Kempe recoloring version of Hadwiger's conjecture

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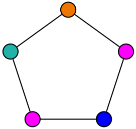
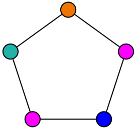
Joint work with Marthe Bonamy, Marc Heinrich and Jonathan Narboni

Recoloring with Kempe changes



Kempe chain (1879)

Maximal bichromatic connected component in G



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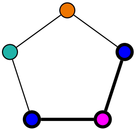

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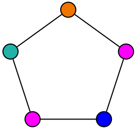

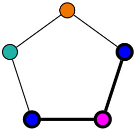

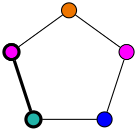

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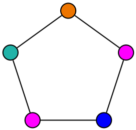
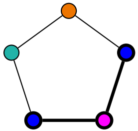
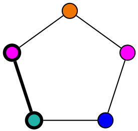
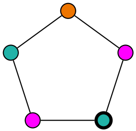








Natural questions in reconfiguration

Reconfiguration graph

- $V(R^k(G)) = k$-colorings of G.
- α and β adjacent if $\alpha \underset{\text{Kempe}}{\longleftrightarrow} \beta$

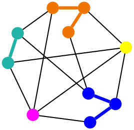
Reachability	Are α and β Kempe-equivalent ?
Connectivity	Is $R^k(G)$ connected ?
Diameter	Estimate $\text{diam}(R^k(G))$?

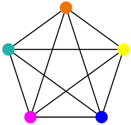
Application

- Powerful tool (ex: Vizing theorem)
- Sampling coloring with a Markov chain

Graph minor

H is a minor of G if H can be obtained be deleting vertices, edges and contracting edges of G





K_k is a minor of G if and only if $V_1 \sqcup \cdots \sqcup V_k \subseteq V(G)$, with V_i connected and $G[V_1, \ldots V_k] = K_k$

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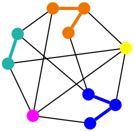

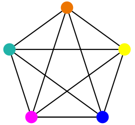

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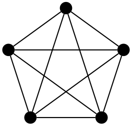
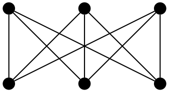
Wagner, Kuratowski 1930

A graph is planar iff K_5-minor and $K_{3,3}$-minor free





Hadwiger's conjecture

Appel, Haken 1976

If G is planar, then $\chi(G) \leq 4$

Robertson, Sanders, Seymour, Thomas 1997

Much simpler proof, but still computer assisted

Hadwiger's conjecture 1943

If G is K_t-minor free then $\chi(G) \leq t - 1$
Proved for $1 \leq t \leq 6$, widely open for $t > 6$

Reconfiguration counterpoint

Meyniel 1978

All 5-colorings of a planar graph are Kempe-equivalent (tight)

Las Vergnas and Meyniel 1981

All 5-colorings of a K_5-minor free graph are Kempe-equivalent

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Conjecture 1 [Las Vergnas and Meyniel 1981]

All the t-colorings of a K_t-minor free graph are Kempe-equivalent

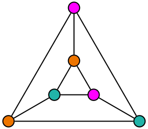
Conjecture 2 [Las Vergnas and Meyniel 1981]

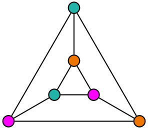
All the t-colorings of a K_t-minor free graph are Kempe-equivalent to a $(t-1)$-coloring

Frozen colorings

Frozen coloring

α is frozen if $\forall i, j \leq k$, the graph induced by colors i and j is connected





Quasi-minor

K_k is quasi-minor of G if there exists $V_1 \sqcup \cdots \sqcup V_k$ such that
$\forall i \neq j, G[V_i \cup V_j]$ is connected and $G[V_1, \ldots V_k] = K_k$

Frozen k-coloring $\Rightarrow$ quasi K_k-minor

Last conjecture

Conjecture 3 [Las Vergnas and Meyniel 1981]

No K_t-minor $\Rightarrow$ No frozen t-coloring

Motivation

If G has no K_t minor and all its t-colorings are Kempe equivalent then

- no frozen t-coloring
- only one t-coloring up to color permutation

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Bonamy, Heinrich, L., Narboni '21+

- Strongly disproved for large t: for every $\varepsilon > 0$ and large enough t, there exists a graph G with a quasi-K_t-minor but no $K_{\left(\frac{2}{3}+\varepsilon\right)t}$-minor

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- All $\frac{t}{2}$-colorings of a K_t-minor free graph are Kempe-equivalent