Recoloring with Kempe changes

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Internship carried out from February 15 to July 28, under the supervision of Marthe Bonamy and Vincent Delecroix

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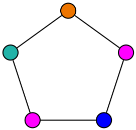
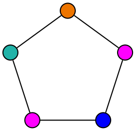


Kempe chain (1879)

Maximal bichromatic connected component in G



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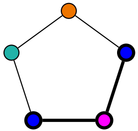

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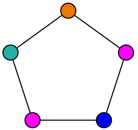

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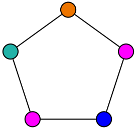

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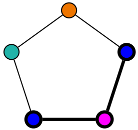
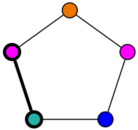
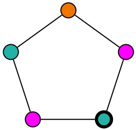




Natural questions

Application

- Powerful tool (ex: Vizing theorem)
- Sampling coloring with a Markov chain

Reconfiguration graph

- $V(R^k(G)) = k$-colorings of G.
- α and β adjacent if $\alpha \underset{\text{Kempe}}{\longleftrightarrow} \beta$

Reachability Are α and β Kempe-equivalent ?

Connectivity Is $R^k(G)$ connected ?

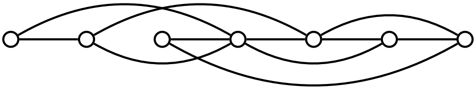
Diameter Estimate $\text{diam}(R^k(G))$?

Warm up

d-degenerate graph

G is d-degenerate if for any $H \subset G$, $\delta(H) \leq d$
Equivalently, G admits an elimination ordering $v_1 \prec v_2 \cdots \prec v_n$ such that

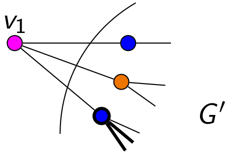
$$\forall i, |N^+(v_i)| \leq d$$



Las Vergnas, Meyniel 1981

All k-colorings of a d-degenerate graph are Kempe-equivalent for $k \geq d + 1$.

Induction: Let α and β be two colorings of G
Consider $G' = G \setminus \{v_1\}$



$$\alpha_{|G'} \underset{K}{\rightsquigarrow} \beta_{|G'}$$

Lift sequence to G then recolor v_1

Fundamental lemma

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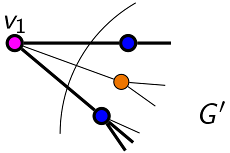

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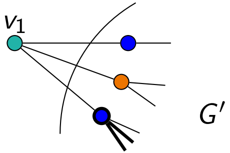

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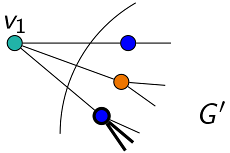

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Lift sequence to G then recolor v_1

Natural question

What is the diameter of the reconfiguration graph in this setting ?

Recoloring via trivial Kempe changes

Cerecedas '07

$R^k_{\text{Glauber}}(G)$ is connected if G is d-degenerate and $k \geq d+2$

Cerecedas' conjecture '07

$\text{diam}(R^k_{\text{Glauber}}(G)) \leq O(n^2)$ if G is d-degenerate and $k \geq d+2$

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Treewidth

Graph parameter that measures how close a graph is from being a tree
$\text{tw}(G) \leq k$ implies G k-degenerate

Bonamy, Bousquet '13

$\text{diam}(R^k_{\text{Glauber}}(G)) \leq O(n^2)$ if $k \geq \text{tw}(G)+2$

Bonamy, Delecroix, L. '21

$\text{diam}(R^k(G)) = O(\text{tw}\, n^2)$ if $k \geq \text{tw}(G)+1$,

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Bousquet, Heinrich '19

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Recoloring with Δ colors

Brooks' theorem

All graphs but odd cycles and cliques are Δ colorable

Mohar's conjecture '07

For all G, for all $k \geq \Delta$, $R^k(G)$ is connected
True if G is not regular.

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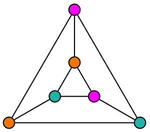

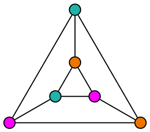

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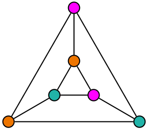

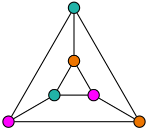

Bonamy, Delecroix, Feghali, L. '21+

For all graph G but the 3-prism, for $k \geq \Delta(G)$, $\mathrm{diam}(R^k(G)) = O(n^2)$

Other work around frozen colorings

Bonamy, Heinrich, Narboni, L. '21

Recoloring version of Hadwiger conjecture:
There exists graphs with no K_t minor but that admit frozen $(\frac{3}{2} - \epsilon)t$-colorings

Bonamy, Heinrich, Narboni, L. '21

Recoloring version of Hadwiger conjecture:
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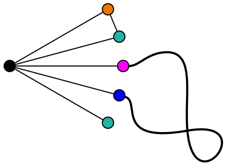
Open Question

Is admitting a frozen coloring the only reason for $R^k(G)$ to be disconneted?



Bonamy, Kaiser, L. '21+

Reed's conjecture for odd hole free graphs and recoloring version for perfect graphs

To be continued...

Connectivity and diameter in various setting

- Diameter for planar graphs ? Bounded genus graph ?
- Diameter for graphs of bounded mad ? d-degenerate graphs ?

New tools

- to prove disconnectivity of $R^k(G)$?
- to prove mixing times upper bounds ?
- to prove lower bounds on the reconfiguration diameter ?

Approximate counting of colorings

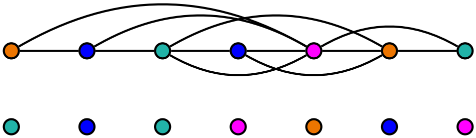
Efficient enumeration

Chordal

- A graph is chordal if it all its induced cycles are triangles
- A graph is chordal iff there exists an ordering of the vertices $v_1 \prec v_2 \cdots \prec v_n$, such that $\forall i, N^+[v_i]$ is a clique
- Chordal graphs are perfect : $\chi(H) = \omega(H)$

Bonamy, Heinrich, Ito, Kobayashi, Mizuta, Mühlenthaler, Suzuki, Wasa '20

If H is chordal, $\mathrm{diam}(R^k(G)) \leq n$ for $k \geq \chi(H)$



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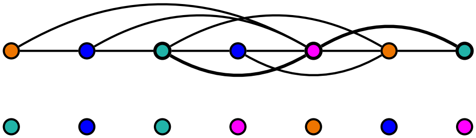

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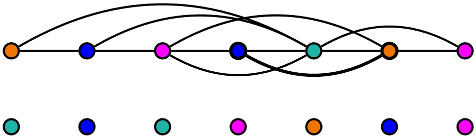

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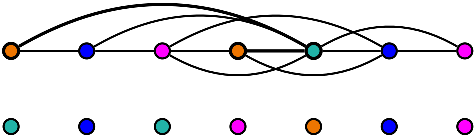

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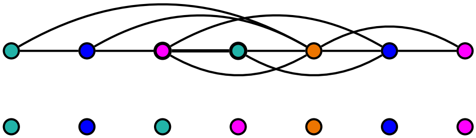

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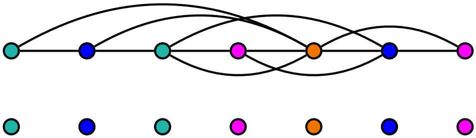

Bonamy, Delecroix, L. '21

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G has treewidth tw iff there exists H chordal with $\omega(H) = \mathrm{tw} + 1$.
Let G and $k \geq \mathrm{tw}(G) + 1$, H an overlying chordal graph with elimination ordering $v_1 \prec v_2 \prec \cdots \prec v_n$

$$\begin{tikzcd}
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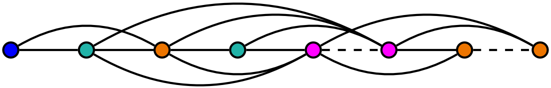

Kempe recoloring with bounded treewidth

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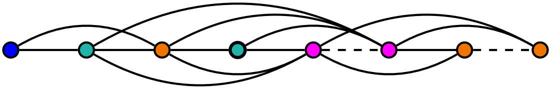

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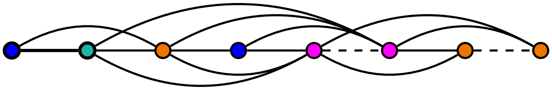

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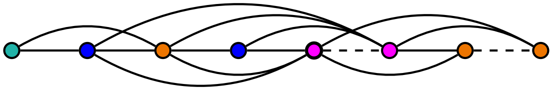

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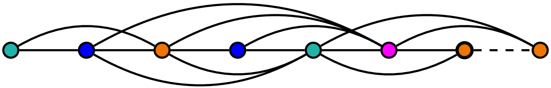

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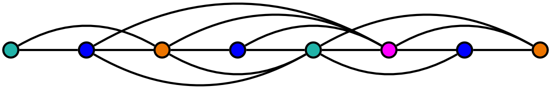

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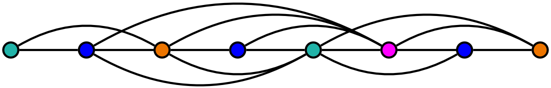
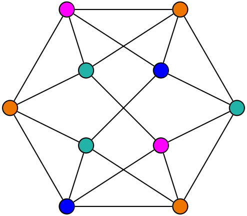


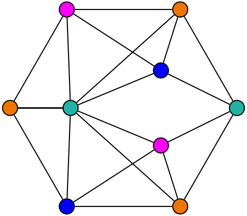
Length of the sequence $(\mathrm{tw}\, n + n + \mathrm{tw}\, n) \times n = O(\mathrm{tw}\, n^2)$

Bonamy, Delecroix, L. '21+

For all graph G but the 3-prism, for $k \geq \Delta(G)$,

$$\mathrm{diam}(R^k(G)) = O(\Delta n^2)$$





Lemma

Key lemma

If G' is $(d-1)$-degenerate, with $\deg(v_i) \leq d$ for all $i < n$, then $\mathrm{diam}(R^k(G')) = O(dn^2)$

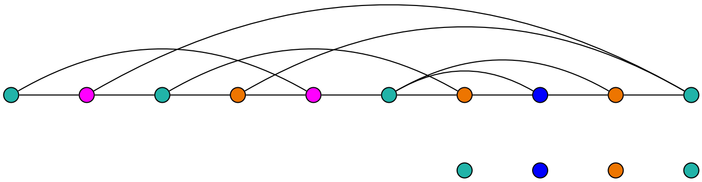
Let u and x be two vertices far away in G, v and w be two non-adjacent neighbors of u
For all $(y, z) \in N(x)$ if y and z are non-adjacent, then there exists a k-coloring α such that $\alpha(v) = \alpha(w)$ and $\alpha(y) = \alpha(z)$

Sketch of proof of the lemma

Induction

If $c \notin \alpha(N^+(u))$, then there exits β such that:

- $\forall v \succ u$, $\alpha(u) = \beta(u)$
- $\beta(u) = c$
- $\alpha \rightsquigarrow_K \beta$ by recoloring each vertex $w \prec u$ at most p times where $p = |\alpha^{-1}(c) \cup N^-(u)|$



Sketch of proof of the lemma

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If $c \notin \alpha(N^+(u))$, then there exits β such that:

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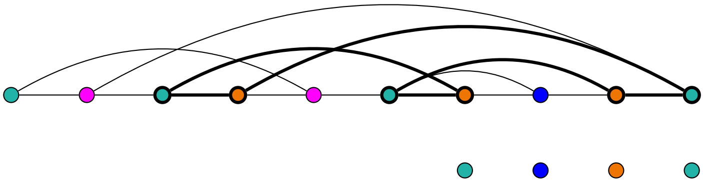

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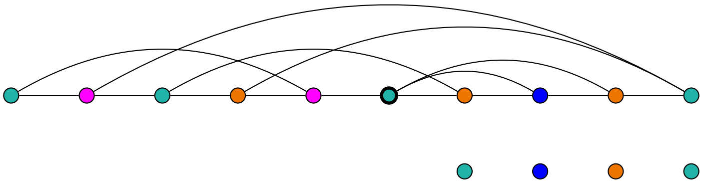

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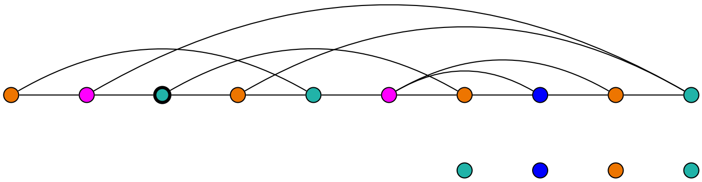

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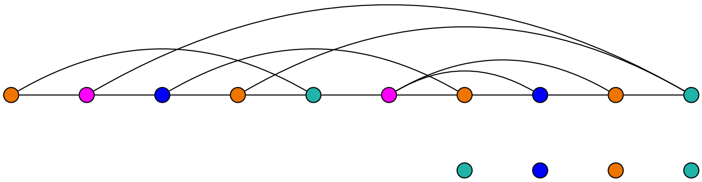
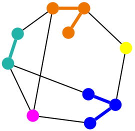
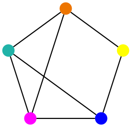


Graph minor

H is a minor of G if H can be obtained be deleting vertices and contracting edges of G





Equivalently, $V_1 \sqcup \cdots \sqcup V_k \subseteq V(G)$, with V_i connected and $G[V_1, \ldots V_k] = H$

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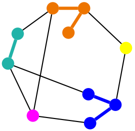

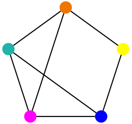

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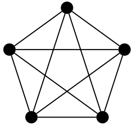
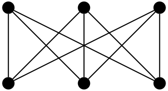
Kuratowski 1930

A graph is planar iff K_5-minor and $K_{3,3}$-minor free





Hadwiger's conjecture

Appel, Haken 1976

If G is planar, then $\chi(G) \leq 4$

Robertson, Sanders, Seymour, Thomas 1997

Much simpler proof, but still computer assisted

Hadwiger's conjecture 1943

If G is K_t-minor free then $\chi(G) \leq t - 1$
Proved for $1 \leq t \leq 6$, widely open for $t > 6$

Reconfiguration counterpoint

Meyniel 1978

All 5-colorings of a planar graph are Kempe-equivalent (tight)

Las Vergnas and Meyniel 1981

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Conjecture 1 [Las Vergnas and Meyniel 1981]

All the t-colorings of a K_t-minor free graph are Kempe-equivalent

Conjecture 2 [Las Vergnas and Meyniel 1981]

All the t-colorings of a K_t-minor free graph are Kempe-equivalent to a $(t-1)$-coloring

Quasi-K_t-minors

Quasi-minor

H is quasi-minor of G if there exists $V_1 \sqcup \cdots \sqcup V_k$ such that $\forall i \neq j, G[V_i \cup V_j]$ is connected and $G[V_1, \ldots V_k] = H$

- K_t-minor $\Rightarrow$ quasi-K_t-minor
- if all V_i are stable sets, $V_1 \sqcup \cdots \sqcup V_k$ is a frozen coloring

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Bonamy, Heinrich, L., Narboni '21+

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