A Recolouring Version of a Conjecture of Reed

Lucas De Meyer* Clément Legrand-Duchesne† Jared León‡ Tim Planken§
Youri Tamitegama¶

January 10, 2025

Abstract

Reed conjectured that the chromatic number of any graph is closer to its clique number than to its maximum degree plus one. We consider a recolouring version of this conjecture, with respect to Kempe changes. Namely, we investigate the largest ε such that all graphs G are k-recolourable for all $k \geqslant \lceil \varepsilon\omega(G) + (1-\varepsilon)(\Delta(G)+1) \rceil + c$, for some fixed c.
For general graphs, an existing construction of a frozen colouring shows that $\varepsilon \leqslant 1/3$. We show that this construction is optimal in the sense that there are no frozen colourings below that threshold. For this reason, we conjecture that $\varepsilon = 1/3$.
In the special case of odd-hole free graphs, we show that $\varepsilon = 1/2$ and $c = 1$, and that this is tight up to one colour.

1 Introduction

The chromatic number χ of any graph G lies between its clique number $\omega(G)$ and the maximum degree $\Delta(G)$ plus one. A natural question is which of these two bounds is closer to the chromatic number. Reed [13] conjectured that the chromatic number of any graph G is at most $\lceil (\omega(G) + \Delta(G) + 1)/2 \rceil$, and in particular, for any $\varepsilon \leqslant 1/2$, the chromatic number of G is at most $\lceil \varepsilon\omega(G) + (1-\varepsilon)(\Delta(G)+1) \rceil$.

In the same article, Reed proved using the probabilistic method that this bound is tight if true (see Theorem 2 in [13]). He also proved that there exists $\varepsilon > 0$ such that for all G, $\chi(G) \leqslant \lceil \varepsilon\omega(G) + (1-\varepsilon)(\Delta(G)+1) \rceil$. More recently, King and Reed [8] gave a significantly shorter proof of this result and proved that for large enough Δ, Reed's conjecture holds for $\varepsilon \leqslant \frac{1}{320e^6}$. This conjecture generated a lot of interest over the past years and similar statement were proved in [13, 4] for increasing values of ε. The best bound known today is due to Hurley, de Joannis de Verclos and Kang [5], who proved that for all graphs G with sufficiently large maximum degree, $\chi(G) \leqslant \lceil 0.119\omega(G) + 0.881(\Delta(G)+1) \rceil$.

*Université Claude Bernard Lyon 1, CNRS, INSA Lyon, LIRIS, UMR5205, Villeurbanne France (lucas.de-meyer@univ-lyon1.fr).
†Theoretical Computer Science Department, Faculty of Mathematics and Computer Science, Jagiellonian University, Kraków, Poland (clement.legrand-duchesne@uj.edu.pl).
CNRS, LaBRI, Université de Bordeaux, Bordeaux, France.
‡Mathematics Institute, University of Warwick, United Kingdom (jared.leon@warwick.ac.uk).
§Institute of Discrete Mathematics and Algebra, Faculty of Mathematics and Computer Science, Technische Universität Bergakademie Freiberg, Germany (tim.planken@math.tu-freiberg.de).
¶Mathematical Institute, University of Oxford, United Kingdom (tamitegama@maths.ox.ac.uk).

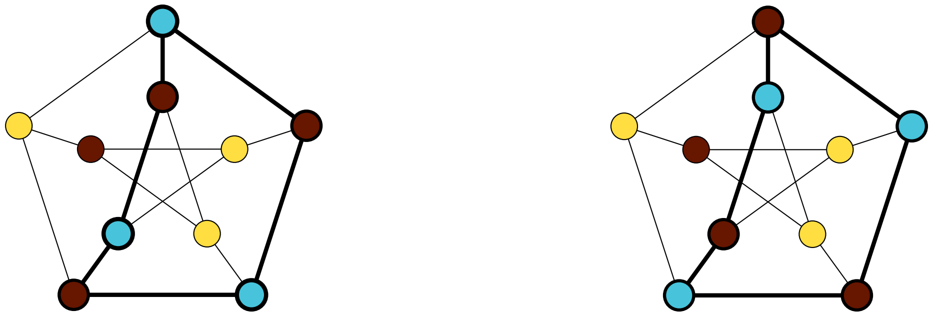


Figure 1. Two 3-colourings of the Petersen graph that differ by one Kempe change. The corresponding {●, ●}-Kempe chain is thickened.

Reed's conjecture has been proven for several hereditary graph classes [7, 1, 14]. Finally, list colouring versions and local strengthenings of Reed's conjecture were also considered in [4, 6] and [6, 7, 1] respectively. More precisely, let $f_\varepsilon(G) = \max_{v \in V(G)} \lceil \varepsilon\omega(v) + (1-\varepsilon)(\deg(v)+1) \rceil$, where $\omega(v)$ is the size of the largest clique containing v, the following local strengthening of Reed conjecture was introduced by King in [7]:

Conjecture 1.1 (Local Reed Conjecture)**.** *For all $\varepsilon \geqslant 1/2$ and all graphs G, $\chi(G) \leqslant f_\varepsilon(G)$ holds.*

1.1 A recolouring version of Reed's conjecture

In this article, we consider a recolouring variation of Reed's conjecture, that was introduced by Bonamy, Kaiser and Legrand-Duchesne [9]. Given a graph and a proper colouring of its vertex set, a *Kempe chain* is a bichromatic component of the graph. A *Kempe change* consists in swapping the two colours within a Kempe chain (see Figure 1), thereby resulting in another proper colouring. This reconfiguration operation was introduced in 1879 by Kempe in an attempt to prove the Four-colour theorem. Kempe changes are decisive in the existing proofs of the Four-Colour theorem and of Vizing edge-colouring theorem. We say that a graph is *k-recolourable* if all its k-colourings are *Kempe equivalent*, that is, connected by a sequence of Kempe changes.

The most common obstruction for k-recolourability is the existence of a *frozen k-colouring*, a k-colouring in which any two colours span a connected subgraph. As a result, the partition induced by the colour classes is invariant under Kempe changes. Beside frozen colourings, the only other known obstruction to recolourability is a topological argument of Mohar and Salas [10, 11] that is specific to 3-colourings of highly regular planar or toroidal graphs.

Bonamy, Kaiser and Legrand-Duchesne asked for the largest ε such that all graphs G are k-recolourable for all $k \geqslant \lceil \varepsilon\omega(G) + (1-\varepsilon)(\Delta(G)+1) \rceil$. It remains open whether this holds for any positive ε. On the other hand, the parameter ε is known to be at most $1/3$. Indeed, Bonamy, Heinrich, Legrand-Duchesne, and Narboni [2] constructed a random graph with degrees concentrating around $\Delta = 3n/4$ and expected clique number $O(\log(n))$, that admits a frozen $n/2$-colouring and is not $n/2$-recolourable with high probability. Thus, this hints that Kempe changes are unlikely to be of any use to prove Reed's conjecture in the range $\varepsilon \in [1/3, 1/2]$.

In Section 4, we prove that this construction is optimal:

Theorem 1.2. *The only connected graphs that admit some frozen k-colouring for $k \leqslant \frac{\omega(G)+2(\Delta(G)+1)}{3}$ are cliques.*

Therefore, Theorem 1.2 strongly motivates the following conjecture.

Conjecture 1.3. *For all* $0 \leqslant \varepsilon < 1/3$*, any graph* G *is* k*-recolourable for all* $k > \lceil \varepsilon\omega(G) + (1-\varepsilon)(\Delta(G)+1)\rceil$.

As a consequence of Theorem 1.2, note that disproving Conjecture 1.3 would require new tools and would significantly improve our understanding of Kempe recolouring and its obstructions. Finally, all advances towards Reed's conjecture rely on the probabilistic method. This proof method does not adapt well to reconfiguration proofs. For this reason, proving Conjecture 1.3 even for small positive ε seems challenging and is likely to either be the first application of the probabilistic method to reconfiguration, or to yield a constructive proof of Reed's colouring question, which would be of independent interest.

1.2 The special case of odd-hole free graphs

A *hole* is an induced cycle of length at least four. The complement of a hole is an *antihole* and odd-(anti)hole is an (anti)hole of odd size. The Strong Perfect graph theorem [3] characterises perfect graphs, that is graphs with chromatic number equal to their clique number, as graphs without odd-holes nor odd-antiholes.

Aravind, Karthick and Subramanian [1] proved the Local Reed conjecture in the class of odd-hole free graphs. Weil [14] then observed this result generalises immediately to the class $\mathcal{H}$ of graphs whose odd-holes all contain some vertex of degree less than $f_{1/2}(G)$. Bonamy, Kaiser and Legrand-Duchesne [9] gave an alternative proof of this result using Kempe changes:

Theorem 1.4 (Bonamy, Kaiser and Legrand-Duchesne)**.** *Every* k*-colouring of any graph* $G \in \mathcal{H}$ *is Kempe equivalent to a* $f_{1/2}(G)$*-colouring.*

This result also implies that the graphs in $\mathcal{H}$, and in particular odd-hole free graphs, have no frozen k-colourings for $k > f_{1/2}(G)$. Again, as frozen colourings are the main known obstruction to recolourability, this motivated the following conjecture:

Conjecture 1.5 (Bonamy, Kaiser and Legrand-Duchesne)**.** *All odd-hole free graphs of maximum degree* Δ *and clique number* ω *are* k*-recolourable for* $k \geqslant \lceil \frac{\omega+\Delta+1}{2} \rceil$.

The same authors proved in [9] a weakening of Conjecture 1.5: all odd-hole free graphs G are k-recolourable for $k > \left\lceil \frac{\chi(G)+\Delta(G)+1}{2} \right\rceil$, which settles the special case of perfect graphs. Moreover, as Reed's conjecture holds for odd-hole free graphs, this also implies that they are k-recolourable for $k > \left\lceil \frac{\omega(G)+3(\Delta(G)+1)}{4} \right\rceil$.

The main contribution of this paper is the following result, confirming Conjecture 1.5 up to using one extra colour.

Theorem 1.6. *Let* G *be graph in which every odd-hole contains a vertex of degree at most* $f(G)$*. Then* G *is* k*-recolourable, for* $k > f_{1/2}(G)$.

A folklore construction shows that this result is almost tight:

Theorem 1.7. *For all* $k \geqslant 3$*, there is a non* k*-recolourable perfect graph* G *with* $k = \lceil \frac{\omega(G)+\Delta(G)+1}{2} \rceil - 1$.

Therefore, the only remaining open case for Conjecture 1.5 is $k = f_{1/2}(G)$. Moreover, this also shows that Theorem 1.4 is tight, even for perfect graphs.

After providing some notations and basic definitions in Section 2, we prove Theorem 1.6 in Section 3 and the lower bounds on both conjectures, namely Theorems 1.7 and 1.2, in Section 4.

2 Preliminaries

We now define the notation that we will be using in our proof. Let G be a graph. A *k-colouring* of G is a function $\gamma : V(G) \to [k]$ such $\gamma(u) \neq \gamma(v)$ for all $uv \in E(G)$. For any subgraph H of G, we let $\gamma(H) := \{\gamma(v) : v \in V(H)\}$ be the set of colours used in the subgraph H, and let $\gamma|_H$ be the colouring γ restricted to H. We say that two colourings α and β *agree* on a set $X \subseteq V(G)$ if $\alpha|_X = \beta|_X$. Similarly, we say that α and β *differ* on X, if X is the set of all vertices $v \in V(G)$ such that $\alpha(v) \neq \beta(v)$. We denote $N(u)$ and $N[u]$ the open and closed neighbourhoods of a vertex u and say that u *misses* the colour c in the colouring α if $c \notin \alpha(N[u])$.

An $\{a, b\}$-*Kempe chain* K of γ is a connected component of the subgraph induced by vertices of colours a or b. We say that a and b are the colours *used* in K. We denote $K_{v,c}(G, \gamma)$ the $\{\gamma(v), c\}$-Kempe chain that contains the vertex v. Let $K(G, \gamma)$ be the set of all Kempe chains in G under the colouring γ, where we also allow the empty Kempe chain (the graph with no vertices) to be part of this set. For an $\{a, b\}$-Kempe chain $K \in K(G, \gamma)$, let $\gamma^{\to}(K)$ be the colouring obtained after interchanging the colours a and b on all the vertices in K. We also refer to this operation as performing the *Kempe change* K in G under γ.

Given an induced subgraph G' of G, a colouring γ of G and an $\{a, b\}$-Kempe chain K of $\gamma|_{G'}$, the *extension* of K is the $\{a, b\}$-Kempe chain of γ that includes K.

3 Recolourability of odd-hole free graphs

This section is dedicated to the proof of Theorem 1.6, that we recall:

Theorem 1.6. *Let G be graph in which every odd-hole contains a vertex of degree at most $f(G)$. Then G is k-recolourable, for $k > f_{1/2}(G)$.*

Let G be an odd-hole free graph. Let $\omega := \omega(G)$ and $\Delta := \Delta(G)$, and let k be an integer satisfying $k > \lceil (\omega + \Delta + 2)/2 \rceil$. Let α and β be two k-colourings of G. We will transform α into β using a sequence of Kempe chains in G. We proceed by induction on the number of vertices of G. Let $v \in V(G)$ be a vertex of G, and let $G' := G - v$. Let $\alpha' := \alpha|_{G'}$ and $\beta' := \beta|_{G'}$ be the restriction of colourings α and β to G', respectively.

By the induction hypothesis, there exists a sequence $\alpha' = \alpha'_0, \alpha'_1, \ldots, \alpha'_h = \beta'$ of k-colourings of G' such that $\alpha'_i = \alpha'^{\to}_{i-1}(K_i)$, for some sequence of Kempe chains $K_1, K_2, \ldots, K_h$ with $K_i \in K(G', \alpha'_{i-1})$ for $i = 1, \ldots, h$.

A natural strategy to recolour α into β would be to apply the Kempe changes $K_1, \ldots K_h$ to α. Unfortunately, any of these Kempe chains might not be a valid Kempe chain in G if it uses the colour of v. To circumvent this issue, the key idea is to allow for some controlled errors at each step. We formalise this error control with the following definition. We say that a colouring γ of G is *faithful* to a colouring γ' of G' if the following conditions both hold:

- ***C1*** There exists two colours a and b such that the colourings $\gamma|_{G'}$ and γ' differ on a set $\mathcal{B}$ of *bad* $\{a, b\}$-Kempe chains under γ'.
- ***C2*** Every Kempe chain in $\mathcal{B}$ contains a neighbour of v coloured a in γ'.

We will construct a sequence of colourings $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_h$ of G such that each α_i is faithful to α'_i and for any $1 \leqslant i \leqslant h$, the colourings α_i and α_{i-1} are Kempe equivalent in G. Assume that there we already constructed the sequence $\alpha_0, \ldots \alpha_{i-1}$ for some $1 \leqslant i \leqslant h$. We use two lemmas to define α_i: Lemma 3.1 in the favourable case where α'_{i-1} and α_{i-1} agree on G', and Lemma 3.2

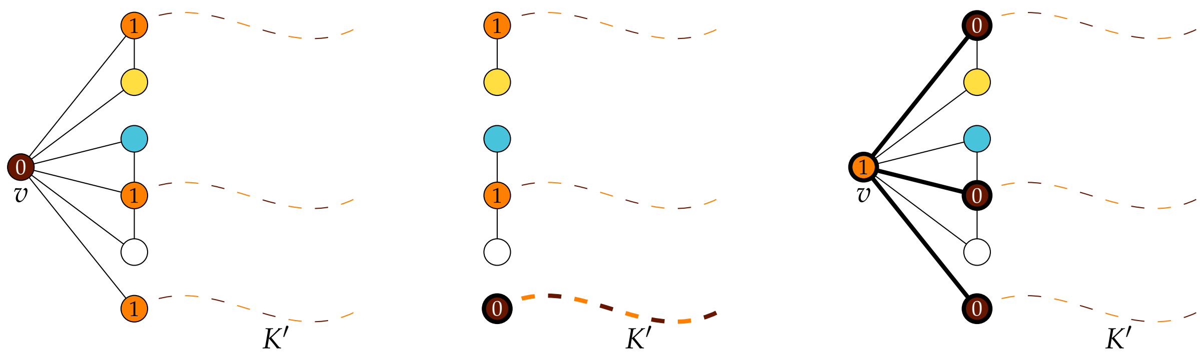
in the more involved case where α_{i-1} is faithful to α'_{i-1} without extending it. These two lemmas will be proved in Section 3.1 and Section 3.2 respectively.

Once the sequence $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_h$ is constructed, we only need to argue that β is Kempe equivalent to α_h, in order to conclude the proof of Theorem 1.6. We first argue that α_h is a Kempe equivalent to a colouring γ that agrees with β on G', then β can be obtained from γ by simply recolouring v to its final colour. Let $\mathcal{B}$ be the set of bad $\{a, b\}$-Kempe chains for α_h and α'_h. If these Kempe chains are also Kempe chains of G, then performing them results in a colouring γ that agrees with β on G'. If not, then v is coloured a or b and as β is a colouring of G, this implies that $\mathcal{B}$ contains all $\{a, b\}$-Kempe chains of α'_h that contain a neighbour of v. Therefore there exists a $\{a, b\}$-Kempe chain K of α'_h whose restriction to G' is $\mathcal{B}$. As a result, $\gamma = \alpha_h{}^{\rightarrow}(K)$ agrees with β on G', which concludes the proof of Theorem 1.6.

3.1 Defining α_i when α'_{i-1} and α_{i-1} agree on G'

Lemma 3.1. *Let γ and γ' be colourings of G and G' respectively, such that $\gamma|_{G'} = \gamma'$. Let K' be a $\{a, b\}$-Kempe chain of γ'. Then the colouring $\delta := \gamma^{\rightarrow}(K)$ is faithful to $\delta' := \gamma'^{\rightarrow}(K')$.*

Proof. If K' is also a Kempe chain in G under γ, then $\delta = \gamma^{\rightarrow}(K')$ and its restriction to G' is δ'. In particular, δ is faithful to δ'. Hence, we can assume that K' contains v. Let K be the extension of K' in γ. Up to relabelling colours, assume that $\gamma(v) = 0$ and the other colour used by K is 1. Let $\mathcal{A}$ be the set of connected components of $K - v$ (see Figure 2a). Note that $\mathcal{A}$ is the set of all $\{0, 1\}$-Kempe chains of γ' that contain a neighbour of v coloured 1. By performing K in G under γ, we obtain a colouring δ whose restriction to G' differs with δ' on the chains of $\mathcal{B} := \mathcal{A} \setminus \{K'\}$ (see Figures 2b and 2c), so **C1** holds. By construction, every Kempe chain in $\mathcal{B}$ contains a neighbour of v coloured 0 in γ, so **C2** also holds. □



(a) The colouring γ. **(b)** The colouring δ'. **(c)** The colouring δ.

Figure 2. The different colourings of Lemma 3.1. The set of Kempe chains in $\mathcal{A}$ are represented by dashed lines. The Kempe chains on which δ and δ' differ with γ are thickened.

In particular, by defining $\alpha_0 = \alpha$, Lemma 3.1 builds a colouring α_1 Kempe equivalent to α_0 and faithful to α'_1.

3.2 Defining α_i when α_{i-1} is faithful to α'_{i-1}

In the more general case where α_{i-1} is only faithful to α'_{i-1}, we either perform a series of Kempe changes that maintains the set $\mathcal{B}$ of bad Kempe chains and does not create any other bad Kempe chains; or we perform a series of Kempe changes to fix the bad Kempe chains of $\mathcal{B}$, before creating a Kempe equivalent colouring α_i faithful to α_i', with a different set of bad Kempe chains. This procedure is handled by the following lemma:

Lemma 3.2. *Let γ_1', γ_2' be two colourings of G' that differ by one Kempe change K'. Let γ_1 be a colouring of G faithful to γ_1'. Then there is a colouring γ_2 of G that is faithful to γ_2' and Kempe equivalent to γ_1.*

Proof. Up to relabelling the colours, we can assume that the set $\mathcal{B}$ of bad Kempe chains under γ_1' are using the colours 0 and 1 and that every Kempe chain in $\mathcal{B}$ contains a neighbour of v coloured 1 in γ_1'. Lemma 3.2 results directly from Claim 3.3, Claim 3.4 and Claim 3.5.

Claim 3.3. *Lemma 3.2 holds if $\gamma_1(v) \neq 0$.*

Proof of Claim. Assume that $\gamma_1(v) \neq 0$. Then the bad Kempe chains of $\mathcal{B}$ are also Kempe chains of G under γ_1. So we can successively perform all the Kempe chains in $\mathcal{B}$ under γ_1 to obtain a colouring δ such that $\delta|_{G'} = \gamma_1'$. By Lemma 3.1, $\gamma_2 = \delta^{\rightarrow}(K')$ is faithful to γ_2' and Kempe equivalent to γ_1. ■

Claim 3.4. *Lemma 3.2 holds if K' does not use the colours 0 and 1.*

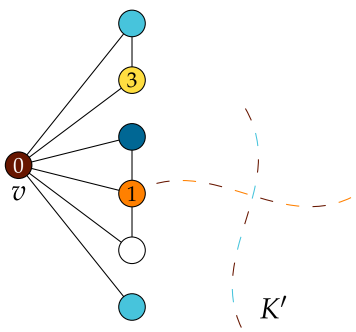
Proof of Claim. Assume that K' uses neither 0 nor 1. By Claim 3.3, we can assume that $\gamma_1(v) = 0$. Then, K' is also a Kempe chain under γ_1 and it does not intersect with the Kempe chains in $\mathcal{B}$. So $\gamma_2 := \gamma_1{}^{\rightarrow}(K')$ satisfies the conditions **C1** and **C2**, by taking identical a, b and $\mathcal{B}$. ■

Claim 3.5. *Lemma 3.2 holds if K' uses 0 or 1 and $\gamma_1(v) = 0$.*

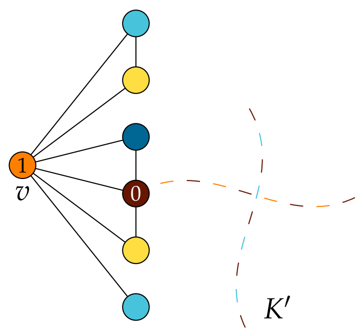
Proof of Claim. Assume that $\gamma_1(v) = 0$ and that K' uses 0 or 1. So, we have $\{0, 1\} \cup \gamma_1'(K') \leqslant 3$.

If v is missing a colour c in γ_1, that is $c \notin \gamma_1(N[v])$, then we can recolour v into c without changing $\gamma_1|_{G'}$ and conclude using Claim 3.3. Thus, we can assume that the neighbourhood of v contains all the colours except 0 in γ_1, i.e. $|\gamma_1(N(v))| = k - 1$.

Likewise, if v has only one neighbour w coloured 1 in γ, then $\gamma_1|_{G'}$ and γ_1' differ only on one bad Kempe chain $B' \in \mathcal{B}$ (see Figure 3a). Let B be the extension of B' and $\delta = \gamma_1{}^{\rightarrow}(B)$. Then $\delta|_{G'} = \gamma_1'$ (see Figure 3b) and γ_2 is given by Lemma 3.1. Thus we can also assume that v has at least two neighbours coloured 1.



(a) The colouring γ_1.



(b) The colouring δ extends γ_1'.

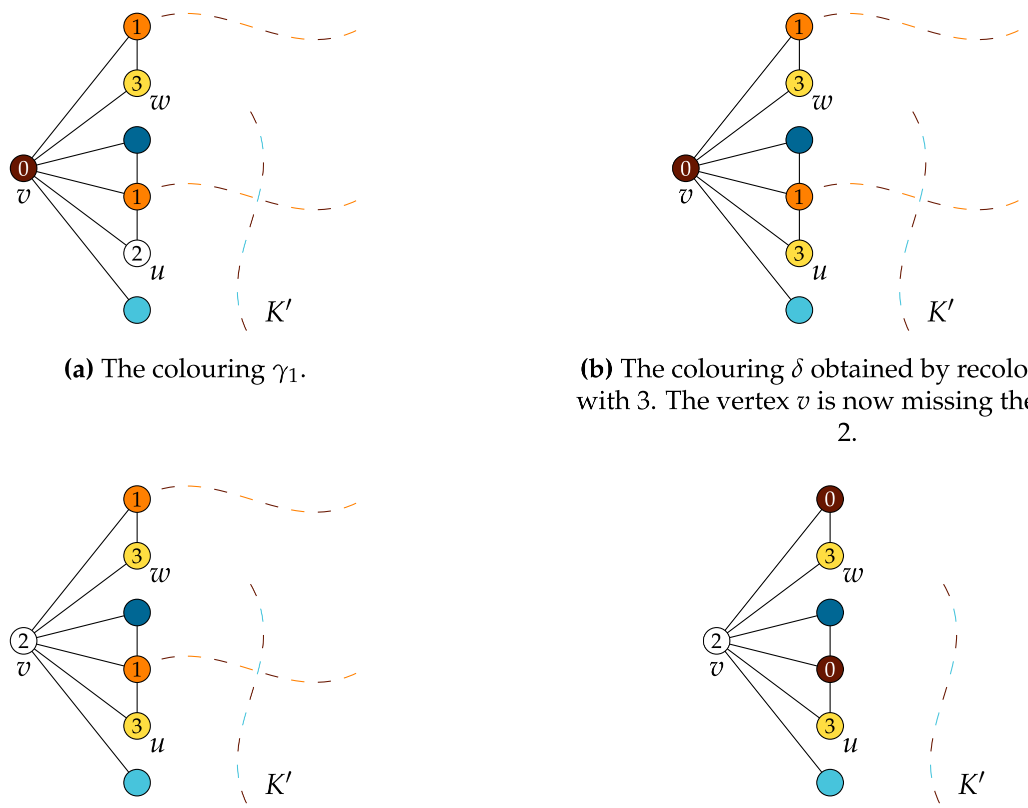
Figure 3. Recoloring sequence when v has only one neighbour coloured 1.

In order to recolour the vertex v and reduce to Claim 3.3, we find a Kempe chain L of γ_1 such that v misses some colour $c \notin \{0,1\} \cup \gamma_1'(K')$ in $\gamma_1{}^{\rightarrow}(L)$. We then recolour v with this c, perform the Kempe changes in $\mathcal{B}$ and K', and maintain L as the next bad Kempe chain (i.e., define $\mathcal{B}$ as $\{L\}$). We now formalise this.

Let S be the set of neighbours of v whose colour appears only once in $\gamma_1(N(v))$. Recall that v has at least two neighbours coloured 1 in γ_1, so these vertices do not belong to S. We have

$$\begin{aligned}\Delta \geqslant |N(v)| &\geqslant 2(k-1-|S|) + |S| \\ &\geqslant 2\left\lceil \frac{\omega+\Delta+1}{2} \right\rceil - |S| \\ &> \omega + \Delta - |S|,\end{aligned}$$

which implies that $|S| \geqslant \omega + 1$. Let T be the subset of vertices S that are not coloured by γ_1 with a colour in $\{0,1\} \cup \gamma_1'(K')$. Since $\{0,1\} \cup \gamma_1'(K') \leqslant 3$ and S contains no vertices coloured 1 or 0, we have $|T| \geqslant \omega$. Thus, T induces a non-edge uw (otherwise, $T \cup \{v\}$ forms a clique of size $\omega + 1$ in G). Up to relabelling, say u and w are respectively coloured 2 and 3 in γ_1 (see Figure 4 for an illustrative example following the consecutive Kempe changes performed on γ_1).



(a) The colouring γ_1.

(b) The colouring δ obtained by recolouring u with 3. The vertex v is now missing the colour 2.

(c) The colouring τ obtained by recolouring v with 2.

(d) The colouring σ obtained by reclouring the bad $\{0,1\}$-Kempe chains

Figure 4. The recolouring sequence of Claim 3.5.

Let L be the $\{2,3\}$-Kempe chains that contains w and δ be the colouring $\gamma_1{}^{\rightarrow}(L)$ (see Figure 4b). Note that L does not contain u, otherwise $L \cup v$ would contain an induced odd cycle, which is also an induced odd cycle of G. As the colours 2 and 3 appear only once in the neighbourhood of v in γ, the vertex v misses the colour 2 in δ. Let σ be the colouring obtained from δ by recolouring v into 2 (see Figure 4c) and τ be the colouring obtained from σ by performing the Kempe chains in $\mathcal{B}$ (see Figure 4d). The colouring $\sigma|_{G'}$ differs from γ'_1 only on the $\{2,3\}$-Kempe chain L, so σ is faithful to γ'_1. Finally, K' uses neither 2 nor 3 so Claim 3.4 applied to σ yields the desired γ_2. ■

□

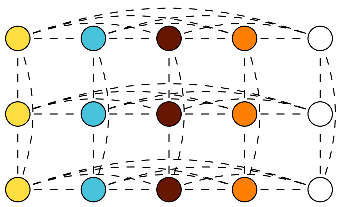
4 Frozen k-colourings of graphs with low degree and clique number

4.1 Odd-hole free graphs

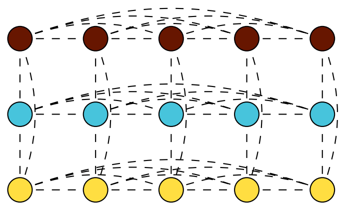
We first prove Theorem 1.7, that we recall:

Theorem 1.7. *For all $k \geqslant 3$, there is a non k-recolourable perfect graph G with $k = \lceil \frac{\omega(G)+\Delta(G)+1}{2} \rceil - 1$.*

Proof. Let G_k be the complement of the Cartesian product of the cliques K_3 and K_k on three and k vertices. The vertices of G_k are indexed by the tuples (u, v) with $u \in \{1, 2, 3\}$ and $v \in \{1, \dots, k\}$, and $(u_1, v_1)(u_1, v_2) \in E(G_k)$ if $u_1 \neq u_2$ and $v_1 \neq v_2$ (see Figure 5 for an example with $k = 5$).



(a) The k-coloring α of G_5



(b) The 3-coloring β of G_5

Figure 5. Colorings of G_5. Non-edges are depicted as dashed lines, while edges are omitted for clarity.

Note that G_k is perfect [12]. Moreover, it is $2(k-1)$-regular and that is has clique number three. So we have that:

$$k = \left\lceil \frac{\omega(G_k) + \Delta(G_k) + 1}{2} \right\rceil - 1$$

Moreover, G_k has a frozen k-colouring $\alpha((u, v)) := v$ and a 3-colouring $\beta(u, v)) := u$, which is a fortiori a k-colouring, so G_k is not k-recolourable. □

This proves that Theorem 1.4 is tight even for perfect graphs and that Theorem 1.6 is almost tight, in the sense that the only open case of Conjecture 1.5 is $k = \lceil \frac{\omega+\Delta+1}{2} \rceil$.

4.2 General graphs

For general graph, it is possible to improve Theorem 1.7 using the random construction of Bonamy, Heinrich, Legrand-Duchesne and Narboni [2], which gives graphs that admit frozen $\lceil \varepsilon\omega + (1-\varepsilon)(\Delta+1) \rceil$-colourings for all $\varepsilon > 1/3$. We prove that this construction is optimal, namely:

Theorem 1.2. *The only connected graphs that admit some frozen k-colouring for $k \leqslant \frac{\omega(G)+2(\Delta(G)+1)}{3}$ are cliques.*

Proof. Let $\mathcal{G}$ be the class of connected graphs that are not cliques. We aim to prove that $\varepsilon^\star = 1/3$, where $\varepsilon^\star$ is the infimum ε such that there exists $G \in \mathcal{G}$ and a frozen k-colouring of G with $k = \varepsilon\omega(G) + (1-\varepsilon)(\Delta(G)+1)$.The construction of Bonamy, Heinrich, Legrand-Duchesne and Narboni [2] shows that $\varepsilon^\star \leqslant 1/3$.

Claim 4.1. *Let α_1 be a frozen k_1-colouring of a graph $G_1 \in \mathcal{G}$ with a colour class of size one. Then $\varepsilon_1 = 1$ or there exists a graph $G_2 \in \mathcal{G}$ with a frozen k_2-colouring and all colour classes of size at least two, such that $\varepsilon_2 \leqslant \varepsilon_1$; where ε_i satisfies $k_i = \varepsilon_i\omega(G_i) + (1-\varepsilon_i)(\Delta(G_i)+1)$.*

Proof of Claim. Let K be the set of vertices that are uniquely coloured in α_1. Let $G_2 = G_1 - K$ and α_2 be the $(k_1 - |K|)$-colouring induced by α_1 on G_2. Since α_1 is frozen, K induces a clique in G_1 and α_2 is frozen. As G_1 is not a clique, there is at least one colour class in α_1 that has size at least two. If there is only one such colour class X, then we have that $k_1 = \omega(G_1) < \Delta(G_1) + 1$, which implies that $\varepsilon_1 = 1$. Thus, we can assume that there are at least two colour classes of size at least two in α_1. Since every pair of colour classes induces a connected graph in G_1, G_2 is connected. And since G_2 has a colour class of size at leas two, G_2 is not a clique and is in $\mathcal{G}$.

The vertices of K dominate G_1, so $\Delta(G_1) = |G_1| - 1$ and $\Delta(G_2) \leqslant \Delta(G_1) - |K|$. On the other hand, $\omega(G_2) = \omega(G_1) - |K|$ and $k_2 = k_1 - |K|$, so $\varepsilon_2 \leqslant \varepsilon_1$. ■

For all G and k, denote $\varepsilon^{(k)}(G)$ the smallest ε such that there exists a frozen k-colouring α of G, with $k = \varepsilon\omega(G) + (1-\varepsilon)(\Delta(G)+1)$ and such that all colour classes of α have size at least two. Denote also $\varepsilon(G) = \inf\{\varepsilon^{(k)}(G) : k \in \mathbb{N}\}$. By Claim 4.1, we have that $\varepsilon^\star = \inf\{\varepsilon(G) : G \in \mathcal{G}\}$.

Claim 4.2. *Let α be a frozen k-colouring of $G \in \mathcal{G}$ with no colour class of size one. Then $\Delta(G) \geqslant \frac{3}{2}(k-1)$. As a result, $\varepsilon^{(k)}(G) > 1/3$.*

Proof of Claim. Note that without loss of generality, we can assume that for any two colour classes X and Y, the subgraph $G[X \cup Y]$ is a tree: it is connected because α is frozen and removing an edge in a cycle can only decrease Δ and ω, which decreases ε.

Given a set X a set of vertices, denote $\deg(X)$ the number of edges adjacent to some vertex in X. Let $X = \alpha^{-1}(c)$ be a colour class of minimal size. We have

$$\begin{aligned}\Delta(G) \geqslant \deg(X)/|X| &= \sum_{c' \neq c} \frac{|X| + |\alpha^{-1}(c')| - 1}{|X|} \\ &\geqslant \sum_{c' \neq c} 2 - \frac{1}{|X|} \geqslant \frac{3}{2}(k-1)\end{aligned}$$

So $k \leqslant 2\Delta(G)/3 + 1$. We have $k = \varepsilon^{(k)}(G)\omega(G) + (1 - \varepsilon^{(k)}(G)) + (\Delta(G)+1)$ and thus

$$\varepsilon^{(k)}(G) = \frac{\Delta(G) + 1 - k}{\Delta(G) + 1 - \omega(G)} > \frac{\Delta(G) + 1 - k}{\Delta(G)} \geqslant \frac{1}{3}$$

■

This proves that $\varepsilon(G)$ and $\varepsilon^\star$ are at least $1/3$ and that $\varepsilon^\star$ and hence $\varepsilon^\star = 1/3$. Moreover, note that $eps^\star$ is unattained, otherwise some graph $G \in \mathcal{G}$ would verify $\varepsilon^{(k)}(G) = 1/3$. As all frozen k-colourings of any graph G must verify $k \geqslant \varepsilon^\star\omega(G) + (1-\varepsilon^\star)(\Delta+1)$, this concludes the proof of Theorem 1.2. □

Acknowledgments

The authors would like to thank Tomáš Kaiser and Marthe Bonamy for the fruitful discussions in the early stages of this project. This research was conducted during the workshop Early Career Researchers in Combinatorics (ECRiC24) in Edinburgh.
The second author was partially supported by the Polish National Science Centre under grant no. 2019/34/E/ST6/00443.
The third author was supported by the Warwick Mathematics Institute Centre for Doctoral Training.
The fourth author was partially supported by DFG, project number 546892829.

References

[1] N. R. Aravind, T. Karthick and C. R. Subramanian. *Bounding χ in terms of ω and Δ for some classes of graphs*. Discrete Mathematics, **311**(12):911–920, 2011.

[2] M. Bonamy, M. Heinrich, C. Legrand-Duchesne and J. Narboni. *On a recolouring version of Hadwiger's conjecture*. J. Combin. Theory Ser. B, **164**:364–370, 2024.

[3] M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas. *The strong perfect graph theorem*. Annals of mathematics, pages 51–229, 2006.

[4] M. Delcourt and L. Postle. On the list coloring version of Reed's conjecture. In *Extended abstracts of the ninth European conference on combinatorics, graph theory and applications, EuroComb 2017, Vienna, Austria, August 28 – September 1, 2017*, pages 343–349. Amsterdam: Elsevier, 2017.

[5] E. Hurley, R. de Joannis de Verclos and R. J. Kang. An improved procedure for colouring graphs of bounded local density. In *Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 135–148. [Society for Industrial and Applied Mathematics (SIAM)], Philadelphia, PA, 2021.

[6] T. Kelly and L. Postle. *A local epsilon version of Reed's conjecture*. Journal of Combinatorial Theory, Series B, **141**:181–222, 2020.

[7] A. D. King. *Claw-Free Graphs and Two Conjectures on Omega, Delta, and Chi*. PhD thesis, McGill University, Montreal, 2009.

[8] A. D. King and B. A. Reed. *A short proof that χ can be bounded ϵ away from $\Delta + 1$ toward ω*. J. Graph Theory, **81**(1):30–34, 2016.

[9] C. Legrand-Duchesne. *Exploring the Space of Colorings of a Graph*. PhD thesis, Université de Bordeaux, Feb. 2024.

[10] B. Mohar and J. Salas. *A new Kempe invariant and the (non)-ergodicity of the Wang–Swendsen–Kotecký algorithm*. Journal of Physics A: Mathematical and Theoretical, **42**(22), 2009.

[11] B. Mohar and J. Salas. *On the non-ergodicity of the Swendsen–Wang–Kotecký algorithm on the kagomé lattice*. Journal of Statistical Mechanics: Theory and Experiment, **2010**(05):P05016, 2010.

[12] G. Ravindra and K. Parthasarathy. *Perfect product graphs*. Discrete Mathematics, **20**:177–186, 1977.

[13] B. Reed. *ω, Δ, and χ*. J. Graph Theory, **27**(4):177–212, 1998.

[14] V. Weil. *A note on Reed's Conjecture about ω, Δ and χ with respect to vertices of high degree*. arXiv e-prints, page arXiv:1611.02063, 2016.