Note on the χ-boundedness of graphs of bounded sim-width

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Abstract

We prove that the class of graphs of bounded sim-width is χ-bounded. This answers an open problem proposed by Abrishami, Briański, Czyżewska, McCarty, Milanič, Rzążewski, and Walczak.

1 Introduction

All graphs in this paper are finite and simple. A class of graphs is *hereditary* if the class contains every induced subgraph of a graph in the class. A hereditary class $\mathcal{C}$ of graphs is *χ-bounded* if there is a function f such that $\chi(G) \leq f(\omega(G))$ for every graph G in $\mathcal{C}$. Such a function f is called the *χ-bounding function* of $\mathcal{C}$. If $\mathcal{C}$ admits a polynomial χ-bounding function, then $\mathcal{C}$ is *polynomially χ-bounded*.

The *sim-width* of a graph is a width parameter introduced by Kang, Kwon, Strømme, and Telle [10]. They showed that chordal graphs and co-comparability graphs have sim-width at most 1. It is known that every class of graphs of bounded tree-width, clique-width [5], mim-width [13], o-mim-width [2], tree-independence number [6], or induced matching tree-width [1] has bounded sim-width, see [1]. For many of these parameters, it was known that the class of graphs whose parameter is bounded is χ-bounded. Here we summarize known χ-boundedness results for those parameters, which result bounded sim-width.

- A class of graphs of tree-width at most k is χ-bounded trivially, because their chromatic number is at most $k + 1$.

- A class of graphs of clique-width at most k is χ-bounded, shown by Dvořák, Kráľ [8]. Later, Bonamy and Pilipczuk [3] showed that it is polynomially χ-bounded.
- A class of graphs of tree-independence number at most k is χ-bounded, because the tree-width of such a graph is bounded by a function of its clique number [7].
- A class of graphs of induced matching tree-width at most k is χ-bounded, shown by Abrishami, Briański, Czyżewska, McCarty, Milanič, Rzążewski, and Walczak [1].

However, it was not known whether mim-width, o-mim-width, or sim-width have such a property.

What we prove is that all of these parameters have the property that having the bounded value implies χ-boundedness. Here is our main theorem, answering the question of Abrishami, Briański, Czyżewska, McCarty, Milanič, Rzążewski, and Walczak [1] on sim-width.

Theorem 1.1. *Every class of graphs of bounded sim-width is χ-bounded.*

Let us recall some definitions. A tree is *subcubic* if every node has degree at most 3. For a finite set V, let $f : 2^V \to \mathbb{Z}$ be a symmetric function, that is $f(A) = f(V \setminus A)$ for every subset A of V. A *branch-decomposition* of f is a pair (T, L) of a subcubic tree T and a bijection L from V to the set of all leaves of T. For a branch-decomposition (T, L) of f, every edge e of T defines a partition (A_e, B_e) of V induced by $L^{-1}(C)$ for components C of $T \setminus e$. We define the *f-width* of $e \in E(T)$ as $f(A_e)$. The *f-width* of a branch-decomposition (T, L) of f is the maximum f-width of edges of T. Then the *f-width* is defined as the minimum f-width of a branch-decomposition of f.

Now let us define the sim-width of a graph G. For a subset A of $V(G)$, let $\mathrm{sim}_G(A)$ be the maximum size of an induced matching M of G where every edge of M has exactly one end in A. The *sim-width* of a graph G is the sim_G-width.

2 Proof

For an integer n, we write $[n]$ to denote the set of positive integers at most n.

Let M_n be the graph on $[2n]$ with n with n edges, joining vertices of difference n in $[2n]$.

Lemma 2.1. *Let d be a positive integer. For every interval I with $2d + 2 \leq |I \cap [6d + 6]| \leq 4d + 4$, M_{3d+3} has at least $d + 1$ edges between I and $[6d + 6] \setminus I$.*

Proof. Let $X = I \cap [6d + 6]$ and $Y = [6d + 6] \setminus I$. For $i = 1, 2, \ldots, 6$, let $B_i = \{(i - 1)d + j : j \in [d]\}$.

If $X \subseteq B_1 \cup B_2 \cup B_3$ or $X \subseteq B_4 \cup B_5 \cup B_6$, then we are done trivially because $|X| \geq 2d + 2$. Thus we may assume that $X \cap (B_1 \cup B_2 \cup B_3) \neq \emptyset$ and $X \cap (B_4 \cup B_5 \cup B_6) \neq \emptyset$. By symmetry, we may assume that $0 < |X \cap (B_1 \cup B_2 \cup B_3)| \leq |X \cap (B_4 \cup B_5 \cup B_6)|$. Since $|X| \leq 4d + 4$, we have $B_1 \cap X = \emptyset$.

If $B_4 \subseteq X$, then we find $d + 1$ edges between B_1 and B_4, proving the lemma. Thus, we may assume that $B_4 \not\subseteq X$ and therefore $B_6 \cap X = \emptyset$.

Since $|X| \geq 2d + 2$ and $B_4 \not\subseteq X$, we deduce that $B_3 \subseteq X$. Thus, we find $d + 1$ edges between B_3 and B_6, proving the lemma. □

The following lemma is well known. For the completeness of the paper, we include the proof.

Lemma 2.2. *If T is a subcubic tree and S is a nonempty set consisting of leaves of T, then there exists an edge e of T such that each component of $T \setminus e$ has at least $\frac{1}{3}|S|$ of the vertices in S.*

Proof. Suppose not. We orient each edge $e = uv$ of T towards v if the component of $T \setminus e$ containing u has fewer than $1/3$ of the vertices in S. Since T has fewer edges than vertices, there is a vertex x of T such that x has no incoming edges. This implies that S is a union of three sets S_1, S_2, S_3 where each S_i has fewer than $1/3$ of the vertices in S. This is a contradiction. □

Theorem 2.3. *Let d be a positive integer. If a graph G has sim-width at most d, then there exists an ordering π of its vertices such that M_{3d+3} is not an ordered induced subgraph of (G, π).*

Proof. Let (T, L) be the branch-decomposition of G certifying that the sim-width of G is at most d. Let us choose a root of T arbitrarily and choose π the ordering of vertices of G according to the ordering of the leaves in the depth-first search ordering of T.

Suppose that an isomorphic copy of M_{3d+3} is an ordered induced subgraph of (G, π). Let U be the set of vertices of the isomorphic copy of M_{3d+3} in (G, π). Every edge e of T induces a partition (A_e, B_e) of $V(G)$ such that $T \setminus e$ splits its leaves into two sets, which induces a partition (A_e, B_e) of G by L. Since T is rooted, at least one of A_e or B_e is an interval of $V(G)$ in π. By Lemma 2.2, there is an edge e of T such that both A_e and B_e contain at least $\frac{1}{3}|U|$ vertices of U. By Lemma 2.1, there is a set M of at least $d + 1$ edges between A_e and B_e. By the definition of M_{3d+3}, this M is an induced matching of G and therefore we found an induced matching of size $d+1$ for (A_e, B_e), contradicting the assumption that (T, L) is a branch-decomposition of G certifying that the sim-width of G is at most d. □

Now we can deduce Theorem 1.1.

Proof of Theorem 1.1. Briański, Davies, and Wolczak [4] proved that the class of ordered graphs forbidding a fixed matching as an ordered induced subgraph is χ-bounded. By Theorem 2.3, every graph of bounded sim-width admits an ordering of its vertices such that it does not contain a fixed matching as an ordered induced subgraph. This completes the proof. □

3 Discussions

Mim-width was introduced earlier by Vatshelle [13]. For a subset A of $V(G)$, let $\mathrm{mim}_G(A)$ be the maximum size of an induced matching M of a subgraph G' of G where the edges of G' are precisely all edges of G having exactly one end in A. Tnd the *mim-width* of a graph G, denoted by $\mathrm{mimw}(G)$, is mim_G-width.

Rank-width was introduced by Oum and Seymour [12]. The *cut-rank function* of a graph G is a function ρ_G on $2^{V(G)}$ such that $\rho_G(A)$ is the rank of an $A \times (V(G) \setminus A)$ 0-1 matrix over the binary field whose entry is 1 if and only if the vertices corresponding to its row and its column are adjacent in G. Then the *rank-width* of a graph G, denoted by $\mathrm{rwd}(G)$, is ρ_G-width. From the definition, it is clear that $\mathrm{sim}_G(A) \le \mathrm{mim}_G(A) \le \rho_G(A)$ and therefore the

$$\mathrm{simw}(G) \le \mathrm{mimw}(G) \le \mathrm{rwd}(G).$$

Thus, Theorem 1.1 implies that every class of graphs of bounded min-width or bounded rank-width is χ-bounded.

A graph G is perfect if $\chi(H) = \omega(G)$ for every induced subgraph H of G. A classical theorem of Lovász [11] states that if a graph is perfect, then so is its complement. Note that a class of perfect graphs admits the identity function as its χ-bounding function. So one may ask whether a class of the complements of graphs in a χ-bounded class is also χ-bounded. For rank-width, a class of the complement of graphs of bounded rank-width is also χ-bounded, because $\mathrm{rwd}(\overline{G}) \le \mathrm{rwd}(G) + 1$, which can be easily observed from the definition. Every graph of mim-width 1 is perfect [13, Corollary 3.7.4] and therefore the class of complements of graphs of mim-width at most 1 is perfect, thus χ-bounded. However, it is not the case for mim-width and sim-width.

Proposition 3.1.

(a) *The class of the complements of graphs of sim-width at most* 1 *is not* χ*-bounded.*

(b) *The class of the complements of graphs of mim-width at most* 2 *is not* χ*-bounded.*

Proof. The *girth* of a graph is the minimum length of a cycle. Erdős [9] showed that for every pair of integers k and g, there is a graph G such that $\chi(G) \ge k$ and the girth of G is at least g.

Now, observe that if a graph G has no cycle of length 4, then $\overline{G}$ has no induced matching of size 2 and so $\overline{G}$ has sim-width at most 1. This proves (a).

For (b), note that if G has no cycle of length 6, then $\mathrm{mim}_{\overline{G}}(A) \le 2$ for every $A \subseteq V(G)$, because the complement of the semi-induced matching of size 3 contains a cycle of length 6. Therefore if G has girth at least 7, then the mim-width of $\overline{G}$ is at most 2. This proves (b). $\square$

Here is a natural follow-up question for Theorem 1.1.

Question 3.2. *Is every class of graphs of bounded sim-width polynomially* χ*-bounded?*

If true, it would imply that every class of graphs of bounded induced matching tree-width or of bounded tree independence number is polynomially χ-bounded, which are still open problems.

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