Kempe changes

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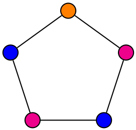
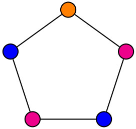
June 15, 2021

Internship carried out from February 15 to July 28, under the supervision of Marthe Bonamy and Vincent Delecroix at LaBRI in Bordeaux



Kempe chain (1879)

Maximal bichromatic component in G



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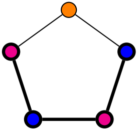

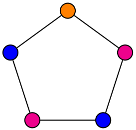

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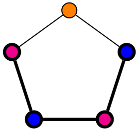

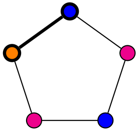

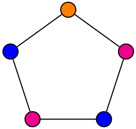

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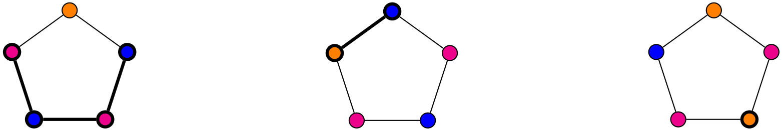

Motivation

Powerful tool

Vizing theorem: The edges of G can be colored with $\Delta(G)+1$ colors

Reconfiguration graph

- $V(R^k(G)) = k$-colorings of G.
- α and β adjacent if $\alpha \underset{\text{Kempe}}{\longleftrightarrow} \beta$

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Reconfiguration graph

- $V(R^k(G)) = k$-colorings of G.
- α and β adjacent if $\alpha \xleftrightarrow[\text{Kempe}]{} \beta$

Reachability	Are α and β Kempe-equivalent ?
Connectivity	Is $R^k(G)$ connected ?
Diameter	Estimate $\mathrm{diam}(R^k(G))$?

Sampling coloring with a Markov chain

- What is the mixing time of the associated Markov chain ?
- Estimate the number of colorings

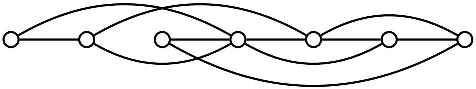
Warm up

k-degenerate graph

G is k-degenerate if there exists an elimination ordering of the vertices $v_1 \prec v_2 \cdots \prec v_n$:

$$\forall i, |N^+[v_i]| \leq k$$

- G has degree less than k implies G $(k+1)$-degenerate
- G not regular of degree less than k implies G k-degenerate



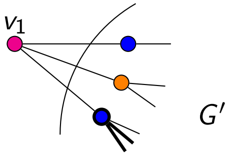
Warm up

Las Vergnas, Meyniel 1981

All ℓ-colorings of a k-degenerate graph are Kempe-equivalent for $\ell > k$.

Induction: Let α and β be two colorings of G
Consider $G' = G \setminus \{v_1\}$

$$\alpha_{|G'} \underset{K}{\rightsquigarrow} \beta_{|G'}$$



Lift sequence to G then recolor v_1

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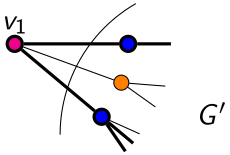

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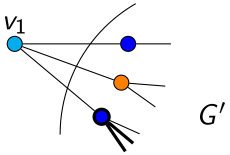

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Natural question

What is the diameter of the reconfiguration graph in this setting ?

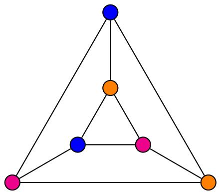
Recoloring with Δ colors

Brooks' theorem

All graphs but cliques and odd cycles can be colored with Δ colors

Mohar conjecture 06

$\forall k \geq \Delta, R^k(G)$ is connected



Feghali, Johnson, Paulusma 16

True for all 3-regular graphs but the 3-prism

Bonamy, Bousquet, Feghali, Johnson 19

True for all k-regular graphs with $k \geq 4$

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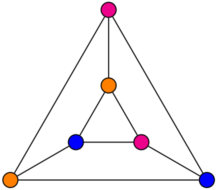

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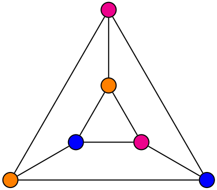

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Bonamy, Delecroix, L. 21+

$R^k(G)$ has diameter $O(\Delta n^2)$ for $k \geq \Delta$

Other result

Treewidth

Parameter that measures how much a graph looks like a tree.
$\mathrm{tw}(G) \leq k \Rightarrow G$ is k-degenerate

Bonamy, Delecroix, L. 21

If G has treewidth tw, then $\forall k \geq \mathrm{tw}+1, \mathrm{diam}(R^k(G)) = O(\mathrm{tw}\, n^2)$

What next ?

- Diameter for k-degenerate graphs
- Sampling and mixing time
- Aproximate counting of the number of colorings
- Glauber dynamics: trivial Kempe changes

Recoloring version of Hadwiger conjecture

Graph minor

H is a minor of G is H can be obtained from G by deleting vertices and contracting along edges

Hadwiger conjecture 1943

For all graph G, with no K_t-minor, $\chi(G) \leq t-1$
True for $t \leq 6$

Las Vergnas and Meyniel conjectures 1981

If G has no K_t minor, $R^k(G)$ is connected for $k \geq t-1$
True for $t \leq 9$

Bonamy, Delecroix, L., Narboni 21

$\forall t, \forall \varepsilon, \exists G$ with a frozen $(\frac{3}{2}+\varepsilon)t$-coloring