Random stuff on graphs: Probabilistic method and randomised algorithms

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- Correction of the exercises on my web page

Monday: 10:05 - 11:45

Thursday: 12:00 - 13:45

Modalities of the course:

- Attendance to the exercise sessions is mandatory.
  - → At most two absences, -1 pt deduction for every additional absence
  - → Exercises of similar difficulty as those from exercise sessions

- 50% of the grade is a final exam with "cheat sheet"
  - → A4 format handwritten notes, no photocopies allowed

- 25% of the grade will be a mid-term test
  - → Easy stuff, close to the lecture content

- 25% of the grade will be exercise sessions
  - → Every exercise sheet will contain harder exercises marked with a ★. You will have one week to try those exercises at home. Twice during the semester, whenever you want, you may hand it over for grading.
  - → Working in groups is encouraged, as long as the writing is individual and you indicate with who you worked.
  - → Internet is authorised for homework, as long as you don't look directly for the solution online and indicate your sources.
  - → AI should be used under no circumstances.

I / A bit of history and motivation

Pioneered by Erdős in the 50's.
Simple counterexamples to many conjectures.

Principle: to show that there exists an object $x \in E$ with a property P, we build a probability distribution on E such that

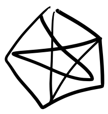
$$\mathbb{P}(x \text{ satisfies } P) > 0.$$

Pros and Cons of the Probabilistic method:

+ Allows to consider large unstructured objects

untractable by computer search

Any explicit construction will by design introduce some structure



K_5

ex: Most graphs have clique number $\omega(G) = O(\log n)$
independance nb $\alpha(G) =$ ——

I_4

For a long time, we did not have an explicit construction of graphs with $\omega(G), \alpha(G) = o(\sqrt{n})$.

+ Easy to use

+ Most graphs are counterexamples

$$\mathbb{P}(x \text{ verifies } P) \xrightarrow{n \to} \begin{matrix} 1 \\ 0 \end{matrix}$$

- Non constructive

Today : $\boxed{\text{The First Moment method}}$

First moment method: If $\mathbb{E}[X] \leq t$, then $\mathbb{P}(X \leq t) > 0$

Linearity of expectation: $\mathbb{E}\left[\sum_i X_i\right] = \sum_i \mathbb{E}[X_i]$.

Markov's inequality: $\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}[X]}{t}$

- We are interested P. Let X be the r.v. that counts the number of obstructions to P

$$\mathbb{E}[X] < 1$$

$$\mathbb{P}(P \text{ is verified}) = \mathbb{P}(X = 0) = \mathbb{P}(X < 1) \underline{> 0}$$

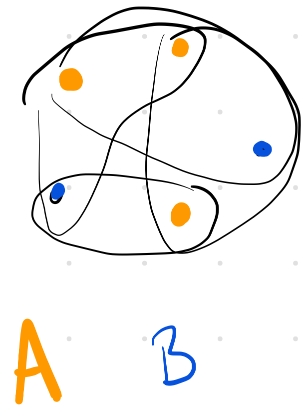
- If X is integer-valued r.v. and $\mathbb{E}[X] < 1$ then

$$\mathbb{P}(X = 0) = 1 - \mathbb{P}(X \geq 1) \geq 1 - \frac{\mathbb{E}[X]}{1} > 0$$

1) Property B or 2-coloring of hypergraphs

Felix Bernstein 1908: A collection $\mathcal{C}$ of subsets of $[n]$ has property B if $[n]$ can be partitioned into $A \sqcup B$ such that for every $C \in \mathcal{C}$, $A \cap C \neq \emptyset$
and $B \cap C \neq \emptyset$

A hypergraph $\mathcal{H} = (V, E)$ E is a collection of subsets of V.



Proper 2-coloring of a hypergraph is
$V \to \{1, 2\}$
such that no edge is monochromatic

$\to$ $\mathcal{H}$ is k-uniform if $|e| = k$ $\forall e \in E$

[Erdős 63] If $\mathcal{H}$ has less than 2^{k-1} hyperedges each of size at least k, than $\mathcal{H}$ is 2-colorable.

<u>Proof</u>: Random uniform 2-coloring.

Let $X_e = \begin{cases} 1 & \text{if } e \text{ is monochromatic} \\ 0 & \text{otherwise} \end{cases}$

$X = \sum_{e \in E(\mathcal{H})} X_e$ we want $\mathbb{P}(X = 0) > 0$

For every e, $\mathbb{E}[X_e] = \mathbb{P}(e \text{ is monochromatic})$
$\leq \underline{\mathbb{P}(e \text{ is blue})} + \mathbb{P}(e \text{ is red})$

$$\leq 2 \cdot 2^{-k} = 2^{1-k}$$

So $\mathbb{E}[X] \underset{L.E}{=} \sum_{e \in E(\mathcal{H})} \mathbb{E}[X_e] \leq |E(\mathcal{H})| \cdot 2^{1-k}$

< 1.

So by First moment method, $\mathbb{P}(X=0) > 0$

$\mathbb{P}(X<1)$ ∎

So there exists a proper 2-colouring of $\mathcal{H}$.

Alternative proof without first moment method.

Same probability distribution

$A_e = \{ e \text{ is monochromatic} \}$.

$\mathbb{P}(\text{the colouring is proper}) = \mathbb{P}\left(\bigcap_{e \in E(\mathcal{H})} \overline{A_e}\right)$

$= 1 - \mathbb{P}\left(\bigcup_{e \in E(\mathcal{H})} A_e\right) > 0$

$\mathbb{P}\left(\bigcup_{e \in E(\mathcal{H})} A_e\right) \underset{\text{subadditivity of probabilities}}{\leq} \sum_{e \in E(\mathcal{H})} \mathbb{P}(A_e)$

$\leq |E(\mathcal{H})| \, 2^{1-k} < 1$.

∎

$m(k) = \min \{ |E(\mathcal{H})| : k \text{ uniform hypergraph } \mathcal{H} \text{ not 2-colourable} \}$

$= \min \{ |\mathcal{C}| : \forall C \in \mathcal{C} \ |C| = k \text{ and } \mathcal{C} \text{ doesn't have property B} \}$

[Erdős 63] $m(k) \geq 2^{k-1}$

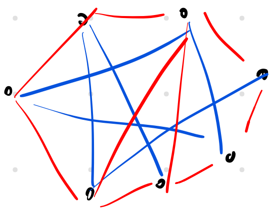
[Radhakrishnan & Srinivasan 2000]: $m(k) \geq \Omega\left(\left(\frac{k}{\ln k}\right)^{1/2} 2^k\right)$

[Erdős 64] $m(k) < (1+o(1)) \frac{e \ln(2)}{4} k^2 2^k$

2) Diagonal Ramsey numbers

$R(s,t) = \min\{n : \text{any 2-edge colouring of } K_n \text{ has a blue } K_s \text{ or a red } K_t\}$

$O(\log n)$



[Erdős 47] If $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$, then $R(k,k) > n$.

So $R(k,k) > \lfloor 2^{k/2} \rfloor \quad \forall k \geq 3$.

Proof: Consider a random uniform 2-colouring of K_n.

For every $S \subseteq V(K_n)$ of size k,

$$X_S = \begin{cases} 1 & \text{if } S \text{ is monochromatic} \\ 0 & \text{otherwise} \end{cases}$$

$$X = \sum_{\substack{S \subseteq V(K_n) \\ |S| = k}} X_S$$

$\mathbb{P}(S \text{ is monochromatic}) = \mathbb{E}[X_S] = 2 \cdot 2^{-\binom{k}{2}} = 2^{1-\binom{k}{2}}$

$$\mathbb{E}[X] \underset{L.E}{=} \sum_S \mathbb{E}[X_S] = \binom{n}{k} 2^{1-\binom{k}{2}} < 1$$

by assumption.

So by First moment method, $\mathbb{P}_n(X=0) > 0$.

$\mathbb{P}$(the 2-colouring has no blue or red K_k)

$$R(k,k) > n$$

- For $k \geq 3$, we take $n = \lfloor 2^{k/2} \rfloor$

$$\binom{n}{k} 2^{1-\binom{k}{2}} \leq \frac{n^k}{k!} 2^{1-\binom{k}{2}} \leq \frac{n^k}{k!} \frac{2^{k/2+1}}{2^{k^2/2}} < 1 \qquad \blacksquare$$

3) Triangle-free graph with high χ

> Sometimes, we need to start with a random construction and then modify it slightly.

[Erdős 59] For all k, there exists a triangle free graph with $\chi(G) > k$.

Proof $\to G \sim G_{n,p}$ n vertices, each edge with probability p independently.

($\to$ Erdős Renyi model, $G_{n,m} \leftarrow$ random uniform graph with n vertices, m edges)

$p = n^{-2/3}$

Sketch of proof: 1. G has no large independent set:

$$\mathbb{P}\left(\alpha(G) < \frac{n}{2k}\right) > 0$$

$$\chi(G) \geq \frac{n}{\alpha(G)}$$

2. G has not too many triangles

3. Delete the triangles and conclude