \{v\}$

$$\alpha_{|G'} \underset{K}{\leadsto} \beta_{|G'}$$

Lift sequence to G then recolor v



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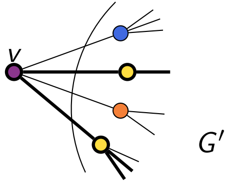

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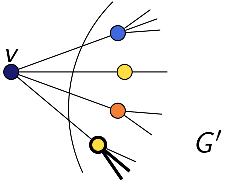
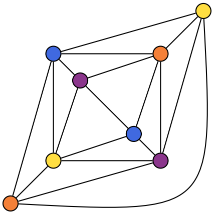
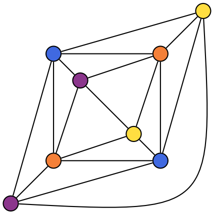


Meyniel 1978

- All 5-colorings of a planar graph are Kempe equivalent
- False for 4-colorings:





Bounding the reconfiguration diameter

Conjecture of Bonamy, Bousquet, Feghali, Johnson 19

$R^k(G)$ has diameter $O(n^2)$, if G is d-degenerate and $k > d$

Bonamy, Delecroix, L. 22+

$R^k(G)$ has polynomial diameter for:

- $k \geq \Delta$ (unless, $k = 3$ and $G = K_2 \square K_3$),
- $k \geq \mathrm{mad}(G) + \varepsilon$
- $k \geq \mathrm{tw}(G) + 1$

Journées GrR 2021

What about planar graphs with $k = 5$ or $k = 6$?

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Deschamps, Feghali, Kardoš, L., Pierron 22+

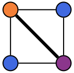
$R^k(G)$ has diameter $O(n^{195})$ if G is planar and $k \geq 5$

Mohar 2007

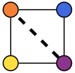
$R_4(G)$ is connected if G is a 3-colorable planar graph

Fisk 1977

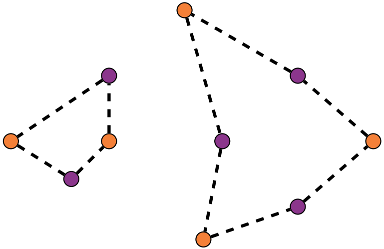
$R_4(G)$ has polynomial diameter if G is a triangulation of the plane



singular



non-singular



Sketch of proof

Theorem

For every 5-coloring α and every 4-coloring β, there is a sequence of Kempe changes going from α to β recoloring each vertex $P(n)$ times

Idea of proof

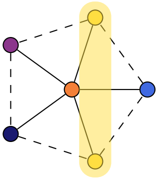
Assumption: there exist I,

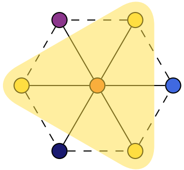
- I linear size independent set
- monochrome in α and in β
- $\forall v \in I, \deg(v) \leq 4$

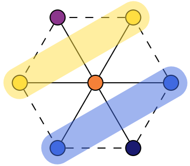
G	$F = G \setminus I$	$H = G \setminus B$
α	$\alpha_{\mid F}$	
	$\downarrow$ Induction	
γ'	γ avoiding $\alpha(I)$	
$\downarrow$ trivial changes		
δ s. t.$\delta(B) = 1$		$\delta_{\mid B}$ 4-col
		$\downarrow$ Fisk
β		$\beta_{\mid B}$ 3-col

Collapsing

Collapsable configurations







Non-collapsable collapsable in at most 3 Kempe changes

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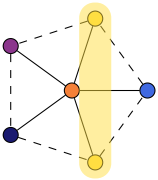

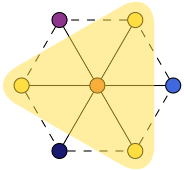

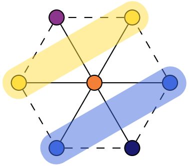

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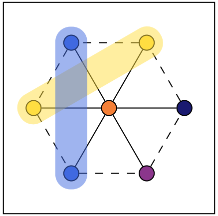

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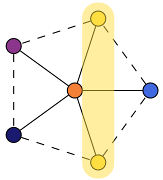

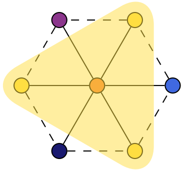

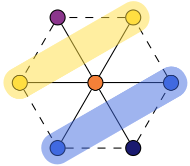

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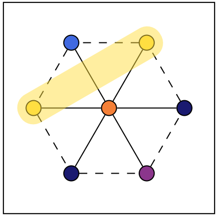

Open questions

Question 1

Let G be a d-degenerate graph. For $k > d$, do we have $\text{diam}(R^k(G)) = O(n^2)$?

Question 2

If $R^k(G)$ is connected, can we have a non-linear lower bound on $\text{diam}(R^k(G))$?

Question 3

Are there other arguments than frozen colorings to prove that $R^k(G)$ is disconnected ?

Thank you !