Dirac's theorem and the reconfiguration geometry of perfect matchings

Ross J. Kang* Clément Legrand-Duchesne†

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Abstract

Dirac's theorem states that all n-vertex graphs with minimum degree $n/2$ contain a perfect matching. This is tight and graphs with minimum degree cn are more generally called c-Dirac graphs. In fact, a phase transition occurs at $c = 1/2$: c-Dirac graphs with $c \geq 1/2$ contain $(\frac{cn}{e+o(1)})^{n/2}$ perfect matchings [Cuckler, Kahn 2009]. Moreover, for any fixed c-Dirac graph G with $c \geq 1/2$, if the edges of G are sampled with probability $O(\log(n)/n)$, the resulting random graph still contains a perfect matching with good probability, thereby witnessing that these matchings are "everywhere" in G [Kang, Kelly, Kühn, Osthus, Pfenninger 2024].

We give a new understanding of this phase transition by describing how the perfect matchings of a Dirac graph cluster. More precisely, we analyse the geometrical properties of the space $\mathcal{H}_k$ of perfect matchings of G, endowed with the graph metric such that perfect matchings equal on all but k edges are at distance one. For any fixed k, we pinpoint sharp thresholds at which this space is shattered into exponentially many components, becomes connected, or becomes an expander. These thresholds are concentrated around $n/2$ for $k \geq 3$, while the connectedness occurs around $2n/3$ for $k = 2$. Moreover, the threshold at which $\mathcal{H}_k$ has positive degree is around $n/2$ for $k = 2$ and is related to the weak Caccetta-Häggkvist conjecture for $k \geq 3$.

We also show analogous results for G a balanced bipartite graph on $2n$ vertices, where n is even.

1 Introduction

Let G be a graph on an even number n of vertices. We are interested in the macroscopic changes in the geometry of the space of perfect matchings of G

*Korteweg–de Vries Institute for Mathematics, University of Amsterdam, Netherlands (r.kang@uva.nl).

†Theoretical Computer Science Department, Faculty of Mathematics and Computer Science, Jagiellonian University, Kraków, Poland (clement.legrand-duchesne@uj.edu.pl).

according to a natural distance, as we vary the minimum degree $\delta(G)$ of G.

It is already a fundamental result in graph theory that the space of perfect matchings undergoes some sort of phase transition at $\delta(G)$ around $n/2$. Dirac's theorem [18] states that if $\delta(G) \geq n/2$, then G must contain a perfect matching, while if $\delta(G) < n/2$, then there are examples of G with no perfect matching. More is already known about this phase transition. Letting $\Phi(G)$ denote the *number* of perfect matchings of G, Cuckler and Kahn [14, 15] (see also the results of Sárközy, Selkow and Szemerédi [35] and Brègman [7]) showed that provided $\delta(G) \geq n/2$,

$$\Phi(G) \geq \left(\frac{\delta(G)}{e + o_{n\to\infty}(1)}\right)^{n/2}. \tag{1}$$

Given that there are so many perfect matchings, even right at the non-emptiness threshold of $\delta(G) = n/2$, a "volume" heuristic suggests it might be natural to expect them to cluster. This intuition is accentuated by the recent result of Kang, Kelly, Kühn, Osthus and Pfenninger [26] showing that perfect matchings are "everywhere" in graphs with large minimum degree: Given an n-vertex graph G with minimum degree at least $n/2$, the random sugbraph obtained by keeping each edge of G independently at random with probability $O(n/\log n)$ still contains a perfect matching with high probability.

In order to test this hypothesis, we consider the collection $\mathcal{M}_G$ of perfect matchings in G as a state space. We permit transitions between states according to their proximity in some metric. A natural metric on $\mathcal{M}_G$ is the Hamming distance, that is, the size of the symmetric difference. Due to the definition of perfect matching, this distance parameter can only take values in even integers at least 4. Two perfect matchings $M_1, M_2 \in \mathcal{M}_G$ have Hamming distance at most $2k$, $k \geq 2$, if and only if there is some *k-switch* between M_1 and M_2 —that is, the addition of j edges and the removal of j other edges that takes M_1 to M_2 for some $j \leq k$. Let $\mathcal{H}_k(G)$ be the state transition graph (also called reconfiguration graph) on $\mathcal{M}_G$ defined by joining two distinct $M_1, M_2 \in \mathcal{M}_G$ if M_1 and M_2 have Hamming distance at most $2k$. We call $\mathcal{H}_k(G)$ the *k-switch graph* on $\mathcal{M}_G$.

Concretely, we cast the "clustering" question above as follows. What is the smallest minimum degree condition on G in terms of n that guarantees connectedness of $\mathcal{H}_k(G)$? A component of size a constant proportion of $|\mathcal{M}_G|$? Positive minimum degree in $\mathcal{H}_k$? Observe that if the minimum degree of G is $n-1$, then G must be the complete graph K_n and it is then easy to see that $\mathcal{H}_k(G)$ is connected. Observe also that if G is the disjoint union of $n/(2k+2)$ cycles of length $2k+2$ (for n a multiple of $2k+2$), then $\mathcal{H}_k(G)$ consists of $2^{n/(2k+2)}$ isolated points. Thus the above questions are all well-defined and non-trivial.

Could we potentially observe some "phase diagram" for $\mathcal{H}_k(G)$ (see Figure 1) similar to those observed in statistical physics and Random Constraint

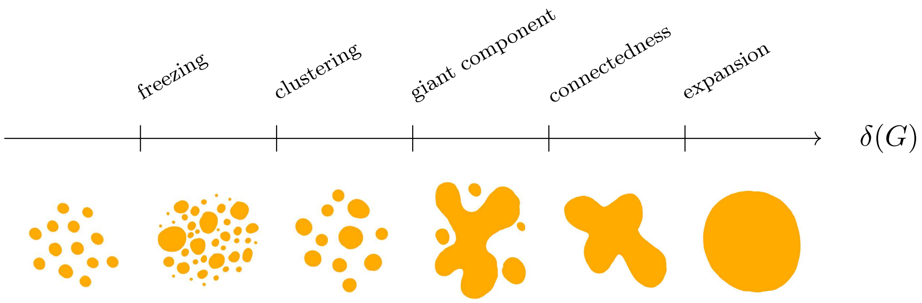


Figure 1: An artist's depiction of a possible phase diagram for the component structure of $\mathcal{H}_k(G)$ as we vary the condition on $\delta(G)$ as a function of $n = |G|$.

Satisfiability Problems (see for example [1, 31]) and, in particular, does something special happen around the non-emptiness threshold at $\delta(G) = n/2$? Our main objective in this work is to sketch the outlines for such a diagram. Our work follows in the line of other recent studies in "emergent combinatorics" [20, 5, 6, 10, 9, 21, 12] where analogous phenomena occur in other combinatorial contexts.

Note that $\mathcal{H}_j(G) \subseteq \mathcal{H}_k(G)$ if $j \leq k$. Due to this monotonicity, it turns out that the case $k = 3$ is essentially representative for any larger k. We discuss the finer subtleties later in the paper, in addition to the results we have obtained for mixing time, and for matchings of some prescribed size $\gamma n/2$, where $0 \leq \gamma \leq 1$. Here are some of our main results.

Theorem 1. *Let G be a graph on an even number n of vertices. If G has minimum degree at most $n/2 - 2$, then the 2-switch graph $\mathcal{H}_2(G)$ may have isolated points. If G has minimum degree at least $n/2 + 1$, then $\mathcal{H}_2(G)$ must have positive minimum degree.*
If G has minimum degree at most $\lfloor (2n-2)/3 \rfloor$, then it may be that $\mathcal{H}_2(G)$ is disconnected. If G has minimum degree at least $\lfloor (2n+3)/3 \rfloor$, then $\mathcal{H}_2(G)$ must be connected.

Theorem 2. *Let G be a graph on an even number n of vertices. If G has minimum degree at most $n/2 - 1$, then it may be that the 3-switch graph $\mathcal{H}_3(G)$ is disconnected. If G has minimum degree at least $n/2 + 2$, then $\mathcal{H}_3(G)$ must be connected.*

Theorem 3. *Let G be a graph on an even number n of vertices. Given any $c > 1$, there exists $\varepsilon > 1$ such that the following holds for any $k \geq 2$ and n. If G has minimum degree at most $n/2 - \varepsilon kn$, then it may be that the k-switch graph $\mathcal{H}_k(G)$ has c^n components and within each of these components, all matchings share a linear number of* frozen *edges.*

Threshold	$k=2$	$k=3$	$k\geq 4$
Freezing	$\left[\frac{n}{2}-\varepsilon n, \frac{n}{2}+1\right]$	$\left[\frac{n}{2}-\varepsilon n, \frac{n}{2}+2\right]$	$\left[\frac{n}{2}-\varepsilon kn, \frac{n}{2}+1\right]$
Clustering	$\left[\frac{n}{2}-\varepsilon n, \left\lfloor\frac{2n+3}{3}\right\rfloor\right]$	$\left[\frac{n}{2}-\varepsilon n, \frac{n}{2}+2\right]$	$\left[\frac{n}{2}-\varepsilon kn, \frac{n}{2}+1\right]$
Giant component	$\left[\frac{n}{2}-\varepsilon, \left\lfloor\frac{2n+3}{3}\right\rfloor\right]$	$\left[\frac{n}{2}-\varepsilon, \frac{n}{2}+2\right]$	$\left[\frac{n}{2}-\varepsilon k, \frac{n}{2}+1\right]$
Isolated points	$\left[\frac{n}{2}-1, \frac{n}{2}+1\right]$		
Connectedness	$\left[\left\lfloor\frac{2n+1}{3}\right\rfloor, \left\lfloor\frac{2n+3}{3}\right\rfloor\right]$	$\left[\frac{n}{2}+1, \frac{n}{2}+2\right]$	$\left[\left\lfloor\frac{n-k+1}{2}\right\rfloor, \frac{n}{2}+1\right]$
Expansion	$\left[\left\lfloor\frac{2n+1}{3}\right\rfloor, \left\lfloor\frac{2n+3}{3}\right\rfloor\right]$	$\left[\frac{n}{2}+1, \frac{n}{2}+2\right]$	$\left[\left\lfloor\frac{n-k+1}{2}\right\rfloor, \frac{n}{2}+1\right]$

Table 1: A summary of the bounds on thresholds for the reconfiguration geometry of perfect matchings in terms of the minimum degree (see Subsection 1.2 for a formal definition of thresholds functions and on how to interpret these bounds).

Finally, the following theorem shows how the threshold for positive degree of $\mathcal{H}_k(G)$ relates to Kelly-Kühn-Osthus Conjecture (a weakening of the celebrated Caccetta-Häggkvist Conjecture [11]) on the minimum semi-degree of an oriented graph of girth $k+1$.

Theorem 4. *For all $k \geq 2$, d and n, there exists a balanced bipartite graph G on $2n$ vertices with $\delta(G) = d+1$ such that $\mathcal{H}_k(G)$ has an isolated vertex, if and only if there exists an oriented n-vertex graph with minimum semidegree d and no directed cycle of length at most k.*

For $k = 3$, Hladký, Král, and Norin [23] proved using flag algebras that any n-vertex digraph with minimum outdegree at least $0.3465n$ contains a directed triangle, so Theorem 4 implies that for any balanced bipartite graph G on $2n$ vertices with minimum degree larger than $\lceil 0.3465n\rceil$, the 3-switch graph $H_3(G)$ has no isolated vertices.

Informally, we can recast these results as follows (see also Table 1). If we only allow 2-switches, then the space $\mathcal{H}_2(G)$ of perfect matchings in G exhibits the following phase transitions in its component structure. Until a little before the non-emptiness threshold $\delta(G) \geq n/2$, the space $\mathcal{H}_2(G)$ can have isolated points and can be "shattered" into exponentially many components with a linear number of frozen edges in each component, that is edges that belong to all matchings of the component. Almost right at the non-emptiness threshold, $\mathcal{H}_2(G)$ must have positive minimum degree (so no isolated points). Significantly past the non-emptiness threshold, at around $\delta(G) = 2n/3$, there is a threshold in the connectivity of $\mathcal{H}_2(G)$: just below $2n/3$ it can be disconnected, while just above $2n/3$ the space must be connected. Note that between $n/2$ and $2n/3$ it remains open as to whether some "large" component in $\mathcal{H}_2(G)$ is guaranteed.

If in addition to 2-switches we allow k-switches for some fixed $k \geq 3$, then the picture narrows. Until a little before the non-emptiness threshold, the space $\mathcal{H}_k(G)$ can be shattered into exponentially many components with a linear number of frozen edges. Almost right at the non-emptiness threshold, $\mathcal{H}_k(G)$ goes from (potentially) having no linear size component to being connected.

One of the motivations for proving that some configuration space is connected is to ensure that Markov chains Monte Carlo algorithms used to sample random configurations converge toward the desired probability distribution. However, connectivity is a far too weak property to ensure that these algorithms are efficient: the rate of convergence of these Markov chains is measured by the mixing time, which relates to the structural expansion properties of the reconfiguration graph. Here, we prove that $\mathcal{H}_k(G)$ becomes an expander right at the connectivity threshold $\delta(G) \geq n/2 + 2$ and that one can sample in polynomial time approximately uniform random perfect matchings by iteratively applying random switches of bounded size to an arbitrary perfect matching (see Subsection 1.1 for a more detailed introduction to random sampling of perfect matchings).

Note that for most choices for the minimum degree below $n/2$ it remains open as to whether $\mathcal{H}_k(G)$ can have isolated points. For δ at most n/k, $\mathcal{H}_k(G)$ may contain isolated points, and we conjecture that $\mathcal{H}_k(G)$ contains no isolated points for $\delta \geq n/k$ as this threshold is related to the celebrated Kelly-Kühn-Osthus Conjecture on directed cycles in oriented graphs of large outdegree (see Section 6 for a discussion on this connection).

Let us next consider G to be a balanced bipartite graph on $2n$ vertices, where n is even. Cuckler and Kahn [14, 15] also showed in this case that, provided $\delta(G) \geq n/2$,

$$\Phi(G) \geq \left(\frac{\delta(G)}{e + o_{n\to\infty}(1)} \right)^n . \tag{2}$$

We have established a completely analogous set of results in this case. In fact, both for balanced bipartite graphs and general graphs, most of our result extend to matchings of a fixed size $\gamma n/2$ for any fixed $\gamma \in (0, 1]$ by replacing $n/2$ by $\gamma n/2$ in the different thresholds.

We offer a word of warning that the "volume" heuristic offered earlier does not necessarily tell the whole story. If G is a balanced bipartite graph on $2n$ vertices that is moreover *d-regular*, then an important result of Schrijver [36] (see also [39] for the case $d = 3$ and [22, 13] for alternative proofs for general $d \geq 3$) gives the following lower bound on the number of perfect matchings in G:

$$\Phi(G) \geq \left(\frac{(d-1)^{d-1}}{d^{d-2}} \right)^n .$$

This asymptotically matches the expression in (2) (see Figure 2), except that it needs no condition like $d \geq n/2$, and so *there are plenty of perfect matchings in G regardless of the value of $d \geq 3$*. By way of contrast, in the above results, we have found for all $d \leq n/2 - 2$ and all $k \geq 2$ that $\mathcal{H}_k(G)$ can be disconnected and have isolated points, as there exists balanced bipartite regular graphs of arbitrarily large girth. Hence an additional relation between d, k and n is required (see Section 7 for a detailed discussion).

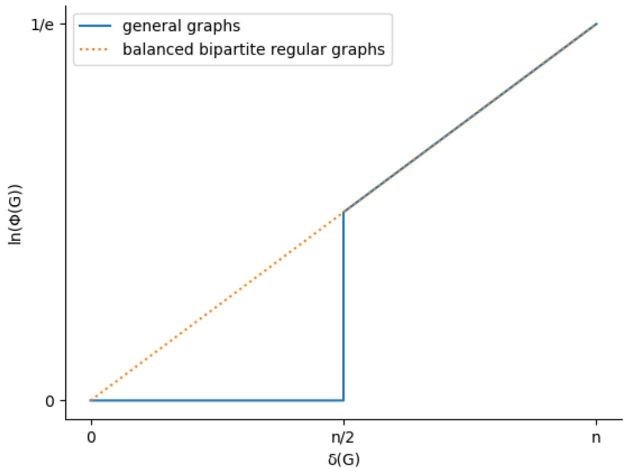


Figure 2: Number of perfect matchings (in logarithmic scale)

1.1 Random sampling and Markov chains on matchings

Counting perfect matchings and computing the permanent The problem of counting the number of perfect matchings of a graph atracted considerable attention over the years. When G is a balanced bipartite graph, this number is counted by the permanent of the adjacency matrix A of G:

$$\mathrm{Perm}(A) = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} A_{i\sigma(i)}.$$

Computing the permanent is a $\#P$-hard problem [38], which motivated the study of approximation methods. To that end, Broder [8] showed that the problem of approximately counting the number of perfect matchings can be reduced to the generation of a random perfect matching from an approximately uniform distribution. To that end, he defined a Markov chain operating on perfect and near-perfect matchings (that is matchings with at most two unmatched vertices), and showed that it mixes rapidly on balanced bipartite graphs with minimum degree at least $n/2$. Since the ratio of near-perfect matchings to perfect matchings is bounded in these dense graphs, this also gives a efficient random approximate sampler of perfect matchings. Broder's rapid mixing proof relied on a coupling argument but contained some error and was therefore withdrawn. Fortunately, Jerrum and Sinclair [25] provided a proof of the rapid mixing of this Markov chain (namely in time $O(n^5 \ln n)$), this time relying on the canonical path method. Their proof generalises to general graphs whose ratio of near-perfect to perfect matchings is polynomial and to arbitrary balanced bipartite graphs. Under the additional assumption that the bipartite graph is γn-regular, perfect matchings can be sampled exactly uniformly, in time $O(n^{1.5+.5/\gamma} \ln n)$ (see [24]).

2-switch Markov chain The 2-switch Markov chain was introduced by Diaconis, Graham and Holmes [16], who proves that this Markov chain is ergodic for convex balanced bipartite graphs, that is such that all ones are consecutive in each row of the adjacency matrix. In [19], Dyer, Jerrum and Müller observe that convex balanced bipartite graphs are a subclass of chordal bipartite graphs, and prove that the 2-switch Markov chain is also ergodic in this wider class. They study a hierarchy of hereditary graphs classes: chain, monotone, biconvex, convex, chordal bipartite graphs and show that the 2-switch Markov chain mixes rapidly on monotone graphs, but exponentially on biconvex graphs.

A direct consequence of Theorem 1 (respectively Theorem 2) is that the 2-switch Markov chain (respectively the 3-switch Markov chain) may not be ergodic on a n-vertex graphs of minimum degree at most $\lfloor (2n-2)/3 \rfloor$ but becomes ergodic if $\delta(G) \geq \lfloor (2n+3)/3 \rfloor$ (respectively at most $n/2-1$ and $n/2+2$). Furthermore, we prove that above these connectedness thresholds, the 2-switch and 3-switch Markov chains actually have polynomial mixing time:

Theorem 5. *Let G be a n-vertex graph with $\delta(G) \geq n/2+2$ (or respectively $\delta(G) \geq \lfloor (2n+3)/3 \rfloor$). Alternatively, let G be a balanced bipartite graph on $2n$ vertices with $\delta(G) \geq \lfloor n/2 \rfloor + 2$ (or respectively $\delta(G) \geq \lfloor (2n+3)/3 \rfloor$). The random walk on $\mathcal{H}_3(G)$ (respectively $\mathcal{H}_2(G)$) has mixing time polynomial in n.*

Although the bounds we obtain for the mixing times of the switch Markov chains are greater than the bounds of Jerrum and Sinclair on the mixing time of Broder's Markov chain, making in suboptimal for sampling purposes, the Switch Markov chains have the merit of operating solely on perfect matchings.

Other Markov chains on matchings On top of the 2-switch Markov chain and Broder's Markov chain, other Markov chains have been studied on matchings. The most famous one is arguably the monomer-dimer model that operates on all matchings and draws connections with some statistical physics model.

1.2 Formal definitions of the thresholds and properties of interest

The formal definitions of the thresholds for our properties of interest (namely being connected, having no isolated vertices, being an expander, having a giant component, having a subexponential number of components, having a sublinear number of frozen variables within each components) are intricate because the last four listed properties depend on some parameter. We call *parametric graph property* any family $(P_c)_{c\in\mathbb{R}_{>1}}$ of graph properties such that for all H and for all $c < c'$ we have $P_c(H) \Rightarrow P_{c'}(H)$. In other words, $(P_c)_{c\in\mathbb{R}_{>1}}$ is monotone with respect to the parameter c. For technical reasons, we will also require our properties P_c to hold on empty graphs. For constant properties (with respect to c), such as being connected or having no isolated vertices, the threshold function for P can be defined as follows: for all n such that $\gamma n/2$ is integral, $\delta^P_{k,\gamma}(n)$ is the minimum δ such that for any n-vertex graph G with $\delta(G) \geq \delta$, the space of γn-matching $\mathcal{H}_{k,\gamma}(G)$ verifies P, or $\delta = n$ if P never holds. More generally, given a parametric graph property $P = (P_c)_{c\in\mathbb{R}_{>1}}$, the *threshold function* $\delta^P_{k,\gamma}$ for P is the collection of functions $(\delta_c)_{c\in\mathbb{R}_{>1}}$ such that for each c, $\delta_c(n)$ is the minimum δ such that for any n-vertex graph G with $\delta(G) \geq \delta$, the space of γn-matching $\mathcal{H}_{k,\gamma}(G)$ verifies P_c, or $\delta = n$ if P_c never holds. Note that the threshold function of any parametric graph property is well defined, because there are finitely many graphs on n vertices and $\delta_c(n) = n$ if P_c holds on none of their k-switch graphs. Moreover, we are implicitely using the fact that P_c holds on empty graph: as graphs of minimum degree less than $\gamma n/2$ may have no γn-matchings, all properties would have $\delta_c(n) \geq n/2$ without this assumption.

We likewise define $\delta^P_{k,\gamma}(\mathcal{B})$ and $\delta^P_{k,\gamma}(\mathcal{R})$ the threshold functions where G is a balanced bipartite graph on $2n$ vertices, and a regular balanced bipartite graph on $2n$ vertices respectively. We are interested in the following properties P:

connectedness $\mathcal{H}_{k,\gamma}(G)$ is connected,

giant $\mathcal{H}_{k,\gamma}(G)$ has a component of size at least $|V(H)|/c$.

isolated $\mathcal{H}_{k,\gamma}(G)$ has no isolated vertices,

clustering $\mathcal{H}_{k,\gamma}(G)$ has less than c^n connected components,

freezing In each component of $\mathcal{H}_{k,\gamma}(G)$, less than n/c edges are frozen, that is common to all the matchings in the component,

expansion For some absolute polynomial P, the random walk on $\mathcal{H}_{k,\gamma}(G)$ mixes in time $O(P(n))$ (see Subsection 5.3 for a discussion on the connections between polynomial mixing time and the expansion properties of $\mathcal{H}_{k,\gamma}(G)$).

For example, the inequality $\gamma n/2 - \varepsilon k \leq \delta^{\text{giant}}_{k,\gamma} \leq \gamma n/2 + 1$, should be read as follows. For any $c > 1$ and any $\gamma \in (0, 1]$, there exists $\varepsilon > 0$ such that for all integer n with $\gamma n/2$ integral, there exists a graph G with minimum degree less than $\gamma n/2 - \varepsilon k$ such that $\mathcal{H}_{k,\gamma}(G)$ contains no component of size at least $|\mathcal{H}_{k,\gamma}(G)|/c$. On the other hand, for all $c > 1$ and all n-vertex graph G with minimum degree at least $n/2 + 1$, $\mathcal{H}_{k,\gamma}(G)$ contains a component of size at least $|\mathcal{H}_{k,\gamma}(G)|/c$.

1.3 Organisation

We prove in Section 3 all our upper bounds on the connectedness thresholds. In Section 4, we prove the lower bounds on the connectedness, giant component, clustering and freezing thresholds for general graphs and balanced bipartite graphs. In Section 5 we prove the polynomial mixing of the switch Markov chain. Section 6 focuses on the threshold for isolated matchings. Finally, we discuss open questions and perspectives in Section 7.

2 Preliminaries

We denote $N(u)$ the neighbourhood a vertex u. Given an even cycle C, an *even chord* of C is a chord splitting the cycle in two even cycles.

2.1 Markov chains and mixing times

A Markov chain is a memoryless stochastic process over a discrete state space Ω. More precisely, a sequence of random variable $(X_0, X_1, \dots)$ is a Markov chain with transition matrix P if for all $t \in \mathbb{N}$ and $x_1, \dots x_{t+1} \in \Omega$,

$$\begin{aligned}\text{Prob}(X_{t+1} = x_{t+1} | \forall i \in [t], X_i = x_i) &= \text{Prob}(X_{n+1} = x_{n+1} | X_n = x_n) \\ &= P(x_n, x_{n+1}).\end{aligned}$$

All Markov chains considered in this article are over a finite state Ω. By representing the distribution of X_t over Ω as a row vector μ_t with $\|\mu_t\|_1 = 1$ and i-th coordinate equal to $\text{Prob}(X_t = x_i)$, we have $\mu_t = \mu_{t-1} P$ and more generally $\mu_t = \mu_0 P^t$. With a slight abuse of notation, we will write $\mu_t(x) = \text{Prob}(X_t = x)$ and $\mu_t(A) = \text{Prob}(X_t \in A)$.

A Markov chain is *ergodic* if $\mu_t(x) > 0$ for all $x \in \Omega$, all initial distribution μ_0 and all large enough t. In particular, every ergodic Markov chain admits a unique stationary distribution π, that is such that $\pi = \pi P$. Note that π is uniform if P is symmetric. Moreover, ergodic Markov chains converge towards their stationary distributions: $\sup_{\mu_0} \left\|\mu_0 P^t - \pi\right\|_{TV}$ tends to zero as t tends to infinity, where $\|\mu - \nu\|_{TV} = \max_{A\subseteq\Omega} |\mu(A) - \nu(A)|$ is the total variation distance. The rate of this convergence is measured by the *mixing time*: $\tau_{mix} = \min\{t : d(t) < 1/4\}$, where $d(t) = \sup_{\mu_0} \left\|\mu_0 P^t - \pi\right\|_{TV}$. The constant $1/4$ might look arbitrary, but using any other constant ε instead affects the mixing time only by a factor of $\log_2(1/\varepsilon)$.

A Markov chain is said to be polynomially mixing if its mixing time is a bounded by a polynomial in $\ln(|\Omega|)$. The mixing time is tied to the expansion of the ground graph H of the Markov chain, that is the graph whose vertex set is Ω and edges corresponding to non-zero probability transitions. Several methods exist to bound the mixing time: couplings, spectral methods and the canonical path method. We will use the latter in this article. Let $\Gamma = (\gamma_{xy})_{x,y\in\Omega}$, be a set of paths called *canonical paths*, joining any two states x and y of Ω in the ground graph. The congestion of Γ is defined as

$$\varrho(\Gamma) = \max_{(a,b)\in E(H)} \left\{ \frac{\sum\limits_{(x,y)\in \mathrm{cp}(a,b)} \pi(x)\pi(y)|\gamma_{x,y}|}{\pi(a)P(a,b)} \right\}.$$

The congestion measures how evenly distributed the set of canonical paths are, and whether the resulting trace on the ground graph has good expansion, and thus bounds the mixing time:

$$\tau_{mix} \leq 2\varrho \ln(|\Omega|).$$

Therefore, the core of the canonical paths methods consists in defining an appropriate set of canonical paths, and bounding its congestion.

2.2 Matchings

A matching is *near-perfect* if all vertices but two are matched. We call γn-matching a matching where exactly γn vertices are matched. Given two matchings M and M', their symmetric difference has maximum degree two. We call *alternating cycle* any cycle of $M\Delta M'$, *alternating path* any maximal path of $M\Delta M'$ and *alternating subpath* any path in $M\Delta M'$. A key argument in the analysis of our Markov chain (and also in the analysis of Border's Markov chain) is that the ratio of perfect matchings to near-perfect matchings is polynomially bounded when the minimum degree of G is sufficiently high:

Lemma 2.1. *Let G be an n-vertex graph in which any two non-adjacent vertices u and v have degrees that sum to at least $n-1$. Alternatively, let G be a balanced bipartite graph on $2n$ vertices such that the degrees of each pair of non-adjacent vertices u and v in different halfs of the bipartition sums up to at least n. The number of near-perfect matchings is at most n^2 times the number of perfect matchings.*

Proof. Let γ be the application that maps near-perfect matchings to a couple composed of a set of two vertices and a perfect matching, as follows. Let N be a near-perfect matching with non-matched vertices u and v. If u and v are adjacent in G, let $\gamma(N) = (\{u, v\}, N \cup uv)$. If u and v are non-adjacent, let $V' = V(G) \setminus \{u, v\}$, let A be the subset of vertices of V' matched by N to a neighbour of v and $B = N(u) \cap V'$ [1]. We have $|A| + |B| = \deg(u) + \deg(v) > n - 2 = |V'|$ so by pigeon hole principle, there exists $x \in A \cap B$. In other words, there exists an edge xy of N such that x is adjacent to u and y to v. Let $\gamma(N) = (\{u, v\}, N \cup \{xu, yv\} \setminus xy)$.

It is straightforward to check that the application γ is injective, from which the result immediately follows. □

3 Upper bounds on the connectedness thresholds

In this section, we show the upper bounds on the connectedness thresholds of Theorems 1 and 2 that we recast here:

Theorem 6. *Let G be an n-vertex graph with $\delta(G) \geq n/2 + 2$ (or a balanced bipartite graph on $2n$-vertices, with $\delta(G) \geq n/2 + 1$). All perfect matchings of G are equivalent via 3-switches.*

Theorem 7. *Let G be an n-vertex graph with $\delta(G) \geq 2n/3 + 1$ (or a balanced bipartite graph on $2n$-vertices, with $\delta(G) \geq 2n/3 + 1$). All perfect matchings of G are equivalent via 2-switches.*

To do so, we first prove in Subsection 3.1 that perfect matchings are connected by 4-swicthes if $\delta \geq n/2 + 1$, before proving in Subsection 3.2 that 4-switch can be realised by a sequence of $O(1)$ 3-switches if $\delta \geq n/2 + 2$, and in Subsection 3.3 that 3-switch can be realised by a sequence of $O(1)$ 2-switches if $\delta \geq \lfloor (2n + 3)/3 \rfloor$, which proves Theorem 6 and Theorem 7 respectively.

3.1 Equivalence via 4-switches

We first prove that perfect matchings are connected by 4-swicthes if $\delta \geq n/2 + 1$:

Theorem 8. *Let $\gamma \in (0, 1]$. Let G be an n-vertex graph in which each pair of non-adjacent vertices have degrees that sum to at least $\gamma n + 1$. Alternatively, let G be a balanced bipartite graph on $2n$ vertices, in which each pair of non-adjacent vertices in different halfs of the bipartition have their degrees that sum up to at least γn. All γn-matchings of G are equivalent via 4-switches.*

[1] If G is a balanced bipartite graph, let $V' = V_1 \setminus \{v\}$, where V_1 is the half of the bipartition containing v. We then have $|A| + |B| = \deg(u) + \deg(v) > n - 1 = |V'|$.

Proof. We distinguish the case of perfect matchings from that of γn-matchings with $\gamma < 1$, the latter one being slightly more complex because of the short alternating paths.

Perfect matchings Let M and M' be two perfect-matchings of G. We proceed by induction on $|M \Delta M'|$ by proving that we can always reduce the size of the symmetric difference by performing a 4-switch in M or in M'.

If there exists $C = u_1 \dots u_{2p}$ an alternating subpath of $M \Delta M'$ with $u_{2i-1}u_{2i} \in M$ and $u_{2i}u_{2i+1} \in M'$, such that $p \in \{2, 3, 4\}$ and $u_1 u_{2p} \in E(G)$, then a 4-switch in M on $u_1 \dots u_{2p}$ results in a perfect matching N such that $|N \Delta M'| \leq |M \Delta M'| - p + 1$. Thus one can assume that no such C exists.

Let $C = u_1 \dots u_6$ be an alternating subpath of $M \Delta M'$ with $u_{2i-1}u_{2i} \in M$ and $u_{2i}u_{2i+1} \in M'$. By assumption, u_1 is neither adjacent to u_4 nor to u_6, and u_6 is not adjacent to u_3. Let $V' = V(G) \setminus \{u_1, \dots u_6\}$. Let A be the subset of V' composed of vertices matched in M to a neighbour of u_1. The only vertices of $\{u_1, \dots u_6\}$ that can be matched in M to neighbours of u_1 are u_1, u_4 and u_6. Therefore, $|A| \geq \deg(u_1) - 3$. Let $B = N(u_6) \cap V'$, we also have $|B| \geq \deg(u_6) - 3$. By pigeonhole principle[2], since V' has size $n - 6$ and is a superset of A and B and $|A| + |B| \geq n - 5$, there exists a vertex x in $A \cap B$. This vertex x is by definition adjacent to u_6, and matched to a vertex y in M, itself adjacent to u_1 (see Figure 3). Performing the 4-switch $u_1 \dots u_6 xy$ on M results in a perfect matching N containing u_2u_3 and u_4u_5, so $|N \Delta M'| \leq |M \Delta M'| - 1$.

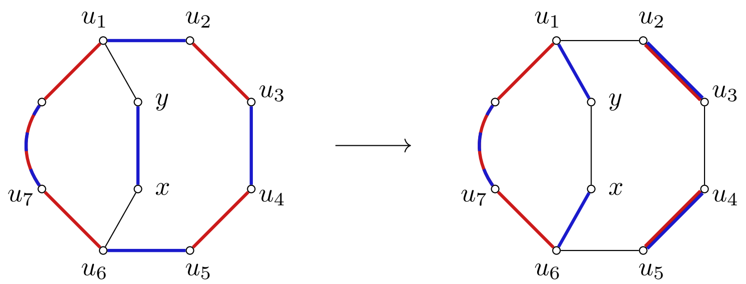


Figure 3: The reconfiguration sequence of Theorem 8. M is represented in blue and M' in red.

γn**-matchings** Let M and M' be two γn-matchings of G. We proceed by induction on $|M \Delta M'|$ by proving that we can always reduce the size of the symmetric difference by performing a 4-switch in M or in M'. We handle separately the following cases (in that order): isolated edges of $M \Delta M'$, even

[2]If G is a balanced bipartite graph on $2n$ vertices, let $V' = V_1 \setminus \{u_1, u_3, u_5\}$, where V_1 is the half of the bipartition that contains u_1. We now have $|A| = \deg(u_1) - 1$ and $|B| = \deg(u_6) - 1$, so $|A| + |B| \geq n - 2 > |V'|$.

alternating paths of length at most 6, odd alternating subpaths of length 3 or 5 with an edge between their extremities, odd alternating subpaths of length 5 with no edges between their extremities and a vertex at distance 3 or 5 in the subpath, and odd alternating paths of length 3 with no edge between their extremities. It is straightforward to check that these cases are exhaustive.

If there exists an edge xy of M such that x and y are both unmatched in M', then removing any edge from $M' \setminus M$ and replacing it by xy yields a γn-matching N such that $|N\Delta M| < |M'\Delta M|$. Thus one can assume that all the edges of M are incident to edges of M' (and likewise that all edges of M' are incident to edges of M). Hence, we can assume that alternating paths have length at least two.

If there exists an even alternating path $u_1, \dots u_{2p+1}$ with $p \in [3]$, such that $u_{2i-1}u_{2i} \in M$ and $u_{2i}u_{2i+1} \in M'$, with u_1 unmatched in M' and u_{2p+1} unmatched in M, then performing the 4-switch on the path $u_1 \dots u_{2p+1}$ results in a γn-matching N with $|N\Delta M'| < |M\Delta M'|$. Thus one can assume $M\Delta M'$ contains no even alternating paths of length at most 6.

Similarly to the perfect matching case, if there exists an alternating subpath $u_1, \dots u_{2p}$ with $p \in \{2, 3, 4\}$, such that $u_{2i-1}u_{2i} \in M$ and $u_{2i}u_{2i+1} \in M'$, and $u_1u_{2p} \in E(G)$, then performing in M the 4-switch on the cycle $u_1 \dots u_{2p}$ results in a γn-matching N with $|N\Delta M'| < |M\Delta M'|$. Thus one can assume that $M\Delta M'$ contains no odd alternating subpaths of length 3 or 5 with an edge between their extremities.

Assume that their exists an odd alternating subpath $P = u_1 \dots u_6$ of length 5 in $M\Delta M'$ with $u_{2i-1}u_{2i} \in M$ and $u_{2i}u_{2i+1} \in M'$. By assumption, u_1 is neither adjacent to u_4, nor to u_6, and u_6 is not adjacent to u_3. If u_1 has a neighbour y that is unmatched in M, then $N = M \cup \{yu_1, u_2u_3, u_4u_5\} \setminus \{u_1u_2, u_3u_4, u_5u_6\}$ is γn-matching such that $|N\Delta M'| \leq |M\Delta M'| - 2$. Likewise, if u_6 has a neighbour x unmatched in M, then $N = M \cup \{u_2u_3, u_4u_5, u_6x\} \setminus \{u_1u_2, u_3u_4, u_5u_6\}$ satisfies the same properties. Thus one can assume that all neighbours of u_1 and u_6 are matched in M. Let V' be the subset of vertices of $V(G) \setminus \{u_1, \dots u_6\}$ matched in M and A be the subset of V' composed of vertices matched in M to a neighbour of u_1. The only vertices of $\{u_1, \dots u_6\}$ that can be matched in M to neighbours of u_1 are u_1, u_4 and u_6. Therefore, $|A| \geq \deg(u_1) - 3$. Let $B = N(u_6) \cap V'$, we also have $|B| \geq \deg(u_6) - 3$. By pigeonhole principle[3], since V' has size $\gamma n - 6$ and is a superset of A and B and $|A| + |B| \geq \gamma n - 4$, there exists a vertex x in $A \cup B$. This vertex x is by definition adjacent to u_6, and matched to a vertex y in M, itself adjacent to u_1 (see Figure 3). Performing the 4-switch $u_1 \dots u_6xy$ on M results in a γn-matching N containing u_2u_3

[3]If G is a balanced bipartite graph on $2n$ vertices, let $V' = V_1 \setminus \{u_1, u_3, u_5\}$, where V_1 is the half of the bipartition containing u_1. We now have $|A| = \deg(u_1) - 1$ and $|B| = \deg(u_6) - 1$, so $|A| + |B| \geq n - 2$.

and u_4u_5, so $|N\Delta M'| \leq |M\Delta M'| - 1$.

Thus, one can assume that $|M\Delta M'|$ is a disjoint union of alternating paths of length 3, each without non-adjacent extremities. Note that exactly half of them contain two edges of M. Let $P = u_1u_2u_3u_4$ and $Q = v_1v_2v_3v_4$ be two alternating paths of $M\Delta M'$, with $A = \{u_1u_2, u_3u_4, v_2v_3\} \subseteq M$ and $B = \{u_2u_3, v_1u_2, v_3v_4\} \subseteq M'$. Replacing A by B in M results in a γn-matching N with $|N\Delta M'| < |N\Delta M'|$.

□

3.2 Equivalence via 3-switches

We now prove Theorem 6, that immediately follows from Theorem 8 and the following lemma:

Lemma 3.1. *Let G be an n-vertex graph with $\delta(G) \geq n/2+2$ (or a balanced bipartite graph on $2n$-vertices, such that $\delta(G) \geq \lfloor (n+2)/2 \rfloor$). Two perfect matchings of G differing by one 4-switch are equivalent via 3-switches.*

Proof. Let M and M' be two perfect matchings of G such that $M\Delta M'$ is a 8-cycle $C = u_1 \dots u_8$, with $u_{2i-1}u_{2i} \in M$ and $u_{2i}u_{2i+1} \in M'$. If C admits an even chords, say u_iu_j with $i < j$ and i even, then M' can be obtained from M after a switch on $u_j \dots u_8u_1 \dots u_i$, followed by a switch on $u_i \dots u_j$. One of these cycles has length 4 and the other length 6, so both are 6-switches.

Hence one can assume that C has no even chord. We first prove in Claim 3.2 that the existence of an edge of $M \cap M'$ with certain adjacencies implies the desired reconfiguration sequence, and secondly that such an edge exists.

Claim 3.2. *Let xy be an edge of $M \cap M'$ such that y is a neighbour of u_1 and x of u_2 and u_6. The matchings M and M' are equivalent via a sequence of at most three 3-switches.*

Proof of Claim. Starting from M, the sequence of 3-switches u_1u_2xy, $yu_2 \dots u_6$ and $yu_6u_7u_8u_1x$ results in M' (see Figure 4). ■

Bipartite case First note that for all integer d, $d \geq \lfloor (n+2)/2 \rfloor$ is equivalent to $d > n/2$. For each $i \in [8]$, let E_i be the set of edges xy of $M \cap M'$ such that either x or y is adjacent to u_i. We have $|E_i| = |N(u_i) \cap (V(G) \setminus V(C))| = |N(u_i) \setminus \{u_{i-1 \mod 8}, u_{i+1 \mod 8}\}|$ because C contains no even chords. So $|E_i| \geq \deg(u_i) - 2 > (n/2 - 2)$. Since $|M \cap M'| = n - 4$, a counting argument shows that there exists an edge $xy \in M \cap M'$ with endpoints adjacent to at least five vertices of C. Otherwise, we have $\sum_{i\in[8]} |E_i| \leq 4|M \cap M'| = 4n - 16$, but $\sum_{i\in[8]} |E_i| > 8(n/2 - 2) = 4n - 16$, a contradiciton.

By pigeonhole principle, there exists $i \in [4]$ such that u_i and u_{i+4} are adjacent to some endpoint of xy, and since G is bipartite, this endpoint is the same for u_i and u_{i+4}, say x. Without loss of generality, say that i is

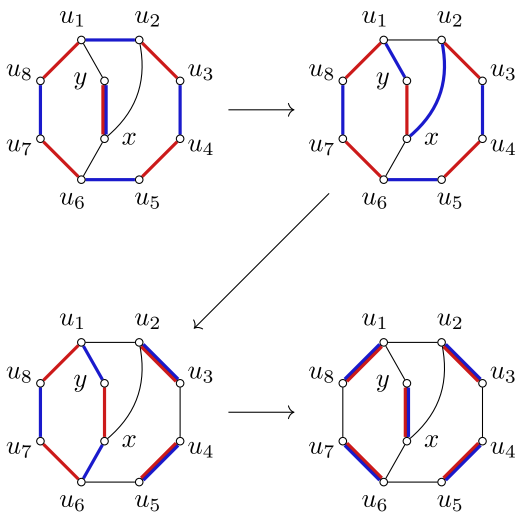


Figure 4: The reconfiguration sequence of Lemma 3.1. M is represented in blue and M' in red.

even. As G is bipartite and xy is adjacent to at least five vertices of C, y is adjacent to at least one vertex u_j with j odd. Up to relabelling i and j, the premise of Claim 3.2 is satisfied, which concludes the proof for the bipartite case.

Non-bipartite case For $i \in [8]$, let $E_i = \{(x, y)\colon xy \in M \cap M'$ and $x \in N(u_i)\}$. Let $V' = V(G) \setminus V(C)$. For each $i \in [8]$, we have $|N(u_i) \cap V'| \geq \deg(u_i) - 5 \geq n/2 - 3$ because C has no even chords. So $|E_i| \geq n/2 - 3$.

Consider the weight function w over all pairs (x, y) such that $xy \in M \cap M'$ defined as follows: $w(x, y)$ is the number of indices $i \in [8]$ such that x is adjacent to u_i (in particular, note that $w(x, y)$ may differ from $w(x, y)$). We have by double counting:

$$\begin{aligned}\sum_{xy \in M \cap M'} w(x,y) + w(y,x) &= \sum_{(x,y):xy \in M \cap M'} w(x,y) \\ &= \sum_{i \in [8]} |E_i| \\ &\geq 8(n/2 - 3)\end{aligned}$$

Since there are $n/2 - 4$ edges in $M \cap M'$, this implies that there exists $xy \in M \cap M'$ with $w(x, y) + w(y, x) > 8$.

For each u_i, let $w'(u_i) = |\{x, y\} \cap N(u_i)|$. We have $\sum_{i=1}^{8} w'(u_i) = w(x, y) + w(y, x) > 8$. We will shift the weights of w' towards eight targets $x_1, \ldots x_4$ and $y_1, \ldots y_4$, to prove that, up to relabelling, the adjacencies required by Claim 3.2 are satisfied by xy.

Consider the following discharging rules:

- For each $i \in [8]$, if u_i is adjacent x, then u_i sends weight $1/2$ to $x_{i \mod 4}$, weight $1/4$ to $y_{i+1 \mod 4}$ and weight $1/4$ to $y_{i+3 \mod 4}$.
- For each $i \in [8]$, if u_i is adjacent y, then u_i sends weight $1/2$ to $y_{i \mod 4}$, weight $1/4$ to $x_{i+1 \mod 4}$ and weight $1/4$ to $x_{i+3 \mod 4}$.

After applying the discharging rules, since the total weight of w' is greater than 8 and there are eight targets, one of them, say x_1 received weight greater than 1. Only four vertices (namely u_i for even i) could send weight $1/4$ to x_1, hence x_1 received a weight of $1/2$ by at least one vertex (u_1 or u_5).

If x_1 received a weight of $1/2$ by exactly one vertex, then it also received weight $1/4$ by at least three vertices, thus there exists $i \in \{2, 4\}$ such that y is adjacent to u_i and u_{i+4}. Since x is also adjacent to u_1 or u_5, the premise of Claim 3.2 is satisfied up to relabelling.

If x_1 received a weight of $1/2$ by two vertices, then x is adjacent to u_1 and u_5. Since x_1 also received a weight of $1/4$ by at least one vertex, y is also adjacent to a vertex u_i with i even. Up to relabelling, this is again the adjacency pattern required by Claim 3.2. □

3.3 Equivalence via 2-Switches

Theorem 7 follows from Theorem 6 combined with the following lemma:

Lemma 3.3. *Let G be a n-vertex graph with $\delta(G) \geq \lfloor (2n+3)/3 \rfloor$ (or a balanced bipartite graph on $2n$-vertices, with $\delta(G) \geq \lfloor (2n+3)/3 \rfloor$). Two perfect matchings of G differing by one 3-switch are equivalent via 2-switches.*

Proof. First note that $d \geq \lfloor (2n+3)/3 \rfloor$ is equivalent to $d > 2n/3$ for all $d \in \mathbb{N}$. Let M and M' be two perfect matchings of G such that $M \Delta M'$ is a 6-cycle $C = u_1, \dots u_6$ with $u_{2i-1}u_{2i} \in M$ and $u_{2i}u_{2i+1} \in M'$. Here again, if C contains an even chord $u_i u_{i+3}$ for some $i \in [3]$, then it divides C into two 4-cycles, and performing a 2-switch on each of these cycles transforms M into M'. Hence one can assume that C has no even chord. We first prove in Claim 3.4 that the existence of an edge of $M \cap M'$ with certain adjacencies implies the desired reconfiguration sequence, and secondly that such an edge exists.

Claim 3.4. *If there exists an edge $xy \in M \cap M'$ with x adjacent to u_1 and u_5, and y to u_2 and u_4, then M is equivalent to M' via a sequence of four 2-Switches.*

Proof of Claim. Starting from M, the sequence of 2-switches u_1u_2yx, $yu_2u_3u_4$, $u_1xu_5u_6$ and yu_4u_5x results in M' (see Figure 5). ■

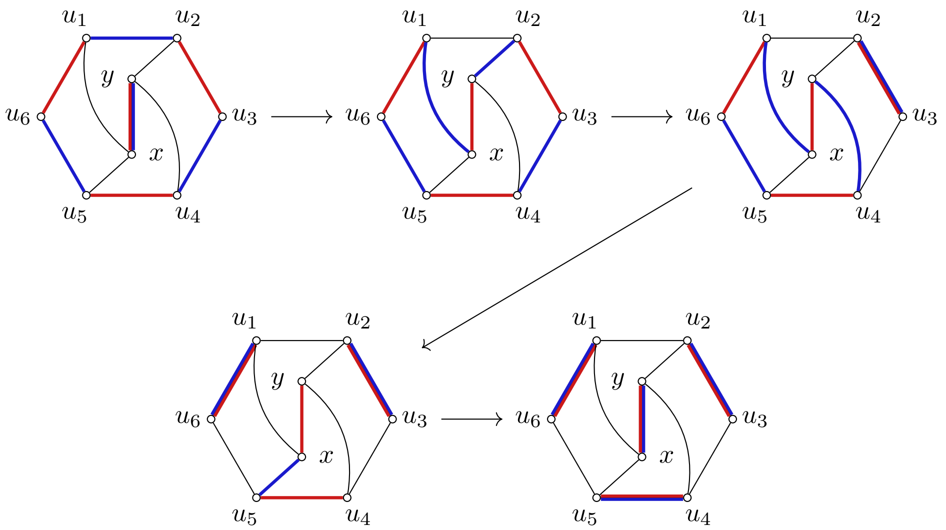


Figure 5: The reconfiguration sequence of Claim 3.4. M is represented in blue and M' in red.

Bipartite case For $i \in \{1, 2, 3\}$, consider E_i the set edges of $M \cap M'$ with opposite endpoints adjacent to u_i and u_{i+3} respectively. Let $V' = V_i \setminus V(C)$ where V_i is the half of the bipartition that contains u_i. Let A_i be the set of vertices of V' that are *not* matched in $M \cap M'$ to a neighbour of u_i. We have $|A_i| = n - 6 - (\deg(u_{i+3}) - 2) < n/3 - 2$, because C contains no even cycle and G is bipartite so the only neighbours of u_i in $V(C)$ are u_{i+1} and $u_{i-1 \mod 6}$. Let $B_i = V' \setminus N(u_{i+3})$. By the same reasoning, we also have $|B_i| < n/3 - 2$. We have $|E_i| = |V' \setminus (A_i \cup B_i)|$ and $|A_i| + |B_i| \geq 4n/3 - 4$, so $|E_i| > n/3 - 2$ for all i. By pigeon hole principle, there exists an edge xy of M that belongs to two of the E_i, say without loss of generality E_1 and E_2, namely x is adjacent to u_1 and u_5, and y to u_2 and u_4, which concludes the proof of the bipartite case by Claim 3.4.

Non-bipartite case For $i \in [6]$, let $E_i = \{(x, y) \colon xy \in M \cap M', x \in N(u_i), y \in N(u_{i+3 \mod 6})\}$. Let A_i be the set of vertices of $V' = V(G) \setminus \{u_1, \ldots u_6\}$ that are *not* matched to a neighbour of u_i. We have $|A_i| \leq n - 6 - (\deg(u_i) - 4)$ because C contains no even chord, so $|A_i| < n/3 - 2$. Let $B_i = V' \setminus N(u_{i+3 \mod 6})$, we also have $|B_i| < n/3 - 2$. So

$$\begin{aligned}
|E_i| &= |V' \setminus (A_i \cup B_i)| \\
&\geq |V'| - (|A_i| + |B_i|) \\
&> n - 6 - 2(n/3 - 2) = n/3 - 2
\end{aligned}$$

Consider the weight function w over all pairs (x, y) such that $xy \in M \cap M'$

defined as follows: $w(x, y)$ is the number of $i \in [6]$ such that x is adjacent to u_i and y to $u_{i+3 \mod 6}$ (in particular, note that $w(x, y)$ may differ from $w(x, y)$). By double counting,

$$\sum_{(x,y)\colon xy \in M \cap M'} w(x, y) = \sum_{i \in [6]} |E_i| > 6(n/3 - 2) = 2(n - 6).$$

Since there are $n - 6$ such couples (x, y), there exists (x, y) with $w(x, y) > 2$. So without loss of generality, x is adjacent to u_1 and u_5 and y to u_2 and u_4, which concludes the proof by Claim 3.4. □

4 Lower bounds

We first give in Subsection 4.1 general constructions yielding lower bounds on the freezing, clustering, giant-component and connectedness thresholds for any k. In Subsection 4.2 we give constructions yielding sharper bounds for the connectedness thresholds when $k \in \{2, 3\}$, as well as lower bounds for the connectedness threshold of regular balanced bipartite graph.

4.1 Lower bounds for generic k

We first define two families of graphs, one composed of balanced bipartite graphs, the other of general graphs, that we will use for several of our lower bounds. For all $\gamma \in (0, 1]$, for all integers $k \geq 2$, $p \geq 1$ and n such that $\gamma n/2$ is integral and at least $p(k+1)$, let $G_{k,p,\gamma,n}$ be the n-vertex graph constructed as follows. We partition the vertex set $V(G_{k,p,\gamma,n})$ into three sets $X \sqcup Y \sqcup Z$ with $|X| = 2p(k+1)$, $|Y| = \gamma n/2 - p(k+1)$ and $|Z| = (1-\gamma/2)n - p(k+1)$. Note that this quantities are integral because we assumed that $\gamma n/2$ is integral so that the graph could have a mathcing of size exactly $\gamma n/2$. The set X induces a collection of p vertex-disjoint cycles of length $2(k + 1)$, Y and Z are independent sets, and all edges between Y and $X \cup Z$ are present (see Figure 6).

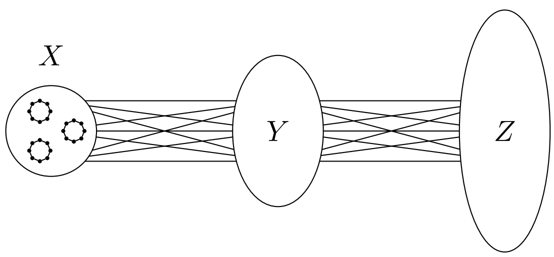


Figure 6: The graph $G_{k,p,\gamma,n}$.

Lemma 4.1. *Let $\gamma \in (0, 1]$ and k, p, n be integers with $k \geq 2$, $p \geq 1$ and $\gamma n/2 \geq p(k + 1)$ an integer. The graph $G_{k,p,\gamma,n}$ has minimum degree*

$\gamma n/2 - p(k+1)$ and under k-switches, the space of γn-matchings of $G_{k,p,\gamma,n}$ has exactly 2^p connected components, which have equal size. Moreover, the restriction to X of the γn-matchings is constant within each of these connected components.

Proof. The vertices in Y have degree $(1-\gamma/2)n+p(k+1) \geq n/2$, the vertices of X have degree larger than those in Z, which have degree $\gamma n/2 - p(k+1)$, so $\delta(G_{k,p,\gamma,n}) = \gamma n/2 - p(k+1)$.

Let M be a γn-matching of $G_{k,p,\gamma,n}$. The number of edges in M is $\gamma n/2 = |Y| + |X|/2$. Since Z and y are independent sets with a complete bipartite graph between them, M has $|Y|$ edges connecting Y to Z and $|X|/2$ edges within X. In other words, M induces a perfect matching on X, and all vertices of Y are matched to a vertex of Z. As $G_{k,p,\gamma,n}[X]$ is a collection of p cycles of length $2(k+1)$, it has exactly 2^p perfect matchings. Any two of these matchings differ on at least one cycle, that is on $2(k+1)$ edges, which proves that $G_{k,p,\gamma,n}[X]$ (and respectively $G_{k,p,\gamma,n}$) have at least 2^p equivalence classes of γn-matchings under k-switches. As $G_{k,p,\gamma,n}[Y \cup Z]$ is a bipartite complete graph, all matchings of $G_p[Y \cup Z]$ saturating Y are equivalent under 2-switches. Therefore, the equivalence classes of the γn-matchings of G_p under k-switches are in one to one correspondance with the restriction of the matchings to X, which proves that the number of equivalence classes is exactly 2^p and that the restriction to X of the γn-matchings is constant within each of these connected components. $\square$

We now construct a balanced bipartite graph similar to $G_{k,p,\gamma,n}$. For all $\gamma \in (0,1]$, for all integer $k \geq 2$, $p \geq 1$ and n such that γn is integral and at least $p(k+1)$, let $G^{(\mathcal{B})}_{k,p,\gamma,n}$ be the balanced bipartite graph on $2n$ vertices constructed as follows. Denote $V_1 \sqcup V_2$ the bipartition of $V(G^{(\mathcal{B})}_{k,p,\gamma,n})$. For each i, we partition V_i in three sets $X_i \sqcup Y_i \sqcup Z_i$ with $|X_i| = p(k+1)$, $|Y_1| = \lceil \gamma n/2 - p(k+1)/2 \rceil$ and $|Z_1| = \lfloor (1-\gamma/2)n - p(k+1)/2 \rfloor$, and Y_2 and Z_2 of size like Y_1 and Z_1 with the floors and ceilings permuted. For each $i \in [2]$ we add all the edges between Y_i and $X_{3-i} \cup Z_{3-i}$. Finally, $G^{(\mathcal{B})}_{k,p,\gamma,n}[X_i \cup X_2]$ is a disjoint collection p cycles of length $2(k+1)$ alternating between X_1 and X_2 (see Figure 7).

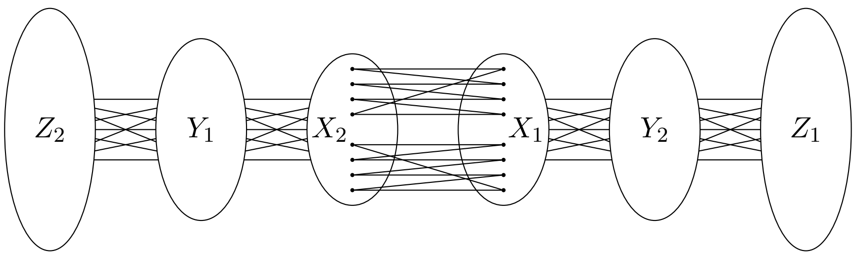


Figure 7: The graph $G^{(\mathcal{B})}_{k,p,\gamma,n}$.

Lemma 4.2. *Let $\gamma \in (0,1]$ and k,p,n be integers with $k \geq 2$, $p \geq 1$ and $\gamma n \geq p(k+1)$ an integer. The graph $G_{k,p,\gamma,n}$ has minimum degree $\lfloor(\gamma n - p(k+1))/2\rfloor$ and under k-switches, the space of γn-matchings of $G^{(\mathcal{B})}_{k,p,\gamma,n}$ has exactly 2^p connected components, which have equal size. Moreover, the restriction to $X_1 \cup X_2$ of the γn-matchings is constant within each of these connected components.*

Proof. The proof is highly similar to that of Lemma 4.1. □

As a result, we have the following three corollaries by adjusting the value of p in $G_{k,p,\gamma,n}$ and $G^{(\mathcal{B})}_{k,p,\gamma,n}$:

Corollary 4.3. *For all $k \geq 2$, for all $\gamma \in (0,1]$, we have the following inequalities:*

$$\delta^{\textit{freezing}}_{k,\gamma}, \delta^{\textit{freezing}}_{k,\gamma}(\mathcal{B}), \delta^{\textit{clustering}}_{k,\gamma}, \delta^{\textit{clustering}}_{k,\gamma}(\mathcal{B}) > \frac{\gamma n}{2} - \varepsilon k n$$

Corollary 4.4. *For all $k \geq 2$, for all $\gamma \in (0,1]$, we have the following inequalities:*

$$\delta^{\textit{giant}}_{k,\gamma}, \delta^{\textit{giant}}_{k,\gamma}(\mathcal{B}) > \frac{\gamma n}{2} - \varepsilon k$$

Corollary 4.5. *For all $k \geq 2$, for all $\gamma \in (0,1]$, we have the following inequalities:*

$$\delta^{\textit{connectedness}}_{k,\gamma} \geq \frac{\gamma n}{2} - k \quad \textit{and} \quad \delta^{\textit{connectedness}}_{k,\gamma}(\mathcal{B}) \geq \left\lfloor \frac{\gamma n - (k-1)}{2} \right\rfloor$$

Proofs of Corollaries 4.3 to 4.5. Let $c > 1$. For the freezing threshold, let $p = cn$, respectively $p = n \log c$ for the clustering threshold, $p = \log c$ for the giant component threshold and $p = 1$ for the connectedness threshold. By Lemma 4.1, the graph $G_{k,p,\gamma,n}$ has minimum degree $\gamma n/2 - p(k+1)n$ and $\mathcal{H}_{k,\gamma}(G_{k,p,\gamma,n})$ verifies the following respective properties: it has cn frozen variables in each connected component, it has $2^{n \log c} = c^n$ connected components, it has c components of equal size so all of them contain at most an $1/c$ fraction of the matchings, it has two connected components. By taking the same value of p in $G_{k,p,\gamma,n}(\mathcal{B})$ we obtain the remaining bounds, for balanced bipartite graphs. □

4.2 Precise bounds for non-connectedness and regular bipartite graphs

We construct two families of graph, one composed of balanced bipartite (regular) graphs, the other of general graphs. For any $k \geq 2$ and any even n, let $F_{k,n}$ be the graph constructed as follows. First assume that n is a multiple of $(k+1)$ to avoid divisibility issues. We partition the vertex set $V(F_{k,n})$ in $(k+1)$ sets $\bigsqcup_{i\in[k+1]} X_i$ such that $|X_i| = n/(k+1)$. The sets X_1

and X_{k+1} each induce a complete graph, all other X_i induce an independent set and $F_{k,n}[X_i \cup X_{i+1}]$ induces a complete bipartite graph for each $i \in [k]$ (Figure 8). Now, if n is not a multiple of $k+1$, we add $n \mod k+1$ remaining vertices in different set X_i, starting by X_2, X_k. In particular, we have $\delta(F_{2,n}) = \lfloor \frac{n-1}{3} \rfloor + \lfloor \frac{n}{3} \rfloor = \lfloor \frac{2n-2}{3} \rfloor$, and $\delta(F_{3,n}) = \lfloor \frac{n-2}{4} \rfloor + \lfloor \frac{n}{4} \rfloor = \frac{n}{2} - 1$ because n is even.

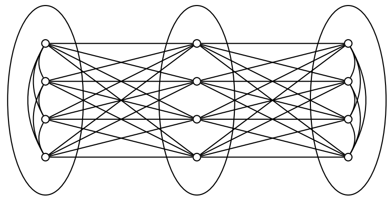


Figure 8: The graph $F_{2,n}$

Lemma 4.6. *Let $k \geq 2$ and $n \geq 4(k+1)$ be an even integer. The switch graph $\mathcal{H}_k(F_{k,n})$ is disconnected.*

Proof. Let E_1 be the subset of edges with both endpoints in X_1 and E_2 the remaining edges. The number of matched edges of E_1 in a perfect matching is invariant under k-switches. Indeed, let M be a perfect of $F_{k,n}$ and C an alternating cycle of M of length at most $2k$ containing some edge of X_1. The distance between X_1 and X_{k+1} is k, so C does not intersect X_{k+1}. Hence, C can be decomposed as the concatenation of paths $P_1, Q_1, \dots P_\ell, Q_\ell$, where the paths P_i use edges edges of E_1 and the paths Q_i use edges of E_2. In particular, each Q_i has even length, because the subgraph on the vertex set $V(F_{k,n}) \setminus X_{k+1}$ containing the edges E_2 is bipartite. So performing a switch on the alternating cycle C preserves the number of edges in E_2 and thus the number of edges in E_1.

There exists a perfect matching with exactly p edges of E_1 for each $p \in \{1, \dots \lfloor n/(2(k+1)) \rfloor\}$, so $\mathcal{H}_k(F_{k,n})$ is disconnected because $n \geq 4(k+1)$. $\square$

Let $k \geq 2$ and $n \geq 2(k+1)$ be two integers. Assume for now that n is a multiple of $(k+1)$. Let $F_{k,n}^{(\mathcal{B})}$ be the balanced bipartite graph on $2n$ vertices obtained by replacing each vertex of a $2(k+1)$ cycle by an independent set of size $n/(k+1)$ (see Figure 8). Note that $F_{k,n}^{(\mathcal{B})}$ is $2n/(k+1)$-regular.

Now if n is not a multiple of $(k+1)$, we construct $F_{k,n}^{(\mathcal{B})}$ similarly, by distributing the additional vertices one-by-one in pairs of consecutive blown-up vertices. The graph we obtain is not regular anymore, but still balanced bipartite and we have $\delta(F_{2,n}) = \lfloor 2n/3 \rfloor$ and $\delta(F_{3,n}) = \lfloor n/2 \rfloor$.

Lemma 4.7. *Let $k \geq 2$ and $n \geq (k+1)$ an integer. The switch graph $\mathcal{H}_k(F_{k,n}^{(\mathcal{B})})$ is disconnected.*

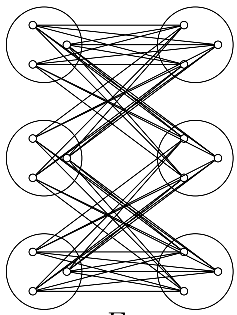


Figure 9: The graph $F^{(\mathcal{R})}_{2,n}$, a blow-up of the vertices of a 6-cycle into independent set of size $n/3$.

Proof. Let E_1 be the subset of edges joining the blow-ups of two fixed vertices of the $2(k+1)$ cycle and E_2 the remaining edges. As in Lemma 4.6, the number of matched edges of E_1 in a perfect matching is invariant under k-switches and for each $p \in \{0, \dots \lfloor n/(k+1) \rfloor\}$, there exists a perfect matching with exactly p edges of E_1, so $\mathcal{H}_k(F^{(\mathcal{R})}_{k,n})$ is disconnected. $\square$

Lemmas 4.6 and 4.7 directly give us some tighter lower bounds than Lemmas 4.1 and 4.2 on the connected thresholds for $k \in \{2, 3\}$, as well as a lower bound on the connectedness threshold for balanced bipartite regular graphs.

Corollary 4.8. *For n large enough, we have the inequalities*

- $\delta_2^{connectedness} \geq \lfloor \frac{2n+1}{3} \rfloor$,
- $\delta_3^{connectedness} \geq \frac{n}{2}$,
- $\delta_2^{connectedness}(\mathcal{B}) \geq \lfloor \frac{2n+3}{3} \rfloor$.
- $\delta_3^{connectedness}(\mathcal{B}) = \lfloor \frac{n+2}{2} \rfloor$.
- *For all $k \geq 2$, if $(k+1)$ divides n,* $\delta_k^{connectedness}(\mathcal{R}) \geq 2\frac{n}{k+1} + 1$.

5 Random sampling

Similarly as for our connectedness results, we first prove in Subsection 5.1 that the 4-switch Markov chain mixes polynomially, before deriving from it the polynomial mixing of the 2-switch and 3-switch Markov chain in Subsection 5.2. Finally, we discuss in Subsection 5.3 the implications of these results on the expansion properties of $\mathcal{H}_k(G)$.

5.1 Polynomial mixing time of the 4-switch Markov chain

In the rest of this section, G is either an n-vertex graph or a balanced bipartite graph on $2n$ vertices. Consider the following Markov Chain Γ on the perfect matchings of G. Given a perfect matching M of G:

1. Select independently and uniformly at random four vertices u_1, u_2, u_3 and u_4, (if G is balanced bipartite, select all them from the same side of the bipartition). Let $u_i v_i$ be the edge of M for each i.

2. If $u_l = u_1$ for some $l \in \{2, 3, 4\}$, let l be the smallest such index and $C = (u_1, v_1, u_2, v_2 \ldots u_{l-1}, v_{l-1})$. Otherwise, let C be the sequence $(u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4)$.

3. If C is a simple cycle, set $M' \leftarrow M \Delta C$ with probability 1/2 and $M' \leftarrow M$ with probability 1/2.

Note that the probability of performing a 4-switch on a fixed cycle C of length $\ell \in \{4, 6, 8\}$ is $\Theta(1/n^{\ell/2})$. Moreover, the transition matrix of Γ is symmetric, so Γ admits the uniform distribution on the space of perfect matchings of G as a stationary distribution. The probability of not performing any switch is at least 1/2 because of laziness of the third step, hence Γ is also aperiodic. Hence, by Theorem 8, Γ is ergodic and converges towards the uniform distribution on perfect matchings if G has sufficiently high minimum degree:

Theorem 9. *Let G be a n-vertex graph in which the degrees of each pair of non-adjacent vertices sum up to at least $n+2$. Alternatively, let G be a balanced bipartite graph on $2n$ vertices in which each pair of non-adjacent vertices in different halfs of the bipartition have their degrees that sum up to at least $n+1$.*
The Markov chain Γ on the perfect matchings of G converges towards the uniform distribution on the set of perfect matchings of G. It has mixing time

$$\tau_{mix} \leq O(n^8 \ln(n))$$

The proof of the mixing time of Theorem 9 is a textbook application of the canonical path method. The canonical paths we will use are close to the reconfiguration sequences used to prove Theorem 8, but operate on the symmetric difference between the endpoints of the path following a precise order.

Proof. To bound the congestion of our Markov chain Γ, we will define a canonical path $\gamma_{S,T}$ connecting each pair of perfect matchings S and T. Consider an arbitrary order on the vertices of G. Given S and T two perfect matchings of G, our canonical path between S and T is defined as follows. Let $C_1, \ldots C_p$ be the alternating cycles in $S \Delta T$, such that for each i, the smallest vertex in C_i is smaller than the smallest vertex in C_{i+1}. Let $M_1, \dots M_{p+1}$ be the perfect matchings of G such that M_i is equal to T on all cycles C_j with $j < i$, equal to S on all cycles C_j with $j \geq i$ and equal to S (and T) around all remaining vertices. Our canonical path will go from $S = M_1$ to $T = M_{p+1}$ and will visit all M_i in increasing order. Thus we only need to explain how to go from M_i to M_{i+1}.

Connecting M_i to M_{i+1}

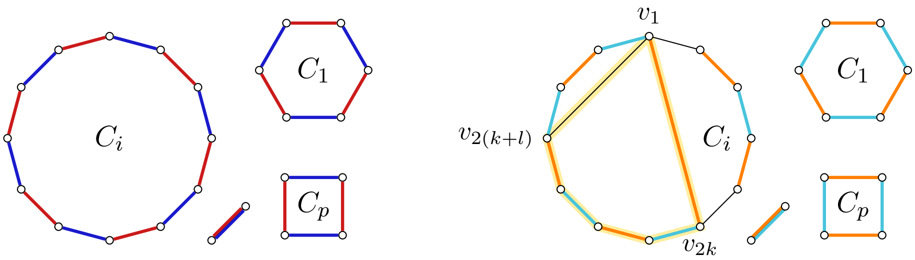
Denote $v_1, v_2, \dots v_{2q}$ the consecutive vertices of C_i, such that v_1 is the smallest vertex and $v_1v_2 \in M_i$.

Claim 5.1. *There exists a sequence $M_i = N_0, \dots N_q = M_{i+1}$ of perfect matchings of G such that for all j, N_{j-1} and N_j differ by a 4-switch and one of the following conditions is verified:*

- *There exists $k \in [q]$ such that N_j is equal to M_i on $G \setminus C_i$ and contains the edges $v_1v_{2k}, v_{2k+1}v_{2k+2}, \dots v_{2q-1}v_{2q}$ and $v_2v_3, \dots v_{2k-2}v_{2k-1}$. In other words, N_j and M_i are equal on all vertices but $\{v_1, \dots v_{2k}\}$, where N_j contains v_1v_{2k} and is equal to M_{i+1} on all other vertices. We will say that N_j is representative and call k its progress index.*
- *There exists $k \in [q]$ and $xy \in N_{j-1}$ such that N_j is equal to M_i on $G \setminus (C_i \cup \{x, y\})$ and contains the edges $v_1x, yv_{2k}, v_{2k+1}v_{2k+2}, \dots v_{2q-1}v_{2q}$ and $v_2v_3, \dots v_{2k-2}v_{2k-1}\}$. In other words, N_j and M_i are equal on all vertices but $\{v_1, \dots v_{2k}, x, y\}$, where N_j contains v_1x and yv_{2k} and is equal to M_{i+1} on all other vertices. Moreover, neither v_{2k} nor v_{2k+2} are adjacent to v_1. We will call such N_j misleading, k its progress index xy the deceptive edge.*

Proof of Claim. We build the sequence $N_0, \dots N_q$ inductively. Suppose we already constructed $N_0, \dots N_{j-1}$. We distinguish several cases, depending on whether N_{j-1} and the matching N_j we construct are representative or misleading.

Case 1 First, assume that N_{j-1} is representative, let k be the progress index of N_{j-1} and assume that there exists some $l \in [3]$ such that $v_{2(k+l)}$ is a neighbour of v_1. Then let l be the largest such integer. Let N_j be the perfect matching obtained by performing the 4-switch on $C = v_1v_{2k}v_{2k+1} \dots v_{2(k+l)}$. The vertices affected by this 4-switch all belong to C_i. Since N_{j-1} was representative and N_j contains the edges $v_1v_{2(k+l)}$, $v_{2(k+l)+1}v_{2(k+l)+2}$, $\dots v_{2q-1}v_{2q}$ and v_2v_3, $\dots v_{2(k+l)-2}v_{2(k+l)-1}$, N_j is also representative with progress index $k + l$ (see Figure 10).



(a) The perfect matchings S, and T are represented in blue and red respectively.

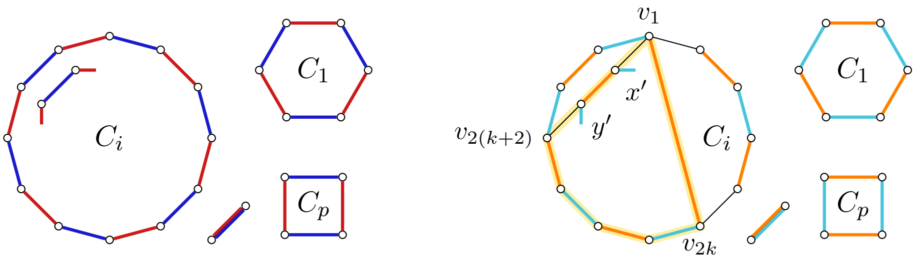
(b) The matchings N_{j-1} and L (defined in Claim 5.2), represented in orange and light blue respectively. N_j is obtained by performing a 4-switch on C, which is here highlighted in yellow.

Figure 10: Constructing N_j when N_{j-1} is representative and v_1 is a neighbour of $v_{2(k+l)}$ for some $l \in [3]$ (here $l = 2$). The resulting N_j is also representative.

Case 2 Assume now no such l exists, but N_{j-1} is still representative. Let A be the subset of vertices of $V' = V(G) \setminus \{v_1, v_{2k}, v_{2k+1}, \dots v_{2k+4}\}$ matched in N_{j-1} to a neighbour of v_1. Since v_{2k+2} and v_{2k+4} are not neighbours of v_1, we have $|A| \geq \deg(v_1) - 3$[4]. Likewise, v_{2k+4} is not a neighbour of v_1 so $B = N(v_{2k+4}) \cap V'$ has size at least $\deg(v_{2k+4}) - 4$. We have $|V'| = n - 6$ and $|A| + |B| \geq n - 5$ because v_1 and v_{2k+4} are not adjacent. So by pigeonhole principle, there exists $x'y' \in N_{j-1}$, with $x', y' \notin \{v_1, v_{2k}, \dots v_{2k+4}\}$, such that x' is a neighbour of v_1 and y' of v_{2k+4}. By performing the 4-switch $C = v_1 v_{2k} \dots v_{2k+4} y' x'$ in N_{j-1}, one obtains the misleading matching N_j with the deceptive edge $x'y'$ and progress index $k + 2$ (see Figure 11). Moreover, note that since l did not exist, neither v_{2k+4} nor v_{2k+6} is adjacent to v_1.

Setup for the remaining cases Now, assume that N_{j-1} is misleading. Let xy be the deceptive edge of N_{j-1} and k the progress index of N_{j-1}. We distinguish cases depending on the value of (x, y): the most generic case is when $\{x, y\} \cap \{v_{2k+1}, \dots v_{2k+4}\} = \emptyset$, otherwise $\{x, y\} = \{v_{2k+1}, v_{2k+2}\}$ or $\{x, y\} = \{v_{2k+3}, v_{2k+4}\}$ because xy is an edge of N_{j-2}. Note that we cannot have $(x, y) = (v_{2k+2}, v_{2k+1})$ because N_{j-1} is deceptive, so v_{2k+2} and v_1 are not adjacent. We handle the case $(x, y) = (v_{2k+4}, v_{2k+3})$ separately, and the other cases all at once by defining an the alternating path P of length 5, going from v_1 to v_{2k+l} for some $l \in \{1, 2\}$, that we will use later to define the appropriate switch:

[4]If G is balanced bipartite, we set $V' = V_1 \setminus \{v_1, v_{2k+1}, v_{2k+3}\}$ where V_1 is the half of the bipartition containing v_1. We then have $|A| \geq \deg(v_1) - 1$. We also have $|B| = |N(v_{2k+4}) \cap V'| \geq \deg(v_{2k+4}) - 3$ because v_1 and v_{2k+4} are not adjacent. Since $|V'| = n - 3$ and $|A| + |B| \geq n - 2$, by pigeonhole principle, A and B intersect.



(a) The perfect matchings S, and T are represented in blue and red respectively.

(b) The matchings N_{j-1} and L (defined in Claim 5.2), represented in orange and light blue respectively. N_j is obtained by performing a 4-switch on C, which is here highlighted in yellow.

Figure 11: Constructing N_j when N_{j-1} is representative and v_1 has no neighbours among $v_{2(k+1)}$, $v_{2(k+2)}$ and $v_{2(k+3)}$. The resulting N_j is misleading.

Case 3 If $(x, y) = (v_{2k+4}, v_{2k+3})$, then since $yv_{2k} = v_{2k+3}v_{2k}$ and $v_{2k+1}v_{2k+2}$ are edges of N_{j-1}, by performing the 2-switch $v_{2k+3}v_{2k}v_{2k+1}v_{2k+2}$, one obtains a matching the representative matching N_j with progress index equal to $k + 2$.

Case 4 If $(x, y) = (v_{2k+3}, v_{2k+4})$, let $P = v_1v_{2k+3}v_{2k+2}v_{2k+1}v_{2k}v_{2k+4}$ and $l = 2$. Note that this case cannot occur if G is bipartite because v_1 and v_{2k+3} are then in the same half of the bipartition.

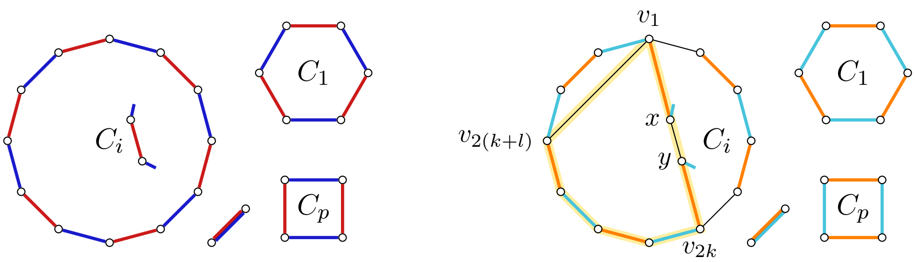
Case 5 If $(x, y) = (v_{2k+1}, v_{2k+2})$, let $P = v_1v_{2k+1}v_{2k}v_{2k+2}v_{2k+3}v_{2k+4}$ and $l = 2$. Note again that this case cannot occur if G is bipartite because v_1 and v_{2k+1} are then in the same half of the bipartition.

Case 6 Otherwise, we are in the generic case $\{x, y\} \cap \{v_{2k+1}, \dots v_{2k+4}\}$. Let $P = v_1xyv_{2k} \dots v_{2k+2}$ and $l = 1$.

As Case 3 is already settled, we assume from now on that $(x, y) \neq (v_{2k+4}, v_{2k+3})$. We handle Cases 4, 5 and 6 simultaneously. In each of these cases, one can check that P is an alternating path of N_{j-1}. We distinguish two subcases, depending on whether there exists some $m \in \{0, 1\}$ such that $v_{2(k+l+m)}$ is a neighbour of v_1.

Cases 4a, 5a and 6a Assume that there exists some $m \in \{0, 1\}$ such that $v_{2(k+l+m)}$ is a neighbour of v_1, and let m be the largest such integer. Let C be the alternating cycle obtained by concatenating P with $v_{2(k+l)} \dots v_{2(k+l+m)}v_1$. The cycle C has length six or eight because P hes length five. Let N_j be the perfect matching obtained by performing the 4-switch on C (see for example Figure 12 for Case 6a, with $P = v_1xyv_{2k} \dots v_{2k+4}$ and $l = 2$ and $m = 0$). The vertices affected by this 4-switch all belong to

$C_i \cup \{x, y\}$. If we had $(x, y) = (v_{2k+3}, v_{2k+4})$ or $(x, y) = (v_{2k+1}, v_{2k+2})$ (that is Cases 4a and 5a respectively), N_j is representative with progress index $k + l + m$: it is equal to M_i on all vertices but $\{v_1, \dots v_{2(k+l+m)}\}$, where N_j contains $v_1 v_{2(k+l+m)}$ and is equal to M_{i+1} on all other vertices. In Case 6a, recall that $P = v_1 x y v_{2k} \dots v_{2k+2}$. Since N_{j-1} was misleading, xy belonged to N_{j-2} and belongs to N_j as well because x and y are consecutive in P. Thus, N_j is representative with progress index $k + l + m$: it is equal to M_i on all vertices but $\{v_1, \dots v_{2(k+l+m)}\}$, where N_j contains $v_1 v_{2(k+l+m)}$ and is equal to M_{i+1} on all other vertices.



(a) The perfect matchings S, and T are represented in blue and red respectively.

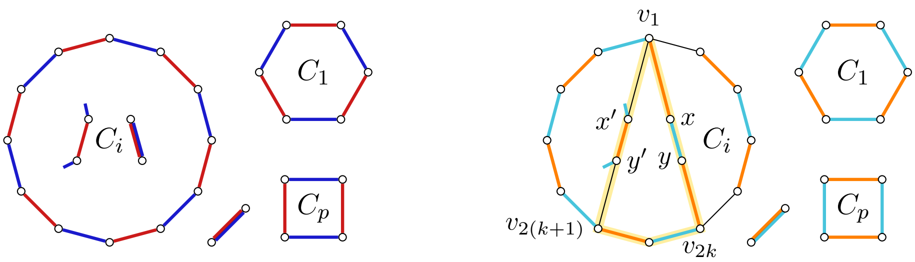
(b) The matchings N_{j-1} and L (defined in Claim 5.2), represented in orange and light blue respectively. N_j is obtained by performing a 4-switch on C, which is here highlighted in yellow.

Figure 12: Constructing N_j when N_{j-1} is misleading and v_1 is adjacent to $v_{2(k+l)}$ for some $l \in [2]$ (here $l = 2$). The resulting N_j is representative.

Cases 4b, 5b, and 6b Finally, assume that no such m exists. Let $V' = V(G) \setminus V(P)$ and A be the set of vertices of V' matched in N_{j-1} to a neighbour of v_1[5]. Since v_1 is not adjacent to v_{2k} and $v_{2(k+l)}$, it has at least $\deg(v_1) - 3$ neighbours among V', so $|A| \geq \deg(v_1) - 3$. Likewise, $v_{2(k+l)}$ has at least $\deg(v_{2(k+l)}) - 4$ neighbours in V' because $v_{2(k+l)}$ and v_1 are not adjacent. So $|A| + |B| \geq n - 5 > |V'|$ and by pigeonhole principle A and B intersect. In other words, there exists $x'y' \in N_{j-1}$, with $x', y' \notin V(P)$, such that x' is a neighbour of v_1 and y' of $v_{2(k+l)}$. Let C be the cycle of length eight obtained by concatenating P with $v_{2(k+l)} y' x' v_1$. By performing the 4-switch on the C in N_{j-1}, one obtains the misleading matching N_j with the deceptive edge $x'y'$ and progress index $k + l$ (see for example Figure 13 for Case 6b where $l = 1$ and $P = v_1 x y v_{2k} \dots v_{2k+2}$). Indeed, if we had $(x, y) = (v_{2k+1}, v_{2k+2})$ or

[5] If G is a balanced bipartite graph, recall that $P = v_1 x y v_{2k} \dots v_{2k+2}$ and $l = 1$. Let $V' = V_1 \setminus \{v_1, y, v_{2k+1}\}$. We have $|A| \geq \deg(v_1) - 1$ because v_1 is not adjacent to v_{2k} or v_{2k+2}. Let $B = N(v_{2k+2} \cap V')$, we have $|B| \geq \deg(v_{2k+2}) - 2$ because v_1 is not adjacent to v_{2k+2}. By assumption, $|A| + |B| \geq n - 2$ so by pigeonhole principle, since $|V'| = n - 3$, A and B intersect.

$(x, y) = (x, y) = (v_{2k+3}, v_{2k+4})$ (that is Cases 4b and 5b respectively), then recall that $P = v_1 v_{2k+1} v_{2k} v_{2k+2} v_{2k+3} v_{2k+4}$ and $v_1 v_{2k+3} v_{2k+2} v_{2k+1} v_{2k} v_{2k+4}$ respectively, so N_j is equal to M_i on all vertices but $\{v_1, \dots v_{2(k+2)}, x', y'\}$, where N_j contains $v_1 v_{2k+2}$ and $x'y'$, and is equal to M_{i+1} on all other vertices. Since m did not exists, neither v_{2k+4} nor v_{2k+6} are adjacent to v_1. In the remaining case, recall that $P = v_1 x y v_{2k} \dots v_{2k+2}$. Since N_{j-1} was misleading with deceptive edge xy, the edge xy belonged to N_{j-2} and also belongs to N_j because x and y are consecutive in P. Thus N_j is equal to M_i on all vertices but $\{v_1, \dots v_{2(k+2)}, x', y'\}$, where N_j contains $v_1 v_{2k+2}$ and $x'y'$, and is equal to M_{i+1} on all other vertices. Finally, note that by assumption, since m did not exists, neither v_{2k+2} nor v_{2k+4} are adjacent to v_1. ■



(a) The perfect matchings S, and T are represented in blue and red respectively.

(b) The matchings N_{j-1} and L (defined in Claim 5.2), represented in orange and light blue respectively. N_j is obtained by performing a 4-switch on C, which is here highlighted in yellow.

Figure 13: Constructing N_j when N_{j-1} is misleading and v_1 has no neighbours among $v_{2(k+1)}$ and $v_{2(k+2)}$. The resulting N_j is misleading.

Note that the progress index increases at each step of our reconfiguration sequence, so the sequence between M_i and M_{i+1} has length at most $|C_i| = |M_i \Delta M_{i+1}|$. So the canonical path between S and T has length at most n.

Analysis of the congestion

Given some transition $M \to M'$, let $\mathrm{cp}(M, M')$ denote the set of couple (S, T) such that the canonical path between S and T includes the transition $M \to M'$. Recall that the congestion of our set of canonical paths is defined by

$$\varrho(\Gamma) = \max_{M \to M'} \left\{ \frac{\sum\limits_{(S,T) \in \mathrm{cp}(M,M')} \pi(S)\pi(T)|\gamma_{S,T}|}{\pi(M)\,\mathrm{Prob}(M \to M')} \right\} \quad \text{where } \pi \text{ is the uniform distribution on perfect matchings}$$

$$\leq n \max_{M \to M'} \frac{\pi(M)|cp(M, M')|}{\mathrm{Prob}(M \to M')} \qquad \text{Because } |\gamma_{S,T}| \leq n$$

Thus we need to bound the number of canonical paths using a transition $M \to M'$. Given some $c \in \{1, 2, 3, 4a, 4b, 5a, 5b, 6a, 6b\}$, denote $\mathrm{cp}_c(M, M')$ the couples $(S, T) \in \mathrm{cp}(M, M')$ such that in the canonical path from S to T, M' is constructed using Case c of Claim 5.1.

Claim 5.2. *For any fixed c and transition $M \to M'$, we have*

$$\frac{|\,\mathrm{cp}_c(M, M')|}{\mathrm{Prob}(M, M')} = O(\pi(M)^{-1} n^6).$$

Proof of Claim. Let $c \in \{1, 2, 3, 4a, 4b, 5a, 5b, 6a, 6b\}$ and $M \to M'$ be a fixed transition and $C = M \Delta M'$ the cycle on which the 4-switch is performed. We define an injective application on $\mathrm{cp}_c(M, M')$ to count it. The Case 3 is somewhat different from the others, because it is the only case where v_1 does not belong to the switch that is performed. For this reason, the definition of our injective application differs slightly for $c = 3$. If $c = 3$, let f_3 be the application that associates to any $(S, T) \in \mathrm{cp}_c(M, M')$ a tuple (L, a, v, w) where L is perfect or near-perfect matching with some additional information $a \in \{0, 1\}$ and $v, w \in V(G)$ to be defined later. Otherwise, let f_c be the application that associates to any $(S, T) \in \mathrm{cp}_c(M, M')$ a tuple (L, a, b), where L is again a perfect or near-perfect matching, with some additional information $a \in \{0, 1\}$ and $b \in [8]$ to be defined later.

Let $(S, T) \in \mathrm{cp}(M, M')$. Recall that $C_1, \dots C_p$ are the alternating cycles in $S \Delta T$ ordered by smallest vertex and that for all i, M_i is the perfect matching that is equal to T on all the cycles C_j with $j < i$ and to S on all the remaining vertices. Let i such that $M \to M'$, was used when connecting M_i to M_{i+1} and denote again $v_1, v_2, \dots v_{2q}$ the consecutive vertices of C_i, such that v_1 is the smallest vertex and $v_1 v_2 \in M_i$. Let k and k' be the progress indices of M and M' respectively. Let L be the matching defined by

$$\begin{aligned} L :=& (S \cap T) \;\cup\; \left(S \cap (\bigcup_{j<i} C_j) \right) \;\cup\; \left(T \cap (\bigcup_{j>i} C_j) \right) \\ & \cup\; \{v_{2i+1} v_{2i+2} : 1 \le i \le k-2\} \;\cup\; \{v_{2i} v_{2i+1} : k \le i \le q\}. \end{aligned}$$

In other words, L is equal to S on all vertices in $\bigcup_{j<i} V(C_j) \cup \{v_3, \dots v_{2k-2}\}$, to T on all vertices in $\bigcup_{j>i} V(C_j) \cup \{v_{2k}, \dots v_{2q}, v_1\}$ and to S (and T) on all vertices of $V(G) \setminus \bigcup_{j=1}^{p} V(C_j)$. Note that L is a perfect matching if and only if $k = 1$ that is if $M = M_i$. Otherwise, L is a near-perfect matching with unmatched vertices v_2 and v_{2k-1}.

Let a denote whether v_2 is greater than v_{2k-1}. If $c = 3$, then v_1 does not belong to the cycle $C = v_{2k} v_{2k+1} v_{2k+2} v_{2k+3}$ that is switched and let $v = v_1$ and $w = v_{2k}$. In all other cases, v_1 belongs to C and we set b to denote the rank within the order of v_1 among the (at most) eight vertices of C.

We will now prove that f_c is injective. Let $(L, a, v) = f_3(S, T)$ or $(L, a, b) = f_c(S, T)$ for some $(S, T) \in \mathrm{cp}(M, M')$. The support of the 4-switch can be recovered: $C = M \Delta M'$. The vertex v_1 can also be recovered: if $c = 3$ it is v, if $c \neq 3$ then it has rank b within the order of the vertices of C.

We now prove that v_2 and the alternating cycles in $S\Delta T$ can be recovered as well. First assume that L is perfect. Then $M = M_i$, so M (respectively L) is equal to T (respectively S) on all vertices in $\bigcup_{j<i} V(C_j)$, to S (respectively T) on all vertices in $\bigcup_{j\geq i} V(C_j)$ and to $S \cap T$) on all other vertices. In other words $L\Delta M = S\Delta T$ and we have recovered the set of alternating cycles. The vertex v_2 is matched to v_1 in M, so we can recover v_2 as well. Thus we can assume that L is not perfect. Let $N = L$ if $c \neq 3$ and $N = L \setminus \{v_1 v_{2k+4}\}$ if $c = 3$ (this is well defined as we have already recovered v_1 and its neighbour in L is v_{2k+4}). Consider $H = N\Delta(M \cap M')$, it is a collection of alternating paths and cycles. Note that the endpoints of the alternating paths of H are the unmatched vertices of N and $M \cap M'$, that is $A = V(C) \cup \{v_2, v_{2k-1}\}$ if $c \neq 3$ and $A = V(C) \cup \{v_2, v_{2k-1}, v_1, v_{2k+4}\}$ if $c = 3$. The matching N is equal to S on $\bigcup_{j<i} V(C_j) \cup \{v_2, \ldots v_{2k-1}\} \setminus A$, to T on $\bigcup_{j<i} V(C_j) \cup \{v_{2k+1}, \ldots v_{2q}\} \setminus A$, and to $S \cap T$ on all remaining vertices of $V(G) \setminus A$. On the other hand, regardless of whether M and M' were representative or misleading in the sequence going from S to T, they were both equal to T on $\bigcup_{j<i} V(C_j) \cup \{v_2, \ldots v_{2k-1}\} \setminus A$, to S on $\bigcup_{j<i} V(C_j) \cup \{v_{2k+1}, \ldots v_{2q}\} \setminus A$, and to $S \cap T$ on all remaining vertices of $V(G) \setminus A$. So all alternating cycles in H are alternating cycles of $S\Delta T$ and the alternating paths in H are subpaths of the alternating paths of $S\Delta T$. By identifying the labels of the vertices in A, the value of c will then be enough information to recover how to combine the alternating paths in H into alternating cycles of $S\Delta T$. Recall that $A = V(C) \cup \{v_2, v_{2k-1}\}$ if $c \neq 3$ and $A = V(C) \cup \{v_2, v_{2k-1}, v_1, v_{2k+4}\}$ if $c = 3$. The vertices v_2 and v_{2k-1} can be recovered using a and the fact that v_2 and v_{2k-1} are the only unmatched vertices of L. If $c \neq 3$, then the labels of $V(C)$ are fully determined by v_1, $M \cap C$ and $M' \cap C$. If $c = 3$, the labels of $V(C)$ are fully determined by $M \cap C$, $M' \cap C$ and the value of $w = v_{2k} \in V(C)$.

Now that we have recovered the alternating cycles in $S\Delta T$, their numbering can be recovered using the ordering on the vertices. S is equal to L on all cycles C_j with $j < i$ and equal to $C_j \Delta L$ on all cycles with $j > i$ (and vice versa for T). Regarding C_i, S and T alternate on it and S contains the edge $v_1 v_2$, which completes the description of S and T.

By Lemma 2.1, the number of values that f_c can take is $O(n^2 \pi(M)^{-1})$, so there are $O(n^2 \pi(M)^{-1})$ canonical path in $\mathrm{cp}_c(M, M')$ for $c \neq 3$ and $O(n^4 \pi(M)^{-1})$ canonical paths in $\mathrm{cp}_c(M, M')$ for $c = 3$. If $c = 3$, then the switch $M \to M'$ is a 2-switch, so $\mathrm{Prob}(M \to M') = \Theta(1/n^2)$. On the other hand $\text{Prob}(M \to M') = \Omega(1/n^4)$ for all 4-switches. So for all c,

$$\frac{|\operatorname{cp}_c(M, M')|}{\text{Prob}(M, M')} = O(\pi(M)^{-1} n^6).$$

■

It follows directly that

$$\varrho(\Gamma) \leq n \max_{M \to M'} \frac{\pi(M)|cp(M, M')|}{\text{Prob}(M \to M')} = O(n^7)$$

As the number of perfect matching of G is $\pi(M)^{-1} \leq n!/(n/2)!$, we have

$$\tau_{\text{mix}}(\Gamma) = O(\varrho(\Gamma) \ln(\pi(M)^{-1})) = O(n^8 \ln(n)).$$

□

5.2 Polynomial mixing time of the 4-switch Markov chain

A direct consequence of Theorem 6 and Theorem 7, is that the 2-switch and 3-switch Markov chains also have polynomial mixing times in these respective regimes, which implies Theorem 5 that we recall:

Theorem 5. *Let G be a n-vertex graph with $\delta(G) \geq n/2+2$ (or respectively $\delta(G) \geq \lfloor(2n+3)/3\rfloor$). Alternatively, let G be a balanced bipartite graph on $2n$ vertices with $\delta(G) \geq \lfloor n/2 \rfloor + 2$ (or respectively $\delta(G) \geq \lfloor(2n+3)/3\rfloor$). The random walk on $\mathcal{H}_3(G)$ (respectively $\mathcal{H}_2(G)$) has mixing time polynomial in n.*

Proof. By Theorem 6 and Theorem 7, 4-switches can be realised by a sequence of length $O(1)$ of 3-switches if $\delta(G) \geq \lfloor n/2 \rfloor + 2$ (respectively by 2-switches if $\delta(G) \geq \lfloor(2n+3)/3\rfloor$). For each 4-switch $(M \to M')$, define the canonical path between M and M' to be an arbitrary such sequence. The number of vertices involved in this sequence is bounded by a constant c, thus the number of canonical paths using any fixed 3-switch or 2-switch is bounded by $n^{O(1)}$. By Markov chain comparison, this proves polynomial mixing time for the random walk on $\mathcal{H}_3(G)$ (respectively on $\mathcal{H}_2(G)$) in the aforementioned regimes. □

5.3 Expansion of $\mathcal{H}_k(G)$

It is well known that the rapid mixing of a random walk on a graph is tied to its expansion properties, as the main obstacle to rapid mixing are bottlenecks, and expanders graphs have bounded positive bottleneck ratio. There are many different definitions of expander graphs, we consider here the spectral one, and more specifically by considering the eigenvalues of its random-walk normalised Laplacian. Given a transition matrix P, let

$$\lambda_\star = \max\{|\lambda|\colon \lambda \text{ is an eigenvalue of } P, \lambda \neq 1\}.$$

The *spectral gap* of P is defined as $\gamma_\star = 1 - \lambda_\star$. Given some constant $c > 0$, we say that a graph is a *c-expander* if the lazy uniform random walk on H has spectral gap at least c. The spectral gap and the mixing time are tied as follows (see [32, Theorems 12.5 and 12.4])

$$\Omega\left(\frac{1}{\gamma_\star}\right) \leq \tau_{\text{mix}} \tag{3}$$

$$\tau_{\text{mix}} \leq O\left(\frac{\ln |H|}{\gamma_\star}\right) \tag{4}$$

In particular, (3) shows that a lazy random walk on a c-expander H mixes in time $O(\ln |H|/c)$. Conversely, (4) shows that in the regime of Theorem 5, $\mathcal{H}_k(G)$ is a $\Omega(n^{-d})$-expander for some fixed d. As n is the number of vertices of G, by (1), $\log |\mathcal{H}_k(G)| = \Theta(n \ln n)$, hence $\mathcal{H}_k(G)$ is a $\tilde{\Omega}(1)$-expander when G satisfies the conditions of Theorem 5.

6 Isolated matchings

The celebrated Caccetta-Häggkvist conjecture [11] asserts that every oriented n-vertex graph with minimum outdegree at least n/k contains a directed cycle of length at most k. The weaker conjecture obtained by replacing the minimum outdegree condition by having minimum semidegree (minimum of outdegrees and indegrees over all vertices) at least n/k is also still open. This weaker conjecture is related to the threshold $\delta_k^{\text{isolated}}(\mathcal{B})$ of appearance of isolated vertices in the k-switch graph of balanced bipartite graphs:

Theorem 10. *For all $k \geq 2$, d and n, there exists an oriented n-vertex graph with minimum semidegree d and no directed cycle of length at most k if and only if there exists a balanced bipartite graph G on $2n$-vertices with $\delta(G) = d + 1$ such that $\mathcal{H}_k(G)$ has an isolated vertex.*

Proof. We first prove the direct implication. Let D be an oriented n-vertex graph with minimum semidegree d and no directed cycle of length at most k. Let us denote $V(D) = \{u_1, \dots, u_n\}$. Let G be the graph on the vertex set $\{a_1, \dots a_n\} \cup \{b_1, \dots b_n\}$ constructed as follows. For each i, connect a_i to b_i. For each distinct i and j, connect a_i to b_j if $u_i \to u_j \in E(D)$. The graph G has minimum degree $d+1$. Let M be the matching $\{a_i b_i \colon i \in [n]\}$. A k-switch on M is a cycle $a_{i_1} b_{i_1} \dots a_{i_\ell} b_{i_\ell}$ of length 2ℓ for some $\ell \leq k$. But no such cycle exists as it would imply that $u_{i_1} \dots u_{i_\ell}$ is a directed cycle of length at most k in D, so M is an isolated vertex of $\mathcal{H}_2(D)$.

Conversely, let G be a balanced bipartite graph on $2n$ vertices with $\delta(G) = d + 1$ and an isolated matching $M = \{a_ib_i : i \in [n]\}$ in $\mathcal{H}_k(G)$. Let D be the oriented n-vertex graph constructed as follows. Denote $V(D) = \{u_1, \dots u_n\}$ and for all distinct i and j, add the arc $u_i \to u_j$ if $a_ib_j \in E(G)$. The minimum semidegree of D is d. A directed cycle $u_{i_1} \dots u_{i\ell}$ in D corresponds to the cycle $a_{i_1}b_{i_1} \dots a_{i_\ell}b_{i_\ell}$. Since M is isolated in $\mathcal{H}_k(G)$, such cycle has length at least $2(k+1)$, so D contains no directed cycle of length at most k (in particular D is oriented because $k \geq 2$. □

This suggest that $\delta_k^{\text{isolated}}(\mathcal{B}) = n/k+1$ and since the bounds we obtained in the balanced bipartite case and the general case were all equal up to $O(1)$, we conjecture the following:

Conjecture 1. *There exists $c > 0$ such that for all k and n,*

$$\delta_k^{isolated} \in [n/k - c, n/k + c].$$

This conjecture holds for $k = 2$, as we obtain a sharp threshold for the appearance of an isolated matching (in balanced bipartite graphs and general graphs):

Theorem 11. *Let G be n-vertex graph (or respectively a balanced bipartite graph on $2n$ vertices). If $\delta(G) \geq n/2+1$ (respectively $\delta(G) \geq \lfloor (n+1)/2 \rfloor + 1$), then for every perfect matching M of G and every $e \in E(M)$, there exists a 2-switch that contains e. On the other hand, $\mathcal{H}_2(G)$ may have isolated vertices if $\delta(G) \leq n/2 - 2$ (respectively $\lfloor (n+1)/2 \rfloor$). In other words,*

$$\frac{n}{2} - 1 \leq \delta_2^{isolated} \leq \frac{n}{2} + 1 \qquad and \qquad \delta_2^{isolated}(\mathcal{B}) = \left\lfloor \frac{n+1}{2} \right\rfloor$$

Proof. We first prove the existence of such a switch if $\delta(G)$ is high enough. Let $V' = V(G) \setminus \{u, v\}$, A be the subset of vertices of V' that are matched in M to a neighbour of u, and $B = N(v) \cap V'$ [6]. We have $|A| \geq \deg(u) - 1 \geq n/2$ and $|B| \geq n/2$. By pigeonhole principle, A and B intersect, so there exists a 2-switch that contains uv.

Conversely, we construct graphs with high minimum degree and isolated matching. To do so, we first construct an auxiliary oriented graph H_p on p vertices with maximal semidegree $\delta(H_p) := \max_{u_i} \max\{\delta^+(u_i), \delta^-(u_i)\}$. The semidegree of an p-vertex oriented graph is at most $\lfloor (p-1)/2 \rfloor$. This bound is attained by taking H_p to be the oriented graph such that there is an arc from u_i to u_j is $j - i \in \{1, \dots, \lfloor (p-1)/2 \rfloor\} \mod p$. By Theorem 10, there exists a balanced bipartite graph $G_n^{(\mathcal{B})}$ on $2n$ vertices with minimum

[6] If G is a balanced bipartite graph, then let $V' = U \setminus \{u\}$, where U is the half of the bipartition containing u and A and B defined identically. Then A and B have size at least $\lfloor (n+1)/2 \rfloor$, so $|A| + |B| \geq n > |V'|$ and by pigeon hole principle, A and B intersect.

degree $\lfloor (n-1)/2 \rfloor + 1 = \lfloor (n+1)/2 \rfloor$, such that $\mathcal{H}_2(G_n^{(\mathcal{B})})$ contains an isolated vertex.

We now give the construction for general graphs. Let n be even and $p = n/2$ and consider the graph G_n on the vertex set $\{a_1, \dots a_p\} \cup \{b_1, \dots b_p\}$ built from H_p as follows. For each i, connect a_i with b_i. For each vertex $u_i \in V(H_p)$, we partition $N^+(u_i)$ into $A_i \cup B_i$ such that $||A_i| - |B_i|| \leq 1$. We connect a_i to a_j and b_j for all $u_j \in A_i$, and b_i to a_j and b_j for all $u_j \in B_i$. The graph G_n constructed this way has minimum degree equal to $1 + 2\lfloor \delta^+(H_p)/2 \rfloor + \delta^-(H_p)$. Moreover, the matching $M = \{a_i b_i \colon i \in [n/2]\}$ is isolated in $\mathcal{H}_2(G_n)$: since H_p is an oriented graph, for each distinct i and j, the subgraph of G_n induced by $\{a_i, b_i, a_j, b_j\}$ is isomorphic to a triangle with an edge attached to one of the vertices, so the edges $a_i b_i$ and $a_j b_j$ cannot be part of a 2-switch. The minimum degree of G_n is

$$\begin{aligned}
\delta(G_n) &= 1 + 2\left\lfloor \frac{p-1}{4} \right\rfloor + \left\lfloor \frac{p-1}{2} \right\rfloor \\
&= 2\left\lceil \frac{p}{4} \right\rceil + \left\lceil \frac{p}{2} \right\rceil - 2 \\
&= 2\left\lceil \frac{n}{8} \right\rceil + \left\lceil \frac{n}{4} \right\rceil - 2 \geq n/2 - 2
\end{aligned}$$

□

7 Open questions and perspectives

7.1 Perfect matchings

There remain three open questions for the geometry of the perfect matchings space: What happens for regular balanced bipartite graphs? What is the threshold for the appearance of a giant component in the 2-switch graph? What is the threshold for the appearance of isolated vertices in the k-switch graph for $k \geq 3$?

Regular graphs For any d, the number $\phi(G)$ of perfect matching of any d-regular balanced bipartite graph G is exponential [36, 39, 22, 13]:

$$\Phi(G) \geq \left(\frac{(d-1)^{d-1}}{d^{d-2}} \right)^n .$$

The "volume" heuristic suggests that for all d, k and n, and any d-regular balanced bipartite graph G on $2n$ vertices, $\mathcal{H}_k(G)$ is connected. This is false as for all d, there exists d-regular balanced bipartite graphs with arbitrarily large girth. This is witnessed by a uniform random d-regular graph, but also by blowing-up of the vertices of a $2(k+1)$-cycle into independent sets of equal size. The resulting graph has a disconnected k-switch graph (see Lemma 4.7) and degree $\frac{2n}{k+1}$, where $2n$ is the number of vertices. We conjecture that this blow-up maximises $\delta(G)$ among regular balanced bipartite graphs on $2n$ vertices with disconnected k-switch graph:

Conjecture 2. *There exists $c \geq 0$ such that for all n and k,*

$$\delta_k^{connectedness}(\mathcal{R}) \in \left[\frac{2n}{k+1} - c, \frac{2n}{k+1} + c\right].$$

Note that Theorem 7 and Theorem 6 confirm the upper bound of this conjecture for $k = 2$ and $k = 3$, and that Corollary 4.8 confirms the lower bound when $k + 1$ divides n. More generally, we believe that the switch graph of d-regular balanced bipartite graphs on $2n$ vertices undergoes a phase transition around $2n/(k + 1)$ that is similar to that described in this article for balanced bipartite graph and general graphs.

Giant component Theorem 7 and Corollary 4.4 place the threshold for the appearance of a giant component in the 2-switch graph between $n/2 - \varepsilon$ and $\lfloor (2n + 3)/3 \rfloor$. We conjecture that the giant component threshold lies close to this lower bound:

Conjecture 3. *For each $c \geq 1$, there exists $\varepsilon \geq 0$ such that for all n, for all n-vertex graphs G (or alternatively all balanced bipartite graphs on $2n$ vertices) with minimum degree at least $n/2 + \varepsilon$, the 2-switch graph $\mathcal{H}_2(G)$ contains a component of size $|\mathcal{H}_2(G)|/c$.*

Isolated matchings For isolated matchings, we have seen in Section 6 that $\delta_k^{\text{isolated}}(\mathcal{B})$ is related to a weaker version of the conjecture of Caccetta and Häggkvist about the maximal minimum semidegree of an oriented graph without directed cycles of length k. This suggests that for all k and n, $\delta_k^{\text{isolated}}(\mathcal{B}) = n/k + 1$ and motivates the following conjecture for general graphs:

Conjecture 1. *There exists $c > 0$ such that for all k and n,*

$$\delta_k^{isolated} \in [n/k - c, n/k + c].$$

7.2 Phase transition for other Dirac-type theorems

Many other structures exhibit phase transitions close to minimum degree at which the structure is guaranteed to exist in G. Most of the existing literature expresses these phase transitions in one of the following terms. Above the non-emptiness threshold, there are $\Omega(n)^n$ embeddings of the structure. The host graph contains the structure in a robust sense: sparsifying the host graph with probability $\Omega(\log n/n)$ preserves the existence of an embedding of the structure (see the survey of Sudakov [37]), thereby witnessing that these embeddings are "everywhere" in the host graph. More recently, this phase transition was also described by the existence of spread distributions, which in particular imply the counting and robustness results we just mentioned.

For example, this depictions of the phase transition were described for Hamiltonian cycles (see [35, 14, 15] for an estimate of the number of Hamiltonian cycles, [30] for the robustness and [27] for the existence of spread distributions), triangle factor or more generally clique factors (see [3, 2] for the robust versions of the Corrádi and Hajnal theorem and [33, 27] for the existence of spread distributions) and spanning trees of bounded degree (see [29] for an counting estimate and [33, 4] for spread distributions).

However, except for Hamiltonian cycles [28] and the present work for perfect matchings, the phase transition has not been described in terms of the geometry of the configuration space. Given a spanning structure $(S_n)_{n\in\mathbb{N}}$, for any n-vertex graph G and integer k, let $\mathcal{H}_k(G)$ be the reconfiguration graph of the embeddings of S_n, where two copies of S_n in G are adjacent if they differ by at most k edges. Motivated by our results for perfect matchings and by the results of [28] for the connectedness threshold of Hamiltonian cycles, we make the following informal conjecture:

Conjecture 4. *For any spanning structure $(S_n)_{n\in\mathbb{N}}$ with non-emptiness threshold δ_n, there exists k such that the geometry of $\mathcal{H}_k(G)$ undergoes a phase transition close to δ_n.*

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