Example : Segment the red region

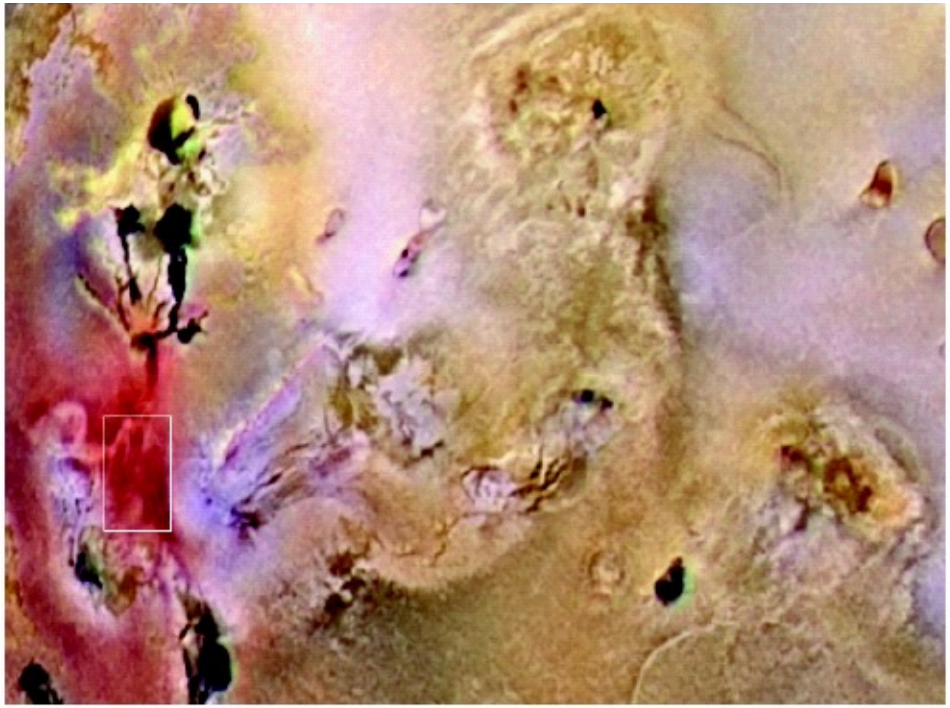

FIGURE 6.44 Segmentation in RGB space. (a) Original image with colors of interest shown enclosed by a rectangle.

Example: Segment the red region

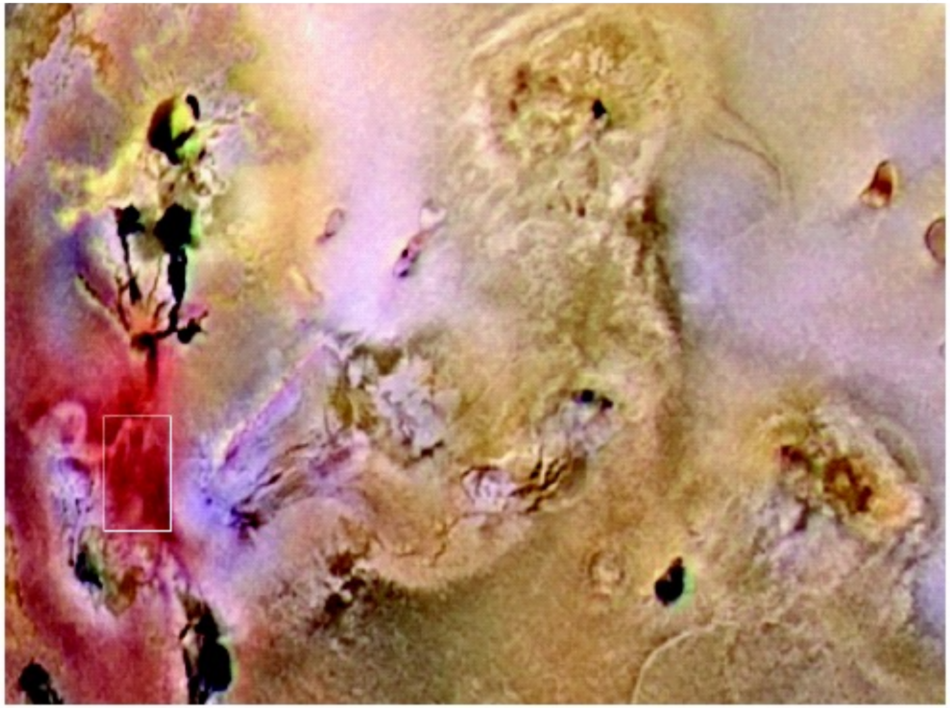

FIGURE 6.44 Segmentation in RGB space. (a) Original image with colors of interest shown enclosed by a rectangle.

Learning: For the pixels $z = [z_r, z_g, z_b]^T$ inside the box compute

mean vector: $[\mu_r, \mu_g, \mu_b]^T$

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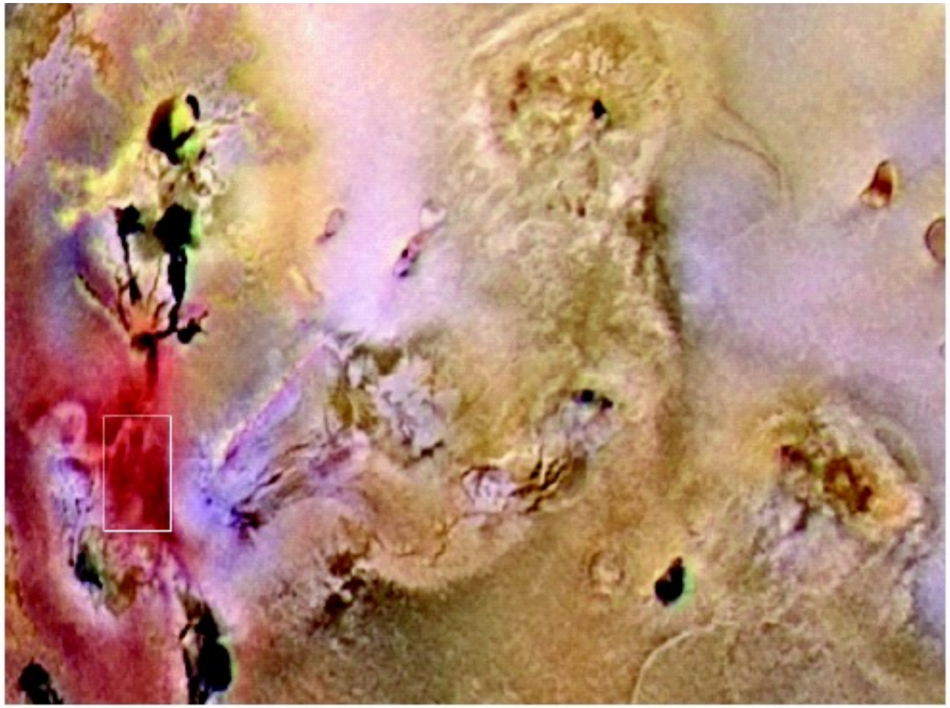

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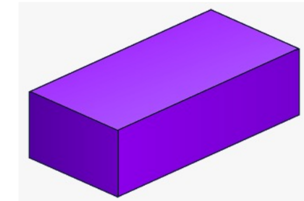
Learning: For the pixels $z = [z_r, z_g, z_b]^T$ inside the box compute

mean vector: $[\mu_r, \mu_g, \mu_b]^T$

standard deviations:

$$\sigma_r = \sqrt{E[(z_r - \mu_r)^2]} \qquad \sigma_g = \sqrt{E[(z_g - \mu_g)^2]} \qquad \sigma_b = \sqrt{E[(z_b - \mu_b)^2]}$$

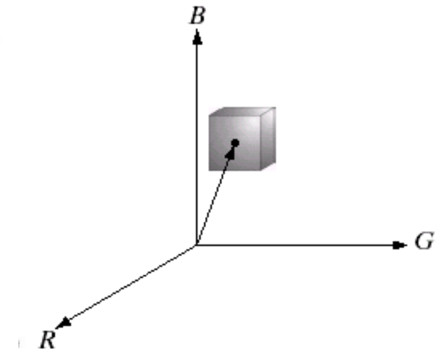
Segmentation: For every pixel $a = [a_r, a_g, a_b]^T$ in the image label a as similar to the pixels in the box if



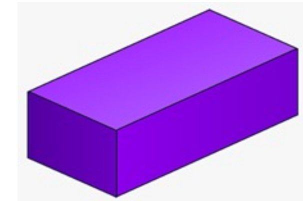
$$\mu_r - 1.25\sigma_r \le a_r \le \mu_r + 1.25\sigma_r \quad \text{AND}$$

$$\mu_g - 1.25\sigma_g \le a_g \le \mu_g + 1.25\sigma_g \quad \text{AND}$$

$$\mu_b - 1.25\sigma_b \le a_b \le \mu_b + 1.25\sigma_b$$



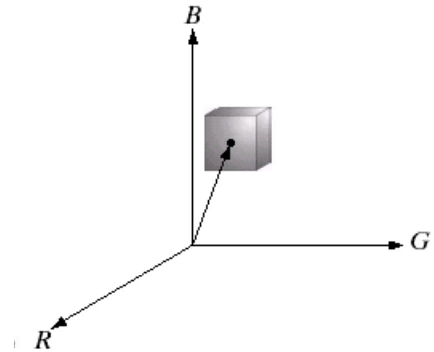
Segmentation: For every pixel $a = [a_r, a_g, a_b]^T$ in the image label a as similar to the pixels in the box if



$\mu_r - 1.25\sigma_r \leq a_r \leq \mu_r + 1.25\sigma_r$ AND

$\mu_g - 1.25\sigma_g \leq a_g \leq \mu_g + 1.25\sigma_g$ AND

$\mu_b - 1.25\sigma_b \leq a_b \leq \mu_b + 1.25 \cdot \sigma_b$



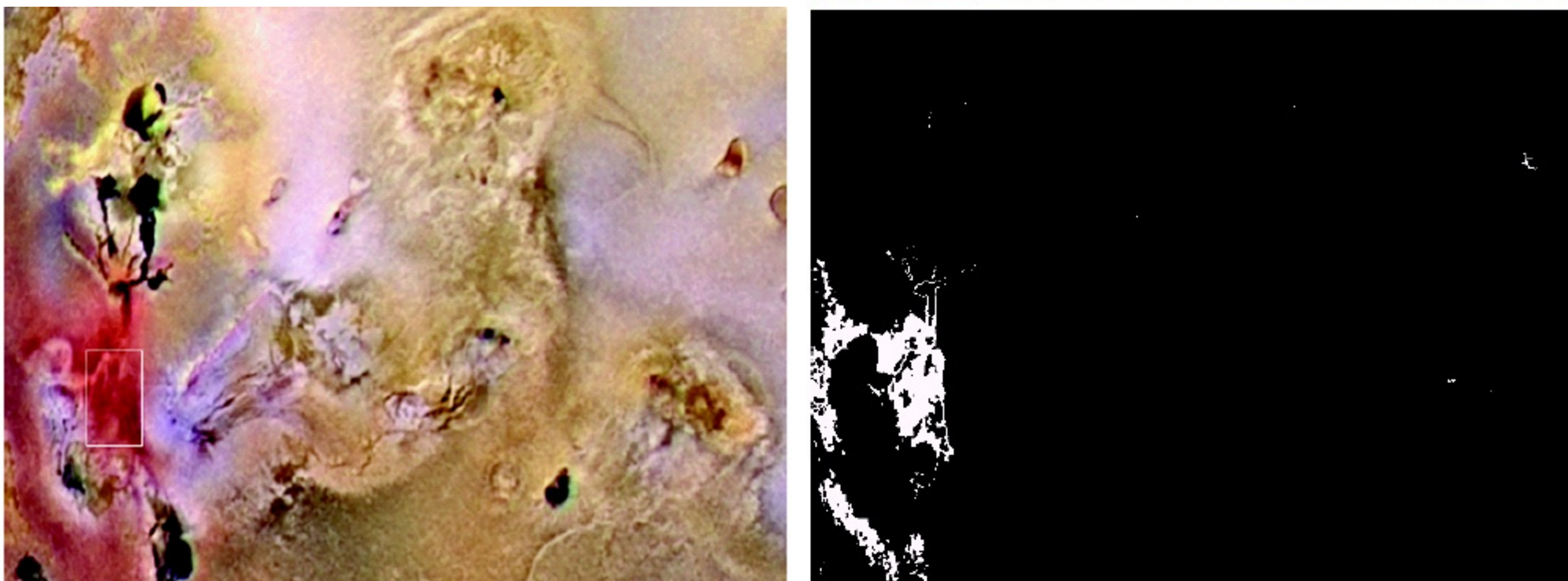


FIGURE 6.44

Noise in Color Images

$$R(x,y) = R_I(x,y) + N_R(x,y)$$
$$G(x,y) = G_I(x,y) + N_G(x,y)$$
$$B(x,y) = B_I(x,y) + N_B(x,y)$$

$R(x,y)$ is noisy red band, $G(x,y)$ is noisy green band, $B(x,y)$ is noisy blue band

$R_I(x,y)$ is ideal noise-free red band

$G_I(x,y)$ is ideal noise-free green band

$B_I(x,y)$ is ideal noise-free blue band

$N_R(x,y)$ is additive noise for red band

$N_G(x,y)$ is additive noise for green band

$N_B(x,y)$ is additive noise for blue band

a b
c d

FIGURE 6.48 (a)–(c) Red, green, and blue component images corrupted by additive Gaussian noise of mean 0 and variance 800. (d) Resulting RGB image. [Compare (d) with Fig. 6.46(a).]

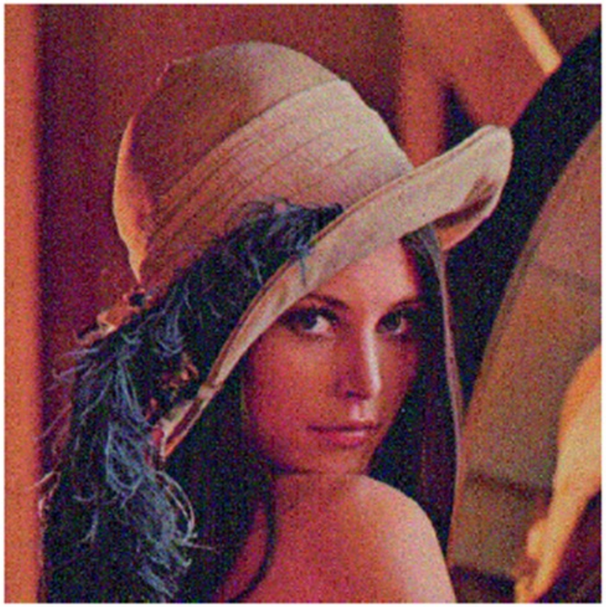


FIGURE 6.48

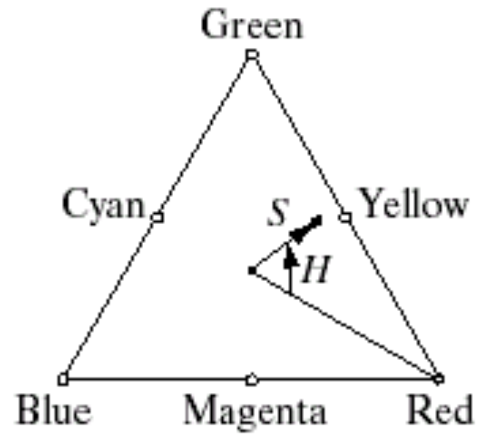


FIGURE 6.13

a b c

FIGURE 6.49 HSI components of the noisy color image in Fig. 6.48(d). (a) Hue. (b) Saturation. (c) Intensity.

a	b
c	d

FIGURE 6.50 (a) RGB image with green plane corrupted by salt-and-pepper noise. (b) Hue component of HSI image. (c) Saturation component. (d) Intensity component.

Principal Component Analysis (PCA)

Consider a multispectral image with B bands and N pixels where each pixel can be represented by the vector

$$f = [f_1 \; f_2 \cdots f_B]^T$$

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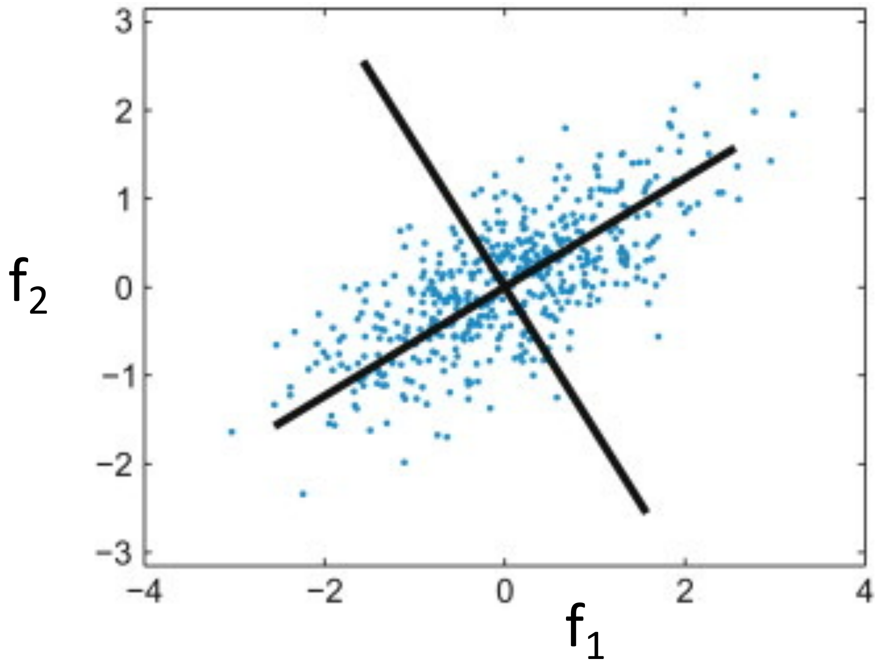
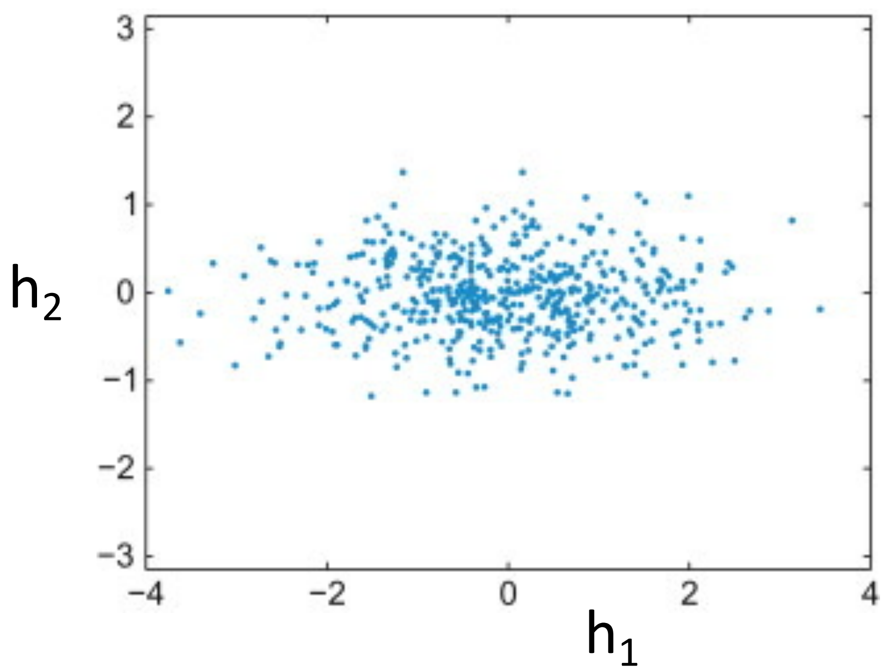
The goal of PCA is to define a single linear transform of each pixel f in an image to generate a new set of B bands that are uncorrelated and sorted according to the variance of the pixels in each band.

PCA can be used to reduce the number of bands employed to represent an image and may reveal interesting features in the image.


f$_2$
f$_1$


h$_2$
h$_1$

Finding the PCA Transform

1. Find a unit vector $e_1 = [e_{11} \ e_{12} \ \cdots \ e_{1B}]^T$ that maximizes the variance of the N pixels in the new band $h_1 = e_{11} f_1 + e_{12} f_2 + \cdots + e_{1B} f_B$

Finding the PCA Transform

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2. Find a unit vector $e_2 = [e_{21}\ e_{22}\ \cdots\ e_{2B}]^T$ that maximizes the variance of the N pixels in the new band $\quad h_2 = e_{21} f_1 + e_{22} f_2 + \cdots + e_{2B} f_B$

and so that h_1 and h_2 are uncorrelated (or equivalently covariance $[h_1, h_2] = 0$)

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   and so that h_1 and h_2 are uncorrelated (or equivalently covariance $[h_1, h_2] = 0$)

3. Continue to find unit vectors $e_3, e_4, \cdots, e_B$ so that $h_1, h_2, \cdots, h_B$ are uncorrelated and the variance is maximized at each step.

Solution

$e_1, e_2, \ldots, e_B$ are the eigenvectors of the $B \times B$ covariance matrix of the N pixels f. The eigenvectors are sorted so that the corresponding eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_B$ satisfy $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_B$

Solution

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PCA transforms each pixel $f = [f_1\ f_2 \cdots f_B]^T$ into a new pixel $h = [h_1\ h_2 \cdots h_B]^T$ according to

$$\begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_B \end{bmatrix} = \underbrace{\begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1B} \\ e_{21} & e_{22} & \cdots & e_{2B} \\ \vdots & & & \vdots \\ e_{B1} & e_{B2} & \cdots & e_{BB} \end{bmatrix}}_{E} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_B \end{bmatrix}$$

Solution

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$$\text{variance}[h_i] = \lambda_i$$

An orthogonal matrix is a square matrix with rows and columns that are orthogonal unit vectors (orthonormal vectors)

If E is an orthogonal matrix then $E^{-1} = E^{T}$

$h = Ef \quad f = E^{-1}h \quad f = E^{T}h$

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The transform preserves the total variance

$$\sum_{i=1}^{B} \text{variance}[h_i] = \sum_{i=1}^{B} \text{variance}[f_i]$$

and moves as much variance as possible to the h_i bands for small i.

Ex $B = 2$ bands
$N = 6$ pixels

pixel	f_1	f_2
1	2	2
2	2	3
3	3	4
4	4	3
5	5	4
6	5	5

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$N = 6$ pixels

pixel	f_1	f_2
1	2	2
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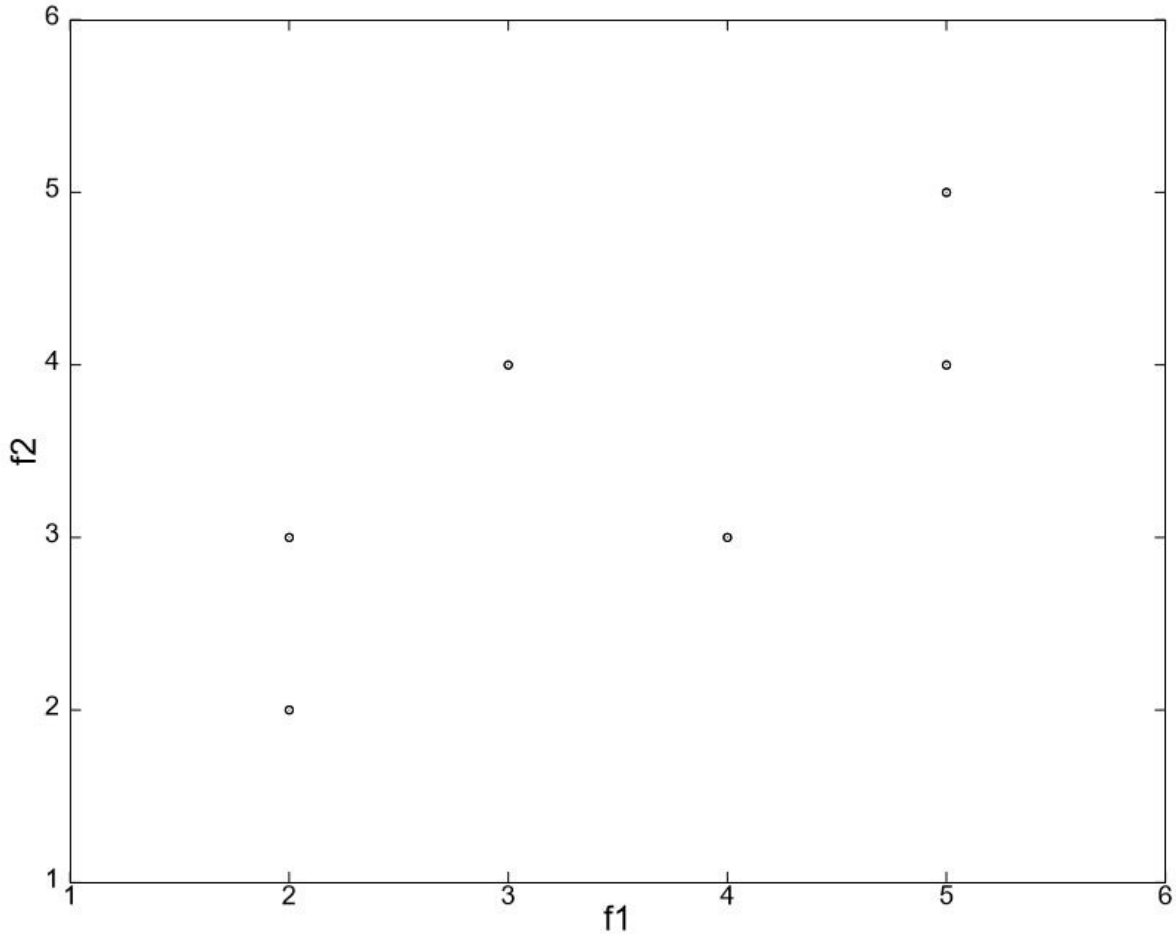

Find the covariance matrix Σ_f of the 6 pixels f.

pixel	f_1	f_2
1	2	2
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3	3	4
4	4	3
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5	5	4
6	5	5

$$\Sigma_f = E\left[(f-\mu_f)(f-\mu_f)^T\right]$$

Find the covariance matrix Σ_f of the 6 pixels f.

pixel	f_1	f_2
1	2	2
2	2	3
3	3	4
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5	5	4
6	5	5

$$\Sigma_f = E\left[(f-\mu_f)(f-\mu_f)^T\right] \qquad \mu_f = \begin{bmatrix} E[f_1] \\ E[f_2] \end{bmatrix} = \begin{bmatrix} 3.5 \\ 3.5 \end{bmatrix}$$

Find the covariance matrix Σ_f of the 6 pixels f.

pixel	f_1	f_2
1	2	2
2	2	3
3	3	4
4	4	3
5	5	4
6	5	5

$$\Sigma_f = E\left[(f-\mu_f)(f-\mu_f)^T\right] \qquad \mu_f = \begin{bmatrix} E[f_1] \\ E[f_2] \end{bmatrix} = \begin{bmatrix} 3.5 \\ 3.5 \end{bmatrix}$$

$$\Sigma_f = \begin{bmatrix} E[(f_1-3.5)(f_1-3.5)] & E[(f_1-3.5)(f_2-3.5)] \\ E[(f_1-3.5)(f_2-3.5)] & E[(f_2-3.5)(f_2-3.5)] \end{bmatrix}$$

Find the covariance matrix Σ_f of the 6 pixels f.

pixel	f_1	f_2
1	2	2
2	2	3
3	3	4
4	4	3
5	5	4
6	5	5

$$\Sigma_f = E\left[(f-\mu_f)(f-\mu_f)^T\right] \qquad \mu_f = \begin{bmatrix} E[f_1] \\ E[f_2] \end{bmatrix} = \begin{bmatrix} 3.5 \\ 3.5 \end{bmatrix}$$

$$\Sigma_f = \begin{bmatrix} E[(f_1-3.5)(f_1-3.5)] & E[(f_1-3.5)(f_2-3.5)] \\ E[(f_1-3.5)(f_2-3.5)] & E[(f_2-3.5)(f_2-3.5)] \end{bmatrix}$$

$$E[(f_1-3.5)^2] = \frac{(2-3.5)^2+(2-3.5)^2+(3-3.5)^2+(4-3.5)^2+(5-3.5)^2+(5-3.5)^2}{6} = 1.583$$

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pixel	f_1	f_2
1	2	2
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5	5	4
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$$\Sigma_f = \begin{bmatrix} E[(f_1-3.5)(f_1-3.5)] & E[(f_1-3.5)(f_2-3.5)] \\ E[(f_1-3.5)(f_2-3.5)] & E[(f_2-3.5)(f_2-3.5)] \end{bmatrix}$$

$$E\left[(f_1-3.5)^2\right] = \frac{(2-3.5)^2+(2-3.5)^2+(3-3.5)^2+(4-3.5)^2+(5-3.5)^2+(5-3.5)^2}{6} = 1.583$$

$$E\left[(f_2-3.5)^2\right] = \frac{(2-3.5)^2+(3-3.5)^2+(4-3.5)^2+(3-3.5)^2+(4-3.5)^2+(5-3.5)^2}{6} = 0.917$$

Find the covariance matrix Σ_f of the 6 pixels f.

pixel	f_1	f_2
1	2	2
2	2	3
3	3	4
4	4	3
5	5	4
6	5	5

$$\Sigma_f = E\left[(f-\mu_f)(f-\mu_f)^T\right] \qquad \mu_f = \begin{bmatrix} E[f_1] \\ E[f_2] \end{bmatrix} = \begin{bmatrix} 3.5 \\ 3.5 \end{bmatrix}$$

$$\Sigma_f = \begin{bmatrix} E[(f_1-3.5)(f_1-3.5)] & E[(f_1-3.5)(f_2-3.5)] \\ E[(f_1-3.5)(f_2-3.5)] & E[(f_2-3.5)(f_2-3.5)] \end{bmatrix}$$

$$E[(f_1-3.5)^2] = \frac{(2-3.5)^2+(2-3.5)^2+(3-3.5)^2+(4-3.5)^2+(5-3.5)^2+(5-3.5)^2}{6} = 1.583$$

$$E[(f_2-3.5)^2] = \frac{(2-3.5)^2+(3-3.5)^2+(4-3.5)^2+(3-3.5)^2+(4-3.5)^2+(5-3.5)^2}{6} = 0.917$$

$$E[(f_1-3.5)(f_2-3.5)] = \frac{\begin{bmatrix} (2-3.5)(2-3.5)+(2-3.5)(3-3.5)+(3-3.5)(4-3.5) \\ +(4-3.5)(3-3.5)+(5-3.5)(4-3.5)+(5-3.5)(5-3.5) \end{bmatrix}}{6} = 0.917$$

$$\Sigma_f = \begin{bmatrix} 1.583 & 0.917 \\ 0.917 & 0.917 \end{bmatrix}$$

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Sorted eigenvectors of Σ_f are

$$e_1 = [0.819 \quad 0.574]^T \quad e_2 = [-0.574 \quad 0.819]^T$$

with eigenvalues $\lambda_1 = 2.226 \quad \lambda_2 = 0.274$

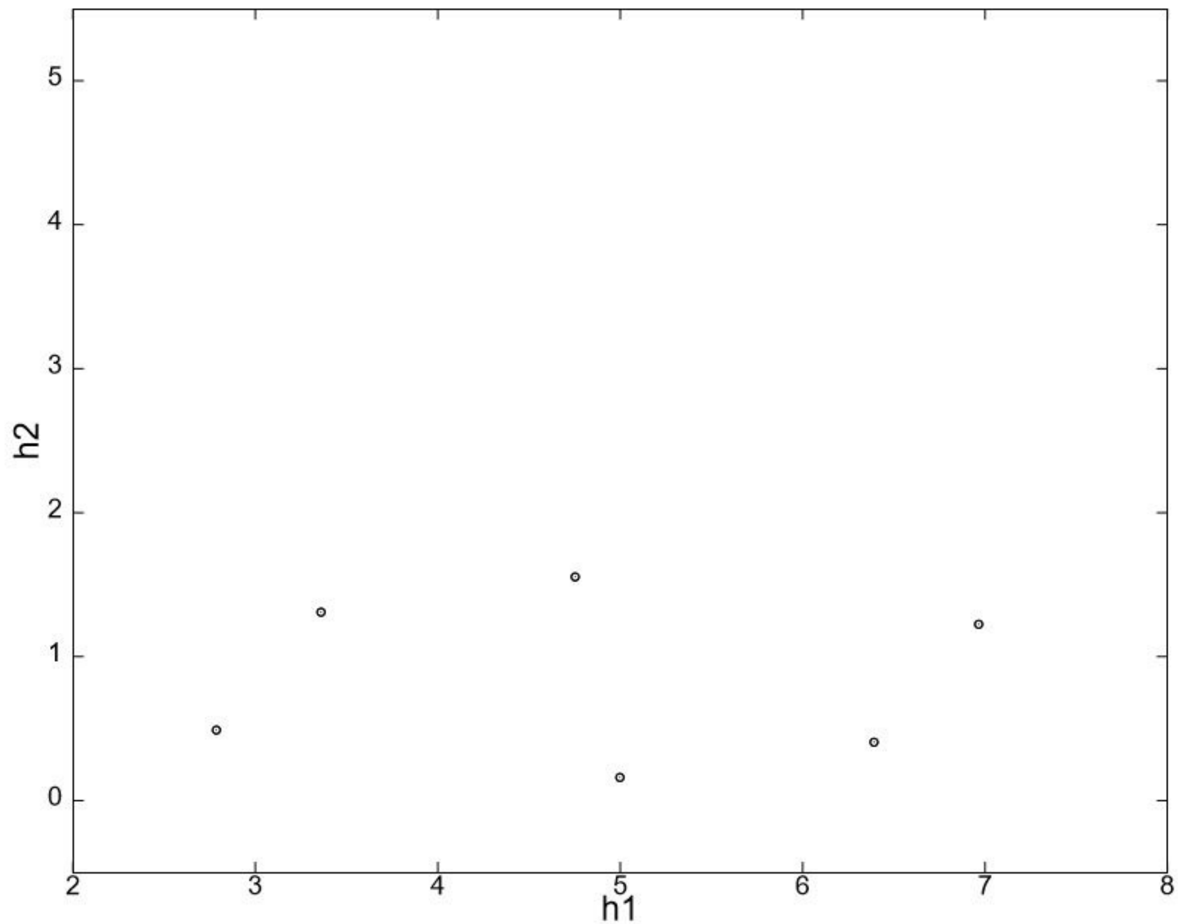
$$\Sigma_f = \begin{bmatrix} 1.583 & 0.917 \\ 0.917 & 0.917 \end{bmatrix}$$

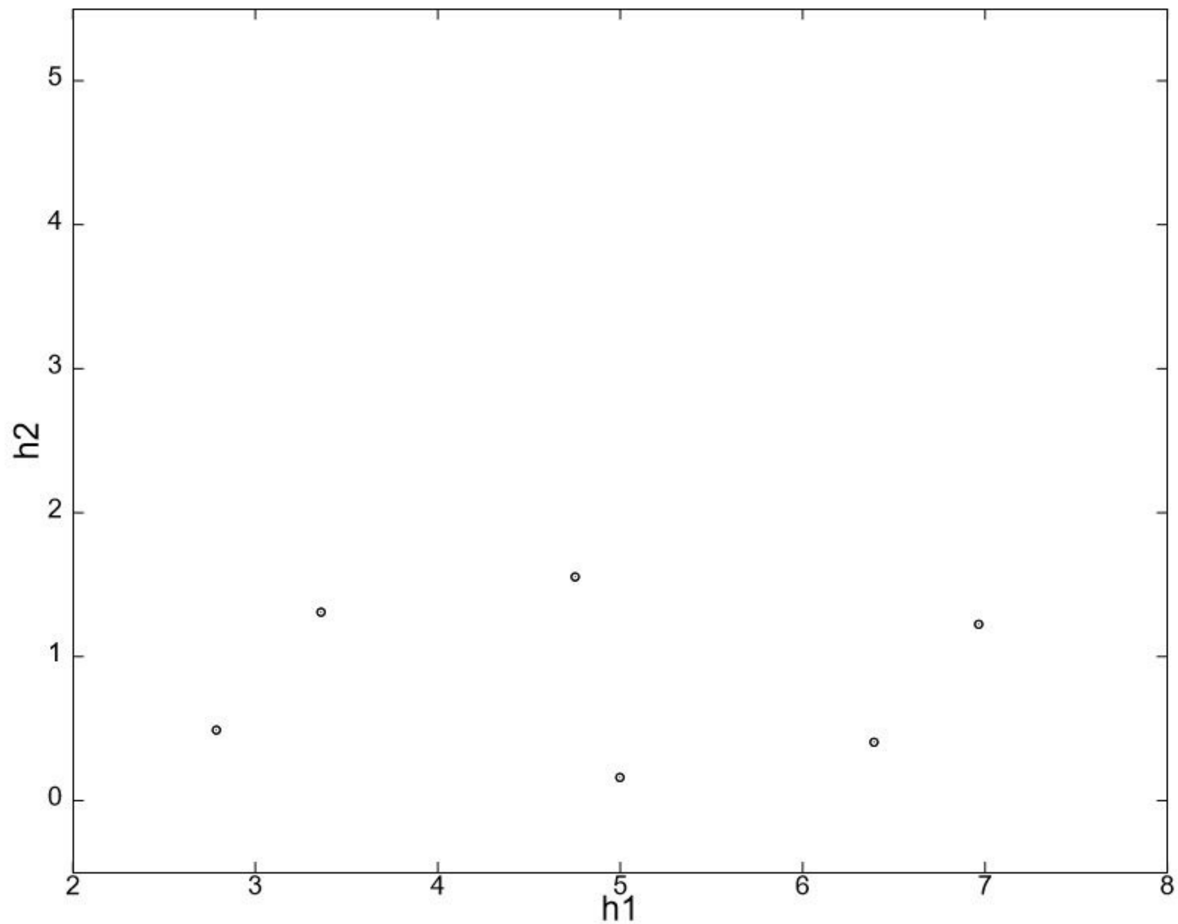
Sorted eigenvectors of Σ_f are

$$e_1 = [0.819 \quad 0.574]^T \qquad e_2 = [-0.574 \quad 0.819]^T$$

with eigenvalues $\lambda_1 = 2.226 \quad \lambda_2 = 0.274$

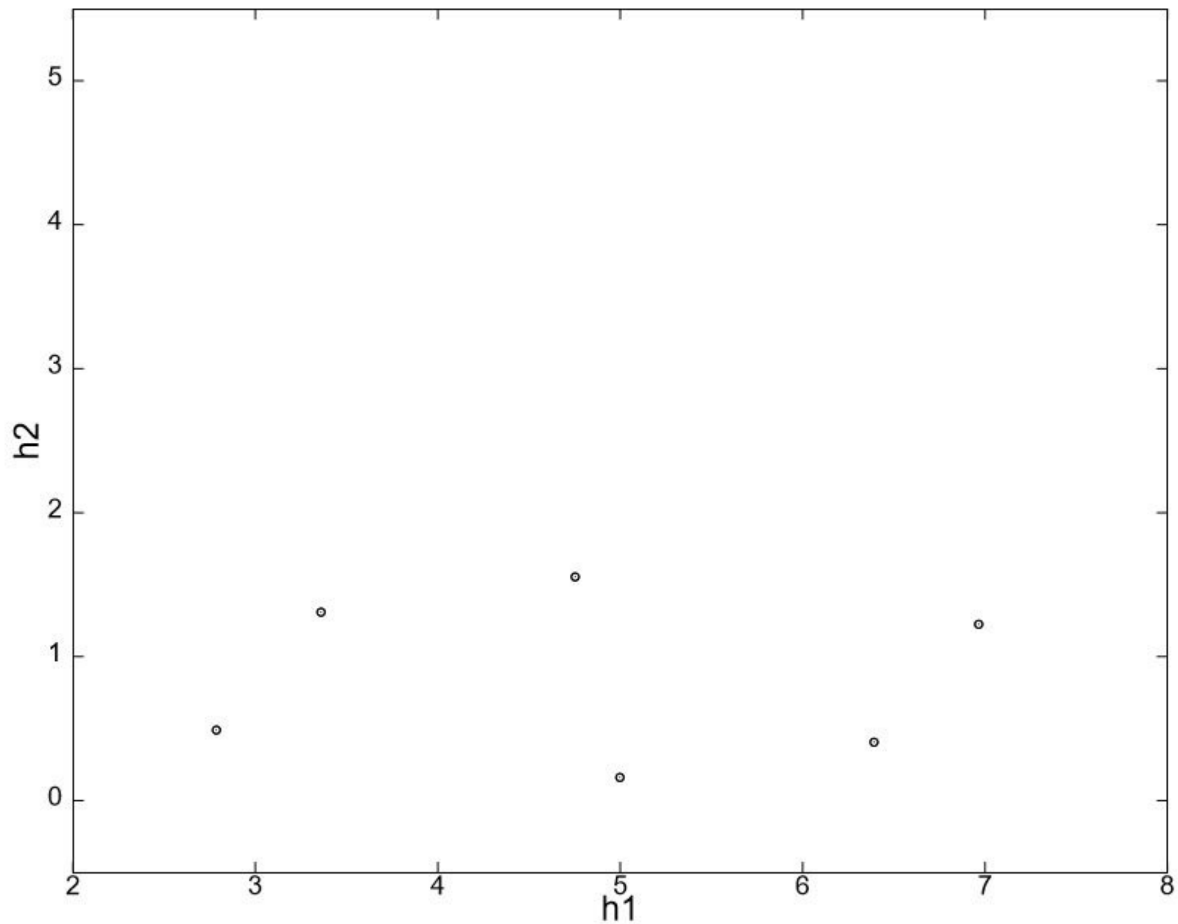
PCA transform is $\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 0.819 & 0.574 \\ -0.574 & 0.819 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$


5
4
3
2
1
0
h2
2
3
4
5
6
7
8
h1



variance $[f_1] = 1.583$ variance $[f_2] = 0.917$

variance $[h_1] = 2.226$ variance $[h_2] = 0.274$

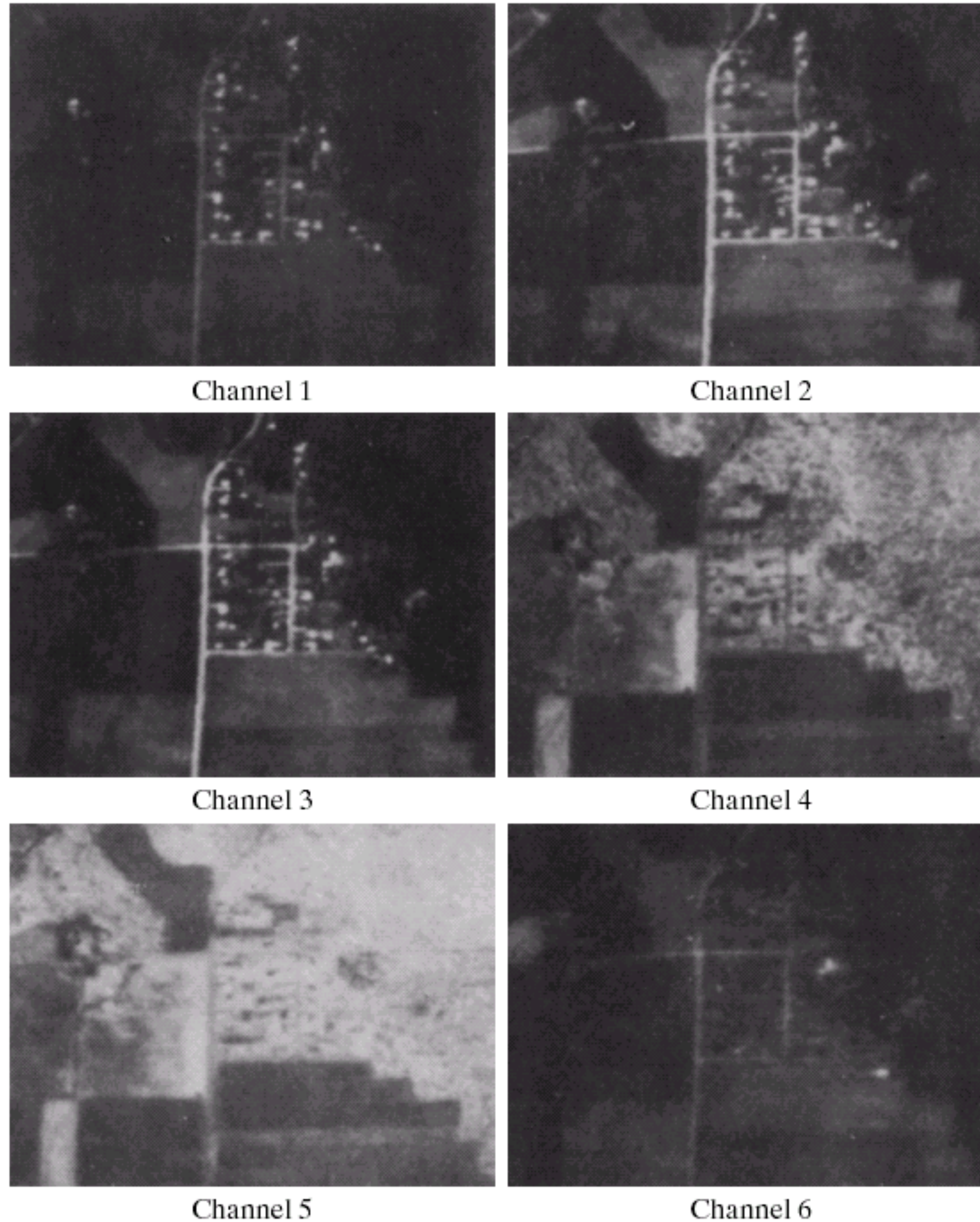


$\text{variance}[f_1] = 1.583$ $\text{variance}[f_2] = 0.917$

$\text{variance}[h_1] = 2.226$ $\text{variance}[h_2] = 0.274$

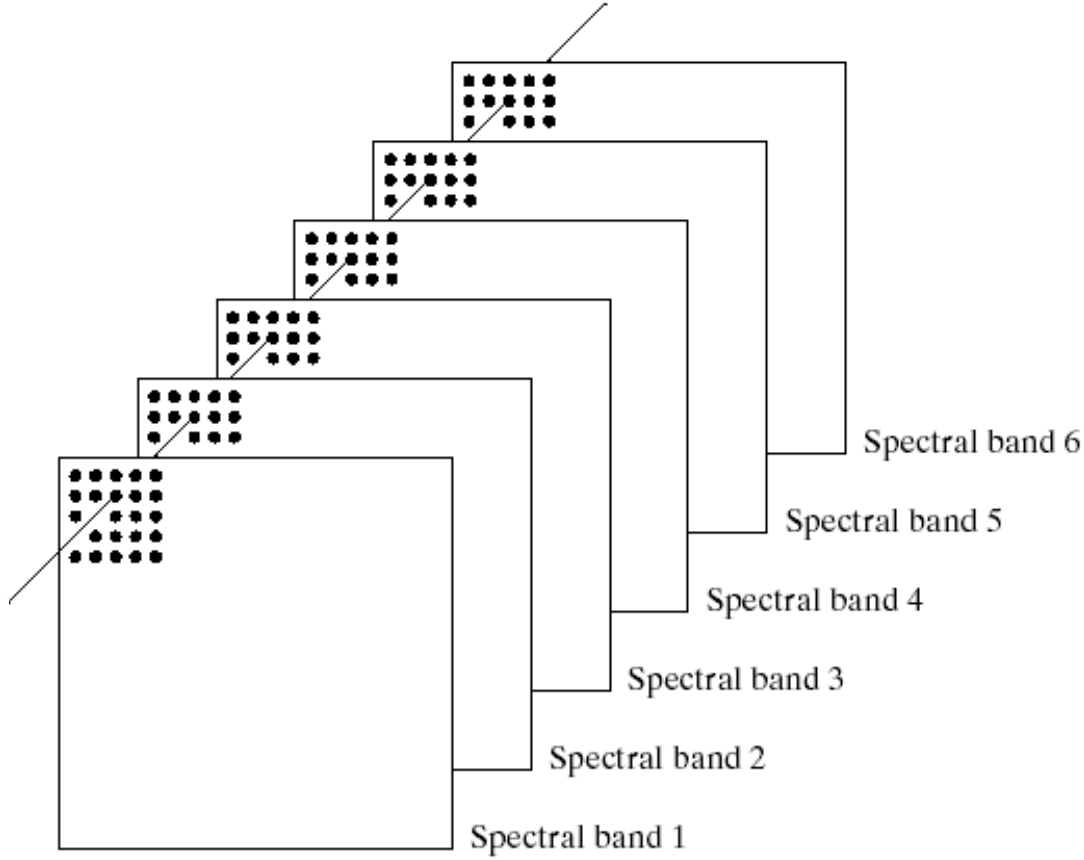
$$\text{variance}[f_1] + \text{variance}[f_2] = \text{variance}[h_1] + \text{variance}[h_2]$$

FIGURE 11.26 Six spectral images from an airborne scanner. (Courtesy of the Laboratory for Applications of Remote Sensing, Purdue University.)



Channel	Wavelength band (microns)
1	0.40–0.44
2	0.62–0.66
3	0.66–0.72
4	0.80–1.00
5	1.00–1.40
6	2.00–2.60

TABLE 11.4 Channel numbers and wavelengths.



$f = [f_1 \ f_2 \cdots f_B]^T$

B = 6

FIGURE 11.27 Formation of a vector from corresponding pixels in six images.

λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
3210	931.4	118.5	83.88	64.00	13.40

TABLE 11.5 Eigenvalues of the covariance matrix obtained from the images in Fig. 11.26.

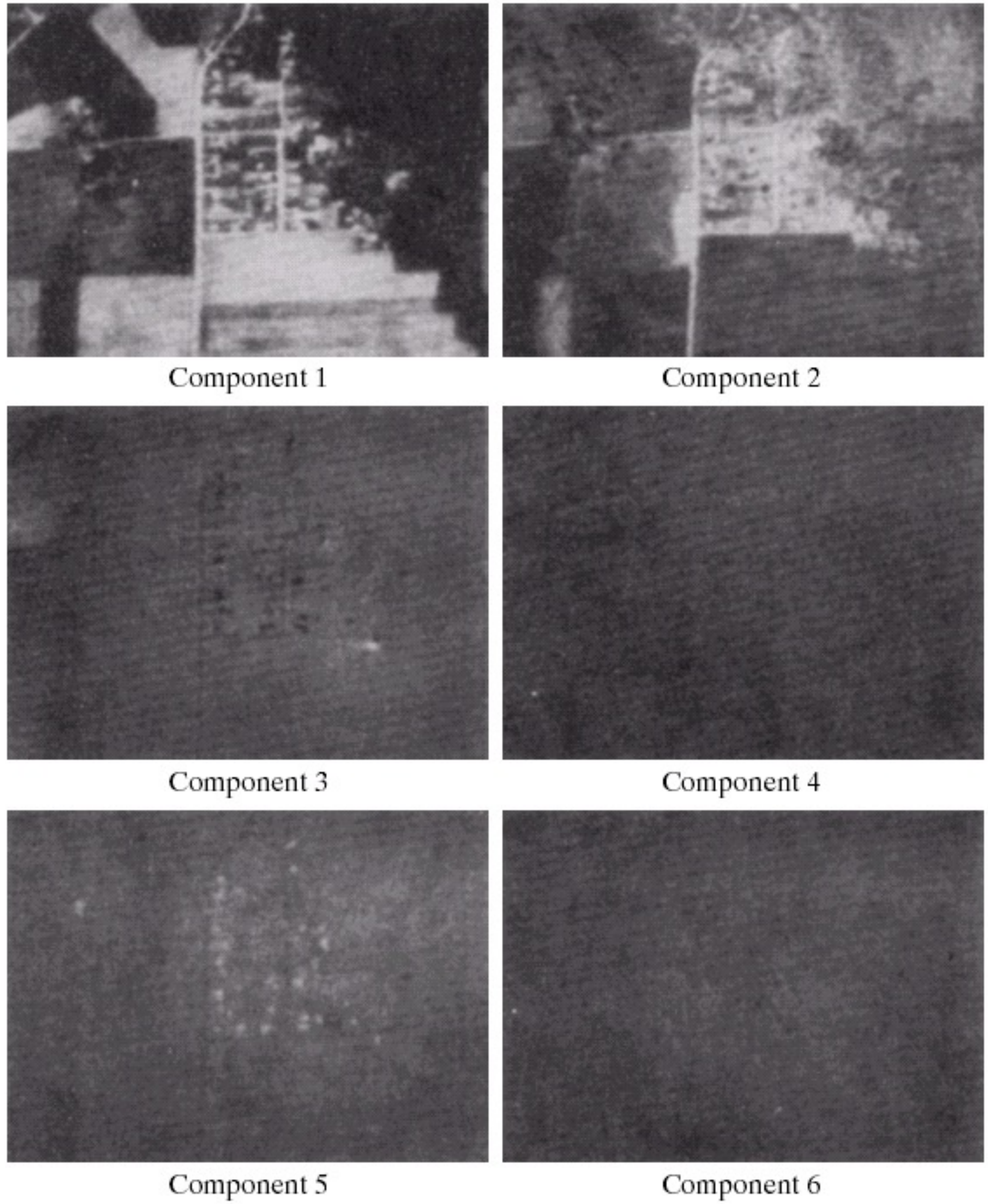


FIGURE 11.28 Six principal-component images computed from the data in Fig. 11.26. (Courtesy of the Laboratory for Applications of Remote Sensing, Purdue University.)

Question 10 (6 points) Suppose that the pixels $z = (z_r\ z_g\ z_b)^T$ in a color image have a 3×1 mean vector μ and a 3×3 covariance matrix Σ given by $\Sigma = E[(z-\mu)(z-\mu)^T]$

where $\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

a) Find an orthogonal 3×3 matrix A that implements the transform

$$w = (w_1\ w_2\ w_3)^T = Az$$

so that w_1, w_2, and w_3 are uncorrelated with each other and V_1 and V_2 are both maximized where $V_1 = \text{variance}[w_1]$ and $V_2 = \text{variance}[w_1] + \text{variance}[w_2]$.

Question 10 (6 points) Suppose that the pixels $z = (z_r\ z_g\ z_b)^T$ in a color image have a 3×1 mean vector μ and a 3×3 covariance matrix Σ given by $\Sigma = E[(z-\mu)(z-\mu)^T]$

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