Jacobian of n-DOF End-Effectors

johnlime

April 28, 2022

Problem . Obtain the Jacobian for the end effector of a 2-dimensional n-DOF arm (`IKPigeon`).

Proof. Let the position of the end effector of the arm be (x, y).

$$x = -\sum_{i}^{n} C_i \cos\left(\sum_{j}^{i} \theta_j\right) \tag{1}$$

$$y = -\sum_{i}^{n} C_i \sin\left(\sum_{j}^{i} \theta_j\right) \tag{2}$$

where C_i is the length of the limb assigned to the ith joint, and $/theta_i$ is its angle.

The Jacobian matrix J is composed of the gradient of each of the positions relative to θ for all joints.

$$J = \begin{bmatrix} \frac{\partial x}{\partial \theta_0} & \frac{\partial x}{\partial \theta_1} & \cdots & \frac{\partial x}{\partial \theta_n} \\ \frac{\partial y}{\partial \theta_0} & \frac{\partial y}{\partial \theta_1} & \cdots & \frac{\partial y}{\partial \theta_n} \end{bmatrix} \tag{3}$$

$\frac{\partial x}{\partial \theta_l}$ and $\frac{\partial y}{\partial \theta_l}$ must be calculated for each joint l.

We first calculate the former gradient.

$$\begin{aligned}
\frac{\partial x}{\partial \theta_l} &= \frac{\partial}{\partial \theta_l}\left(-\sum_{i}^{l-1} C_i \cos\left(\sum_{j}^{i} \theta_j\right) - \sum_{l}^{n} C_i \cos\left(\theta_l + \sum_{j \neq l}^{i} \theta_j\right)\right) \\
&= \frac{\partial}{\partial \theta_l}\left(-\sum_{l}^{n} C_i \cos\left(\theta_l + \sum_{j \neq l}^{i} \theta_j\right)\right) && \because \frac{\partial x}{\partial \theta_l} \cos\left(\sum_{j}^{i} \theta_j\right) = 0 \text{ when } i < l \\
&= -\frac{\partial}{\partial \theta_l}\left(\sum_{l}^{n} C_i \left(\cos\theta_l \cos\left(\sum_{j \neq l}^{i} \theta_j\right) + \sin\theta_l \sin\left(\sum_{j \neq l}^{i} \theta_j\right)\right)\right) \\
&= -\sum_{l}^{n} C_i \frac{\partial}{\partial \theta_l}\left(\cos\theta_l \cos\left(\sum_{j \neq l}^{i} \theta_j\right) + \sin\theta_l \sin\left(\sum_{j \neq l}^{i} \theta_j\right)\right) \\
&= -\sum_{l}^{n} C_i \left(\left(\left(\frac{\partial}{\partial \theta_l} \cos\theta_l\right) \cos\left(\sum_{j \neq l}^{i} \theta_j\right) + \cos\theta_l \left(\frac{\partial}{\partial \theta_l} \cos\left(\sum_{j \neq l}^{i} \theta_j\right)\right)\right)\right. \\
&\quad + \left.\left(\left(\frac{\partial}{\partial \theta_l} \sin\theta_l\right) \sin\left(\sum_{j \neq l}^{i} \theta_j\right) + \sin\theta_l \left(\frac{\partial}{\partial \theta_l} \sin\left(\sum_{j \neq l}^{i} \theta_j\right)\right)\right)\right)
\end{aligned}$$

Since $\frac{\partial}{\partial \theta_l} \cos\left(\sum_{j \neq l}^{i} \theta_j\right) = 0$ and $\frac{\partial}{\partial \theta_l} \sin\left(\sum_{j \neq l}^{i} \theta_j\right) = 0$,

$$
\begin{aligned}
\frac{\partial x}{\partial \theta_l} &= -\sum_{l}^{n} C_i \left(\left(\left(\frac{\partial}{\partial \theta_l} \cos\theta_l \right) \cos\left(\sum_{j \neq l}^{i} \theta_j \right) \right) + \left(\left(\frac{\partial}{\partial \theta_l} \sin\theta_l \right) \sin\left(\sum_{j \neq l}^{i} \theta_j \right) \right) \right) \\
&= -\sum_{l}^{n} C_i \left((-\sin\theta_l) \cos\left(\sum_{j \neq l}^{i} \theta_j \right) + (\cos\theta_l) \sin\left(\sum_{j \neq l}^{i} \theta_j \right) \right) \\
&= \sum_{l}^{n} C_i \left(\sin\theta_l \cos\left(\sum_{j \neq l}^{i} \theta_j \right) - \cos\theta_l \sin\left(\sum_{j \neq l}^{i} \theta_j \right) \right)
\end{aligned}
$$

Therefore, the former gradient $\frac{\partial x}{\partial \theta_l}$ can be expressed as the following.

$$
\frac{\partial x}{\partial \theta_l} = \sum_{l}^{n} C_i \left(\sin\theta_l \cos\left(\sum_{j \neq l}^{i} \theta_j \right) - \cos\theta_l \sin\left(\sum_{j \neq l}^{i} \theta_j \right) \right) \tag{4}
$$

Similarly, the latter gradient $\frac{\partial y}{\partial \theta_l}$ can be expressed as the following.

$$
\frac{\partial y}{\partial \theta_l} = \sum_{l}^{n} C_i \left(\sin\theta_l \sin\left(\sum_{j \neq l}^{i} \theta_j \right) - \cos\theta_l \cos\left(\sum_{j \neq l}^{i} \theta_j \right) \right) \tag{5}
$$

□