A Gray code for arborescences of tournaments

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Abstract

We consider the following question of Knuth: given a directed graph G and a root r, can the arborescences of G rooted in r be listed such that any two consecutive arborescences differ by only one arc? Such an ordering is called a pivot Gray code and can be formulated as a Hamiltonian path in the reconfiguration graph of the arborescences of G under arc flips, also called flip graph of G. We give a positive answer for tournaments and explore several conditions showing that the flip graph of a directed graph may contain no Hamiltonian cycles.

A *Gray code* is a linear or cyclic order on the elements of a fixed set (usually bit representations of numbers between 0 and $2^n - 1$), such that consecutive elements differ on exactly one bit. Gray codes were originally considered to avoid the errors introduced by unperfectly synchronised physical switches, causing a period of transition between two consecutive binary numbers. A Gray code can also be seen as a Hamiltonian path or even a Hamiltonian cycle in the reconfiguration graph: the graph whose vertices are the enumerated objects, with an edge between any two objects at Hamming distance one (see Mütze's extensive survey on Gray codes for combinatorial structures [Müt23]).

In the context of spanning trees of a graph G, two types of Gray codes can be considered. The most general one is an order with the *revolving door* property: each spanning tree in the sequence is obtained from the previous one by an edge exchange, that is by removing an edge and adding another one. The first such algorithm was given by Cummins in 1966 [Cum66]. In fact, Cummins showed that the reconfiguration graph of spanning trees under edge exchanges of any graph G, also called *flip graph of G*, is edge-Hamiltonian. Namely, for each edge of the flip graph there exists a Hamiltonian cycle passing through this edge. Shank [Sha68] then gave a short and simple proof of this result. As of today, the most efficient known Gray code enumeration algorithm for spanning trees runs in constant delay between consecutive outputs in average and was given by Smith [Smi97]. Finally, the set of spanning trees of graph G are the bases of the matroid formed by the edges of G and Holzmann and Harary [HH72] generalised

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Shank's proof to the reconfiguration graph of the bases of any matroid under element exchanges.

The second type of Gray codes for spanning trees are those with the *strong revolving door* property, that is those in which the edges added and deleted at each step share a common endpoint. These Gray codes are also referred to as *pivot Gray code*.

Problem 1. *Does every graph G admit a pivot Gray code on its spanning trees?*

Although it is rather simple to prove that the corresponding reconfiguration graphs is connected for any graph G, it remains open whether all graphs admit a pivot Gray code for spanning trees. Indeed, one of the main techniques used to construct a Gray code for spanning trees consists in partitioning the spanning trees in two sets: those containing a specific edge e and those avoiding it. One can then apply induction on both sets by considering the graph G/e where e is contracted, and the graph $G - e$ where e is removed respectively. However, the uncontraction of the edge e does not preserve the strong revolving door property as all edges adjacent to e are pairwise agjacent in G/e, but not in G. Nevertheless, pivot Gray codes for spanning trees have been constructed in some graph classes, namely for fan graphs [CGS24] and more generally for outerplanar graphs [BM24].

As Knuth noted [Knu11, Answer to Exercise 7.2.1.6–102], Problem 1 would immediately follow from the existence of a Gray code on the rooted arborescences of a directed graph, by replacing each edge by two arcs going in both directions. More precisely, the *arborescences* of a directed graph G rooted in r are all the spanning trees of G directed away from r. Given two arborescences that differ on exactly one arc, the arcs added and deleted must point to the same vertex u, for this vertex u to be accessible in both arborescences from r. Therefore, Gray codes on the arborescences of G naturally have the strong revolving door property. The problem of designing such a Gray code was proposed by Knuth, with an estimated difficulty of 46/50:

Problem 2 (Exercise 7.2.1.6–102 in [Knu11])**.** *Does every directed graph G admit a pivot Gray code on its arborescences rooted in any fixed vertex?*

In 1967, Chen [Che67, Theorem 1] claimed that for any directed graph G and vertex $r \in V(G)$, the flip graph on the arborescences of G rooted in r contains a Hamiltonian cycle, provided there are at least three arborescences. A counterexample to this claim was provided by Rao and Raju [RR72] in 1972: they constructed a family of directed graphs whose flip graph is a path. These digraphs are not just a sporadic exception: we characterize the cases where an arborescence has degree 1 in the flip graph. In Section 2.2, we construct a greater variety of counterexamples of directed graphs with unbalanced bipartite flip graphs, hence without Hamiltonian cycle. However, the imbalance we obtain for these counterexamples is only equal to one, thus they do not contradict the existence of a Hamiltonian path in the flip graph of any directed graph. Finally, our main result is the construction of a pivot Gray code on the arborescences of any tournament (see Section 3):

Theorem 1. *Let G be a tournament and a vertex $r \in V(G)$. The flip graph of the arborescences of G rooted in r admits a Hamiltonian path.*

1 Preliminaries

1.1 Notations and glossary

Given a directed graph $G = (V, E)$, we denote by $u \to v$ or uv an arc going from u to v. We denote by $N^+(v) = \{u \colon v \to u\}$ the *outneighbourhood* of v and $\deg^+(v) = |N^+(v)|$ its *outdegree*. Analogously, we denote by $N^-(v) = \{u \colon u \to v\}$ its *inneighbourhood* and $\deg^-(v) = |N^-(v)|$ its *indegree*. A vertex of outdegree zero is a *sink*. The support of a directed graph G is the undirected graph obtained by removing the orientations of the arcs of G. (If G contains two vertices u and v with $u \to v$ and $v \to u$, then its support contains only one edge between u and v.) Given a fixed vertex r of G called *root*, we call *arborescence* any spanning directed tree of G in which all arcs are oriented away from r. Two arborescences that are equal on all arcs but one are said to differ by an *arc flip*. *Flipping in* the arc uv in an arborescence T corresponds to removing the unique arc entering v and replacing it by the arc uv, thereby obtaining another arborescence that differs from T by an arc flip. The *flip graph* of G rooted in r, is the undirected graph $\mathcal{F}_r(G)$ whose vertices are the arborescences of G rooted in r, with an edge between each pair of arborescences that differ by an arc flip.

Given a directed graph G rooted in r and one of its vertices u, denote $D_G(u)$ the *descendants* of u, that is the vertices v such that all paths from r to v pass by u (u included). Given an arborescence A of G, an arc $uv \in E(G) \setminus E(A)$ can be flipped in if and only if v is not a descendant of u in A. We now prove an auxiliary lemma allowing us to extend sub-arborescences into arborescences.

Lemma 2. *Let G be a directed graph and r a vertex of G, such that G admits at least one arborescence rooted in r. Any directed subtree T of G rooted in r can be completed into an arborescence.*

Proof. We proceed by induction of $|V(T)|$. Let A be an arborescence of G and T a directed subtree of G, both rooted in r. Let X be the set of vertices of G that do not belong to T. Let P be a minimal path from r to a vertex in X, such a path exists because A is an arborescence rooted in r. Let v and u be the last and second to last vertices in P. We have $u \to v$ and u belongs to $V(T)$ by minimality of P, therefore, adding the arc uv to T extends T. $\square$

1.2 Hamiltonicity of ladders and hypercubes

We start with some easy constructions of Hamiltonian paths in ladders and hypercubes, which will be used in other constructions. A *ladder* of length n is the undirected graph consisting of two paths $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ of length n, with an additional edge between a_i and b_i for all i. A vertex in the ladder has *level* i if $v \in \{a_i, b_i\}$.

Lemma 3. *Let F be the ladder of length n, and let $i, j \leq n$ be integers with $i \neq j$. Let u be a vertex of level i. Then F has a Hamiltonian path from u to some vertex at level j.*

Note that we cannot prescribe *which* of the two vertices at level j is the endpoint of this Hamiltonian path. In fact, since F is bipartite, a parity argument shows that only one of the two vertices at level j can potentially be the endpoint of a Hamiltonian path starting at u.

Proof. Without loss of generality, assume that $j > i$ and $u = a_i$. Consider the paths P_1, P_2 and P_3, where $P_1 = a_i, \ldots, a_1, b_1, \ldots, b_i$ and $P_3 = b_j, \ldots, b_n, a_n, \ldots, a_j$, and P_2 goes from b_i to some vertex $v \in \{a_j, b_j\}$ by zigzagging (see Figure 1):

$$P_2 = b_i, b_{i+1}, a_{i+1}, a_{i+2}, b_{i+2}, b_{i+3}, \ldots, v$$

Concatenating the three paths P_1, P_2, and P_3 results in a Hamiltonian cycle, ending at a vertex $w \neq v$ of level j. Note that w depends on the parity of $j - i$. $\square$

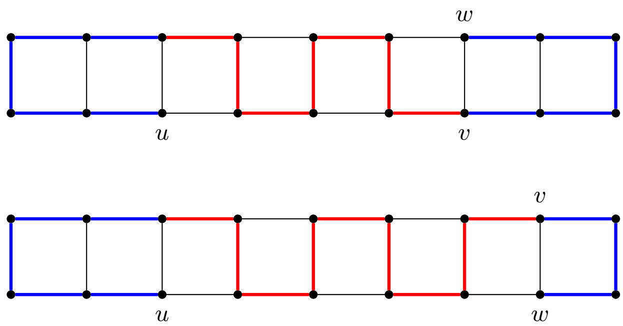


Figure 1: Two Hamiltonian paths in a ladder with extremities at different levels. The paths P_1 and P_3 are drawn in blue, the path P_2 in red.

The hypercube of dimension d is the graph on 2^d vertices indexed by $\{0,1\}^d$ in which two vertices are adjacent if they differ on at most one coordinate. Hypercubes of dimension at least 2 are edge-Hamiltonian:

Lemma 4. *Let $G = K_2^d$ be the hypercube of dimension $d \leq 2$ and $uv \in E(G)$. The hypercube G has a Hamiltonian cycle containing the edge uv.*

Listing binary strings was the original problem considered by Gray, so many different proofs of the Hamiltonicity of the hypercube exist, such as the binary reflected Gray code [Gra53] for example. From it, one can deduce that the hypercube is edge-Hamiltonian simply by the edge-transitivity of the hypercube.

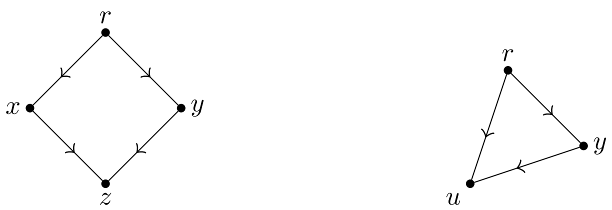
1.3 Reductions

We now prove several reductions rules that preserve the flip graph of a directed graph, or at least the existence of an Hamiltonian path in it. The operations we consider are the removal, the subdivision and the contraction of an arc.

Arc deletion. Given a fixed directed graph G, we denote $G - uv$ the directed graph obtained by removing the arc $u \to v$.

Observation 5. *Let uv be an arc of a directed graph G rooted in r. The flip graph of $G - uv$ is isomorphic to the subgraph of $\mathcal{F}_r(G)$ induced by the arborescences not containing uv. A direct consequence of this is that all arcs that appear in none of the arborescences of G can be removed from G without affecting its flip graph.*

Arc contraction. Denote G/uv the directed graph obtained by contracting the arc $u \to v$, that is replacing u and v by a vertex w with $N^-(w) = N^-(u) \cup N^-(v) \setminus \{u, v\}$ and $N^+(w) = N^+(u) \cup N^+(v) \setminus \{u, v\}$. Arc contractions do not behave as well as edge deletions with respect to the flip graph. Indeed, the flip graph can be affected by contracting an arc, even if this arc belongs to all arborescences of G. For example, the graph G on three vertices r, u and v, rooted in r and containing the arcs ru, uv and rv, has two arborescences: one containing ru and uv, the other containing ru and rv. However contracting ru results in a single arc, hence the number of arborescences of G was not preserved. Recall also form the introduction that another problem might occur. The reconfiguration sequences on the contracted graph do not correspond to reconfiguration sequences in the original graph. For example, consider the graph G on four vertices r, x, y, z, rooted in r, with four arcs $r \to x$, $r \to y$, $x \to z$ and $y \to z$ (see Figure 2a). When contracting xz into a single vertex u, the arc ru can be flipped to yu (see Figure 2b). However, r and y have disjoint outneighbourhoods in G so this flip does not correspond to a flip in the original graph.



(a) G (b) G/xz

Figure 2: A directed graph G for which contracting xz into u creates a flip unfeasible in G.

However, some contractions still preserve the existence of an Hamiltonian path. Let G be a directed graph, r its root and rx an outgoing arc of r. Let $G' = G/rx$ be the directed graph obtained after the contraction of rx into the new root r'. Let $\mathcal{T}_{/rx}$ the set of arborescences of G that contain the arc rx.

Lemma 6. *Let A' be an arborescence of G'. If $\mathcal{F}_{r'}(G')$ contains a Hamiltonian path starting from A', then $\mathcal{F}_r(G)[\mathcal{T}_{/rx}]$ contains a Hamiltonian path starting from any arborescence A such that contracting rx results in A'.*

Proof. Let ϕ be the map that associates to any subgraph H of G containing the arc rx the subgraph $H_{/rx}$ of G' where rx was contracted.

Let T' be an arborescence of G' and N be the set of vertices in $N^+(x) \cap N^+(r)$ that are outneighbours of r' in T'. We first prove that the set $\phi^{-1}(T')$ spans a hypercube of dimension $|N|$ in $\mathcal{F}_r(G)$. For all arcs $r'z \in E(T')$ with $z \in N$, we have $\phi^{-1}(r'z) = \{rz, xz\}$. For any other arc in T', there is a unique arc in G mapped to it by ϕ. Hence, $\phi^{-1}(T')$ is in one-to-one correspondence with the subsets of N. Moreover, two arborescences T_0 and T_1 in $\phi^{-1}(T')$ differ by a flip if only if there exists some $z \in N$ such that $rz \in T_i$ and $xz \in T_{1-i}$, and $T_0 - z = T_1 - z$. Therefore, $\phi^{-1}(T')$ is isomorphic to the Hasse diagram of the poset $(N, \subset)$ and induces an hypercube of dimension $|N|$ in $\mathcal{F}_r(G)$.

We now show that for any S', T' adjacent in $\mathcal{F}_{r'}(G')$, there is a path in $\mathcal{F}_r(G)[\mathcal{T}_{/rx}]$ starting from any arboresence S in $\phi^{-1}(S')$, that visits exactly $\phi^{-1}(S')$ before ending at some arboresence in $\phi^{-1}(T')$. The lemma then follows from applying this iteratively on the edges of the Hamiltonian path of $\mathcal{F}_{r'}(G')$. Let $S \in \phi^{-1}(S')$ and N the set of vertices of $N^+(x) \cap N^+(r)$ that are outneighbours of r' in S. As $\phi^{-1}(S')$ induces a hypercube of dimension $|N|$, it contains a Hamiltonian path starting at S and ending at some $S_2 \in \phi^{-1}(S')$. Let $a'b'$ be the arc flipped in in S' to obtain T'. Note that b' cannot be equal to r', because r' is the root of G', hence $\phi^{-1}(b') = \{b'\}$ and we denote $b := b'$. If $a' = r'$, then by definition of contraction, there exists $a \in \{r, x\}$ such that $b' \in N^+_G(a)$. Otherwise, a' was not the result of the contraction and we set $a := a'$. In both cases, there is a path from r' to a' in S' that avoids b', otherwise the arc $a'b'$ could not be flipped in, so there is also a path from r to a avoiding b in S_2 because $b \notin \{r, x\}$. Thus, a is not a descendant of b in S_2 and the arc ab can be flipped in in S_2. Denote T the resulting arborescence. As $\phi(ab) = a'b'$, we have $\phi(T) = T'$, which concludes the proof. $\square$

Arc subdivision. The following observation shows that we can restrict Problem 2 to oriented graphs by subdiving one arc in each bigon of some directed graph.

Observation 7. *Subdividing an arc does not modify the flip graph.*

Proof. Let G be a directed graph rooted in r and G' be the directed graph obtained by subdividing an arc uw, denote v the vertex introduced in the operation. We build a bijection ϕ from the arborescences of G to the arborescences of G'. Given an arborescence A containing uw, let $\phi(A)$ be the arborescence of G' obtained by subdividing uw by introducing the vertex v. Given en arborescence A that does not contain uw, let $\phi(A) = A \cup \{uv\}$. It is clear that ϕ is a bijection and that ϕ preserves the Hamming distance. Moreover, note that all arborescences of G contain the arc uv because v has indegree one, thus ϕ^{-1} also preserves the Hamming distance, which proves that $\mathcal{F}_r(G)$ and $\mathcal{F}_r(G')$ are isomorphic. $\square$

1.4 Auxiliary lemmas

Finally, we prove two auxiliary lemmas that we will use to characterise degree one vertices in the flip graph, and cut-vertices separating a clique from the rest of the flip graph of a tournament, which we will use in our proof of Theorem 1. We will say that a directed graph H is *built on* G if it has the same vertex set and each arc $u \to v$ in $E(H) \setminus E(G)$ is such that u is a descendant of v in G. We will call each such arc a *backedge* of H.

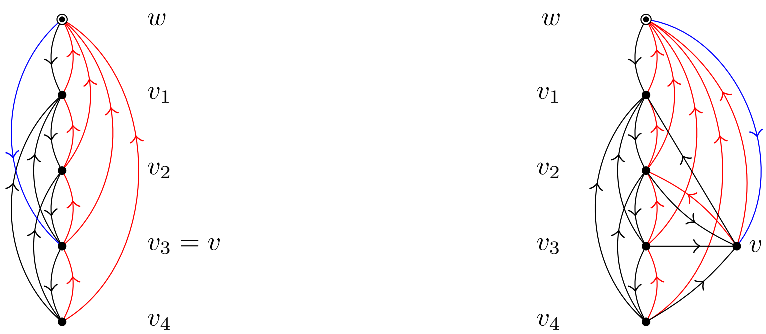
Lemma 8. *If H is built on G, then $\mathcal{F}_r(H)$ is isomorphic to $\mathcal{F}_r(G)$.*

Proof. The arcs of $E(H) \setminus E(G)$ do not appear in any arborescence of H. Otherwise, let A be an arborescence of H with $uv \in E(A) \setminus E(G)$. Without loss of generality, assume that uv was chosen such that uv is the only arc of $E(A) \setminus E(G)$ on the path from the root r to v. Then there exists a path from r to u that uses arcs of G and avoids v, which contradicts the fact that u is a descendant of v in G. Thus the arborescences of H are exactly the arborescences of G and the flip graphs of H and G are isomorphic. $\square$

Finally, we characterise the structure of the tournaments whose flip graphs satisfy some technical conditions arising in our proof of Theorem 1.

Lemma 9. *Let G be a directed graph rooted in w, such that the support of $G \setminus \{w\}$ is a clique. Let $v \in V(G)$ such that flipping in wv in any arborescence rooted in w that does not contain wv, results in the same arborescence A. Then the following hold:*

1. *$\mathcal{F}_w(G - wv)$ is a clique,*
2. *all vertices other v have a single path from w to them in $G - wv$,*
3. *all paths going from w to v in $G - wv$ have length at least 2,*
4. *$G - wv$ has the following structure:*
   - *(a) If $\mathcal{F}_w(G - wv)$ is a single vertex, then $G - wv$ is built on the directed path $w, v_1, \ldots, v_n$, with $v = v_k$ for some fixed $k > 1$. In other words, G contains the graph $L_{k,n}$ (drawn on Figure 3a in black) as a subgraph, and the arcs in $E(G) \setminus E(L_{k,n})$ are either ending at w or of the form $v_{i+1} \to v_i$ (drawn on Figure 3a in red).*
   - *(b) If $\mathcal{F}_w(G - wv)$ is a clique on at least two vertices, then $G - wv$ is built on the directed path $w, v_1, \ldots, v_n$ with an additional vertex v and all arcs $v_i v$ with $i \geq k - 1$ for some fixed $1 < k \leq n$. In other words, G contains the graph $M_{k,n}$ (drawn on Figure 3b in black) as a subgraph, and the arcs in $E(G) \setminus E(M_{k,n})$ are either $v \to v_{k-1}$, or ending at w, or of the form $v_{i+1} \to v_i$ (drawn on Figure 3b in red).*



(a) Illustration of Item 4a with $n = 4$ and $k = 3$

(b) Illustration of Item 4b with $n = 4$ and $k = 3$

Figure 3: The structure of tournaments G, such that all arborescences of $G - wv$ rooted in w produce the same tree when flipping in wv. The arc wv is drawn in blue, the black arcs are present in G while the red arcs are optional.

Proof. All arborescences of $G - wv$ rooted in w have in common all arcs but the one entering v because they result in the same arborescence when flipping in wv. Thus $\mathcal{F}_w(G - wv)$ is a clique, which proves Item 1.

Perform a BFS from w in $G - wv$. If each layer of the BFS contains only one vertex, then $G - wv$ consists of a directed path $w, v_1, \ldots, v_n$ and some backedges, in other words $G - wv$ is built on a directed path. In fact, G contains all arcs v_iv_j with $i > j + 1$ because $G - w$ has complete support (see Figure 3a). The additional possible backedges are either ending at w, or arcs of the form $v_{i+1} \to v_i$, which proves Item 4a. Moreover, note that $G - wv$ has only one arborescence rooted in w.

If some layer of the BFS contains more than one vertex, assume towards contradiction that some vertex x different from v has two inneighbours with depth no larger than that of x (that is either in the same layer or the previous one). Thus, there are at least two paths from w to x in $G - wv$. By Lemma 2, each of these paths can be completed into an arborescence of $G - wv$ rooted in w. These two arborescences have different arc entering x, hence flipping in wv results in different arborescences, a contradiction. So all layers are either reduced to one single vertex, or contain two vertices v_k and v with the arc v_kv. Hence all vertices but v have at most one inneighbour at smaller depth in the BFS. Let k be the depth of v. All other vertices at depth at least k are inneighbours of v because G has complete support. Hence $G - wv$ contains as a spanning subgraph the path $w, v_1, \dots, v_n$ with one additional vertex v such that $v_i \to v$ if $i \geq k - 1$. In addition, since $G - w$ has complete support, $G - w$ also contains all backedges of the form $v_j \to v_i$ with $i > j + 1$, or $v \to v_i$ with $i < k - 1$. In other words, the graph $M_{k,n}$ drawn on Figure 3b is a subgraph of G. Finally, G can also contain other backedges, namely $v \to v_{k-1}$, the arcs ending at w, or the arcs of the form $v_{i+1} \to v_i$, which proves Item 4b.

Note that in both cases, $v \neq v_1$ because $wv \notin G - wv$, so $k > 1$, which implies Item 3. Moreover, as backedges by definition never belong to a path from the root to a vertex, all vertices but v have only one path from w to them in $G - wv$, which proves Item 2. $\square$

2 Directed graphs with no Hamiltonian cycle in their flip graph

In this section, we give several counterexamples witnessing that the flip graph of a directed graph may not contain a Hamiltonian cycle.

2.1 Flip graph can be paths

In [RR72], Rao and Raju showed that the flip graph of bidirected cycles is a path. We recall their proof here:

Lemma 10 ([RR72])**.** *Let G be the directed graph obtained by replacing each edge of a n-cycle by a bigon. The flip graph of G is a path on n vertices.*

Proof. Let $V = \{v_1, \dots, v_n\}$ be the vertices of G, such that v_1 is the root and each vertex v_i is incident to $v_{i-1 \bmod n}$ and $v_{i+1 \bmod n}$. There are only two paths form v_1 to v_i in G: the first one is $v_1, \dots, v_i$ and the second $v_1, v_n, \dots, v_i$. So every arborescence of G rooted in r corresponds to a unique interval $[1, i]$ of indices such that v_i is reached through a path using the arc v_1v_2. In each of these arborescences, only the arcs entering the leafs can be swapped. Thus the reconfiguration graph of G is path on n vertices (see Figure 4).

$\square$

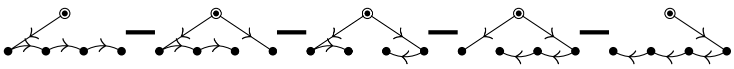


Figure 4: The flip graph of a bidirected 5-cycle

2.2 Unbalanced bipartite flip graphs

Theorem 11. *Let G be a directed graph rooted in r, in which all vertices have indegree at most two. Then the flip graph $\mathcal{F}_r(G)$ is bipartite.*

Proof. We can express the parity of trees as a multiplicative weight, as follows: If xz and yz are two incoming arcs of the same vertex z, we arbitrarily assign weight $w_{wz} = +1$ to one of these arcs and $w_{yz} = -1$ to the other arc. Every arc that is the single incoming arc of a vertex gets weight $+1$. The weight $w(A) \in \{+1, -1\}$ is then the product of the weights of its arcs. Flipping an arc changes the sign, and thus the weight defines a bipartition of the flip graph. $\square$

If the two bipartition classes differ in size by at least one, then this constitutes yet another counterexample of directed graph without Hamiltonian cycles in their flip graph. For example, the graph in Figure 5 has a bipartite flip graph with thirteen vertices, which is shown in Figure 6. We will also see another example with seven arborescences in the proof of Theorem 1, see Figure 8.

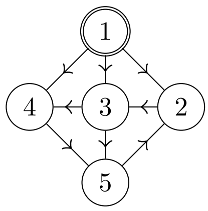


Figure 5: A graph with 13 arborescences

If the two bipartition classes differ in size by at least 2, this would immediately be an obstacle to the existence of a Hamiltonian path in the flip path. However, we can prove that in the flip graph of a directed graph with indegree at most two, the size of the two parts of the bipartition differs by at most 1 (and it is easy to test whether they differ by 1, i.e. the total number of trees is odd).

Theorem 12. *Let G be a directed graph rooted in r, in which all vertices have indegree at most two. The two bipartition classes differ in size by at most 1.*

Proof. Denote $w(A)$ the weight function defined in Theorem 11. We claim that

$$\left|\sum_A w(A)\right| \leq 1.$$

The sum $\sum_A w(A)$ over all arborescences A expresses the difference between the number of "positive" and "negative" trees. This sum can be calculated by the weighted matrix-tree theorem. Let $L = (L_{ij})$ be the weighted Laplace matrix, (or Kirchhoff matrix?) which has entry $L_{ij} = -w_{ij}$ for each arc ij, and $L_{ij} = 0$

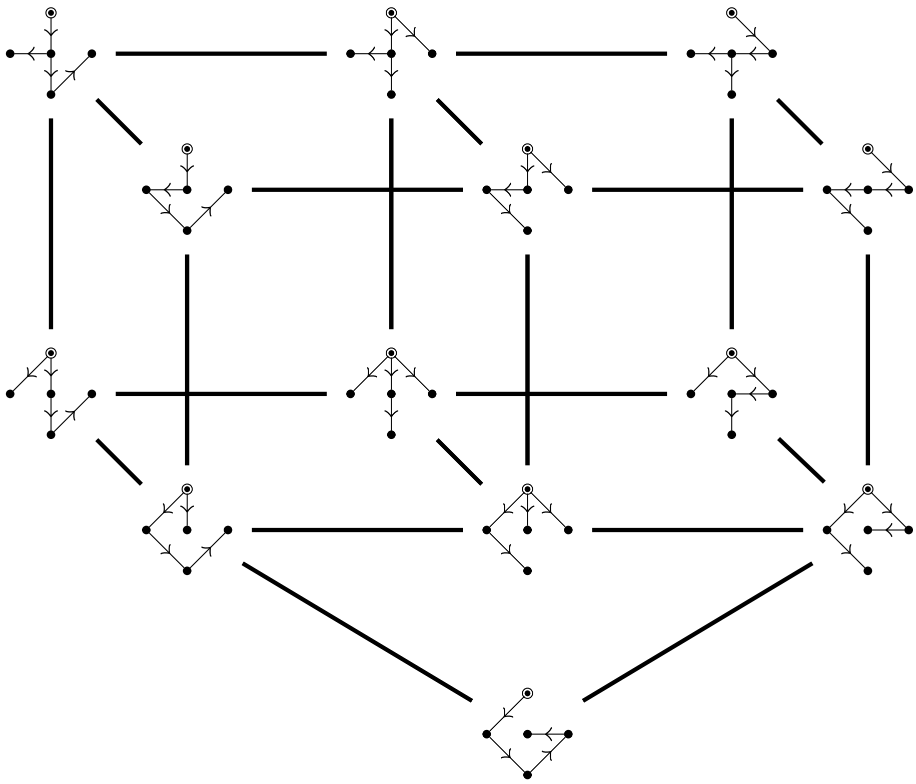


Figure 6: The flip graph of the graph drawn on Figure 5

for all off-diagonal entries that don't correspond to an arc. The diagonal entries L_{ii} are chosen to make the column sums zero. Let $\tilde{L}$ be the matrix L after removing the row and column corresponding to the root. Then

$$\sum_A w(A) = \det \tilde{L}.$$

In our case, for a vertex with two incoming arcs, the corresponding column of L has two entries $+1$ and -1, and hence the diagonal entry in this column, which is supposed to balance the column sum, is zero. For a vertex with a single incoming arc, the corresponding column of L has an entry $+1$, and hence the diagonal entry is -1.

The matrix $\tilde{L}$ has therefore the following properties.

- All entries are 0, $+1$, or -1.
- There are at most two nonzeros per column.
- If a column has two nonzeros, they are of opposite sign.

The matrices arising from network flow problems have the same properties. The Theorem of Heller and Tompkins characterizes the totally unimodular matrices among the matrices with the first two properties, and it implies in this case that $\tilde{L}$ is totally unimodular, and in particular, it has determinant 0 or ± 1.

It is easy to see this directly: If there is a zero column, the determinant is zero. If there is a column with a single nonzero, we expand the determinant by this column, and we obtain a smaller matrix that fulfills the same conditions. Repeating this process, we either find a zero column at some point, or we reduce the matrix to a trivial matrix of size 1×1, or to a matrix where each column contains a $+1$ and a -1. In the first and third cases, $\det \tilde{L} = 0$; in the second case, $\det \tilde{L} = \pm 1$. □

The procedure can easily be carried out combinatorially. Deleting the row of the root means that all neighbours of the root lose an incoming arc. A vertex j with a single incoming arc corresponds to a column j with a single nonzero entry, say L_{ij}. Expanding this column deletes row i and column j. In the graph, this corresponds to removing all edges out of i, which means that the outneighbours of i will have their indegree reduced, and the process can continue.

If the process stops before dismantling the whole graph, the determinant is zero, otherwise it is ± 1.

$$L = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

2.3 Arborescences of degree one in the flip graph

Another property preventing the existence of Hamiltonian cycles in the flip graph is the existence of degree-one vertices, which might occur even when the flip graph is not bipartite (and thus not a path). For example, the graph drawn on Figure 7 is an oriented graph with this property.

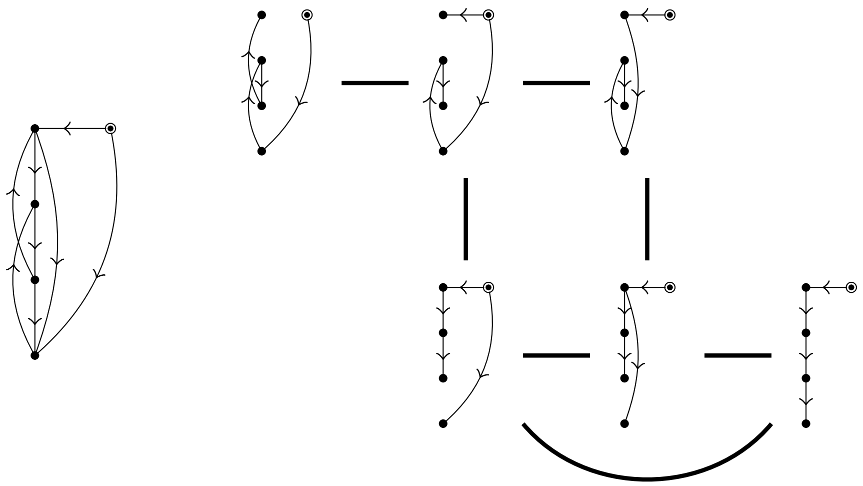


Figure 7: An oriented graph and its flip graph, which contains a degree-one vertex

The degree-one vertices in the flip graph can be characterised as follows:

Observation 13. *Let G be a directed graph rooted in r and A an arborescence of G such that A has degree one in the flip graph $\mathcal{F}_r(G)$, by flipping in some arc uv. Then $G - uv$ is built on A.*

Proof. Let $xy \in E(G - uv) \setminus E(A)$. Assume towards contradiction that $x \notin D_A(y)$. So in A, the arc xy can be flipped in as x is accessible from r without passing through y in A. Thus $xy = uv$, a contradiction. $\square$

3 A Gray code for arborescences in tournaments

We prove the following refinement of Theorem 1:

Theorem 14. *Let G be a directed graph and r be a vertex of G such that the support of $G-r$ is a clique (in particular, G can have arcs in opposite directions). Then the flip graph $\mathcal{F}_r(G)$ is empty or has a Hamiltonian path.*

First, we observe that Theorem 14 is optimal in the sense there exists tournaments with a flip graph that contains a Hamiltonian path but no Hamiltonian cycle. The graph G drawn on Figure 8 with its flip graph is an example of such tournament, and note also that G falls into the conditions of Theorem 11.

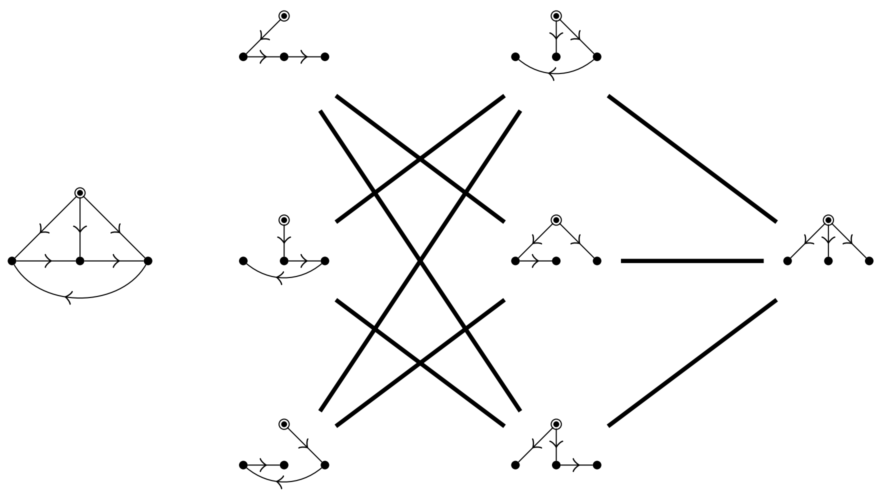


Figure 8: An oriented graph with seven arborescences and whose flip graph is bipartite, so without Hamiltonian cycle.

The rest of this section consists in the proof of Theorem 14. Let $\mathcal{G} = \{(G, r) :$ the support of $G - r$ is a clique$\}$. Note that $\mathcal{G}$ is stable under the following operations:

- deletion of arcs incident to r: $\forall u, (G - ru, r) \in \mathcal{G}$
- contraction of arcs ru incident to r: let G' be the graph obtained from G by removing r and u and replacing them by a vertex w with arcs wx for all $x \in N^+(r) \cup N^+(u)$. Then $(G', w) \in \mathcal{G}$

We proceed by induction. Let $(G, r) \in \mathcal{G}$. The arcs ending in r do not belong to any arborescence, hence we assume without loss of generality that $N^-(r) = \emptyset$. If r has only one outneighbour u, then the arc ru is part of all arborescence of G. Hence, by Lemma 6 $\mathcal{F}_r(G)$ is isomorphic to $\mathcal{F}_u(G-r)$ and by induction both admit a Hamiltonian path or are empty.

If r has at least two outneighbours u and v, then they are connected by an arc, say $u \to v$, because $G - r$ is a tournament. The vertices u and v might also be connected by the arc $v \to u$, but we do not care at this stage of the proof. We choose u and v as follows: among all pairs $(u, v) \in N^+(r)^2$ with $uv \in G$, we choose one such that u has an outneighbourhood of maximal size.

The set of arborescences of G can be partitioned into four *types* of arborescences: those that do not contain the arc $e = ru$, those that contain ru and $f = rv$, those that contain ru and $g = uv$ and those that ru but neither rv nor rv. These types partition $\mathcal{F}_r(G)$ into four subsets, that we will denote $\mathcal{T}_{-e}$, $\mathcal{T}_{/e/f}$, $\mathcal{T}_{/e/g}$ and $\mathcal{T}_{-f-g/e}$ respectively.

3.1 Structure of each type

Lemma 15. *If the flip graph of G rooted in r is non-empty, the different types of arborescences induce the following structures in $\mathcal{F}_r(G)$:*

- *$\mathcal{F}_r(G)[\mathcal{T}_{-e}]$ is empty or contains a Hamiltonian path P_{-e},*
- *$\mathcal{F}_r(G)[\mathcal{T}_{-f-g/e}]$ is empty or contains a Hamiltonian path $P_{-f-g/e}$,*
- *$\mathcal{F}_r(G)[\mathcal{T}_{/e/f} \cup \mathcal{T}_{/e/g}]$ contains a spanning ladder.*

Proof. As a direct application of Observation 5, we get that $\mathcal{F}_r(G)[\mathcal{T}_{-e}]$ is isomorphic to $\mathcal{F}_r(G-e)$, so by induction $\mathcal{F}_r(G)[\mathcal{T}_{-e}]$ is empty or also admits a Hamiltonian path P_{-e}.

Using Observation 5 and Lemma 6, we get that $\mathcal{F}_r(G)[\mathcal{T}_{-f-g/e}]$ is isomorphic to $\mathcal{F}_w(H)$, where H is obtained from G by contracting e into w and deleting wv. By induction, $\mathcal{F}_r(G)[\mathcal{T}_{-f-g/e}]$ is empty or admits a Hamiltonian path $P_{-f-g/e}$.

Finally, note that $\mathcal{F}_r(G)[\mathcal{T}_{/e/f}]$ and $\mathcal{F}_r(G)[\mathcal{T}_{/e/g}]$ are isomorphic via flipping the arc f into the arc g. They are also isomorphic to $\mathcal{F}_w(H)$, where H is obtained from G by contracting e and g into w. Moreover, note that $\mathcal{F}_r(G)[\mathcal{T}_{/e/f} \cup \mathcal{T}_{/e/g}]$ is not empty, because H admits a arborescence if and only G does. Combining all these arguments with Lemma 6, we deduce that $\mathcal{F}_r(G)[\mathcal{T}_{/e/f} \cup \mathcal{T}_{/e/g}]$ contains a spanning ladder (see Figure 9).

□

3.2 Assembling the pieces

Proof of Theorem 14. We first argue that one can assume without loss of generality that $\mathcal{T}_{-f-g/e}$ and $\mathcal{T}_{-e}$ are non-empty. If both are empty, then by Lemma 15 $\mathcal{F}_r(G) = \mathcal{F}_r(G)[\mathcal{T}_{/e/f} \cup \mathcal{T}_{/e/g}]$ contains a spanning ladder and Theorem 14 follows directly from Lemma 3. If $\mathcal{T}_{-f-g/e}$ is empty and $\mathcal{T}_{-e}$ is not, then $\mathcal{F}_r(G)[\mathcal{T}_{-e}]$ contains a Hamiltonian path P_{-e} by Lemma 15. By flipping in e in one of the extremities of P_{-e}, one obtains a arborescence in $\mathcal{T}_{/e/f} \cup \mathcal{T}_{/e/g}$. By Lemma 15 combined with Lemma 3, $\mathcal{F}_r(G)[\mathcal{T}_{/e/f} \cup \mathcal{T}_{/e/g}]$ contains a Hamiltonian cycle, which can be used to extend P_{-e} into a Hamiltonian path of $\mathcal{F}_r(G)$. Similarly, we can assume that $\mathcal{T}_{-e}$ is non-empty.

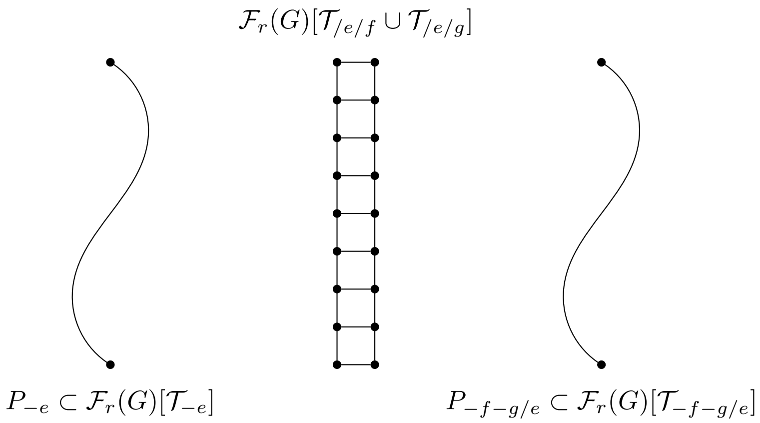


Figure 9: Spanning structures within the different types of arborescences.

For A in $\mathcal{T}_{-f-g/e}$, let A' and A'' be the arborescences in $\mathcal{T}_{/e/f}$ (respectively $\mathcal{T}_{/e/g}$) obtained from A by flipping in f (respectively g). Note that for $A_1, A_2 \in \mathcal{T}_{-f-g/e}$, if $A'_1 = A'_2$, then A_1 and A_2 differ by a flip of an arc entering v.

Assume for now that the extremities A_1 and A_2 of $P_{-f-g/e}$ have distinct A'_i. By Lemma 3, $\mathcal{F}_r(G)[\mathcal{T}_{/e/f} \cup \mathcal{T}_{/e/g}]$ contains a Hamiltonian path going from A'_1 to A'_2 or A''_2. Along with $P_{-f-g/e}$ (obtained from Lemma 15), one obtains a Hamiltonian cycle of $\mathcal{F}_r(G)[\mathcal{T}_{/e/f} \cup \mathcal{T}_{/e/g} \cup \mathcal{T}_{-f-g/e}]$. Finally, all arborescences in $\mathcal{T}_{-e}$ are adjacent to some arborescence in $\mathcal{T}_{/e/f} \cup \mathcal{T}_{/e/g} \cup \mathcal{T}_{-f-g/e}$ by flipping in the edge e, so $\mathcal{F}_r(G)$ contains a Hamiltonian path starting in $\mathcal{T}_{-e}$ (see Figure 10).

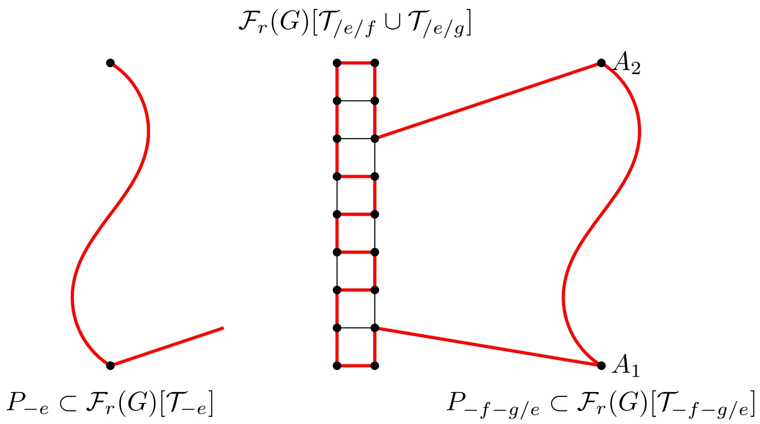


Figure 10: Hamiltonian path if $A'_1 \neq A'_2$.

Thus we can assume without loss of generality that the extremities A_1 and A_2 of $P_{-f-g/e}$ have $A'_1 = A'_2$ (and $A''_1 = A''_2$). So A_1 and A_2 only differ on one arc entering v, so the path $P_{-f-g/e}$ from Lemma 15 is in fact a Hamiltonian cycle of $\mathcal{F}_r(G)[\mathcal{T}_{-f-g/e}]$. Either $A' = A'_1$ for all arborescences in $\mathcal{T}_{-f-g/e}$ or there are two arborescences A and B, consecutive on $P_{-f-g/e}$, such that $A' \neq B'$. In the second case, $\mathcal{F}_r(G)[\mathcal{T}_{-f-g/e}]$ contains a Hamiltonian path whose extremities are A and B, so we are back in the case of the previous paragraph.

Thus we can assume that for any $A \in \mathcal{T}_{-f-g/e}$, we have $A' = A'_1$ and $A'' = A''_1$. By Lemma 9 applied to G/e with w being the contraction or r with u and $v := v$, this implies that G/e has the structure described in Item 4 for some k and n and that $\mathcal{F}_r(G)[\mathcal{T}_{-f-g/e}]$ is a clique (by Item 1).

For $B \in \mathcal{F}_r(G)[\mathcal{T}_{-e}]$, let $\tilde{B}$ be the arborescence of $\mathcal{F}_r(G)[\mathcal{T}_{/e/f} \cup \mathcal{T}_{/e/g} \cup \mathcal{T}_{-f-g/e}]$ obtained by flipping in e. Let B_1 and B_2 be the extremities of P_{-e}. We now do a case analysis on the types of $\tilde{B}_1$ and $\tilde{B}_2$.

3.2.1 $\tilde{B}_1$ or $\tilde{B}_2$ belongs to $\mathcal{F}_r(G)[\mathcal{T}_{-f-g/e}]$.

Then $\mathcal{F}_r(G)$ contains a Hamiltonian path that first visits $\mathcal{F}_r(G)[\mathcal{T}_{-e}]$ via P_{-e}, then all the vertices of $\mathcal{F}_r(G)[\mathcal{T}_{-f-g/e}]$ and finally those of $\mathcal{F}_r(G)[\mathcal{T}_{/e/f} \cup \mathcal{T}_{/e/g}]$ (see Figure 11).

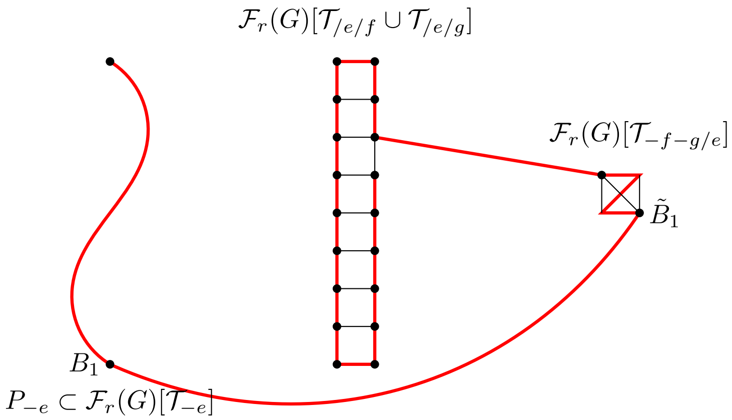


Figure 11: Hamiltonian path if $\tilde{B}_1 \in \mathcal{F}_r(G)[\mathcal{T}_{-f-g/e}]$.

3.2.2 $\tilde{B}_1$ or $\tilde{B}_2$ belongs to $\mathcal{F}_r(G)[\mathcal{T}_{/e/f} \cup \mathcal{T}_{/e/g}] \setminus \{A', A''\}$.

Without loss of generality, assume that $\tilde{B}_1$ belongs to $\mathcal{F}_r(G)[\mathcal{T}_{/e/f} \cup \mathcal{T}_{/e/g}] \setminus \{A', A''\}$. By Lemma 3, $\mathcal{F}_r(G)[\mathcal{T}_{/e/f} \cup \mathcal{T}_{/e/g}]$ contains a Hamiltonian path starting at $\tilde{B}_1$ and ending at A' or A''. As a result, $\mathcal{F}_r(G)$ contains a Hamiltonian path visiting first P_{-e}, then $\mathcal{F}_r(G)[\mathcal{T}_{/e/f} \cup \mathcal{T}_{/e/g}]$ and finally $\mathcal{F}_r(G)[\mathcal{T}_{-f-g/e}]$ (see Figure 12).

3.2.3 $\tilde{B}_1$ and $\tilde{B}_2$ belong to $\{A', A''\}$.

Recall that G/e is of the form described by Item 4 of Lemma 9 because $A' = A'_1$ and $A'' = A''_1$ for all $A \in \mathcal{T}_{-f-g/e}$. Recall also that w is the vertex of G/e resulting from the contraction of r and u. We say that a vertex has *depth* i if it lies at distance i from w in $G - f - g/e$ (that is at distance i from $\{r, u\}$ in $G - f - g$). We label the vertices in $V(G) \setminus \{r, u, v\}$ as in Lemma 9: let v_i be the only vertex of $V(G - f - g/e) \setminus \{w, v\}$ at depth i in the BFS starting from the root w in $G - f - g/e$.

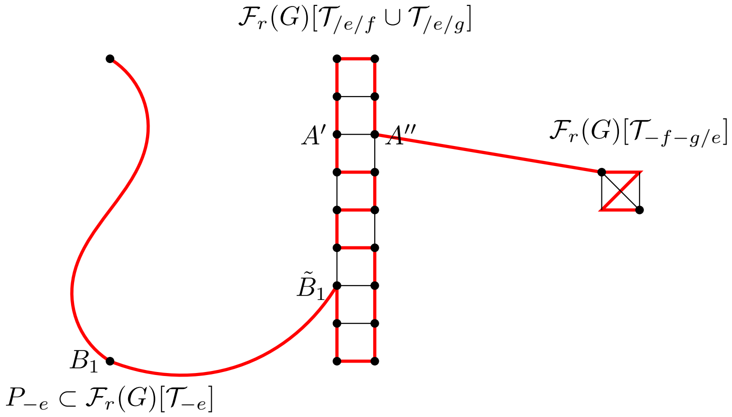


Figure 12: Hamiltonian path if $\tilde{B}_1 \in \mathcal{F}_r(G)[\mathcal{T}_{/e/f} \cup \mathcal{T}_{/e/g} \setminus \{A', A''\}]$.

Case 1: $\tilde{B}_1 \neq \tilde{B}_2$. Without loss of generality, assume that $\tilde{B}_1 = A'$ and $\tilde{B}_2 = A''$.

Claim 15.1. *The outneighbourhood of r in G is composed of u at depth 0, v_1 at depth one, and v at depth at least two.*

Proof of Claim. By definition, u is an outneighbour of r and has depth 0 (and thus corresponds to w in Figure 3). By Item 3 of Lemma 9, the depth of v is at least two. On the other hand, other outneighbours of r in G are outneighbours of w in $G/e - f - g$ and thus have depth one, so $\{u, v\} \subseteq N^+(r) \subseteq \{u, v_1, v\}$. If v_1 is not an outneighbour of r in G, then r has outdegree 1 in G_{-e}, so all arborescences in $\mathcal{T}_{-e}$ contain the edge $f = rv$. Hence $\tilde{B}_1$ and $\tilde{B}_2$ both belong to $\mathcal{T}_{/e/f}$ so $\tilde{B}_1 = \tilde{B}_2 = A'$ which is a contradiction. ∎

Assume that $uv_1 \notin E(G)$. Let x be an outneighbourhood of u in G different from v. We have $x \in N^+_{G-f-g/e}(w)$, hence x has depth one, that is $x = v_1$, a contradiction. So $N^+_G(u) = \{v\}$. Recall that u was chosen such that $N^+(u)$ is maximal among all couples $(u, v) \in N^+(r)^2$ with $uv \in E(G)$, so $N^+(v)$ and $N^+(v_1)$ have size at most one. As the support of $G \setminus \{r\}$ is a clique, this implies that G is a directed triangle on the vertices u, v, v_1 with r an inneighbour of all three vertices. So the only inneighbours of v are r and u, which implies that $\mathcal{T}_{-f-g/e}$ is empty, a contradiction (moreover, note this graph is the graph drawn on Figure 8 and has a Hamiltonian path in its flip graph).

So we can assume that $uv_1 \in E(G)$ and thus v_1 is an outneighbour of both r and u. Consider the arborescence A of $G - f - g/e$ obtained by performing a BFS staring at w. There are two arborescences A_1 and A_2 of G that are mapped to A when contracting e: one contains the arcs ru and rv_1, the other contains the arcs ru and uv_1. So both of them contain e but contain neither f nor g. However, flipping f in them results in two different arborescences: as $v \neq v_1$, the arc uv_1 is contained in A'_1 but not in A'_2. This contradicts the fact that A' is constant over $\mathcal{T}_{-f-g/e}$, so $\tilde{B}_1 = \tilde{B}_2$.

Case 2: $\tilde{B}_1 = \tilde{B}_2$. Since $\tilde{B}_1 = \tilde{B}_2$, the arborescences B_1 and B_2 are adjacent and P_{-e} is in fact a Hamiltonian cycle. We now proceed similarly as after the deduction that $\mathcal{F}_r(G)[\mathcal{T}_{-f-g/e}]$ contains a Hamiltonian cycle. Either $\tilde{B} = \tilde{B}_1$ for all arborescences in $\mathcal{T}_{-e}$ or there are two arborescences B_3 and B_4, consecutive on P_{-e}, such that $\tilde{B}_3 \neq \tilde{B}_4$. In the second case, $\mathcal{F}_r(G)[\mathcal{T}_{-e}]$ contains a Hamiltonian path whose extremities are B_3 and B_4, so we are back in the case of one of the previous sections (Section 3.2.2 or Section 3.2.3) or in the case of the previous paragraph.

Thus we can assume that that for any $B \in \mathcal{T}_{-e}$, we have $\tilde{B} = \tilde{B}_1 \in \{A', A''\}$ and thus that $\mathcal{F}_r(G)[\mathcal{T}_{-e}]$ is a clique. Lemma 9 can now be applied to G with $w := r$ and $v := u$ and to G/e with $w := w$ and $v := v$ to describe precisely the structure of G.

Claim 15.2. *The outneighbourhood of r in G consists of u and v, which has depth three. Moreover, all arborescences of $\mathcal{T}_{-e}$ contain f.*

Proof of Claim. We proceed similarly as in Claim 15.1. By Item 4 of Lemma 9 applied to G/e with $w := w$ and $v := v$, we have $\{u, v\} \subseteq N_G^+(r) \subseteq N_{G-f-g/e}^+(w) \cup \{u\} \subseteq \{u, v, v_1\}$ and u has depth 0 while v has depth at least 2 (by Item 3).

If v has depth at least four, then by Item 4 of Lemma 9 applied to $G_{/e}$ with $w := w$ and $v := v$, the graph $G_{-f-g/e}$ has two paths from the root w to v_1: w, v, v_1 and w, v, v_2, v_3, v_1. Since $v \in N_G^+(r)$, the graph $G - e$ also contains two paths from the root r to v_1: r, v, v_1 and r, v, v_2, v_3, v_1. By Item 2 of Lemma 9 applied to G with $w := r$ and $u := v$, this contradicts the fact that $\tilde{B}$ is constant for $B \in \mathcal{T}_{-e}$. So the depth of v is either two or three.

As f can always be flipped in (because it is incident to the root), one arborescence of $\mathcal{T}_{-e}$ contains f, which implies that for all $B \in \mathcal{T}_{-e}$, $\tilde{B} = A'$. So all arborescences of $\mathcal{T}_{-e}$ contain f, because flipping in e results in A'. So v_1 is not an outneighbour of r, otherwise the path $r, v_1, \ldots, v_{k-1}, v$ can be completed into a arborescence that contains neither e nor f, a contradiction, which proves that $N_G^+(r) = \{u, v\}$.

We can now prove that v has depth exactly three. Assume towards contradiction that v has depth two. Then, G/e is of the form described by Item 4a of Lemma 9 for some n (see Figure 3a), otherwise v has no outneighbours and since r has only u and v as outneighbours, $\mathcal{T}_{-e}$ is empty, a contradiction.

First, assume that n (the maximal depth in G) is equal to two. So by Item 4a of Lemma 9 applied to G/e, G is one of the graphs drawn on Figure 13. In fact,

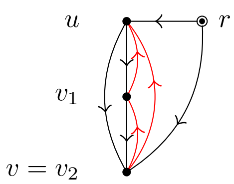


Figure 13: The graph G if v has depth two and $n = 2$. The black arcs are contained in G, while the red arcs are optionnal.

G cannot contain the arc $v \to v_1$, otherwise G contains two paths from r to v_1 (r, u, v_1 and r, u, v, v_1) which contradicts Item 2 of Lemma 9 applied to G with $w := r$ and $v := u$. So G must contain the arc $v \to v_2$, otherwise v is a sink of G, so $\mathcal{T}_{-e}$ is empty, a contradiction. Finally, v_1 has indegree 1, so the arc $v_1 \to u$ is part of no arborescence and by Observation 5, we can assume that G does not contain $v_1 \to u$. There is only one remaining possibility for G and its flip graph contains a Hamiltonian path, as one can see on Figure 14.

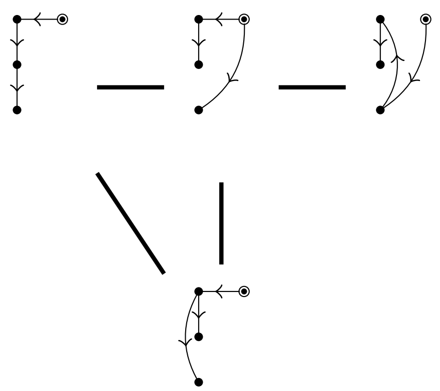


Figure 14: The flip graph of G if v has depth 2 and $n = 2$

So we can assume that the maximal depth n is at least three and recall that v has depth two. Recall also that $G_{/e}$ is of the form described by Item 4a of Lemma 9. Since $N^+_G(r) = \{u, v\}$, this implies that $uv_1 \in E(G)$. So there are two paths from r to v_1: r, v, v_3, v_1 and r, v, v_3, u, v_1), which contradicts Item 2 of Lemma 9 applied to $G - e$ with $w := r$ and $v := u$. So v has depth exactly three, which concludes the proof of Claim 15.2. ■

We first recall what we know about the outneighborhoods of r, u, v and v_1. We have $N^+(r) = \{u, v\}$ by Claim 15.2, $N^+(u) = \{v_1, v\}$ because v_1 has depth 1 and $v_1 \notin N^+(r)$. By Item 4 of Lemma 9 applied to G/e with $w := w$ and $v := v$, we have $\{v_1\} \subseteq N^+(v) \subseteq \{u, v_1, v_2\}$ and $N^+(v_1) = \{v_2\}$. So none of the vertices in $N^+(\{r, u, v, v_1\}) \cap (\{v_i : i \geq 3\} \setminus \{v\}) = \emptyset$ and in any arborescence of $\mathcal{T}_{/e/g}$, there are only two possibilies for the path from r to v_2: either r, u, v, v_2 or r, u, v, v_1, v_2. By Item 4 of Lemma 9 applied to G/e with $w := w$ and $v := v$ again, for all $i \geq 3$, $N^-(v_i) \cap (\{r, u, v\} \cup \{v_j : j < i\} = \{v_{i-1}\}$, a straightforward induction shows in any arborescence of $\mathcal{T}_{/e/g}$, there are only two possibilities for the path from r to v_i: $r, u, v, v_2, \ldots, v_i$ or $r, u, v, v_1, v_2, \ldots, v_i$. As a result, there are at most two arborescences in $\mathcal{T}_{/e/g}$. This implies that $\mathcal{F}_r(G)[\mathcal{T}_{/e/f} \cup \mathcal{T}_{/e/g}]$ is either an edge or a four cycle. In either case, there is a Hamiltonian path $\mathcal{F}_r(G)$, that first visits $\mathcal{T}_{-e}$, then $\mathcal{T}_{/e/f}$ (because by Claim 15.2 all arborescences of $\mathcal{T}_{-e}$ contain f, so flipping in e gives A'), then $\mathcal{T}_{/e/g}$ and finally $\mathcal{T}_{-f-g/e}$.

□

Acknowledgements

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