A Recolouring Version of a Conjecture of Reed

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Abstract

Reed conjectured that the chromatic number of any graph is closer to its clique number than to its maximum degree plus one. We consider a recolouring version of this conjecture, with respect to Kempe changes. Namely, we investigate the largest ε such that all graphs G are k-recolourable for all $k \geqslant \lceil \varepsilon\omega(G) + (1-\varepsilon)(\Delta(G)+1) \rceil$.
For general graphs, an existing construction of a frozen colouring shows that $\varepsilon \leqslant 1/3$. We show that this construction is optimal in the sense that there are no frozen colourings below that threshold. For this reason, we conjecture that $\varepsilon = 1/3$. For triangle-free graphs, we give a construction of frozen colouring that shows that $\varepsilon \leqslant 4/9$, and prove that is it also optimal. In the special case of odd-hole-free graphs, we show that $\varepsilon = 1/2$, and that this is tight up to one colour.

1 Introduction

The chromatic number $\chi(G)$ of any graph G lies between its clique number $\omega(G)$ and the maximum degree $\Delta(G)$ plus one. A natural question is which of these two bounds is closer to the chromatic number. Reed [13] conjectured that the chromatic number of any graph G is at most $\lceil (\omega(G) + \Delta(G) + 1)/2 \rceil$, and in particular, that for any $\varepsilon \leqslant 1/2$, the chromatic number of G is at most $\lceil \varepsilon\omega(G) + (1-\varepsilon)(\Delta(G)+1) \rceil$.

In the same article, Reed proved using the probabilistic method that this bound is tight if true (see [13, Theorem 2]). He also proved that there exists $\varepsilon > 0$ such that for all G, $\chi(G) \leqslant \lceil \varepsilon\omega(G) + (1-\varepsilon)(\Delta(G)+1) \rceil$. More recently, King and Reed [8] gave a significantly shorter proof of this result and proved that for large enough Δ, Reed's conjecture holds for $\varepsilon \leqslant \frac{1}{320e^6}$. This conjecture generated a lot of interest over the past years and similar statements were proved in [13, 4] for increasing values of ε. The best bound known today is due to Hurley, de Joannis de Verclos and Kang [5], who proved that for all graphs G with sufficiently large maximum degree, $\chi(G) \leqslant \lceil 0.119\omega(G) + 0.881(\Delta(G)+1) \rceil$.

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Reed's conjecture has been proven for several hereditary graph classes [7, 1, 14]. Finally, list colouring versions and local strengthenings of Reed's conjecture were also considered in [4, 6] and [7, 1, 6] respectively. More precisely, let $f_\varepsilon(G) = \max_{v \in V(G)} \lceil \varepsilon\omega(v) + (1-\varepsilon)(\deg(v)+1) \rceil$, where $\omega(v)$ is the size of the largest clique containing v, the following local strengthening of Reed conjecture was introduced by King in [7]:

Conjecture 1 (Local Reed Conjecture)**.** *For all $\varepsilon \leqslant 1/2$ and all graphs G, $\chi(G) \leqslant f_\varepsilon(G)$ holds.*

1.1 A recolouring version of Reed's conjecture

In this article, we consider a recolouring variation of Reed's conjecture, that was introduced by Bonamy, Kaiser and Legrand-Duchesne (see [9, Section 2.3.1.2]). Given a graph and a proper colouring of its vertex set, a *Kempe chain* is a connected bichromatic component of the graph. A *Kempe change* consists in swapping the two colours within a Kempe chain (see Figure 1), thereby resulting in another proper colouring. This reconfiguration operation was introduced in 1879 by Kempe in an attempt to prove the Four-Colour theorem. Kempe changes are decisive in the existing proofs of the Four-Colour theorem and of Vizing's edge-colouring theorem. We say that a graph is *k-recolourable* if all its k-colourings are *Kempe equivalent*, that is, if any k-colouring can be obtained from any other through a series of Kempe changes.

Figure 1. Two 3-colourings of the Petersen graph that differ by one Kempe change. The corresponding { ●, ● }-Kempe chain is thickened.

The most common obstruction for k-recolourability is the existence of a *frozen k-colouring*, a k-colouring in which any two colours span a connected subgraph. As a result, the partition induced by the colour classes is invariant under Kempe changes, and if this colouring is not unique (up to colour permutation), then the graph is not k-recolourable. Besides frozen colourings, the only other known obstruction to recolourability is a topological argument of Mohar and Salas [10, 11] that is specific to 3-colourings of highly regular planar or toroidal graphs.

Bonamy, Kaiser and Legrand-Duchesne asked for the largest ε such that all graphs G are k-recolourable for all $k \geqslant \lceil \varepsilon\omega(G) + (1-\varepsilon)(\Delta(G)+1) \rceil$. It remains open whether this holds for some positive ε. On the other hand, the parameter ε is known to be at most $1/3$. Indeed, Bonamy, Heinrich, Legrand-Duchesne, and Narboni [2] constructed a random graph with degrees concentrating around $3n/4$ and expected clique number $\Theta(\log(n))$, that admits a frozen $n/2$-colouring and is not $n/2$-recolourable with high probability. In particular, this hints that Kempe changes are unlikely to be of any use to prove Reed's conjecture in the range $\varepsilon \in [1/3, 1/2]$.

In Section 4, we prove that this construction is optimal in general:

Theorem 1. *For any $\eta \leqslant 1/3$, if G admits a frozen k-colouring for $k \leqslant \lceil \eta\omega(G) + (1-\eta)(\Delta(G)+1) \rceil$, then it is unique up to permuting colours.*

Since the random construction of Bonamy, Heinrich Legrand-Duchesne and Narboni admits large cliques of size $\Theta(\log k)$, it is natural to ask whether similar obstructions occur in graphs of bounded clique number. We fully resolve this question for triangle-free graphs by giving an optimal construction of a triangle-free graph with a frozen non-unique k-colouring and maximum degree at most δk for any $\delta > 9/5$:

Theorem 2. *For any $\eta > 4/9$, there exist triangle-free graphs that are not k-recolourable, with $k = \lceil 2\eta + (1-\eta)(\Delta+1) \rceil$.*
Moreover, for any $\eta \leqslant 4/9$, if a triangle-free graph admits a frozen k-colouring for $k = \lceil 2\eta + (1-\eta)(\Delta+1) \rceil$, then it is unique up to permuting colours.

Since Theorems 1 and 2 guarantee that the main obstructions to recolourability do not occur in these ranges, they strongly motivate the following conjectures.

Conjecture 2. *Any graph G is k-recolourable for all $k > \lceil \frac{1}{3}\omega(G) + \frac{2}{3}(\Delta(G)+1) \rceil$.*

Conjecture 3. *Any triangle-free graph G is k-recolourable for all $k > \lceil \frac{4}{9} \cdot 2 + \frac{5}{9}(\Delta(G)+1) \rceil$.*

As a consequence of Theorem 1 (respectively Theorem 2), note that disproving Conjecture 2 (respectively Conjecture 3) would require new tools and would significantly improve our understanding of Kempe recolouring and its obstructions. Finally, all advances towards Reed's conjecture rely on the probabilistic method. This proof method does not adapt well to reconfiguration proofs. For this reason, proving Conjecture 2 or Conjecture 3 even for small positive ε seems challenging and is likely to either be the first application of the probabilistic method to reconfiguration, or to yield a constructive proof of Reed's colouring question, which would be of independent interest.

1.2 Recolouring odd-hole-free graphs

A *hole* is an induced cycle of length at least four. The complement of a hole is an *antihole*, and an odd-(anti)hole is an (anti)hole of odd size. The Strong Perfect Graph Theorem [3] characterises perfect graphs, that is graphs with chromatic number equal to their clique number, as graphs that neither have odd-holes nor odd-antiholes.

Aravind, Karthick and Subramanian [1] proved the Local Reed Conjecture in the class of odd-hole-free graphs. Weil [14] then observed this result generalises immediately to the class $\mathcal{H}$ of graphs whose odd-holes all contain some vertex of degree less than $f_{1/2}(G)$. Bonamy, Kaiser and Legrand-Duchesne [9] gave an alternative proof of this result using Kempe changes:

Theorem 3 (Bonamy, Kaiser and Legrand-Duchesne [9])**.** *Every k-colouring of any graph $G \in \mathcal{H}$ is Kempe equivalent to a $f_{1/2}(G)$-colouring.*

This result also implies that the graphs in $\mathcal{H}$, and in particular odd-hole-free graphs, have no frozen k-colourings for $k > f_{1/2}(G)$. Again, as frozen colourings are the main known obstruction to recolourability, this motivated the following conjecture:

Conjecture 4 (Bonamy, Kaiser and Legrand-Duchesne [9])**.** *All odd-hole-free graphs of maximum degree Δ and clique number ω are k-recolourable for $k \geqslant \lceil \frac{\omega+\Delta+1}{2} \rceil$.*

In [9], the same authors proved a weakening of Conjecture 4: all odd-hole-free graphs G are k-recolourable for $k > \lceil (\chi(G) + \Delta(G) + 1)/2 \rceil$, which settles the special case of perfect graphs. Moreover, as Reed's conjecture holds for odd-hole-free graphs, this also implies that they are k-recolourable for $k > \lceil(\omega(G) + 3(\Delta(G) + 1))/4\rceil$. The main contribution of this paper improves this result and confirms Conjecture 4 up to using one extra colour:

Theorem 4. *Let G be graph in which every odd-hole contains a vertex of degree at most $f_{1/2}(G)$. Then G is k-recolourable for all $k > f_{1/2}(G)$.*

A folklore construction of a frozen colouring shows that this result is almost tight:

Theorem 5. *For all $k \geqslant 3$, there is a non-k-recolourable perfect graph G with $k = \left\lceil \frac{\omega(G)+\Delta(G)+1}{2} \right\rceil - 1$.*

Therefore, the only remaining open case for Conjecture 4 is $k = f_{1/2}(G)$. Moreover, this also shows that Theorem 3 is tight up to one colour, even for perfect graphs.

1.3 Summary

Given a class $\mathcal{G}$ of connected graphs, let $\varepsilon^\star(\mathcal{G})$ be the supremum of all ε such that there exists $c \geqslant 0$ such that all graphs in $\mathcal{G}$ are k-recolourable for all $k \geqslant \lceil \varepsilon\omega + (1-\varepsilon)(\Delta+1)\rceil + c$. Denote $\eta^\star(\mathcal{G})$ be the infimum η such that there exists a non-k-recolourable graph in $\mathcal{G}$ with a frozen k-colouring and $k = \lceil \eta\omega + (1-\eta)(\Delta+1)\rceil$. We have $\varepsilon^\star(\mathcal{G}) \leqslant \eta^\star(\mathcal{G})$ for any graph class $\mathcal{G}$. Our results can be summarised as follows:

$\mathcal{G}$	all graphs	triangle-free	odd-hole-free
$\varepsilon^\star$	Conjecture 2	Conjecture 3	1/2 (Thm 4)
$\eta^\star$	1/3 (Thm 1)	4/9 (Thm 2)	1/2 (Thm 5)

After providing some notations and basic definitions in Section 2, we prove Theorem 4 in Section 3 and the bounds on $\eta^\star$ in Section 4.

2 Preliminaries

We now define the notation that we will be using in our proofs. Let G be a graph. A *k-colouring* of G is a function $\gamma : V(G) \to [k]$ such $\gamma(u) \neq \gamma(v)$ for all $uv \in E(G)$. For any subgraph H of G, we let $\gamma(H) := \{\gamma(v) : v \in V(H)\}$ be the set of colours used in the subgraph H, and let $\gamma|_H$ be the colouring γ restricted to H. We say that two colourings α and β *agree* on a set $X \subseteq V(G)$ if $\alpha|_X = \beta|_X$. Similarly, we say that α and β *differ* on X, if X is the set of all vertices $v \in V(G)$ such that $\alpha(v) \neq \beta(v)$. We denote $N(u)$ and $N[u]$ the open and closed neighbourhoods of a vertex u and say that u *misses* the colour c in the colouring γ if $c \notin \gamma(N[u])$. Given a vertex v and subset U of the vertices, we denote $\deg(v, U) = |N(v) \cap U|$. Likewise, given two subset U_1 and U_2 of vertices, we denote $\deg(U_1, U_2)$ the number of edges between U_1 and U_2, and denote $\deg(U) = \deg(U, V(G) \setminus U)$.

An *$\{a, b\}$-Kempe chain* K of γ is a connected component of the subgraph induced by vertices of colours a or b (see Figure 1). We say that a and b are the colours *used* in K. We denote $K_{v,c}(G, \gamma)$ the $\{\gamma(v), c\}$-Kempe chain that contains the vertex v. Let $K(G, \gamma)$ be the set of all Kempe chains in G under the colouring γ, where we also allow the empty Kempe chain (the graph with no vertices) to be part of this set. For an $\{a, b\}$-Kempe chain $K \in K(G, \gamma)$, let $\gamma^{\to}(K)$ be the colouring obtained after interchanging the colours a and b on all the vertices in K. We also refer to this operation as performing the *Kempe change* K in G under γ.

Given an induced subgraph G' of G, a colouring γ of G and an $\{a, b\}$-Kempe chain K of $\gamma|_{G'}$, the *extension* of K is the $\{a, b\}$-Kempe chain of γ that includes K.

We now state a lemma that will be used repeatedly in Section 4 to lower bound the maximum degree of a graph:

Lemma 2.1. *Any colour class U of a frozen k-colouring of an n-vertex graph G has average degree*

$$\frac{\deg(U)}{|U|} \geqslant \frac{n - |U| + (k-1)(|U|-1)}{|U|}.$$

Proof. For each other color class U', the graph $G[U \cup U']$ is connected because α is frozen. Hence $G[U \cup U']$ has at least $|U| + |U'| - 1$ edges and

$$\frac{\deg(U)}{|U|} \geqslant \sum_{U'} \frac{|U| + |U'| - 1}{|U|} \geqslant \frac{n - |U| + (k-1)(|U|-1)}{|U|}.$$

□

Finally we will also use the Chernoff bound in Section 4, to prove that the maximum degree of a random construction concentrates around the average degree.

Lemma 2.2 (Chernoff bound)**.** *For any $t \geqslant 0$, any binomial random variable $X \sim \mathcal{B}(n,p)$ satisfies*

$$\mathbb{P}(|X - np| > t) < 2e^{-\frac{t^2}{3np}}.$$

3 Recolourability of odd-hole-free graphs

This section is dedicated to the proof of Theorem 4, that we recall:

Theorem 4. *Let G be graph in which every odd-hole contains a vertex of degree at most $f_{1/2}(G)$. Then G is k-recolourable for all $k > f_{1/2}(G)$.*

Recall also that $f_{1/2}(G) = \max_{v \in V(G)} \lceil (\omega(v) + \deg(v) + 1)/2 \rceil$. Let G be a graph in which all odd-hole contain some vertex of degree at most $f_{1/2}(G)$ and let k be an integer satisfying $k > f_{1/2}(G)$. We proceed by induction on the number of vertices of G.

If G contains an odd-hole, let v be a vertex of degree at most $f_{1/2}(G)$ in it. A classical result (namely Lemma 2.3 in [?], used in a variety of other Kempe recolouring proofs) shows that G is k-recolourable, as $k > f_{1/2}(G) \geqslant \deg(v)$ and $G \setminus \{v\}$ is k-recolourable by induction. Therefore one can assume that G has no odd-holes.

Let α and β be two k-colourings of G. We will transform α into β using a sequence of Kempe chains in G. Let $v \in V(G)$ be a vertex of G, and let $G' := G - v$. Let $\alpha' := \alpha|_{G'}$ and $\beta' := \beta|_{G'}$ be the restriction of colourings α and β to G', respectively. By the induction hypothesis, there exists a sequence $\alpha' = \alpha'_0, \alpha'_1, \ldots, \alpha'_h = \beta'$ of k-colourings of G' such that $\alpha'_i = \overrightarrow{\alpha'_{i-1}}(K_i)$, for some sequence of Kempe chains $K_1, K_2, \ldots, K_h$ with $K_i \in K(G', \alpha'_{i-1})$ for $i = 1, \ldots, h$.

A natural strategy to recolour α into β would be to apply the Kempe changes $K_1, \ldots, K_h$ to α. Unfortunately, any of these Kempe chains might not be a valid Kempe chain in G if it uses the colour of v. To circumvent this issue, the key idea is to allow for some controlled errors at each step. We formalise this error control with the following definition. We say that a colouring γ of G is *faithful* to a colouring γ' of G' if there exist two colours a and b such that the following conditions hold:

C1 The colourings $\gamma|_{G'}$ and γ' differ on a set $\mathcal{B}$ of $\{a,b\}$-Kempe chains under γ', and the vertices of $\mathcal{B}$ only use colours a, b under γ. We call $\mathcal{B}$ the *bad* Kempe chains for γ and γ'.

C2 Every Kempe chain in $\mathcal{B}$ contains a neighbour of v coloured a in γ'.

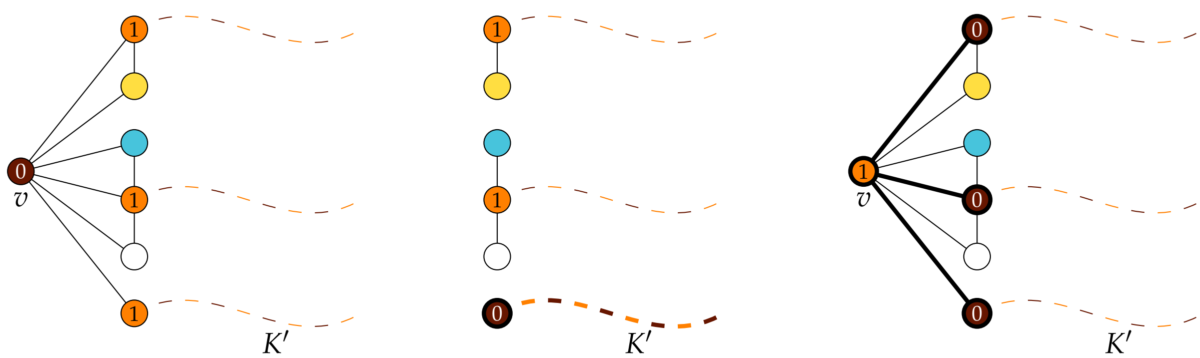
We will construct a sequence of colourings $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_h$ of G such that each α_i is faithful to α'_i and for any $1 \leqslant i \leqslant h$, the colourings α_i and α_{i-1} are Kempe equivalent in G. Assume that we have already constructed the sequence $\alpha_0, \ldots, \alpha_{i-1}$ for some $1 \leqslant i \leqslant h$. We use two lemmas to define α_i: Lemma 3.1 in the favourable case where α'_{i-1} and α_{i-1} agree on G', and Lemma 3.2 in the more involved case where α_{i-1} is faithful to α'_{i-1} without extending it. These two lemmas will be proved in Subsection 3.1 and Subsection 3.2 respectively.

Once the sequence $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_h$ is constructed, we only need to argue that β is Kempe equivalent to α_h, in order to conclude the proof of Theorem 4. We first argue that α_h is Kempe equivalent to a colouring γ that agrees with β on G', then β can be obtained from γ by simply recolouring v to its final colour. Let $\mathcal{B}$ be the set of bad $\{a, b\}$-Kempe chains for α_h and α'_h. If these Kempe chains are also Kempe chains of G, then performing them results in a colouring γ that agrees with β on G'. If not, then v is coloured a or b and as β is a colouring of G, this implies that $\mathcal{B}$ contains all $\{a, b\}$-Kempe chains of α'_h that contain a neighbour of v. Therefore there exists an $\{a, b\}$-Kempe chain K of α_h whose restriction to G' is $\mathcal{B}$. As a result, $\gamma = \alpha_h{}^{\rightarrow}(K)$ agrees with β on G', which concludes the proof of Theorem 4.

3.1 Defining α_i when α_{i-1} and α'_{i-1} agree on G'

Lemma 3.1. *Let γ and γ' be colourings of G and G' respectively, such that $\gamma|_{G'} = \gamma'$. Let K' be a $\{a, b\}$-Kempe chain of γ', and let K be the extension of K' in γ. Then the colouring $\delta := \gamma^{\rightarrow}(K)$ is faithful to $\delta' := \gamma'^{\rightarrow}(K')$.*

Proof. If K' is also a Kempe chain in G under γ, i.e. $K = K'$, then $\delta = \gamma^{\rightarrow}(K')$ and its restriction to G' is δ'. In particular, δ is faithful to δ'. Hence, we can assume that K contains v. Up to relabelling colours, assume that $\gamma(v) = 0$ and the other colour used by K is 1. Let $\mathcal{A}$ be the set of connected components of $K - v$ (see Figure 2a). Note that $\mathcal{A}$ is the set of all $\{0, 1\}$-Kempe chains of γ' that contain a neighbour of v coloured 1. By performing K in G under γ, we obtain a colouring δ whose restriction to G' differs with δ' on the chains of $\mathcal{B} := \mathcal{A} \setminus \{K'\}$ (see Figures 2b and 2c), so **C1** holds. By construction, every Kempe chain in $\mathcal{B}$ contains a neighbour of v coloured 0 in γ, so **C2** also holds. □



(a) The colouring γ. **(b)** The colouring δ'. **(c)** The colouring δ.

Figure 2. The different colourings of Lemma 3.1. The set of Kempe chains in $\mathcal{A}$ are represented by dashed lines. The Kempe chains on which δ and δ' differ with γ are thickened.

In particular, by defining $\alpha_0 = \alpha$, Lemma 3.1 builds a colouring α_1 Kempe equivalent to α_0 and faithful to α'_1.

3.2 Defining α_i when α_{i-1} is faithful to α'_{i-1}

In the more general case where α_{i-1} is only faithful to α'_{i-1}, we either perform a series of Kempe changes that maintains the set $\mathcal{B}$ of bad Kempe chains and does not create any other bad Kempe chains; or we perform a series of Kempe changes to fix the bad Kempe chains of $\mathcal{B}$, before creating a Kempe equivalent colouring α_i faithful to α'_i, with a different set of bad Kempe chains. This procedure is handled by the following lemma:

Lemma 3.2. *Let γ'_1, γ'_2 be two colourings of G' that differ by one Kempe change K'. Let γ_1 be a colouring of G faithful to γ'_1. Then there is a colouring γ_2 of G that is faithful to γ'_2 and Kempe equivalent to γ_1.*

Proof. Up to relabelling the colours, we can assume that the set $\mathcal{B}$ of bad Kempe chains under γ'_1 are using the colours 0 and 1 and that every Kempe chain in $\mathcal{B}$ contains a neighbour of v coloured 1 in γ'_1. Lemma 3.2 results directly from Claim 3.3, Claim 3.4 and Claim 3.5.

Claim 3.3. *Lemma 3.2 holds if $\gamma_1(v) \neq 0$.*

Proof of Claim. Assume that $\gamma_1(v) \neq 0$. Then the bad Kempe chains of $\mathcal{B}$ are also Kempe chains of G under γ_1. So we can successively perform all the Kempe chains in $\mathcal{B}$ under γ_1 to obtain a colouring δ such that $\delta|_{G'} = \gamma'_1$. By Lemma 3.1, $\gamma_2 = \delta^{\rightarrow}(K')$ is faithful to γ'_2 and Kempe equivalent to γ_1. ■

Claim 3.4. *Lemma 3.2 holds if K' does not use the colours 0 and 1.*

Proof of Claim. Assume that K' uses neither 0 nor 1. By Claim 3.3, we can assume that $\gamma_1(v) = 0$. Then, K' is also a Kempe chain under γ_1 and it does not intersect with the Kempe chains in $\mathcal{B}$. So $\gamma_2 := \gamma_1{}^{\rightarrow}(K')$ satisfies the conditions **C1** and **C2**, by taking identical a, b and $\mathcal{B}$. ■

Claim 3.5. *Lemma 3.2 holds if K' uses 0 or 1 and $\gamma_1(v) = 0$.*

Proof of Claim. Assume that $\gamma_1(v) = 0$ and that K' uses 0 or 1. So, we have $|\{0,1\} \cup \gamma'_1(K')| \leqslant 3$.

If v is missing a colour c in γ_1, that is $c \notin \gamma_1(N[v])$, then we can recolour v into c without changing $\gamma_1|_{G'}$ and conclude using Claim 3.3. Thus, we can assume that the neighbourhood of v contains all the colours except 0 in γ_1, i.e. $|\gamma_1(N(v))| = k - 1$.

Likewise, if v has only one neighbour w coloured 1 in γ_1, then $\gamma_1|_{G'}$ and γ'_1 differ only on one bad Kempe chain $B' \in \mathcal{B}$ (see Figure 3a). Let B be the extension of B' and $\delta = \gamma_1{}^{\rightarrow}(B)$. Then $\delta|_{G'} = \gamma'_1$ (see Figure 3b) and γ_2 is given by Lemma 3.1. Thus we can also assume that v has at least two neighbours coloured 1. In order to recolour the vertex v and reduce to Claim 3.3, we find a Kempe chain L of γ_1 such that v misses some colour $c \notin \{0,1\} \cup \gamma'_1(K')$ in $\gamma_1{}^{\rightarrow}(L)$. We then recolour v with this c, perform the Kempe changes in $\mathcal{B}$ and K', and maintain L as the next bad Kempe chain (i.e., define $\mathcal{B}$ as $\{L\}$). We now formalise this.

(a) The colouring γ_1.

(b) The colouring δ extends γ_1'.

Figure 3. Recolouring sequence when v has only one neighbour coloured 1.

Let S be the set of neighbours of v whose colour appears only once in $\gamma_1(N(v))$. Recall that v has at least two neighbours coloured 1 in γ_1, so these vertices do not belong to S. We have

$$\begin{aligned}\deg(v) = |N(v)| &\geqslant 2(k-1-|S|) + |S| \\ &\geqslant 2\left\lceil \frac{\omega(v)+\deg(v)+1}{2} \right\rceil - |S| \\ &> \omega(v) + \deg(v) - |S|,\end{aligned}$$

which implies that $|S| \geqslant \omega(v)+1$. Let T be the subset of vertices in S that are not coloured by γ_1 with a colour in $\{0,1\} \cup \gamma_1'(K')$. Since $|\{0,1\} \cup \gamma_1'(K')| \leqslant 3$ and S contains no vertices coloured 1 or 0, we have $|T| \geqslant \omega(v)$. Thus, T induces a non-edge uw (otherwise, $T \cup \{v\}$ forms a clique of size $\omega(v)+1$ in G). Up to relabelling, say u and w are respectively coloured 2 and 3 in γ_1 (see Figure 4 for an illustrative example following the consecutive Kempe changes performed on γ_1).

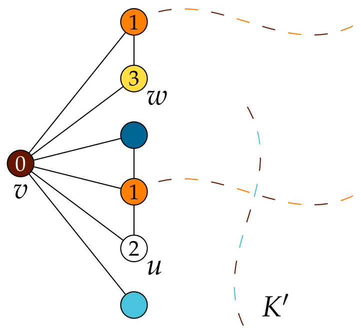
Let L be the $\{2,3\}$-Kempe chain in γ_1 that contains w and δ be the colouring $\gamma_1{}^{\rightarrow}(L)$ (see Figure 4b). Note that L does not contain u, otherwise $L \cup \{v\}$ would contain an induced odd cycle, which is also an induced odd cycle of G. As the colours 2 and 3 appear only once in the neighbourhood of v in γ, the vertex v misses the colour 2 in δ. Let σ be the colouring obtained from δ by recolouring v into 2 (see Figure 4c) and τ be the colouring obtained from σ by performing the Kempe chains in $\mathcal{B}$ (see Figure 4d). The colouring $\sigma|_{G'}$ differs from γ_1' only on the $\{2,3\}$-Kempe chain L, so σ is faithful to γ_1'. Finally, K' uses neither 2 nor 3 so Claim 3.4 applied to σ yields the desired γ_2. ■

□

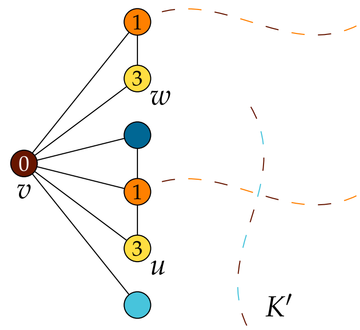
4 Frozen k-colourings of graphs with low degree and clique number

Recall that given a class $\mathcal{G}$ of connected graphs, we denote $\eta^\star(\mathcal{G})$ the infimum η such that there exists a non k-recolourable in $\mathcal{G}$ with a frozen k-colouring and $k = \lceil \eta\omega + (1-\eta)(\Delta + 1) \rceil$. Note that for all fixed graph G, the function $\eta \mapsto \eta\omega(G) + (1-\eta)(\Delta(G)+1)$ is non-increasing, so the ceiling in the definition of $\eta^\star$ is redundant and $\eta^\star(\mathcal{G}) = \inf\{\eta : \exists G \in \mathcal{G}$ with a frozen non-unique k-colouring and $k = \eta\omega(G) + (1-\eta)(\Delta(G)+1)\}$.

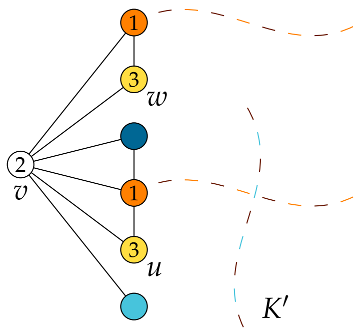
We give the value of $\eta^\star$ for odd-hole-free graphs, general graphs and triangle-free graphs in Subsection 4.1, Subsection 4.2 and Subsection 4.3 respectively.



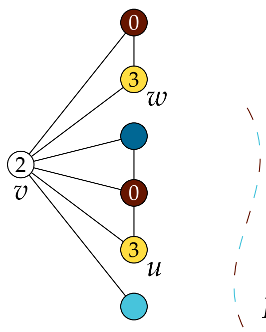
(a) The colouring γ_1.



(b) The colouring δ obtained by recolouring u with 3. The vertex v is now missing the colour 2.



(c) The colouring τ obtained by recolouring v with 2.



(d) The colouring σ obtained by reclouring the bad $\{0,1\}$-Kempe chains

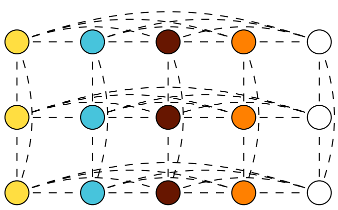
Figure 4. The recolouring sequence of Claim 3.5.

4.1 Odd-hole-free graphs

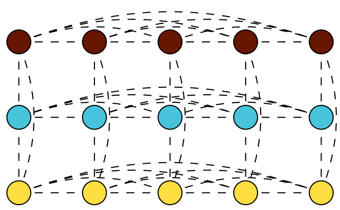
Theorem 5, that we recall, shows that $\eta^\star(\{\text{odd-hole-free graphs}\}) = 1/2$:

Theorem 5. *For all $k \geqslant 3$, there is a non-k-recolourable perfect graph G with* $k = \left\lceil \frac{\omega(G)+\Delta(G)+1}{2} \right\rceil - 1$.

Proof. Let G_k be the tensor product of the cliques K_3 and K_k on three and k vertices, i.e. G_k is the complement of the $3 \times k$ Rook graph $K_3 \square K_k$. The vertices of G_k are indexed by the tuples (u, v) with $u \in \{1, 2, 3\}$ and $v \in \{1, \dots, k\}$, and $(u_1, v_1)(u_1, v_2) \in E(G_k)$ if $u_1u_2 \in E(K_3)$ and $v_1v_2 \in E(K_k)$ (see Figure 5 for an example representing G_5).



(a) The k-colouring α of G_5



(b) The 3-colouring β of G_5

Figure 5. Colourings of G_5. Non-edges are depicted as dashed lines, while edges are omitted for clarity.

Note that G_k is perfect [12]. Moreover, it is $2(k-1)$-regular and has clique number three. So we have that:

$$k = \left\lceil \frac{\omega(G_k) + \Delta(G_k) + 1}{2} \right\rceil - 1$$

Moreover, G_k has a frozen k-colouring α with $\alpha((u,v)) := v$ and a 3-colouring β with $\beta(u,v)) := u$, which is a fortiori a k-colouring, so G_k is not k-recolourable. $\square$

This proves that Theorems 3 and 4 are tight up to one colour even for perfect graphs, in the sense that the only open case of Conjecture 4 is $k = f_{1/2}(G)$.

4.2 General graphs

For general graphs, it is possible to improve Theorem 5 using the random construction of Bonamy, Heinrich, Legrand-Duchesne and Narboni [2], which gives graphs that admit a frozen non-unique $\lceil \eta\omega + (1-\eta)(\Delta+1) \rceil$-colouring for all $\eta > 1/3$. We prove that this construction is optimal, namely:

Theorem 1. *For any $\eta \leqslant 1/3$, if G admits a frozen k-colouring for $k \leqslant \lceil \eta\omega(G) + (1-\eta)(\Delta(G)+1) \rceil$, then it is unique up to permuting colours.*

Proof. Let $\mathcal{G}$ be the class of all connected graphs. As cliques have only one colouring up to colour permutation, $\eta^\star(\mathcal{G}) = \eta^\star(\mathcal{G}')$, where $\mathcal{G}' = \mathcal{G} \setminus \{K_t : t \geqslant 1\}$. Thus, we aim to prove that $\eta^\star := \eta^\star(\mathcal{G}') = 1/3$. The construction of Bonamy, Heinrich, Legrand-Duchesne and Narboni [2] shows that $\eta^\star \leqslant 1/3$.

Claim 4.1. *Let α_1 be a frozen k_1-colouring of a graph $G_1 \in \mathcal{G}'$ with a colour class of size one. Then $\eta_1 = 1$ or there exists a graph $G_2 \in \mathcal{G}'$ with a frozen k_2-colouring and all colour classes of size at least two, such that $\eta_2 \leqslant \eta_1$; where η_i satisfies $k_i = \eta_i\omega(G_i) + (1-\eta_i)(\Delta(G_i)+1)$.*

Proof of Claim. Let X be the set of vertices that are uniquely coloured in α_1. Let $G_2 = G_1 - X$ and α_2 be the $(k_1 - |X|)$-colouring induced by α_1 on G_2. Since α_1 is frozen, X induces a clique in G_1 and α_2 is frozen. As G_1 is not a clique, there is at least one colour class in α_1 that has size at least two. If there is only one such colour class, then we have that $k_1 = \omega(G_1) < \Delta(G_1) + 1$, which implies that $\eta_1 = 1$. Thus, we can assume that there are at least two colour classes of size at least two in α_1. Since every pair of colour classes induces a connected graph in G_1, G_2 is connected and $\alpha_1|_{G_2}$ is frozen. And since G_2 has a colour class of size at least two, G_2 is not a clique and is in $\mathcal{G}'$.

The vertices of X dominate G_1, so $\Delta(G_1) = |G_1| - 1$ and $\Delta(G_2) \leqslant \Delta(G_1) - |X|$. On the other hand, $\omega(G_2) = \omega(G_1) - |X|$ and $k_2 = k_1 - |X|$. Hence, from G_1 to G_2, the number of colours of the frozen colouring and the clique number decreased exactly by $|X|$, while the maximum degree decreased by at least $|X|$, so $\eta_2 \leqslant \eta_1$. $\blacksquare$

For all G and k, denote $\eta^{(k)}(G)$ the smallest η such that there exists a frozen k-colouring α of G, with $k = \eta\omega(G) + (1-\eta)(\Delta(G)+1)$ and such that all colour classes of α have size at least two. Denote also $\eta(G) = \inf\{\eta^{(k)}(G) : k \in \mathbb{N}\}$. By Claim 4.1, we have that $\eta^\star = \inf\{\eta(G) : G \in \mathcal{G}'\}$.

Claim 4.2. *Let α be a frozen k-colouring of $G \in \mathcal{G}'$ with no colour class of size one. Then $\Delta(G) \geqslant \frac{3}{2}(k-1)$. As a result, $\eta^{(k)}(G) > 1/3$.*

Proof of Claim. Let U be a colour class of minimal size. We have $n \geqslant 2k$ and by Lemma 2.1,

$$\begin{aligned}\Delta(G) \geqslant \deg(U)/|U| &\geqslant \frac{n-|U|+(k-1)(|U|-1)}{|U|}\\ &\geqslant \frac{3}{2}(k-1)\end{aligned}$$

So $k \leqslant 2\Delta(G)/3+1$. We have $k = \eta^{(k)}(G)\omega(G) + (1-\eta^{(k)}(G))(\Delta(G)+1)$ and thus

$$\eta^{(k)}(G) = \frac{\Delta(G)+1-k}{\Delta(G)+1-\omega(G)} > \frac{\Delta(G)+1-k}{\Delta(G)} \geqslant \frac{1}{3}$$

■

This proves that $\eta(G)$ and $\eta^\star$ are at least $1/3$, and hence $\eta^\star = 1/3$. Moreover, note that $\eta^\star$ is unattained in $\mathcal{G}$, otherwise some graph $G \in \mathcal{G}$ would verify $\eta^{(k)}(G) = 1/3$. As all frozen non-unique k-colourings of any graph G must verify $k > \eta^\star\omega(G) + (1-\eta^\star)(\Delta+1)$, this concludes the proof of Theorem 1. □

4.3 Triangle-free graphs

The random construction of Bonamy, Heinrich Legrand-Duchesne and Narboni has clique number $\Theta(\log k)$. We show here that $\eta^\star$ changes when restricting to graphs of bounded clique number: Theorem 2, that we recall below, shows that $\eta^\star(\{\text{triangle-free graphs}\}) = 4/9$.

Theorem 2. *For any $\eta > 4/9$, there exist triangle-free graphs that are not k-recolourable, with $k = \lceil 2\eta + (1-\eta)(\Delta+1)\rceil$.*
Moreover, for any $\eta \leqslant 4/9$, if a triangle-free graph admits a frozen k-colouring for $k = \lceil 2\eta + (1-\eta)(\Delta+1)\rceil$, then it is unique up to permuting colours.

We prove the two parts of Theorem 2 separately, starting with the first part.

Existence of frozen triangle-free graphs with $\eta > 4/9$

To prove the first part of Theorem 2 and show that $\eta^\star(\{\text{triangle-free graphs}\}) \leqslant 4/9$, we construct triangle-free graphs with a frozen non-unique k-colouring and small maximum degree:

Lemma 4.3. *For any $\delta > 9/5$, for large enough k, there exists a triangle-free graph G_k with maximum degree δk, that admits a frozen k-colouring and is not k-recolourable.*

The first part of Theorem 2 follows directly from Lemma 4.3:

Proof of the first part of Theorem 2 assuming Lemma 4.3. Let η such that the graph G_k in Lemma 4.3 verifies $k = 2\eta + (1-\eta)(\Delta(G_k)+1)$. We have

$$\eta = \frac{\Delta(G_k)+1-k}{\Delta(G_k)-1} = \frac{(\delta-1)k+1}{\delta k - 1}.$$

As this equality holds for any $\delta > 9/5$ and large enough k, this proves Theorem 2 and $\eta^\star(\{\text{triangle-free graphs}\}) \leqslant 4/9$. □

Proof of Lemma 4.3. Let $k \geqslant 0$. We first build a $2(k-2)$-regular triangle-free graph H_k with a frozen non-unique k-colouring. Then, we will define G_k as a random subgraph of H_k to reduce its maximum degree while preserving the other properties of H_k.

Consider the graph H_k on $5k$ vertices defined as follows (the construction is illustrated on Figure 6). Partition the vertices into k colour classes $V_1, \dots V_k$ of size 5 and number the vertices from 1 to 5 within each colour class. For any integers $i < j$, for any $x \in [5]$, connect the vertex labelled x in V_i to the vertices labelled $x + 2 \mod 5$ and $x + 3 \mod 5$ in V_j. This results in a graph H_k with degree $2k - 2$ and a frozen k-colouring α whose colour classes are the sets V_i.

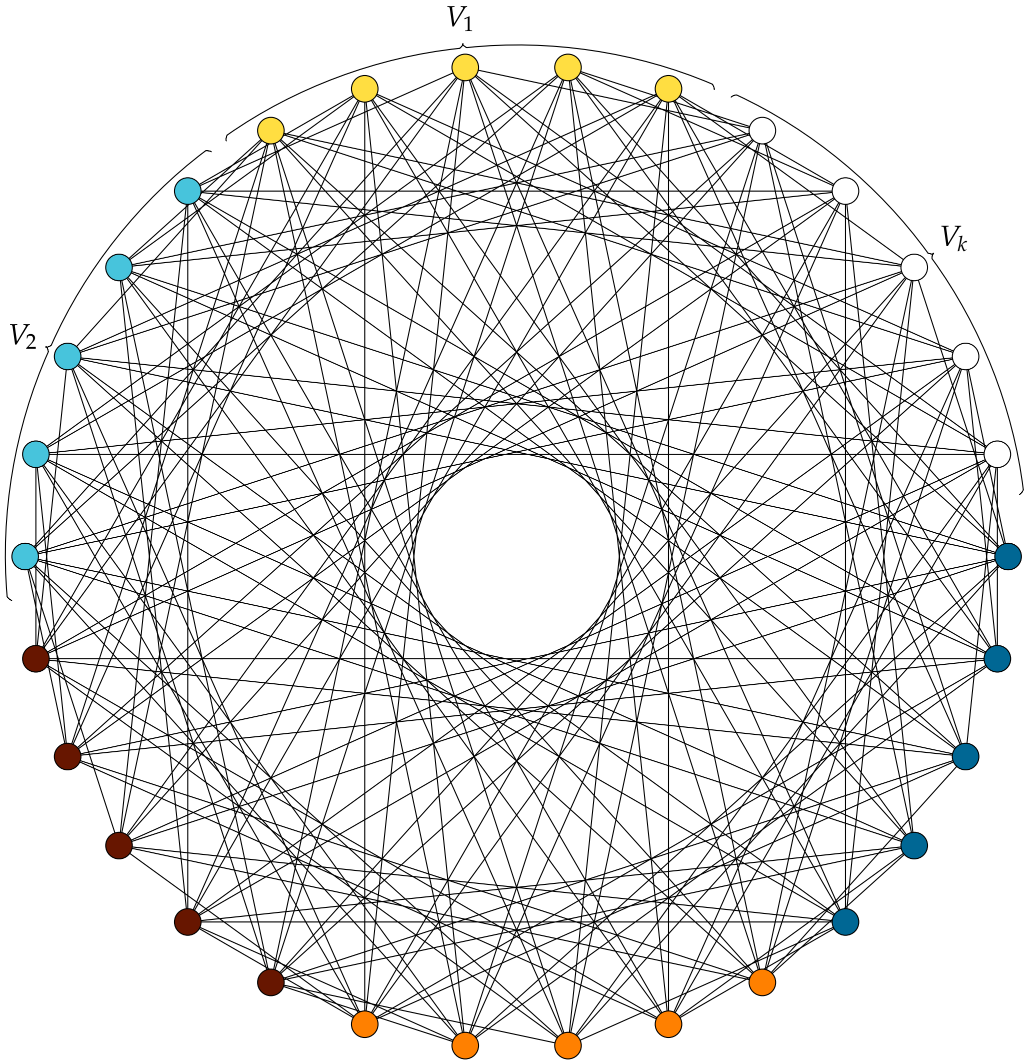


Figure 6. The graph H_6. Within each set V_i the vertices are labelled in anti-clockwise order.

We first argue that H_k is triangle-free. Indeed, any triangle would have to use vertices in three different colour classes, say V_i, V_j and V_ℓ, with $i < j < \ell$. Denote x the label of the vertex of V_i in this triangle. By construction, this would imply the following relation $x + a_{ij} + a_{j\ell} = x + a_{i\ell} \mod 5$, where $a_{ij}, a_{j\ell}, a_{i\ell} \in \{2, 3\}$. In other words, $0 \in \{1, \dots, 4\} \mod 5$, which is a contradiction.

Let G_k be the random graph obtained from H_k as follows. For each pair (i, j) with $i < j$, delete independently at random an edge in the cycle alternating between V_i and V_j. Note that α

remains frozen in G_k as any pair of colours induces a path on ten vertices. Furthermore, for any distinct i and j, each vertex in V_i has probability $1/5$ of loosing one of its edges connecting it to V_j. For any vertex u, denote X_u the binomial variable counting the number of edges incident to u deleted by this process. We have $X_u \sim \mathcal{B}(k-1, 1/5)$. By Chernoff bound, we have

$$\forall t > 0, \mathbb{P}(|X_u - \frac{k-1}{5}| > t) \leqslant 2e^{-\frac{5t^2}{3(k-1)}}$$

The probability that $\Delta(G_k) > 9/5(k-1) + t$ can be bounded:

$$\begin{aligned}
\mathbb{P}(\Delta(G_k) \leqslant 9/5(k-1) + t) &= \mathbb{P}(\forall u, X_u \geqslant (k-1)/5 - t) \\
&\leqslant 1 - \mathbb{P}(\exists u, |X_u - (k-1)/5| > t) \\
&\leqslant 1 - \sum_{u \in V(G_k)} \mathbb{P}(|X_u - (k-1)/5| > t) && \text{By Union bound} \\
&\leqslant 1 - 2(k-1)e^{-\frac{5t^2}{3(k-1)}} \xrightarrow[k\to\infty]{} 1 && \text{if } t = \omega(\sqrt{k})
\end{aligned}$$

Therefore, for any $\eta > 9/5$, for any large enough k, the random graph G_k has maximum degree at most ηk with high probability.

It remains only to prove that G_k admits a different k-colouring, which follows from the fact that $H_k[V_1 \cup V_2 \cup V_3]$ admits another 3-colouring (see Figure 7).

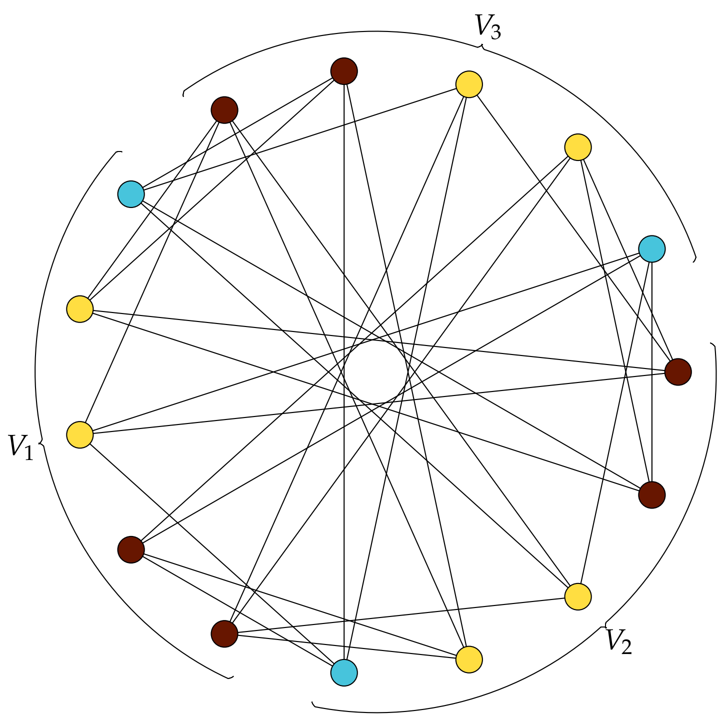


Figure 7. Another 3-colouring of $H_k[V_1 \cup V_2 \cup V_3]$. Within each set V_i the vertices are labelled in anti-clockwise order.

□

Non-existence of triangle-free graphs with $\eta \leqslant 4/9$

Proof of the second part of Theorem 2. Let G be a triangle-free graph with a frozen k-colouring α that is non-unique (hence G is not a clique), with $k = \lceil 2\eta + (1-\eta)(\Delta+1) \rceil$ for some $\eta \leqslant 4/9$. Here again, one can assume without loss of generality that $k = 2\eta + (1-\eta)(\Delta+1)$. Since G is not a clique and α is frozen, we have $\Delta(G) + 1 \geqslant 3$ and since $\eta \leqslant 4/9$, note that $k \geqslant 3$. Furthermore, note that $\Delta(G) = (k-1-\eta)/(1-\eta) < 9(k-1)/5$.

We first bound the number of colour classes of size m in α, for all $m \leqslant 4$.

Claim 4.4. *The colouring α has no colour class of size one or two, and at most two colour classes of size three and at most three colour classes of size four.*

Proof of Claim. We first prove that no colour class of G is dominated by a vertex. Assume otherwise and let u be a vertex dominating a colour class U. Let v be a neighbour of u not in U (such a vertex exists because $k \geqslant 3$) and w be a neighbour of v in U. The vertices u, v and w form a triangle. In particular, α has no colour class of size one or two as it would be dominated by some vertex in any other colour class.

Suppose that α has three colour classes V_1, V_2 and V_3 of size three. Since no vertex dominates a colour class, each pair of spans a path on 6 vertices $u_1, w_2, v_1, v_2, w_1, u_3$ with u_i, v_i and w_i in V_i for $i \in \{1,2\}$. For each $i \leqslant 2$, $X_i = \{u_i, w_i\}$ dominates V_{3-i}, so u_i and w_i cannot have a common neighbour x in V_3, otherwise any neighbour of x in V_{3-i} forms a triangle with x and one of the vertices in X_i. Thus, for each $i \leqslant 2$ either u_i or w_i has degree one in V_3. In particular $\deg(v_i, V_3) = 2$, so v_1 and v_2 have a common neighbour in V_3, but as v_1 and v_2 are adjacent, this creates a triangle.

Suppose now that α has four colour classes of size four. One can assume without loss of generality that each pair of them spans a tree. As no vertex dominates another colour class, the only possible adjacencies between any two of these colour classes are those depicted on Figure 8. To avoid a tedious case analysis, we computer checked that it is not possible to build a frozen four-colouring using those (see generating_frozen_four_colourings.py attached).

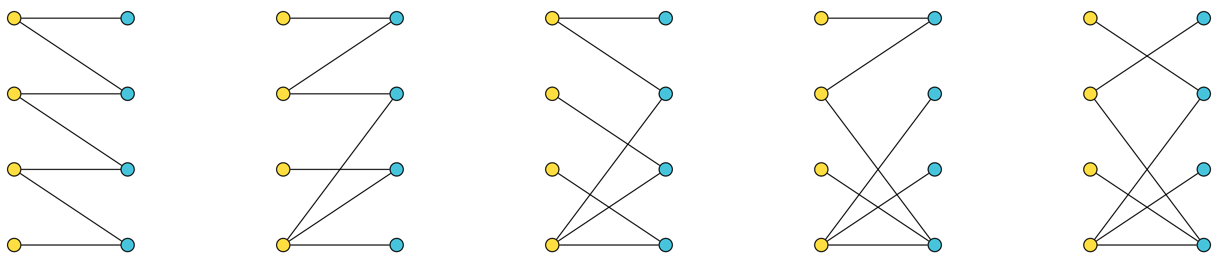


Figure 8. All the possible adjacencies between two colour classes of size four in α, up to isomorphism

■

We can now improve on Claim 4.4 and prove that in fact all colour classes have size at least five.

Claim 4.5. *α has no colour class of size less than five.*

Proof of Claim. Let U be one of the smallest colour classes. We have $\Delta(G) \geqslant \lceil \deg(U)/|U| \rceil$ and by Lemma 2.1

$$\frac{\deg(U)}{|U|} \geqslant \frac{n - |U| + (k-1)(|U|-1)}{|U|} = \frac{n + k(|U|-1) + 1 - 2|U|}{|U|}$$

We will first determine a lower bound on $\deg(U)/|U|$, before raising a contradiction. If $|U| = 3$, then by Claim 4.4 we have $n \geqslant 6 + 4(k-2)$ and thus $\deg(U)/|U| \geqslant (6k-7)/3$. If $|U| = 4$, then by Claim 4.4 we have $n \geqslant 12 + 5(k-3)$ and thus $\deg(U)/|U| \geqslant (8k-10)/4$.

In any of theses cases, $\Delta(G) \geqslant \lceil \deg(U)/|U| \rceil \geqslant 2k-2$. This contradicts $\Delta(G) < 9(k-1)/5$, as $k \geqslant 3$. ■

Let U be one of the smallest colour classes. So $n \geqslant k|U|$ and by Claim 4.5 $|U| \geqslant 5$. By Lemma 2.1,

$$\begin{aligned}
\frac{\deg(U)}{|U|} &\geqslant \frac{n + k(|U|-1) + 1 - 2|U|}{|U|} \\
&\geqslant 2(k-1) - \frac{k-1}{|U|} \\
&\geqslant \frac{9}{5}(k-1)
\end{aligned}$$

Hence $\Delta(G) \geqslant \frac{9}{5}(k-1)$ which is a contradiction. □

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