Kempe changes on Δ-colorings

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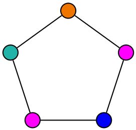
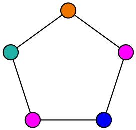
Joint work with Marthe Bonamy and Vincent Delecroix at LaBRI in Bordeaux

Kempe chain (1879)

Maximal bichromatic connected component in G





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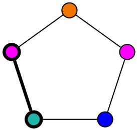

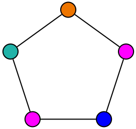

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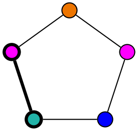
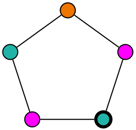
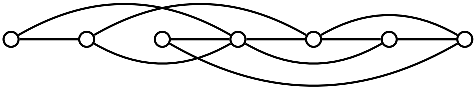




d-degenerate graph

G is d-degenerate if for any $H \subset G$, $\delta(H) \leq d$
Equivalently, G admits an elimination ordering $v_1 \prec v_2 \cdots \prec v_n$ such that

$$\forall i, |N^+(v_i)| \leq d$$



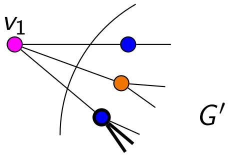
Fundamental lemma

Las Vergnas, Meyniel 1981

All k-colorings of a d-degenerate graph are Kempe-equivalent for $k \geq d+1$

Induction: Let α and β be two colorings of G
Consider $G' = G \setminus \{v_1\}$

$$\alpha_{|G'} \underset{K}{\rightsquigarrow} \beta_{|G'}$$



Lift sequence to G then recolor v_1

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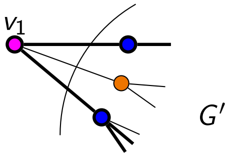

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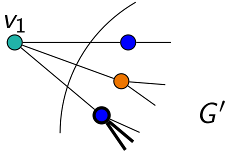

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Recoloring with Δ colors

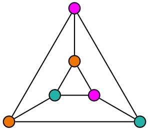
Brook's theorem

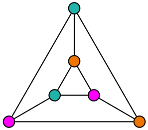
All graphs but odd cycles and cliques are Δ-colorable

Mohar conjecture '07

For all G, for all $k \geq \Delta$, $R^k(G)$ is connected
True if G is not regular

Feghali, Johnson, Paulusma '17 True for all cubic graphs but the 3-prism
Bonamy, Bousquet, Feghali, Johnson '19 True for $\Delta \geq 4$





Bonamy, Delecroix, L. '21+

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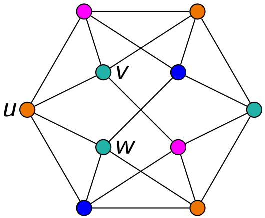
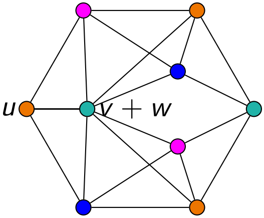


G



G_{v+w}

$$V(R^k(G)) \simeq \bigcup_{(v,w)\in N(u)^2\setminus E} V(R^k(G_{v+w}))$$

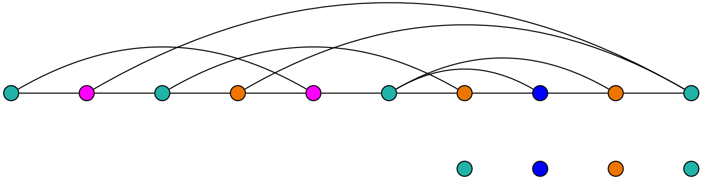
Key lemma

If G' is $(d-1)$-degenerate, with $\deg(v_i) \leq d$ for all $i < n$, then $\mathrm{diam}(R^k(G')) = O(dn^2)$

Induction

If $c \notin \alpha(N^+(u))$, then there exits β such that:

- $\forall v \succ u$, $\alpha(u) = \beta(u)$
- $\beta(u) = c$
- $\alpha \rightsquigarrow_K \beta$ by recoloring each vertex $w \prec u$ at most p times where $p = |\alpha^{-1}(c) \cup N^-(u)|$



Sketch of proof of the lemma

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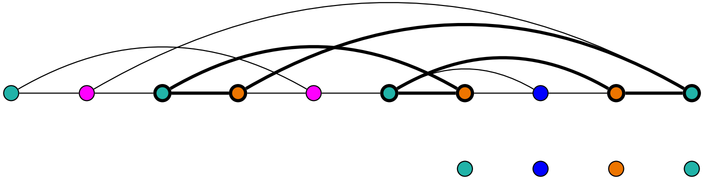

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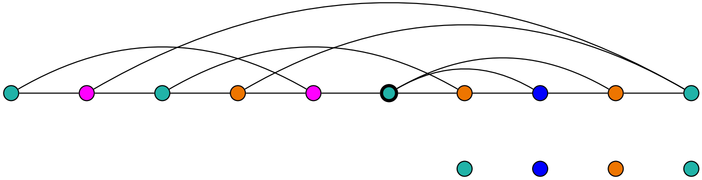

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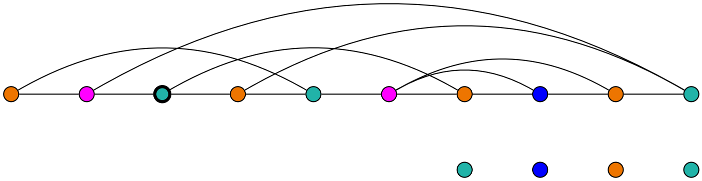

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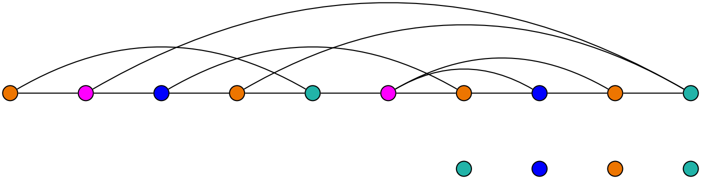

Avoiding the Δ^2 factor: first case

G is 3-connected of diameter more than three

Let u and x be two vertices far away in G, let $(v, w) \in N(u)^2 \setminus E$
For all $(y, z) \in N(x)^2 \setminus E$, $\exists \alpha$ such that $\alpha(v) = \alpha(w)$ and $\alpha(y) = \alpha(z)$

Avoiding the Δ^2 factor: second case

G is not 3-connected

If G_1 and G_2 are $d-1$ degenerate, of degree a most d and $S = G_1 \cup G_2$ is a complete graph, $\mathrm{diam}(R^k(G_1 \cap G_2)) = O(dn^2)$ for $k \geq d$