Shift graph recognition is NP-complete

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Abstract

Shift graphs are one of the classical constructions of triangle-free graphs with arbitrarily large chromatic number. We investigate the computational complexity of the recognition, Maximum independent set and 3-Colouring problems on shift graphs and prove that all three problems are NP-complete.

1 Introduction

A class of graphs is *hereditary* if it closed under taking induced subgraphs. A hereditary class $\mathcal{C}$ of graph is *χ-bounded* if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for each graph $G \in \mathcal{C}$, $\chi(G) \leqslant f(\omega(G))$. A well-known fact is that not all hereditary classes of graphs are χ-bounded. Even more, there exist triangle-free graphs with arbitrarily large chromatic number. The first such constructions are due to Zykov [24] and Blanche Descartes [8], and among other constructions with such properties, one can also mention Mycielski graphs [16], shift graphs [9], Burling graphs [6], or some subfamilies of Kneser and Schrijver graphs [13, 22]. A common feature of all these graphs is that they all admit explicit constructions, allowing to derive a many interesting structural properties. For example, a construction due to Burling turned out to be a counter-example to several conjectures about algorithms in non-χ bounded classes [17, 19].

In Table 1, we review the simplest and most well-known constructions of triangle-free graphs of arbitratily large chromatic number. These constructions are often presented by generating a sequence of triangle-free graphs $(G_k)_{k\in\mathbb{N}}$ such that $\chi(G_k) = k$ for all k. We turn this into a hereditary class by considering all induced subgraphs of the graphs in the sequence $(G_k)_{k\in\mathbb{N}}$. Thus, by abuse of vocabulary, when we say a "shift/Burling/etc. graph", we mean in particular an induced subgraph of one of the graphs from the sequence $(G_k)_{k\in\mathbb{N}}$ of shift/Burling/etc. graphs (see Section 2 for precise definitions).

In view of the explicit constructions of each of the aforementioned classes, a natural question is to determine for each of them, whether there exists an efficient (i.e. polynomial) algorithm solving the associated membership problem, i.e. deciding whether a given graph belongs or not to them. This

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question has been recently investigated for some of these classes, as well as the complexity of two other natural problems, when restricted to these classes: Maximum Independent Set (or MIS for short) and 3-Colouring (see Table 1 for references). The goal of the present paper is to complement these works, and to prove that all three problems are NP-complete for shift graphs.

Graph class	Recognition	Maximum Independent set	3-Colouring
Mycielski	P	NPC [18]	NPC [13, 14]
Zykov	NPC [15]	NPC [15]	NPC [15]
Blanche Descartes	NPC [15]	NPC [15]	NPC [15]
Burling	P [19]	P [19]	NPC [23]
Twincut	P [5]	NPC [5]	NPC [5]
Shift	NPC [*]	NPC [*]	NPC [*]

Table 1. Complexity of Recognition, Maximum independent set and 3-Colouring of the main constructions of triangle-free graph of large chromatic number

Surprisingly, one can observe from Table 1 that 3-Colouring is NP-complete in all known constructions of triangle-free graphs of high chromatic number, which motivates the following quesiton:

Question 1. *Is 3-Colouring NP-complete on all non-χ-bounded class (in particular in all triangle-free hereditary graph classes of unbounded chromatic number)?*

The most natural candidates for possible counterexamples to Question 1 are the classes of Kneser and Schrijver graphs.

For most of the constructions from Table 1 of triangle-free graphs with high chromatic number, there exist variants producing graphs with arbitrarily large girth or odd-girth, and unbounded chromatic number. More precisely, there exist shift graphs, Twincut and Mycielski graphs of arbitrarily large odd-girth and chromatic number, and Zykov and Blanche Descartes graphs with arbitrarily large girth and chromatic number. The complexity results of Table 1 might also hold with this additional condition in these classes. In the particular case of shift graphs, k-iterated shift graphs have odd-girth at least $2k + 3$ but may contain four-cycles. Each of our NP-completeness results extends to the class of k-iterated shift graphs. Moreover, we prove that shift graphs of girth at least five are 3-colourable, but that maximum independent set is still NP-complete on shift graphs of arbitrarily large girth.

We give in Section 2 all basic definitions concerning shift graphs, as well as some known characterisations. We prove in Section 3 that Maximum independent set (MIS) is NP-complete in shift graphs. Section 4 contains a proof that 3-Colouring (3-COL) is NP-complete when restricted to shift graphs and Section 5 shows that recognising shift graphs is also NP-complete.

2 Preliminaries

Unless stated otherwise, all graphs considered in this paper are finite, simple without loops. For every integer $n \in \mathbb{N}$, we use the notation $[n]$ to denote the set $\{1, \ldots, n\}$ of integers.

2.1 Notations and basic definitions

A digraph D consists of a set $V(D)$ of vertices and a set $A(D)$ of directed edges denoted (u,v), uv or $u \to v$ and called *arcs*. Given an arc $a = u \to v$, the *head* of a is v and its *tail* is u. We denote with $N^-(u) = \{v : v \to u\}$ and $N^+(u) = \{v : u \to v\}$ the in-neighbourhood and out-neighbourhood of u. If $N^-(u) = \varnothing$ then u is a *source*, while if $N^+(u) = \varnothing$ then u is a *sink*. An *oriented graph* is a digraph such that for each pair u, v of distinct vertices, $A(D)$ contains at most one of the two arcs $u \to v$ and $v \to u$. The *support* of a digraph D is the undirected graph G obtained when removing the orientations of the arcs of D, that is $V(G) = V(D)$ and $E(G) = \{\{x,y\} : (x,y) \in A(D)\}$. For the sake of notation, we will often use the notation $\overrightarrow{G}$ to denote an oriented graph whose support is the unoriented graph G. We say that a vertex u of a digraph D is *transitive* if it is neither a source nor a sink. For $k \in \mathbb{N}$, a *k-walk* (resp. *k-path*) is a sequence $(v_0, \ldots, v_k)$ of $k+1$ (distinct) vertices such that $v_i \to v_{i+1}$ for each $0 \leqslant i \leqslant k-1$. Then *length* of a path is defined as its number of edges. Again, we call v_k the head of the k-path $(v_0, \ldots, v_k)$ and v_0 its tail. The *k-prefix* (resp. *k-suffix*) of a path P is the subpath obtained when keeping only its $k+1$ first (resp. last) vertices.

Let D be a digraph and let P and Q be two k-paths $P = (u_0, \ldots, u_k)$ and $Q = (v_0, \ldots, v_k)$ such that P and Q are vertex disjoint and there are no arcs between $V(P)$ and $V(Q)$. We then denote $D/\{P, Q\}$ the digraph obtained by identifying P and Q, that is by replacing for each i, u_i and v_i by a new vertex w_i whose in-neighbourhood is $N^-(u_i) \cup N^-(v_i)$ and out-neighbourhood is $N^+(u_i) \cup N^+(v_i)$. Formally, the vertex set of $D/\{P,Q\}$ is $(V(D) \setminus \{u_0, \ldots, u_k, v_0, \ldots, v_k\}) \sqcup \{w_0, \ldots, w_k\}$ and for all $xy \in A(D)$, we have $p(x)p(y) \in A(D/\{P,Q\})$, where $p(x) = x$ if $x \in V(D) \setminus \{u_0, \ldots, u_k, v_0, \ldots, v_k\}$ and $p(x) = w_i$ if x is equal to u_i or v_i. This operation is well-defined: each vertex u_i or v_i corresponds to a unique w_i because $V(P)$ and $V(Q)$ are disjoint, and $D/\{P,Q\}$ contains no loop because there are no arcs between $V(P)$ and $V(Q)$.

For each $n \in \mathbb{N}$, the *transitive tournament of order n* is the oriented graph $\overrightarrow{T_n}$ with vertex set $[n]$, where there is an arc $i \to j$ for each $1 \leqslant i < j \leqslant n$. In this article, the *comb* of length k is the graph Comb_k obtained from a path on k edges (the *jaw* of Comb_k) to which we attach to each vertex an additional pendant edge (such edges are the *teeth* of Comb_k). The comb of length 0 is a single tooth. The teeth attached to each extremity of the jaw are called the *molars*, while the other teeth are called the *incisors*.

For a graph G, we let $\alpha(G)$ denote the maximum possible size of an independent set of G, and we let $\mathrm{g}(G)$ and $\mathrm{og}(G)$ denote respectively its *girth* (minimim over the sizes of its cycles) and its *odd girth* (minimum over the sizes of its odd cycles). If D is a digraph, we will write $\chi(D)$ to denote the chromatic number of its support graph.

2.2 Line digraphs

Given a digraph D, the *line digraph* $L(D)$ of D is the digraph whose vertices are the arcs of D and in which there is an arc from a to b if the head of a is the tail of b in D, in other words when a and b are consecutive. We refer to D as a *root* digraph of $L(D)$. For $k \in \mathbb{N}$, the *k-iterated line digraph* $L^k(D)$ of D is then defined recursively by setting $L^k(D) = L(L^{k-1}(D))$ when $k \geqslant 1$, and $L^0(D) = D$. Equivalently, when $k \geqslant 1$, the vertices of $L^k(D)$ are the directed k-walks in D and $L^k(D)$ contains an arc from a to b if the corresponding directed walks overlap on the $(k-1)$-suffix of a and the $(k-1)$-prefix of b, namely $a = (v_0, \ldots, v_k)$ and $b = (v_1, \ldots, v_{k+1})$ with $(v_i, v_{i+1}) \in A(D)$ for all

$0 \leqslant i \leqslant k$. Observe that if D is acyclic, then for each $k \geqslant 1$, the vertex set of $L^k(D)$ also corresponds to the set of k-paths of D.

The following result of Beineke characterises line digraphs of oriented graphs.

Lemma 2 (Beineke [3])**.** *A digraph D is the line digraph of an oriented graph $\overrightarrow{G}$ if and only if the two following conditions are satisfied (see Figure 1):*

1. *If D contains three arcs a, b and c such that a and b have the same tail, and b and c have the same head, then D also contains an arc from the tail of c to the head of a.*
2. *D does not contain four arcs a, b, c and d such that the tails of a and c are identical, the heads of d and b are identical, and the head of a (resp. c) is the tail of b (resp. d).*

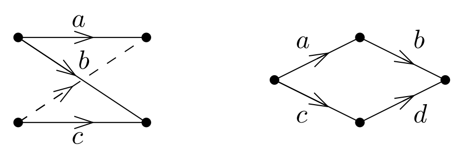


Figure 1. The two configurations depicted in Lemma 2.

Note that the second condition forbids parallel arcs in $\overrightarrow{G}$, while the first one ensures that all the arcs of $\overrightarrow{G}$ entering a fixed vertex have identical out-neighbourhood in D. Moreover, note that these two conditions ensure that the only allowed orientation of a triangle is the cyclic one and the only allowed orientations of the 4-cycles of D are the cyclic ones and the ones alternating at each vertex.

Beineke's characterisation gives in particular a simple polynomial time algorithm to recognise line digraphs. On the other hand, Chvátal and Ebenegger proved that deciding if an unoriented graph is the support of a line digraph is NP-complete [7].

2.3 Shift graphs

The *shift graph sequence* is the sequence of graphs $(G_n)_{n\in\mathbb{N}}$, where G_n is the graph whose vertices are all the ordered pairs (a_1, a_2) of $[n]$ such that $1 \leqslant a_1 < a_2 \leqslant n$, in which two vertices $a = (a_1, a_2)$ and $b = (b_1, b_2)$ are adjacent if $b_1 = a_2$ (or $a_1 = b_2$). Note that the vertex set of G_n corresponds equivalently to the set of all closed intervals between two distinct integers in $[n]$, with edges between consecutive intervals. From this definition, it is straightforward to check that G_n is the support of the line digraph of the transitive tournament $\overrightarrow{T_n}$.

The class of *shift graphs* is the hereditary closure of the graphs $(G_n)_{n\in\mathbb{N}}$, that is, all induced subgraph of some G_n. Note that in the litterature, shift graphs might refer to what we will call later *iterated shift graphs* in this paper. Observe that every shift graph admits a natural acyclic orientation, obtained when considering the arc $(a, b) \to (b, c)$ for each $1 \leqslant a < b < c \leqslant n$. We call this orientation the *shift orientation* and denote it with $\overrightarrow{G_n}$. Since for every oriented graph $\overrightarrow{G}$, its line graph $L(\overrightarrow{G})$ is acyclic if and only if $\overrightarrow{G}$ is acyclic, Beineke's characterisation immediately implies the following equivalence:

Lemma 3. *A graph is a shift graph if and only if it admits an acyclic orientation which is the line digraph of an oriented (acyclic) graph.*

As a result all 4-cycles in the natural orientation of a shift graph are alternating at each vertex and shift graphs are triangle-free. The next lemma is a folklore result that shows that shift graphs also have unbounded chromatic number (see for example [11, Theorem 4] or [12, Lemma 2.21] for the first part). We include its proof as it is simple.

Lemma 4 (Folklore)**.** *For any oriented acyclic graph* $\overrightarrow{G}$, $\chi(L(G)) \geqslant \log \chi(G)$ *and* $\mathrm{og}(L(G)) > \mathrm{og}(G)$.

Proof. To prove the first inequality, consider a proper colouring c of $L(\overrightarrow{G})$ using k colours. Then assign to every vertex v of $\overrightarrow{G}$ the set S_v of all colours with of arcs entering v (with respect to c). If (u,v) is an arc of $\overrightarrow{G}$, then as c is a proper colouring, $c(u,v) \notin S_u$, so $S_u \neq S_v$. It follows that the mapping $u \mapsto S_u$ defines a proper colouring of $\overrightarrow{G}$ using at most 2^k colours, showing $\chi(G) \leqslant 2^{\chi(L(G))}$ as desired.

For the second inequality, let C be a shortest odd cycle of $L(G)$. Let $Q_1, \dots, Q_{2m}$ be directed paths such that for each i, the head of Q_{2i-1} and Q_{2i} are equal, we denote it u_{2i}, the tail of Q_{2i} and $Q_{2i+1 \mod 2m}$ are equal, we denote it u_{2i+1}, and such that C is the concatenation of the paths P_i. As $\overrightarrow{G}$ is acyclic, so it $L(\overrightarrow{G})$ and $Q_1, \dots, Q_{2m}$ are well defined. For each i, let ℓ_i be the length of Q_i. We have $\ell_i \geqslant 2$ for each i, because otherwise C would contain an alternating path of lentgh three, which by Lemma 2 implies that there exists a shorter odd cycle in $L(G)$. Note that this argument fails if C is a cycle of length four, which is why the assumption that C has odd length is needed. By definition of the line digraph, each of the path Q_i corresponds to a directed path P_i of $\overrightarrow{G}$ of length $\ell_i + 1$, whose first and last arcs are u_i and u_{i+1} (or u_i and u_{i+1} depending on the parity of i). So $\overrightarrow{G}[\bigcup_i V(P_i)]$ contains a cycle B of length $\sum_i(\ell_i - 1) = |C| - 2m$, which is odd and shorter that the length of C. This proves the second inequality. □

Iterated shift graphs. For $k \geqslant 1$, the *k-iterated shift graph* $G_{n,k}$ (sometimes referred to as generalised shift graph, or shift graph) is the graph whose vertices are ordered $(k+1)$-tuples $(a_0, \dots, a_k)$ of $[n]$ such that $1 \leqslant a_0 < a_1 < \dots < a_k \leqslant n$, in which two vertices $a = (a_0, \dots, a_k)$ and $b = (b_0, \dots, b_k)$ are adjacent if $b_i = a_{i+1}$ for all $i \in [k-1]$ (or $a_i = b_{i+1}$ for all $i \in [k-1]$). From this definition, it is straightforward to check that $G_{n,k}$ is the support of the k-iterated line digraph $L^k(\overrightarrow{T_n})$, where $\overrightarrow{T_n}$ is the transitive tournament on n vertices. Note that for each $n \in \mathbb{N}$, $G_{n,2}$ corresponds exactly to the shift graph G_n defined earlier, and that as before, $G_{n,k}$ admits a natural acyclic orientation, which we call its *shift orientation*, and denote by $\overrightarrow{G_{n,k}}$. Note also that for each $k \geqslant 1$, the graph $G_{n,k}$ is a shift graph with respect to the definition we gave above, i.e. it is an induced subgraph of one of the graphs $(G_n)_{n \in \mathbb{N}}$ because $\overrightarrow{G_{n,k}}$ is the line digraph of the acyclic oriented graph $L^{k-1}(\overrightarrow{T_n})$.

The main interest of considering the graphs $G_{n,k}$ is that they form a family of graphs with unbounded chromatic number and arbitrarily large odd girth. More precisely, by Lemma 4, for every $k \geqslant 1$ and $n \in \mathbb{N}$, $G_{n,k}$ has odd-girth at least $2k + 3$ and chromatic number at least $\log^{(k)}(n)$, where $\log^{(k)}$ denotes the k-iterated logarithm function. The class of *k-iterated shift graphs* is the hereditary closure of $(G_{n,k})_{n \in \mathbb{N}}$, or equivalently, the class of all induced subgraphs of k-line digraphs of acyclic digraphs.

Given an acyclic digraph $\overrightarrow{G}$ and an induced subdigraph $\overrightarrow{H}$ of $L^k(\overrightarrow{G})$, we call *k-shift embedding* of $\overrightarrow{H}$ in $\overrightarrow{G}$ any injective map φ from the vertices of $\overrightarrow{H}$ to the k-paths of $\overrightarrow{G}$ such that for all $u, v \in V(\overrightarrow{H})$,

$u \to v$ in $\overrightarrow{H}$ if and only if the $(k-1)$-prefix of $\varphi(v)$ is the $(k-1)$-suffix of $\varphi(u)$. We say that φ is *convex* if for every $u, v, w \in V(\overrightarrow{H})$ such that v is on a (non-directed) path from u to w, then $V(\varphi(u)) \cap V(\varphi(w)) \subseteq V(\varphi(v))$. The next lemma shows that k-shift embeddings commute with identification:

Lemma 5. *Let $\overrightarrow{H}$ and $\overrightarrow{G}$ be two oriented acyclic graphs and φ be a k-shift embedding of $\overrightarrow{H}$ in $\overrightarrow{G}$. Let u, v be two vertices of $\overrightarrow{H}$, such that $\varphi(u)$ and $\varphi(v)$ are vertex disjoint, and there are no arcs $\varphi(u)$ between $\varphi(v)$, and no directed path from u to v or vice-versa. Then there exists a k-shift embedding of $\overrightarrow{H}/\{u, v\}$ in $\overrightarrow{G}/\{\varphi(u), \varphi(v)\}$, which are both oriented acyclic graphs.*

Proof. First, note that the identification is well-defined because $\varphi(u)$ and $\varphi(v)$ are vertex-disjoint and have no arc between them. Moreover, these identifications create no directed cycles because there are no directed paths between $V(\varphi(u))$ and $V(\varphi(v))$ in $\overrightarrow{G}$, and in particular between u and v in $\overrightarrow{H}$. Hence $\overrightarrow{H}/\{u, v\}$ and $\overrightarrow{G}/\{\varphi(u), \varphi(v)\}$ are oriented acyclic graphs.

Denote $\varphi(u) = (u_1, \ldots, u_k)$, $\varphi(v) = (v_1, \ldots, v_k)$ and $(w_1, \ldots w_k)$ be the vertex resulting from the identification of $\varphi(u)$ with $\varphi(v)$ in $\overrightarrow{G}$. Let $p : V(\overrightarrow{G}) \to V(\overrightarrow{G}/\{\varphi(u), \varphi(v)\})$ such that $p(x) = x$ for all $x \notin V(\varphi(u)) \cup V(\varphi(v))$ and $\varphi(y) = w_i$ if $y \in \{u_i, v_i\}$ for some i. As $\varphi(u)$ and $\varphi(v)$ are vertex-disjoint, there are no arcs between u and v, so p is a morphism. Let ψ be the map from $\overrightarrow{H}/\{u, v\}$ to the k-paths of $\overrightarrow{G}/\{\varphi(u), \varphi(v)\}$ such that for all $x \in V(G)$, we let $\psi(p(x)) = (p(x_1), \ldots, p(x_k))$ where $(x_1, \ldots, x_k) = \varphi(x)$. First, note that $\psi(x)$ is a k-path for each $x \in V(\overrightarrow{H}/\{u, v\})$ because p is a morphism. Moreover, for each arc xy in $\overrightarrow{H}/\{u, v\}$, the $(k-1)$-suffix of $\psi(x) = \varphi(p^{-1}(x))$ and the $(k-1)$-prefix of $\psi(y) = \varphi(p^{-1}(y))$ are equal, so ψ is a k-shift embedding of $\overrightarrow{H}/\{u, v\}$ in $\overrightarrow{G}/\{\varphi(u), \varphi(v)\}$. □

Shift trees. We now prove a technical lemma that we will use to construct the root graph of some k-iterated shift graphs. Informally, it states that for trees, k-iterated shift graphs and shift graphs are the same.

Lemma 6. *Let T be a shift graph which is a tree, with shift orientation $\overrightarrow{T}$. Then, for all $k \geqslant 1$, T is also a k-iterated shift graph.*
More precisely, there exists an acyclic oriented graph $\overrightarrow{G}$ and a convex k-shift embedding φ of $\overrightarrow{T}$ in $\overrightarrow{G}$.

Proof. We proceed by induction on $|V(T)|$. Clearly, if $|V(T)| \leqslant 1$, then the result is immediate, as if $V(T)$ is empty, then $\overrightarrow{G}$ can be chosen as the empty digraph, and if T has a single vertex, then $\overrightarrow{G}$ can be chosen as the oriented path of length k. We now assume that $|V(T)| \geqslant 2$.

Let a be a leaf of T, and assume without loss of generality that for some $b \in V(T)$ we have $a \to b$ in D, the case $b \to a$ being symmetric. Let $T' := T - a$, and $\overrightarrow{T'} := \overrightarrow{T} - a$. By induction hypothesis, there exists an acyclic oriented graph $\overrightarrow{G'}$ and a convex k-shift embedding φ' of $\overrightarrow{T'}$ in $\overrightarrow{G'}$. Let v, w, x, y be the first, second, penultimate and last vertex of $\varphi(b)$ respectively (with $w = x$ if $k = 2$). Let Q_b be the $(k-1)$-prefix of $\varphi'(b)$.

If b is the only vertex of T' with Q_b as the $(k-1)$-prefix of its image by φ', then let $\overrightarrow{G}$ be the oriented graph obtained from $\overrightarrow{G'}$ by adding a new vertex u with $u \to v$. This creates a k-path P_a whose $(k-1)$-suffix is the $(k-1)$-prefix of $\varphi'(b)$ but of no other k-path in $\varphi'(V(\overrightarrow{T}'))$. Hence, by defining $\varphi(a) := P_a$ and $\varphi(c) := \varphi'(c)$ for all other vertices, φ is a k-shift embedding of $\overrightarrow{T}$ in $\overrightarrow{G}$. The convexity of φ comes from the fact that φ' was also convex, and that for each $u \in V(T')$, $\varphi(u) \cap \varphi(a) \subseteq V(Q_b)$. Thus the second part of the lemma holds as well and we can now assume that there exists $c \in V(T') \setminus \{b\}$ such that $\varphi(c)$ has Q_b as $(k-1)$-prefix and ends at $z \in N^+_{G'}(x)$ (see Figure 2).

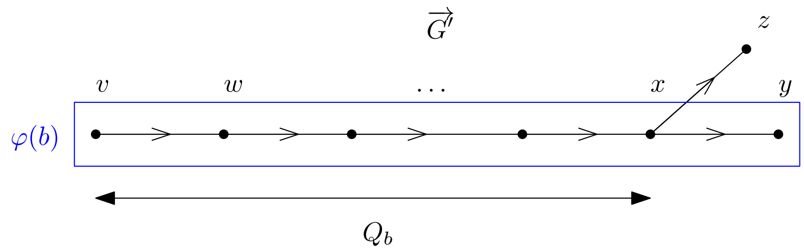


Figure 2. Configuration in the proof of Lemma 6.

First, observe that for each $t \in V(G')$ such that $t \to v$ in $\overrightarrow{G'}$, we have $tQ_b \notin \varphi(V(T'))$, thus b is a source in $\overrightarrow{G'}$. Indeed, note that otherwise there would exist some node $d \in V(T')$ such that we both have $d \to c$ and $d \to b$ in $\overrightarrow{T'}$ (and thus also in $\overrightarrow{T}$). Lemma 2 would then imply that $a \to b$ in $\overrightarrow{T}$, contradicting that a is a leaf of T. We now let $\overrightarrow{G''}$ be the oriented graph obtained after duplicating the vertex v by creating a vertex v' that is a false twin of v in $\overrightarrow{G'}$, i.e. v' has exactly the same out- and in-neighbourhoods as v in $\overrightarrow{G''}$ (note that such operation preserves acyclicity). We define ψ as the map from $V(T')$ to the set of k-walks of G'' by setting $\psi_{|V(T')\setminus\{b\}} := \varphi_{|V(T')\setminus\{b\}}$ and define $\psi(b)$ as the k-path $\varphi(b)$ where v was replaced by v'. We claim that for any two vertices i, j of $V(T')$ we have $i \to j$ in $\overrightarrow{T'}$ if and only if the $(k-1)$-suffix of $\psi(i)$ equals to the $(k-1)$-prefix of $\psi(j)$. This is immediate if i and j are different from b, or if $i = b$ because $\psi(i)$ and $\varphi(b)$ have the same $(k-1)$-suffix. Since b is source in $\overrightarrow{G'}$, the case $b = j$ also trivially follows, and as φ is convex, note that ψ is also convex. Thus, we just proved that ψ is a convex k-shift embedding of $\overrightarrow{T'}$ in $\overrightarrow{G''}$ such that b is the only vertex of $\overrightarrow{T'}$ whose corresponding k-path $\psi(b)$ starts in v'. In particular, we are back in the case of the previous paragraph, and can thus conclude the proof. $\square$

3 Maximum independent set problem

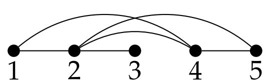
Given a graph G and an integer $s \in \mathbb{N}$, denote G^{*s} the graph obtained by subdividing s times each edge of G. In particular, we have $G^{*0} = G$.

Lemma 7 ([18]). *For every graph G and every even integer $s \geqslant 0$, we have $\alpha(G^{*s}) = \alpha(G) + \frac{s}{2}|E(G)|$.*

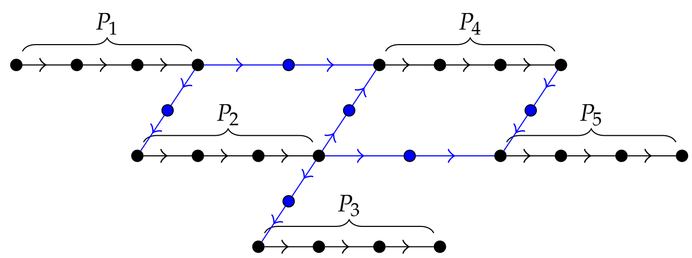
Proposition 8. *For any integers k and ℓ,* MIS *is NP-complete when restricted to the class of k-iterated shift graphs of girth at least ℓ.*

Proof. We show that MIS on general graphs admits a polynomial time reduction to MIS when restricted to the class of k-iterated shift graphs of girth at least ℓ. Let s be an even integer such that $3(s+1) \geqslant \ell$ and $s \geqslant k+1$. Let G be a connected graph and assume that $[n] = V(G)$. We claim that G^{*s} is a k-iterated shift graph of girth at least $3(s+1) \geqslant \ell$. By Lemma 7, this will imply directly the result.

To prove our claim, consider the acyclic graph H constructed as follows. Let $(P_u)_{u\in V(G)}$ be a collection of disjoint directed paths of length k. For each u, we denote u^- and u^+ the first and last vertex of P_u. For each edge $uv \in E(G)$ with $u < v$, connect u^+ with v^- by a directed path P_{uv} of length $s+1-k > 0$ and let H be the graph obtained. We now prove that the graph $L^k(H)$ is isomorphic to G^{*s}. Each vertex u of G is mapped to the k-path P_u. In $L^k(H)$, for each $uv \in E(G)$ with $u < v$, the vertices P_u and P_v are connected by a path $(P_u, Q_1, \ldots Q_k, R_1, \ldots R_{s-k}, P_v)$, where each Q_i is the k-path starting at the $(i+1)^{\text{th}}$ vertex of P_u and ending in $V(P_{uv}) \cup V(P_v)$, and each R_i is the k-path starting at the $(i+1)^{\text{th}}$ vertex of P_{uv} and ending in $V(P_{uv}) \cup V(P_v)$. Thus G^{*s} is a subgraph of $L^k(H)$ and this subgraph is spanning because all k-paths of H are of the form described above. All vertices in $V(H) \setminus \{u^+ \colon u \in V(G)\}$ have out-degree equal to one and all vertices in $V(H) \setminus \{u^- \colon u \in V(G)\}$ have in-degree equal to one. So all the k-paths of H that are not some P_u have degree exactly two in $L^k(H)$. Combining this with the fact that the k-paths P_u have degree $\deg^+_H(u^+) + \deg^-_H(u^-) = \deg_G(u)$, this proves that G^{*s} is an induced subgraph of $L^k(H)$, so $L^k(H) = G^{*s}$.



(a) The graph G



(b) The graph H for $k = 3$ and $s = 4$. The black directed paths are the paths $(P_u)_{u\in V(G)}$ and the blue paths are the connections P_{uv} of length $s-k+1$ between them.

Figure 3. An example of our reduction

Let C be a cycle of H and u the smallest vertex of G such that C passes by u^+ (such vertex exists because $H - \bigcup_{u\in V(G)} u^+$ is a forest of subdivided stars with central vertices $(v^-)_{v\in V(G)}$). The cycle C alternates at u^+ so H is acyclic. Finally, $L^k(H) = G^{*s}$ has girth $g(s+1) \geqslant 3(s+1) \geqslant \ell$, where g is the girth of G.

$\square$

4 3-Colouring shift graphs

Theorem 9. *For any $k \geqslant 1$, 3-*COL *is NP-complete when restricted to the class of shift graphs.*

Proof. We design a polynomial time reduction from 3-COL on general graphs to 3-COL on k-iterated shift graphs. For simplicity, we first describe our reduction for shift graphs before adapting it to

k-iterated shift graphs, as the latter case is more technical, but the ideas are essentially identical. We will equivalently view proper colourings of $L(\overrightarrow{G})$ as arc-colourings of $\overrightarrow{G}$, that is an assignment of colors to $A(G)$ such that $u \to v$ and $v \to w$ have distinct colors for all $u, v, w \in V(G)$.

The gadget H. We construct an acyclic oriented graph $\overrightarrow{H}$ such that $\chi(L(\overrightarrow{H})) = 3$, having a marked vertex x with the following property: in any 3-arc-colouring of $\overrightarrow{H}$ the set of arcs entering x uses exactly two colours.

Consider the minimum value of $n \in \mathbb{N}$ such that the transitive n-vertex tournament $\overrightarrow{T_n}$ satisfies $\chi(L(\overrightarrow{T_n})) = 4$. Let v_{n-1} and v_n respectively denote the vertices of $\overrightarrow{T_n}$ with respective in-degrees $n-2$ and $n-1$. Let $\overrightarrow{H}$ be the graph obtained from $\overrightarrow{T_n}$ by iteratively removing arcs entering v_{n-1} until the chromatic number of its line digraph drops to three. Note that $\overrightarrow{S_n} := \overrightarrow{T_n} - \{uv_{n-1} : u \in V(T_n) \setminus \{v_{n-1}, v_n\}\}$ is the transitive tournament on the vertices $V(T_n) \setminus \{v_{n-1}\}$, with an additional vertex v_{n-1} whose only incident arc ends in the sink v_n. Thus $\overrightarrow{S_n}$ is 3-arc-colourable, and $\overrightarrow{H}$ is well defined.

Let uv_{n-1} be the last arc removed in the construction of $\overrightarrow{H}$. We thus have $\chi(L(\overrightarrow{H} + uv_{n-1})) = 4$ and $\chi(L(\overrightarrow{H})) = 3$. Let α be a proper 3-colouring of $L(\overrightarrow{H})$, where $v_{n-1}v_n$ is without loss of generality coloured 1. The arcs entering v_{n-1} cannot use the colour 1, so they use at most two colours. Suppose that they are all using the same colour, say 2 (see Figure 4). Let $c := \alpha(uv_n)$ (note that c is well defined as $u \to v_n$ is an arc of $\overrightarrow{H}$) and β be the 3-colouring of $L(\overrightarrow{H} + uv_{n-1})$ such that $\beta(uv_{n-1}) := c$, $\beta(v_{n-1}v_n) \in \{1, 3\} \setminus \{c\}$ and $\beta(xy) = \alpha(xy)$ for all other arcs. The arc-colouring β uses three colours like α and we claim that it is a proper colouring of $L(\overrightarrow{H} + uv_n)$. As α is proper, it suffices to check that the colourings of the arcs uv_{n-1} and $v_{n-1}v_n$ do not create a monochromatic edge. As v_n is a sink, and as uv_n was colored with c by α, note that the arc uv_{n-1} cannot have a neighbour of colour c in $L(\overrightarrow{H} + uv_n)$. Moreover, $v_{n-1}v_n$ is the only arc of $L(\overrightarrow{H} + uv_n)$ starting in v_{n-1}, and uses a colour different from 2 and c, showing that β is a proper 3-colouring. This contradicts $\chi(L(\overrightarrow{H} + uv_n)) = 4$. We thus deduce that the set of arcs entering v_{n-1} in H use both the colours 2 and 3. To sum up, $\overrightarrow{H}$ is 3-colourable and in each 3-arc-colouring of $\overrightarrow{H}$, the arcs entering the marked vertex $x := v_{n-1}$ use two different colours.

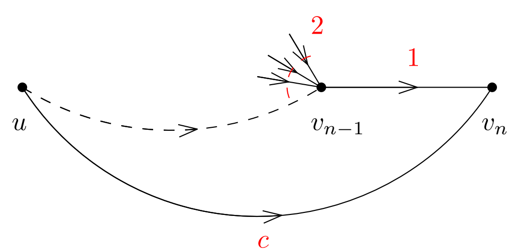
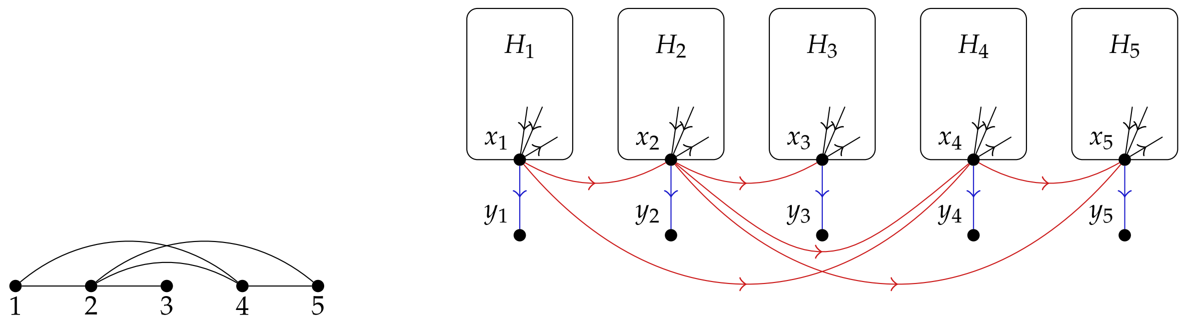


Figure 4. In red, the coloration α depicted in the first part of the proof of Theorem 9.

Constructing the shift graph. Let G be a graph. We contruct an acyclic oriented graph $\overrightarrow{G'}$ such that $\chi(G) \leqslant 3$ if and only if $\chi(L(\overrightarrow{G'})) \leqslant 3$ as follows. Fix an arbitrary total order $<$ on the vertex set of G and for each vertex u of G, take a copy $\overrightarrow{H_u}$ of $\overrightarrow{H}$, with an arc $e_u := x_u \to y_u$ starting in the marked vertex x_u of H_u and ending in a new pendant vertex y_u. For each edge $uv \in E(G)$, with $u < v$, add the arc $x_u \to x_v$ (see Figure 5). The oriented graph $\overrightarrow{G'}$ constructed is acyclic because each copy $\overrightarrow{H_u}$ of $\overrightarrow{H}$ is acyclic and separated from the rest of G' by the cut vertex x_u, and the additional edges can be ordered acyclically following the order on the vertices of G. Therefore $L(\overrightarrow{G'})$ is a shift graph.

We will now show that G is 3-colourable if and only if $L(\overrightarrow{G'})$ is 3-colourable.



(a) A graph G

(b) The oriented graph $\overrightarrow{G'}$ constructed. The blue arcs $x_i \to y_i$ correspond to the vertices of G and the red arcs to its edges.

Figure 5. Constructing an equivalent shift graph.

Equivalence of the instances. Assume that there exists proper 3-colouring α of G. Colour each arc of $\overrightarrow{G'}$ starting in x_u with the colour $\alpha(u)$ of u. At this stage it remains only to colour the copies of $\overrightarrow{H}$. Let β be a 3-arc-colouring of $\overrightarrow{H}$. For each copy $\overrightarrow{H_u}$, permute the colours of β such that the arcs entering x_u use the two colours in $[3]$ different from $\alpha(u)$. We claim that the resulting colouring is a 3-arc-colouring of $\overrightarrow{G'}$. Indeed, at each x_v, all the arcs starting in x_v have an identical colour $\alpha(v)$, and the arcs entering x_v are either coming from the copy $\overrightarrow{H_v}$ or of the form $x_u \to x_v$ for some $u < v$, in both case they use colours different from $\alpha(v)$. Thus $L(\overrightarrow{G'})$ is 3-colourable.

Conversely, in a proper 3-arc-colouring β of $\overrightarrow{G'}$, observe that by the aforementioned property of $\overrightarrow{H}$, for each $u \in V(G)$, the set of arcs entering the marked vertex x_u uses exactly 2 colors, hence all the arcs starting in x_u in $\overrightarrow{G'}$ are coloured identically. Colour u with this colour for each $u \in V(G)$. We claim that this gives a proper 3-colouring that we denote with α. Indeed, given any edge uv in $E(G)$ with $u < v$, the colour of $u \in V(G)$ with respect to α is the colour of $x_u \to x_v$ with respect to β, which must differ from the colour of $x_v \to y_v$ with respect to β. In particular, note that the colour of $x_v \to y_v$ with respect to β is the colour of $v \in V(G)$ with respect to α. Thus G is 3-colourable.

□

In the next proof, an *in-forest* (resp. *out-forest*) is an oriented forest in which each vertex has in-degree (resp. out-degree) at most 1. Observe that in particular, every component of an in-forest (resp. out-forest) has exactly one vertex with in-degree (resp. out-degree) equal to 0, which we call the *root* of its component.

We now prove that if one restricts to shift graphs of large girth (and not large odd-girth as in Theorem 9), 3-colouring becomes a trivial problem.

Proposition 10. *All shift graphs of girth at least five are 3-colourable.*

Proof. Let G be a shift graph of girth at least five and denote $\overrightarrow{G}$ its shift orientation. Let $\overrightarrow{H}$ be an oriented acyclic graph which is a root digraph of $\overrightarrow{G}$. We will identify $V(G)$ with $A(\overrightarrow{H})$. Observe that each vertex of $\overrightarrow{H}$ has either in-degree or out-degree at most one, as a vertex with both in- and out-degree at least two would produce a four-cycle in the line digraph. We partition the vertices of $\overrightarrow{H}$ in two subsets: let X be the set of vertices of $\overrightarrow{H}$ with in-degree at most one, and $Y := V(H) \setminus X$ be the remaining vertices, which all have out-degree at most one. We now partition the arcs of $\overrightarrow{H}$ in four subsets. We let A_X and A_Y respectively denote the sets of arcs having both extremities in X and Y respectively. We let $A_{X\to Y}$ (resp. $A_{Y\to X}$) denote the set of arcs of $\overrightarrow{H}$ having their tail in X (resp. Y) and their head in Y (resp. X). Note that $A_{X\to Y}$ and $A_{X\to Y}$ both form independent sets in G. We color all vertices of $A_{X\to Y}$ with one colour, say 1.

We now show that the arcs of $A_X \cup A_Y \cup A_{Y\to X}$ form a forest in G, and therefore can be coloured with two additional colours, which results in a proper 3-arc-colouring of $\overrightarrow{H}$. First, observe that as X does not contain vertices with in-degree 2, $\overrightarrow{H}[X] = (X, A_X)$ must be an out-forest. Symmetrically, $\overrightarrow{H}[Y] = (Y, A_Y)$ is an in-forest. Observe also that the arcs of $A_{Y\to X}$ must form a matching, and can only connect a root of some component of $\overrightarrow{H}[Y]$ to a root of some component of $\overrightarrow{H}[X]$. It follows that the induced subgraph $G[A_X \cup A_Y \cup A_{Y\to X}]$ is a forest. More precisely, one can observe this graph is isomorphic to the graph obtained when contracting each edge of the matching $A_{Y\to X}$, and when removing all roots of those components of $\overrightarrow{H}[X]$ and $\overrightarrow{H}[Y]$ which are not incident to some arc in $A_{Y\to X}$. □

5 Recognising shift graphs

Recall that a 3-CNF formula is a boolean formula φ in conjonctive normal form (CNF), in which each clause contains 3 litterals. Such a formula φ is called *monotone* if all variables always appear positively, i.e. no litteral has the form $\neg x$ for some variable x. Lovász [13] proved that the following problem is NP-complete.

Monotone Not all equal 3-SAT (MNAE-3SAT):
Given as input a monotone 3-CNF formula φ, decide if there exists a valuation of the variables such that each clause of φ contains both variables set ot true and false.

Note that MNAE-3SAT is equivalent to the problem of deciding whether a given 3-regular hypergraph is 2-colourable. In particulra, this hypergraph formulation was the one used in [13]. In order to prove the next result, we will design a polynomial time reduction to MNAE-3SAT, inspired by the one from [7].

Theorem 11. *For each $k \geqslant 1$, the problem of recognising whether a graph is a k-iterated shift graph (or equivalently, the support of the k-iterated line digraph of an acyclic oriented graph) is NP-complete.*

Proof. Let $k \in \mathbb{N}$. We reduce MNAE-3SAT to the problem of recognising a k-iterated shift graph. Let φ be a monotone 3-CNF formula with m clauses and n variables. We construct a graph G_φ of size $O(mnk)$ which is a k-iterated shift graph if φ is a valid instance of MNAE-3SAT, but is not a shift graph (and *a fortiori* not a k-iterated shift graph) if φ is not a valid instance of MNAE-3SAT.

Gadgets. We first describe a gadget that will be used later for encoding the variables. Let H be the 4-sun, that is the graph on eight vertices composed of a 4-cycle with one pendant edge attached to each of the vertices on the cycle. For each variable x, consider the gadget composed of a path U_x on $8m+1$ vertices. Denote $u_x^0, \dots u_x^{4m-1}$ every other vertex of U_x, starting from the second vertex of U_x. For each $i \in \{0, \dots, 4m-1\}$ we connect u_x^i to the first vertex of the jaw of a comb Q_i of length $k+1$. Denote v_x^i the last vertex of the jaw of this comb and w_x^i the vertex adjacent to v_x^i via the last tooth.Connect each u_x^i to a distinct copy H_x^i of H via one of its vertices of degree three that we will call t_x^i (see Figure 9). For each variable x, let L_x denote a disjoint copy of this graph, and let M_x be the subgraph of L_x obtained by removing all vertices of the graphs H_x^i, except the vertices $(t_x^i)_{0 \leqslant i \leqslant 4m-1}$. In particular, M_x is connected and contains $\bigcup_{i=0}^{4m-1}\{t_x^i, v_x^i, w_x^i\} \cup V(U_x)$.

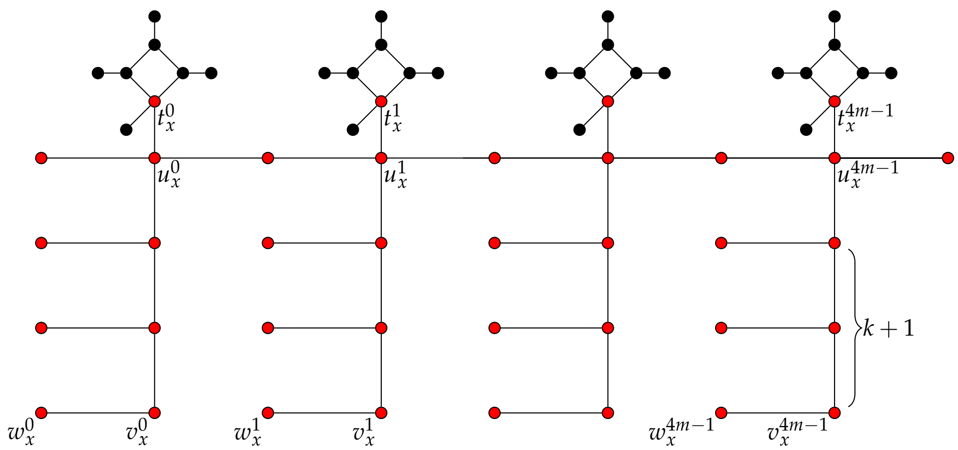


Figure 6. The variable gadget L_x. The red vertices induce the subgraph M_x.

For each $i \in \{0, \dots, m-1\}$, if x, y, z denote the three variables appearing in the i^{th} clause of φ, we attach the gadgets L_x, L_y and L_z as follows. Add three edges to form a path $a_ib_ic_id_i$, where $a_i = w_x^{4i}$, $b_i = w_y^{4i+1}$, $c_i = w_y^{4i+3}$ and $d_i = w_z^{4i}$, so that the vertex set $\{w_x^{4i}, w_y^{4i+1}, w_y^{4i+3}, w_z^{4i}, v_x^{4i}, v_y^{4i+1}, v_y^{4i+3}, v_z^{4i}\}$ forms a comb C_i of length three, with jaw $w_x^{4i}w_y^{4i+1}w_y^{4i+3}w_z^{4i}$. Denote G_φ the constructed graph.

Valid orientations of G_φ. We call an orientation of a graph *valid* if it is a line digraph of an acyclic oriented graph. In other words, valid orientations are exactly the acyclic orientations satisfying the two conditions of Lemma 4. Before proving that there is a valuation such that φ is a positive instance of MNAE-3SAT if and only if G_φ admits a valid orientation, we prove several lemmas describing valid orientations of G_φ and some of its subgraphs.

Claim 12. *Up to isomorphism, there are only two valid orientations of the 4-sun, which are opposite from each other (see Figure 7). In these orientations, the 4-cycle is alternating and all vertices of degree three are transitive.*

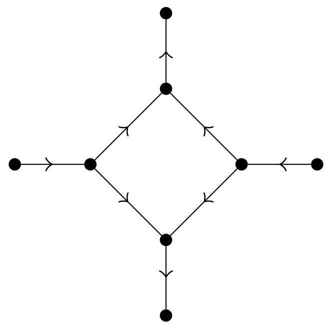


Figure 7. The only valid orientation of the 4-sun (and its opposite).

Proof of the Claim. Consider $\overrightarrow{H}$ a valid orientation of H. Recall that by Lemma 3, the 4-cycles are alternating at each vertex in a shift orientation of shift graph.

Now let v be a vertex of degree three in H. Without loss of generality, assume that v has no in-neighbour in C, i.e. that it is a source of C. Let u be the unique neighbour of v which is not on C, and w and x be two vertices of C such that w is adjacent to v, while x is not. As the orientation of C in $\overrightarrow{H}$ alternates, we must have $v \to w$ and $w \to x$ in $\overrightarrow{H}$. In particular, the first condition of Lemma 2, then implies that we have $u \to v$ in $\overrightarrow{H}$, hence v is transitive. As this reasoning is independent of the choice of v on C, we deduce that $\overrightarrow{H}$ has the desired form. ■

Claim 13. *Consider a valid orientation of L_x for some fixed variable x. Then there are only two possible orientations for its restriction to M_x, which are opposite from each other, and such that*

- *if $w_x^0 \to v_x^0$ then for all i, we have $w_x^i \to v_x^i$ if i is even and $v_x^i \to w_x^i$ if i is odd,*
- *if $v_x^0 \to w_x^0$ then for all i, we have $v_x^i \to w_x^i$ if i is even and $w_x^i \to v_x^i$ if i is odd.*

Proof of the Claim. Fix a valid orientation $\overrightarrow{L_x}$ of L_x and without loss of generality, assume that $w_x^0 \to v_x^0$ in $\overrightarrow{L_x}$, the other case being symmetric. For each $i \in \{0, \dots, 4m-1\}$, we let $\overrightarrow{H_x^i}$ denotes the orientation of the subgraph H_x^i induced by $\overrightarrow{L_x}$. By Claim 12, in each $\overrightarrow{H_x^i}$ all the vertices of degree three are transitive (however, each subgraph H_x^i may be oriented in any of its two opposite orientations independently of any other copy).

We first show that for each $i \in \{0, \dots, 4m-1\}$, we have the following

1. if $u_x^i \to t_x^i$ in $\overrightarrow{L_x}$, then u_x^i is a sink in $\overrightarrow{L_x} - t_x^i$.
2. if $t_x^i \to u_x^i$ in $\overrightarrow{L_x}$, then u_x^i is a source in $\overrightarrow{L_x} - t_x^i$,

We only show 1, as 2 is symmetric. Let $i \in \{0, \dots, 4m-1\}$ and assume that $u_x^i \to t_x^i$ in $\overrightarrow{L_x}$. By Claim 12, t_x^i is transitive in $\overrightarrow{H_x^i}$, and thus has an in-neighbour in $\overrightarrow{L_x}$ distinct from u_x^{2i}. In particular, note that the first condition of Lemma 2 then implies that u_x^{2i} cannot have an out-neighbour in $\overrightarrow{L_x} - t_x^{2i}$, showing 2.

Using once again the first condition of Lemma 2, observe that the vertices of degree two in $U_x \setminus \{u_0, \dots, u_{4m-1}\}$ are transitive and that the vertices u^i are alternatively sources or sinks in

$\overrightarrow{L_x} - \bigcup_i t_x^i$. In order to conclude the proof of the claim, observe that for i such that u_x^i is a sink in $\overrightarrow{L_x} - t_x^i$ (respectively a source), applying iteratively the first condition of Lemma 2 implies that the jaw of the comb Q_i is oriented towards u_x^i and each teeth st where s belongs to the jaw is oriented $t \to s$ in $\overrightarrow{L_x}$ (resp. away from u_x^i and $s \to t$). So for each i, u_x^i is a sink if and only if $w_x^i \to v_x^i$. In particular, because the edge $w_x^0 v_x^0$ is oriented $w_x^0 \to v_x^0$ and the vertices u_x^i are alternatively sources and sinks in $\overrightarrow{L_x} - \bigcup_i t_x^i$, this proves that we have $w_x^i \to v_x^i$ for all even i and $v_x^i \to w_x^i$ for all odd i.

■

For each $i \in \{0, \dots, 4m-1\}$, and every orientation $\overrightarrow{C_i}$ of the comb C_i, we say that a teeth e of is

- *positive* if e is a molar and is oriented away from the jaw, or if e is an incisor and is oriented towards the jaw.
- *negative* if e is a molar and is oriented towards the jaw, or if e is an incisor and is oriented away from the jaw.

Claim 14. *For each $i \in \{0, \dots, 4m-1\}$, in any valid orientation of C_i, the teeth of C_i cannot be all positive, or all negative. Conversely, any orientation of the teeth of C_i in which the incisors are both positive (respectively both negative), and in which at least one molar is negative (respectively positive) can be extended to a valid orientation of C_i in which the jaw of C_i is a directed path.*

Proof of the Claim. We let $aa', \dots, dd'$ denote the four teeth of C_i with jaw $abcd$. Assume towards a contradiction that there exists a valid orientation $\overrightarrow{C_i}$ of C_i making all teeth positive. That is, the molars aa' and dd' are oriented away from a and d, while the incisors bb' and cc' are oriented towards b and c. Without loss of generality, assume that $b \to c$. By Lemma 2, as c has both an in-neighbour and an out-neighbour, the edge ab must be oriented $a \to b$ (see Figure 9). We then obtain a contradiction as the orientation of the induced path $a'abb'$ contradicts the first condition of Lemma 2.

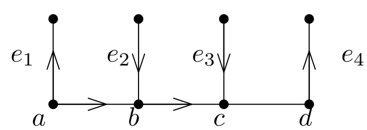


Figure 8. The orientation depicted in the first part of the proof of Claim 14.

Conversely, consider an orientation of the teeth $aa', \dots, dd'$, such that bb' and cc' are say positive, but not all four edges are positive. Say without loss of generality aa' is negative. Then, no matter which orientation dd' has, orienting the jaw of C_i as a directed path from a to d (as depicted on Figure 9) gives an orientation of C_i avoiding the forbidden configurations of Lemma 2, and thus which is valid. ■

Equivalence of the instances. We are now ready to prove that φ admits a valuation such that each clause contains both variables set to true and false if and only if G_φ is a k-iterated shift graph. In fact for the reverse direction, we will show the stronger property that if φ is not a positive instance of MNAE-3SAT, then G_φ is not even a shift graph.

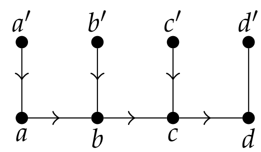


Figure 9. A valid orientation of C_i. The edge a_1a_2 is negative, b_1b_2 and c_1c_2 are negative, and d_1d_2 can either be positive or negative.

We start proving the reverse implication and consider a valid orientation $\overrightarrow{G_\varphi}$ of G_φ. We now consider the following variable assignment: for each variable x, set x to true if $w_x^0 \to v_x^0$ in $\overrightarrow{G_\varphi}$ and to false otherwise. We show that with this variable assignment, each clause of φ contains both variables set to true and false. Let $i \in \{0, \dots, m-1\}$ and $x \vee y \vee z$ be the i^{th} clause of φ. Recall that the corresponding comb C_i of length 3 in G_φ has jaw $a_ib_ic_id_i := w_x^{4i}w_y^{4i+1}w_y^{4i+3}w_z^{4i}$, and denote $e_1 = w_x^{4i}v_x^{4i}$, $e_2 = w_y^{4i+1}v_y^{4i+1}$, $e_3 = w_y^{4i+3}v_y^{4i+3}$ and $e_4 = w_z^{4i}v_z^{4i}$ the teeth of C_i. Note that Claim 13 implies that e_1, e_2, e_4 are respectively positive in $\overrightarrow{G_\varphi}$ if and only if x, y, z are respectively set to true, and that e_3 is positive if and only if e_2 is positive. Claim 14 then implies that at least one teeth of C_i is positive, and one is negative in $\overrightarrow{G_\varphi}$. So the variable assignment we defined satisfies the desired properties.

We now prove the direct implication and consider a variable assignment for which each clause of φ contains both variables set to true and false. For each variable x, let $\overrightarrow{M_x}$ denote the (unique) valid orientation of M_x given by Claim 13 such that $w_x^0 \to v_x^0$ in $\overrightarrow{M_x}$ if x is set to true, and such that $v_x^0 \to w_x^0$ in $\overrightarrow{M_x}$ if x is set to false. Then, for each $i \in \{0, \dots, 4m-1\}$, we consider for each copy H_x^i of the four-sun the (unique) valid orientation $\overrightarrow{H_x^i}$ given by Claim 12, such that in $\overrightarrow{M_x} \cup \overrightarrow{H_x^i}$, the vertex t_x^i has both in- and out-degrees equal to two. We let $\overrightarrow{L_x}$ denote the orientation obtained after orienting this way M_x and all subgraphs H_x^i of L_x. See Figure 10.

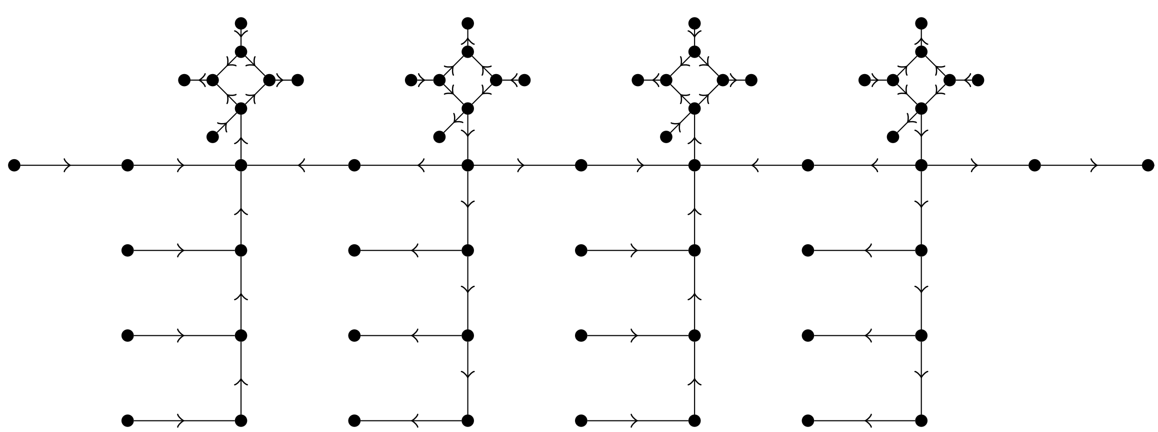


Figure 10. The orientation of L_x when x is true (for the false variables, the orientation is reversed)

In order to obtain an orientation of G_φ, the only edges that remain to be oriented are the edges forming the jaw of each clause gadget. For each $i \in \{0, \dots, m-1\}$, consider the comb C_i corresponding to the i^{th} clause $x \vee y \vee z$ of φ, and denote with $e_1, \dots, e_4$ its teeth, again with respect to the order in which they attach to the jaw. As $\overrightarrow{M_y}$ is a valid orientation of M_y, and as $4i+1$ and $4i+3$ are odd, by Claim 13, e_2 and e_3 are either both negative or both positive. Moreover, observe that by definition of the orientations $\overrightarrow{M_x}, \overrightarrow{M_y}, \overrightarrow{M_z}$, Claim 13 implies that e_1, e_2, e_4 are respectively positive if and only if x, y, z are respectively set to true. Since the clause $x \vee y \vee z$ contains both variables set to true and false, we can apply Claim 14 to extend our orientation to a valid orientation $\overrightarrow{C_i}$ of C_i, in which the jaw is a directed path. Fixing such an orientation of each C_i, we now obtain an orientation $\overrightarrow{G_\varphi}$ of G_φ, whose restriction to each subgraph of the form L_x or C_i is valid.

In order to prove that $\overrightarrow{G_\varphi}$ is a valid orientation, we only need to check the first condition of Lemma 2 at the junctions between the clause and the variable gadgets, and the orientation of the cycles using these junctions. Note that any such cycle must pass by some u_x^i, at which it alternates. Finally, as each v_x^i is a transitive vertex, the first condition of Lemma 2 is satisfied and the orientation of G_φ is that of an acyclic line digraph. In other words, G_φ is a shift graph. This concludes the proof for shift graphs, but not for iterated shift graphs.

To prove that G_φ is a k-iterated shift graph, we now construct an oriented acyclic graph $\overrightarrow{G'_\varphi}$ such that $\overrightarrow{G_\varphi}$ is an induced subdigraph of $L^k(\overrightarrow{G'_\varphi})$. For each variable x, by Lemma 6, there exists an acyclic oriented graph $\overrightarrow{M'_x}$ with a convex k-shift embedding ψ_x of $\overrightarrow{M_x} - \bigcup_j w_x^j$ in $\overrightarrow{M'_x}$. Similarly, for each clause comb C_i, there exists an acyclic oriented graph graph $\overrightarrow{C'_i}$ with a convex k-shift embedding ψ_i of $\overrightarrow{C_i}$ in $\overrightarrow{C'_i}$. Note that by identifying the leaves of the clause combs C_i with the approriate vertices v_x^j of the subgraphs $\overrightarrow{M_x} - \bigcup_j w_x^j$, one obtains the graph $\overrightarrow{G_\varphi}$.

Denote ψ the convex k-shift embedding of $\bigsqcup_x(\overrightarrow{M_x} - \bigcup_j w_x^j) \sqcup \bigsqcup_i \overrightarrow{C_i}$ in $\bigsqcup_x \overrightarrow{M'_x} \sqcup \bigsqcup_i \overrightarrow{C'_i}$ obtained by taking the union of all these k-shift embeddings. More precisely, for each vertex u in some C_i, we let $\psi(u) = \psi_i(u)$ and for each vertex u in some $(\overrightarrow{M_x} - \bigcup_j w_x^j)$, $\psi(u) = \psi_x(u)$.

In order to obtain a k-shift embedding of $\overrightarrow{G_\varphi}$, we apply iteratively Lemma 5 on ψ by identifying each leaf of the comb clauses to the appropriate vertex v_x^j of one of the subgraphs $\overrightarrow{M_x} - \bigcup_j w_x^j$. To do so, we need to justify that during this process, the images by the k-shift embedding of the vertices to be identified are vertex disjoint and have no arcs between them. At first, the leaves of the combs C_i and the vertices $(v_x^j)_{j,x}$ lie in different components of $\bigsqcup_x(\overrightarrow{M_x} - \bigcup_j w_x^j) \sqcup \bigsqcup_i \overrightarrow{C_i}$, so this hypothesis is verified. Moreover, $(\psi(v_x^i))_{i,x}$ forms a collection of pairwise disjoint k-paths that have no arcs between them because any distinct $v_{x_2}^{i_1}$ and $v_{x_2}^{i_2}$ are either in different component of $(\overrightarrow{M_x} - \bigcup_j w_x^j)$, or connected by a unique path P containing a directed subpath of length $k+2$ between $v_{x_1}^{i_1}$ and $u_{x_1}^{i_1}$. As result, $N(\psi(v_{x_1}^{i_1})))$ and $\psi(u_{x_1}^{i_1})$ are vertex-disjoint and since ψ is convex, so are $N(\psi(v_{x_1}^{i_1})))$ and $\psi(v_{x_2}^{i_2})$. So there are also no arcs between $\psi(v_{x_1}^{i_1})$ and $\psi(v_{x_2}^{i_2})$.

, one obtains the graph $\overrightarrow{G_\varphi}$. During this identification process,

images of the vertices to be indentified remain disjoint. At the end of this identification process, we have a k-shift embedding of G_φ minus the 4-suns.

Finally, we identify each $\psi(t_x^{4i})$ with the path of length k starting in the vertex t of the graph F drawn on Figure 11. It is straightforward to check that the k-line digraph of the graph drawn on

Figure 11 is the 4-sun and the resulting graph G'_φ is such that G_φ is an induced subgraph of G'_φ. $\square$

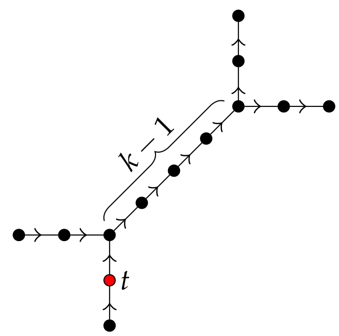


Figure 11. A graph F whose k-line digraph is the 4-sun.

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