Kempe recoloring in bounded treewidth graphs

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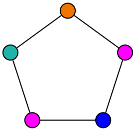
joint work with Marthe Bonamy and Vincent Delecroix

Recoloring with Kempe changes

Kempe chain (1879)

Maximal bichromatic component in G



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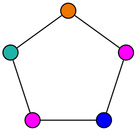

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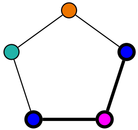

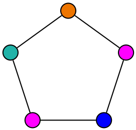

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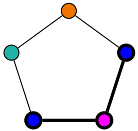

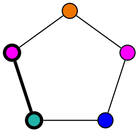

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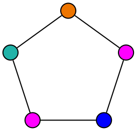

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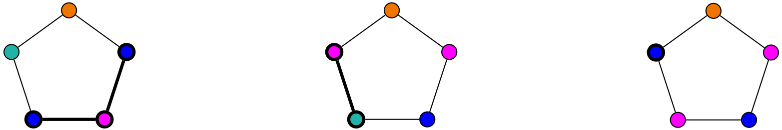


Natural questions

Application

- Powerful tool (ex: Vizing theorem)
- Sampling coloring with a Markov chain

Reconfiguration graph

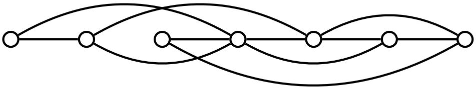
- $V(R^k(G)) = k$-colorings of G.
- α and β adjacent if $\alpha \xleftrightarrow[\text{Kempe}]{} \beta$

Reachability	Are α and β Kempe-equivalent ?
Connectivity	Is $R^k(G)$ connected ?
Diameter	Estimate $\mathrm{diam}(R^k(G))$?

d-degenerate graph

G is d-degenerate if for any $H \subset G$, $\delta(H) \leq d$
Equivalently, G admits an elimination ordering $v_1 \prec v_2 \cdots \prec v_n$ such that

$$\forall i, |N^+(v_i)| \leq d$$

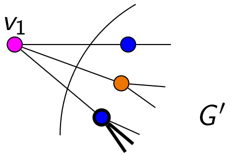


Warm up

Las Vergnas, Meyniel 1981

All k-colorings of a d-degenerate graph are Kempe-equivalent for $k \geq d+1$.

Induction: Let α and β be two colorings of G
Consider $G' = G \setminus \{v_1\}$



$$\alpha_{|G'} \underset{K}{\rightsquigarrow} \beta_{|G'}$$

Lift sequence to G then recolor v_1

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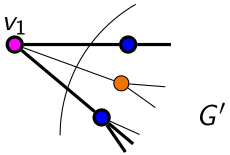

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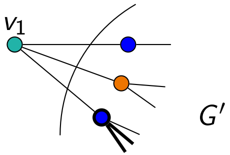

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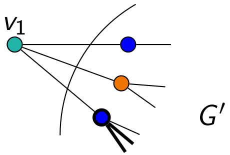

Lift sequence to G then recolor v_1

Natural question

What is the diameter of the reconfiguration graph in this setting ?

Recoloring via trivial Kempe changes

Cerecedas '07

$R^k_{\text{Glauber}}(G)$ is connected if G is d-degenerate and $k \geq d+2$

Cerecedas' conjecture '07

$\text{diam}(R^k_{\text{Glauber}}(G)) \leq O(n^2)$ if G is d-degenerate and $k \geq d+2$

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Treewidth

Graph parameter that measures how close a graph is from being a tree
$\text{tw}(G) \leq k$ implies G is k-degenerate

Bonamy, Bousquet '13

$\text{diam}(R^k_{\text{Glauber}}(G)) \leq O(n^2)$ if $k \geq \text{tw}(G)+2$

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Bousquet, Heinrich '19

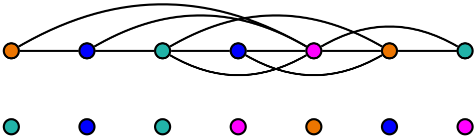
$\text{diam}(R^k_{\text{Glauber}}(G)) \leq O(n^{d+1})$ if G is d-degenerate and $k \geq d+2$

Chordal

- A graph is chordal if it all its induced cycles are triangles
- A graph is chordal iff there exists an ordering of the vertices $v_1 \prec v_2 \cdots \prec v_n$, such that $\forall i, N^+[v_i]$ is a clique
- Chordal graphs are perfect : $\chi(H) = \omega(H)$

Bonamy, Heinrich, Ito, Kobayashi, Mizuta, Mühlenthaler, Suzuki, Wasa '20

If H is chordal, $\text{diam}(R^k(G)) \leq n$ for $k \geq \chi(H)$



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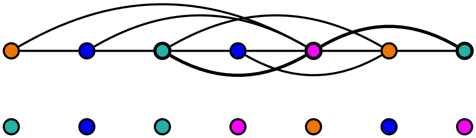

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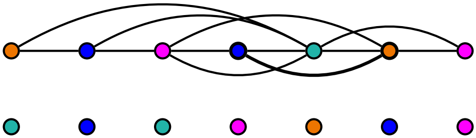

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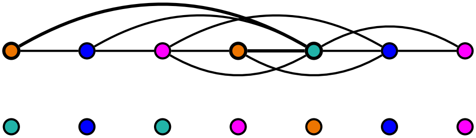

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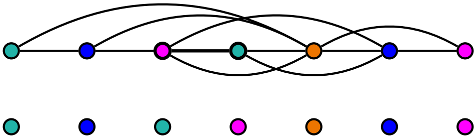

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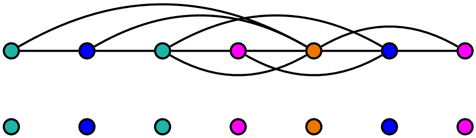

Kempe recoloring with bounded treewidth

Bonamy, Delecroix, L. '21

$\mathrm{diam}(R^k(G)) \leq O(\mathrm{tw}\, n^2)$ if $k \geq \mathrm{tw}(G) + 1$

G has treewidth tw iff there exists H chordal with $\omega(H) = \mathrm{tw} + 1$.
Let G and $k \geq \mathrm{tw}(G) + 1$, H an overlying chordal graph with elimination ordering $v_1 \prec v_2 \prec \cdots \prec v_n$

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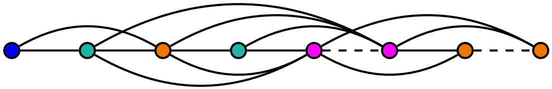

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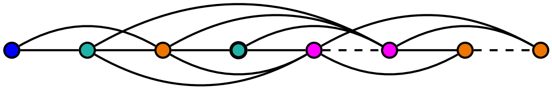

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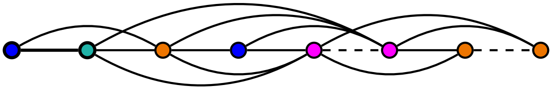

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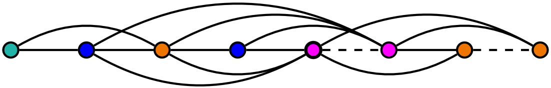

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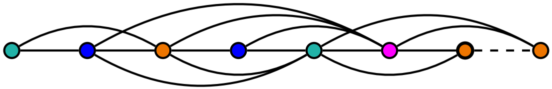

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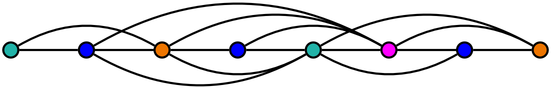

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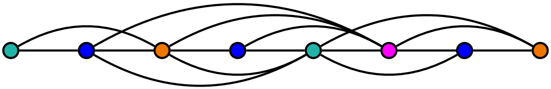

Length of the sequence $(\mathrm{tw}\, n + n + \mathrm{tw}\, n) \times n = O(\mathrm{tw}\, n^2)$

Bonamy, Delecroix, L. '21

G d-degenerate, with all vertices but one of degree at most $d + 1$, then for $k \geq d + 1$, $\mathrm{diam}(R^k(G)) = O(dn^2)$

Open question

Can we bound the diameter without making additional assumptions ?