EL2520 – Control Theory and Practice
Classical Loop-shaping

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Abstract

This report was drafted for KTH's EL2520 - Control Theory and Practice, Advanced course during VT16. It is split into two parts: in the first one, we design a lead-lag controller in order to control a process under specific requirements. The second part assumes the first one and furthers the work into attenuation of disturbances as well as reference tracking.

1 Basics

The process to be controlled is modelled by the transfer function $G(s)$:

$$G(s) = \frac{3(1-s)}{(5s+1)(10s+1)}$$

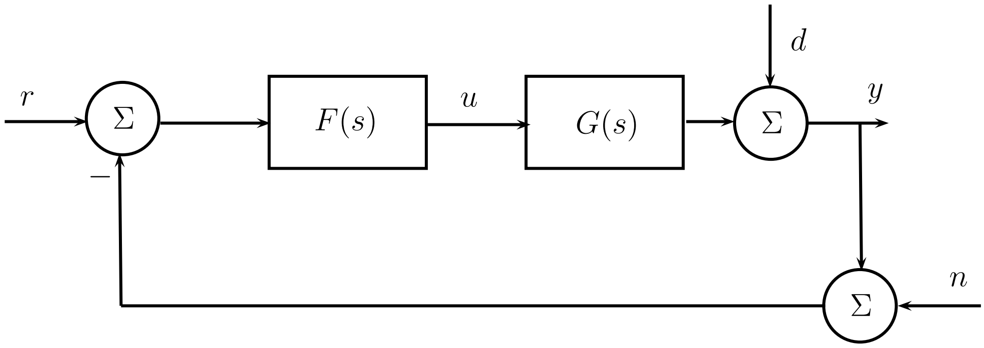

Figure 1: Closed loop block diagram, where $F-$controller, $G-$system, $r-$reference signal, $u-$control signal, $d-$disturbance signal, $y-$output signal, $n-$measurement noise.

1.1 Exercise 1

In order for the closed-loop system to have

- a phase margin of 30°,
- a crossover frequency of 0.4 rad/s, and
- zero steady-state error for a step response in the reference signal

we consider a lead-lag controller of the form

$$F(s) = K \cdot \frac{\tau_D s + 1}{\beta \tau_d s + 1} \cdot \frac{\tau_I s + 1}{\tau_I s + \gamma}$$

whose K, τ_D, β, τ_I and γ coefficients shall be configured in such a way that the closed-loop system fulfills the above requirements.

We first consider the third requirement: the error $E(s)$ is given by

$$E(s) = R(s) - Y(s) = \frac{1}{1+F(s)G(s)} R(s) = \frac{1}{1+F(s)G(s)} \cdot \frac{1}{s}$$

for a step reference. The steady state error will thus be

$$e(\infty) = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \frac{1}{1+F(0)G(0)} = \frac{\gamma}{\gamma+3K}$$

For $e(\infty)$ to be zero, either $\gamma = 0$ or $K \to \infty$. Sensibly, we choose $\gamma = 0$.

If we set $\tau_I = 1$, then the lag component becomes $F_g(s) = \dfrac{s+1}{s}$, and the phase margin of $F_g(s)G(s)$ is equal to $\phi_m^0 = 130.6013° - 180° = -49.3987°$. Thus, β, τ_d and K can be obtained by the following equations:

- $\beta = \dfrac{1 - sin(30 - \phi_m^0)}{1 + sin(30 - \phi_m^0)} = 0.0086$
- $\tau_D = \dfrac{1}{\omega_c\sqrt{\beta}} = \dfrac{1}{0.4\sqrt{\beta}} = 26.9459$
- $K = \dfrac{\sqrt{\beta}}{|F_g(j\omega_c)G(j\omega_c)|} = \dfrac{\sqrt{\beta}}{0.9436} = 0.0983$

Finally, now that the value of each coefficient has been identified, the controller is identified as

$$F(s) = 0.0983 \cdot \frac{26.9459s + 1}{0.2317s + 1} \cdot \frac{s + 1}{s}$$

We can verify that all three requirements have been met by plotting the bode diagram (figure 2) and the step response (figure 3) for the initial, uncontrolled process $G(s)$ and the final, controlled process $F(s)G(s)$.

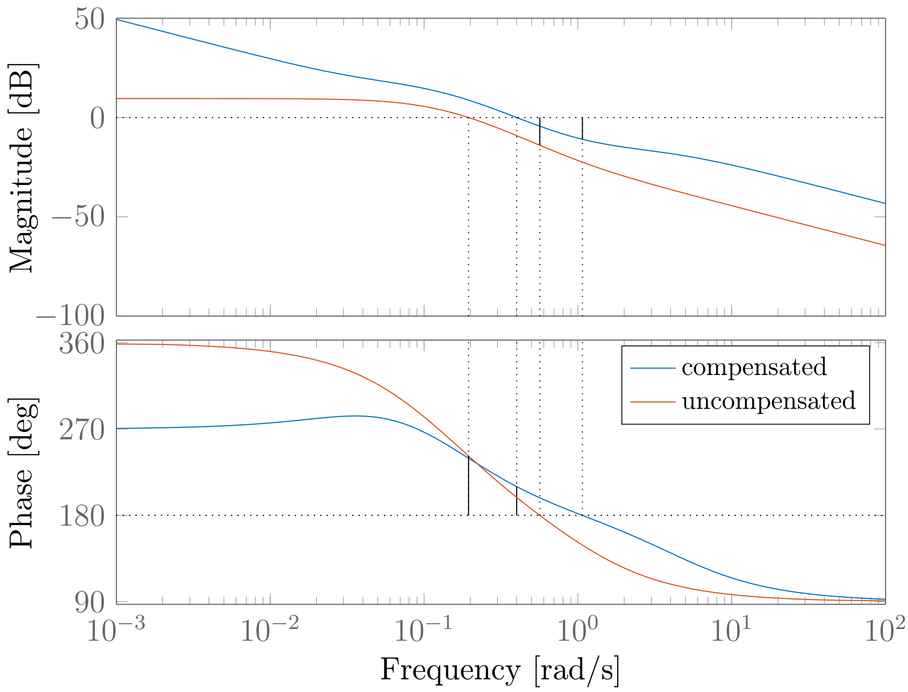

Figure 2: The frequency response of the initial, uncontrolled process (`red`) and final, controlled process (`blue`).

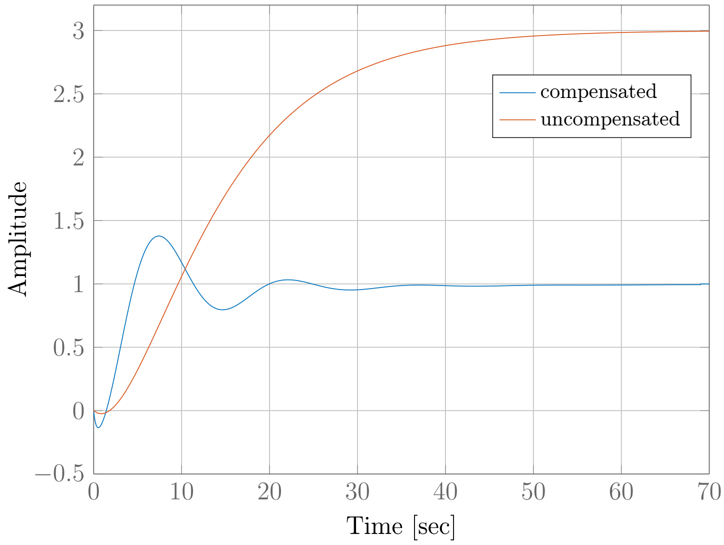

Figure 3: The step response of the initial, uncontrolled process (`red`) and final, controlled process (`blue`).

1.2 Exercise 2

The bandwidth and the resonance peak of the closed-loop system, along with the rise time and the overshoot of a step response are found in table 1.

ω_B $[rad/s]$	M_T $[dB]$	T_r $[sec]$	M %
0.7474	5.8247	2.4583	37.7922

Table 1: Closed loop system characteristics for a phase margin of 30°.

1.3 Exercise 3

If we now require a phase margin of 50°, and the crossover frequency to remain unchanged at $\omega_c = 0.4$ rad/s, then the τ_I coefficient needs to be increased. For $\tau_I = 2$, the lag component becomes $F_g(s) = \dfrac{2s+1}{2s}$. In this case, the phase margin of $F_g(s)G(s)$ is equal to $\phi_m^0 = 147.4597° - 180° = -32.5403°$. Thus, β, τ_d and K, obtained the same way as before, become:

- $\beta = \dfrac{1 - sin(50 - \phi_m^0)}{1 + sin(50 - \phi_m^0)} = 0.0042$
- $\tau_D = \dfrac{1}{\omega_c\sqrt{\beta}} = \dfrac{1}{0.4\sqrt{\beta}} = 38.3493$
- $K = \dfrac{\sqrt{\beta}}{|F_g(j\omega_c)G(j\omega_c)|} = \dfrac{\sqrt{\beta}}{0.5610} = 0.1162$

Finally, now that the value of each coefficient has been identified, the controller is identified as

$$F(s) = 0.1162 \cdot \frac{38.3493s + 1}{0.1611s + 1} \cdot \frac{2s + 1}{2s}$$

We can verify that all three requirements have been met by plotting the bode diagram (figure 4) and the step response (figure 5) for the initial, uncontrolled process $G(s)$ and the final, controlled process $F(s)G(s)$.

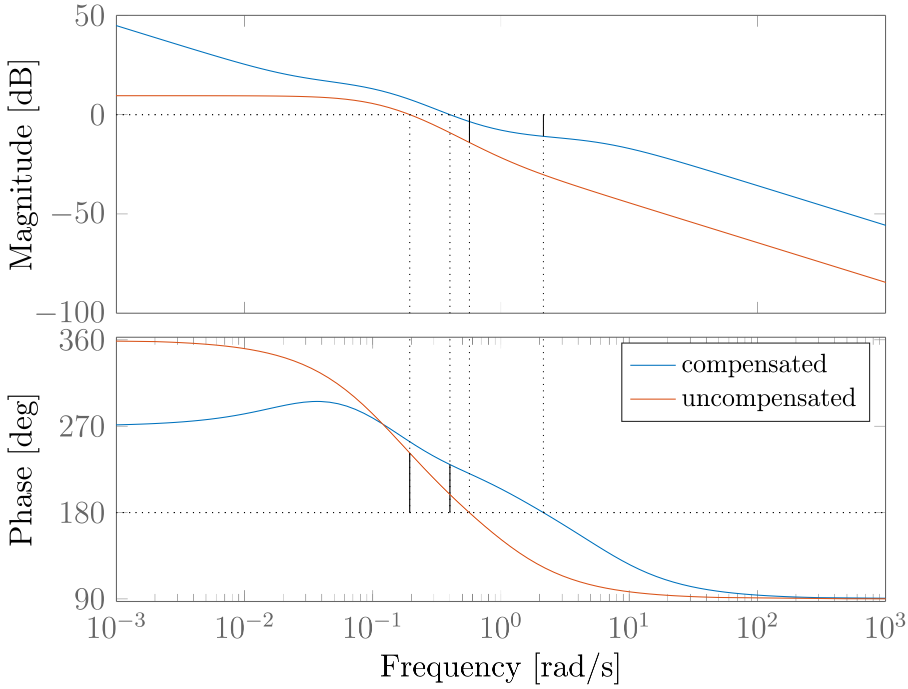

Figure 4: The frequency response of the initial, uncontrolled process (`red`) and final, controlled process (`blue`).

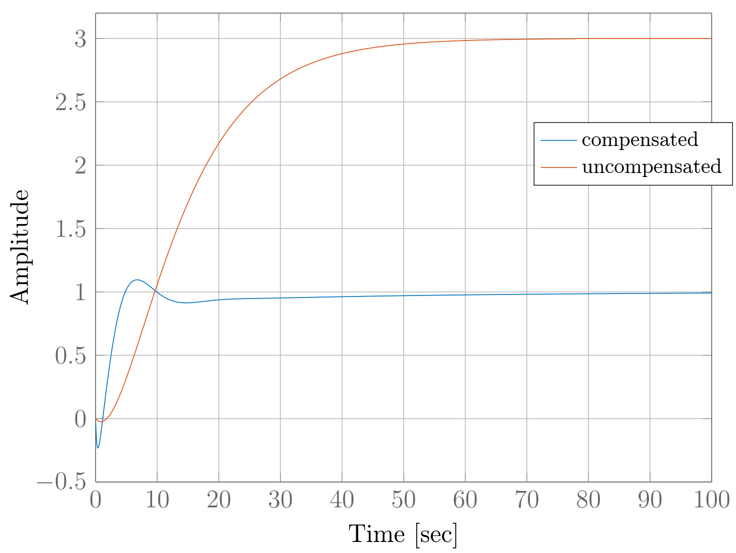

Figure 5: The step response of the initial, uncontrolled process (`red`) and final, controlled process (`blue`).

The bandwidth and the resonance peak of the closed-loop system, along with the rise time and the overshoot of a step response are found in table 2.

ω_B $[rad/s]$	M_T $[dB]$	T_r $[sec]$	M %
0.86	1.4757	2.7	9.5649

Table 2: Closed loop system characteristics for a phase margin of 50°.

2 Disturbance Attenuation

This exercise deals with the construction of a controller whose function is twofold: it tracks the reference signal and attenuates disturbances. The requirements of the system are such that:

- The rise time for a step change in the reference signal is less than 0.2 s
- The overshoot is less than 10%
- For a step in the disturbance, $|y(t)| \leq 1 \; \forall t$ and $|y(t)| \leq 0.1$ for $t > 0.5$ s
- Since the signals are scaled, the control signal obeys $|u(t)| \leq 1 \; \forall t$

The transfer functions of the plant $G(s)$ and the disturbance $G_d(s)$ have been estimated to expressions 1 and 2 respectively. The target of this exercise is to construct the $F_r(s), F_y(s)$ transfer functions in such a way that all four of the above requirements are met.

$$G(s) = \frac{20}{(s+1)((\frac{s}{20})^2 + \frac{s}{20} + 1)} \tag{1}$$

$$G_d(s) = \frac{10}{s+1} \tag{2}$$

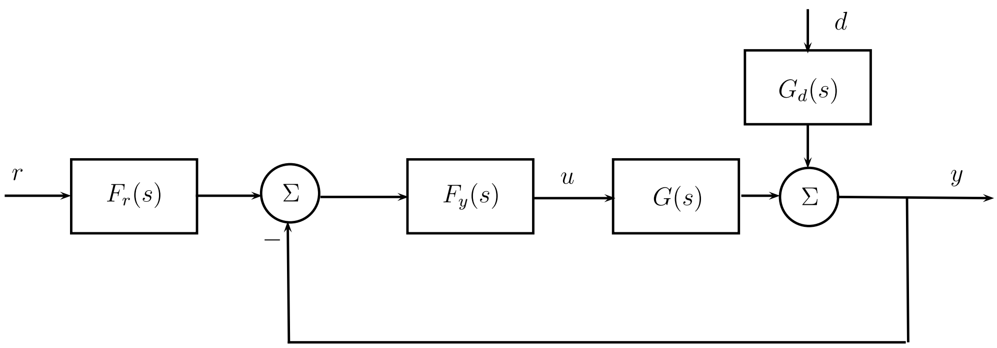

Figure 6: F_r- prefilter, F_y- feedback controller, $G-$ system, G_d- disturbance dynamics, $r-$ reference signal, $u-$ control signal, $d-$ disturbance signal, $y-$ measurement signal.

2.1 Exercice 1

Control action is needed at least at frequencies where $|G_d(j\omega)| > 1$; this expression is valid for all $\omega < \omega_c = 9.9473$ rad/s[1]. Hence, the minimal frequency interval where control is needed is $[0, 9.9473]$ rad/s. The controller $F_y(s)$ that shall be designed here will be such that

$$L(s) = F_y(s)G(s) = \frac{\omega_c}{s}$$

We will approach the design of F_y in two ways: one where it F_y is improper, and one where it is proper.

[1]this is the crossover frequency of G_d

2.1.1 F_y is improper

The first approach is the most simplistic one:

$$F_y(s) = \frac{\omega_c}{sG(s)} = \frac{\omega_c(s+1)((\frac{s}{20})^2 + \frac{s}{20} + 1)}{20s}$$

This means that F_y is not proper since the number of its poles is less than the number of its zeros. Although this controller displays good disturbance attenuation (figure 8), the amplitude of the output for a step disturbance does not meet the third requirement for $|y(t)| \leq 0.1$ for $t > 0.5$ s.

2.1.2 F_y is proper

In order for F_y to be proper, we add the minimum number of poles required, which is two, p_1, p_2, such that

$$F_y(s) = \frac{\omega_c}{sG(s)} \cdot \frac{p_1 \cdot p_2}{(s+p_1)(s+p_2)}$$

and in order for $L(s) \approx \frac{\omega_c}{s}$ we choose their location so that the frequency response of the closed-loop transfer function from d to y matches that of the previous case, where F_y was improper. In this case, the two poles should be positioned away from ω_c. Such a case was found to be valid experimentally for $p_1 = p_2 \geq 50\omega_c$. Thus, the poles were chosen as $p_1 = p_2 = 50\omega_c$. As is evident in figures 8 and 9, the frequency response of the closed loop transfer function from d to y and the response of the output for a unit disturbance are equivalent in the frequency band where control action is appropriate.

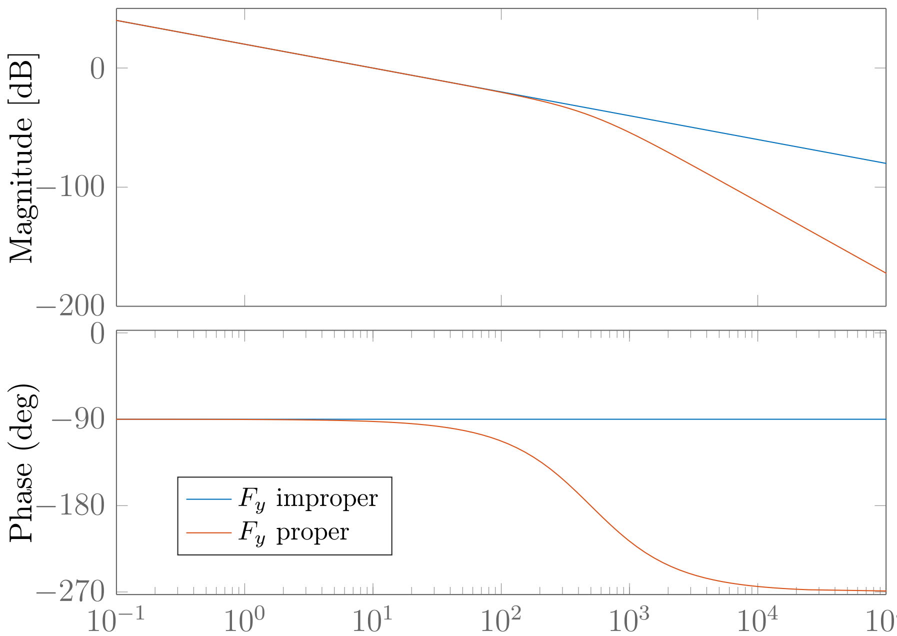

Figure 7: The frequency response of $L(s) = F_y(s)G(s)$ when F_y is proper (red) and when F_y is improper (blue). $p_1 = p_2 = 50\omega_c$

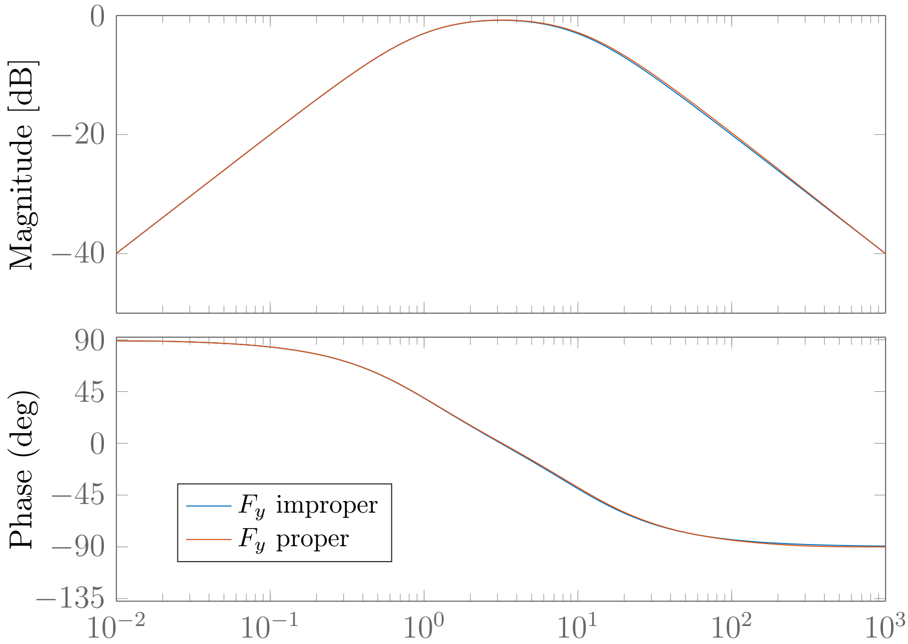

Figure 8: The frequency response of the closed-loop transfer function from d to y when F_y is proper (`red`) and when F_y is improper (`blue`). $p_1 = p_2 = 50\omega_c$

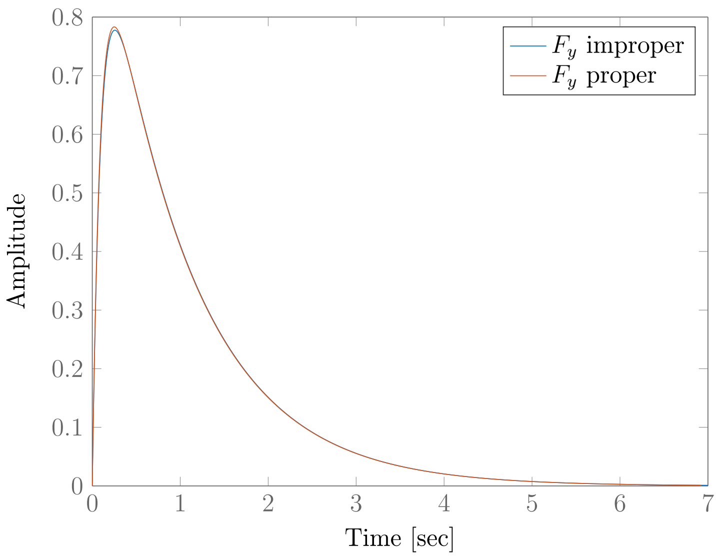

Figure 9: The response of the transfer function from d to y for a step in the disturbance when F_y is proper (`red`) and when F_y is (`blue`). $p_1 = p_2 = 50\omega_c$

2.2 Exercice 2

Since the previous controller did not exhibit adequate performance and did not meet the specified requirements, we now attempt to tune it in accordance to a desire to include integral action. We will again consider two possible designs.

2.2.1 F_y is improper

We take F_y to be of the form $F_y(s) = \frac{s+\omega_I}{s} G^{-1}(s) G_d(s)$ as a starting point. Although F_y is now improper, we can design ω_I so that our third requirement is met. Indeed, if we choose $\omega_I = 0.5\omega_c$, then $|y(t)| \leq 0.1, \forall t \geq 0.5\text{s}$ and, $\forall t : |y(t)| < 1$. This behaviour is illustrated in figure 10 in blue colour.

2.2.2 F_y is proper

Since F_y needs to be proper, we need to add two poles to it; hence F_y is now formed as

$$F_y(s) = \frac{s+\omega_I}{s} \cdot \frac{p_1 \cdot p_2}{(s+p_1)(s+p_2)} \cdot G^{-1}(s)G_d(s)$$

The location of these poles should be such that when $G_d \approx 1$, e.g. for $\omega < \omega_c$, the controller should contain the inverse of the system. This means that p_1, p_2 should be larger than ω_c. A choice of $p_1 = p_2 = 5\omega_c = 10\omega_I$ is appropriate since the poles are adequately far from the crossover frequency of the disturbance and the third requirement is satisfied, as can be seen in figure 10 in red colour.

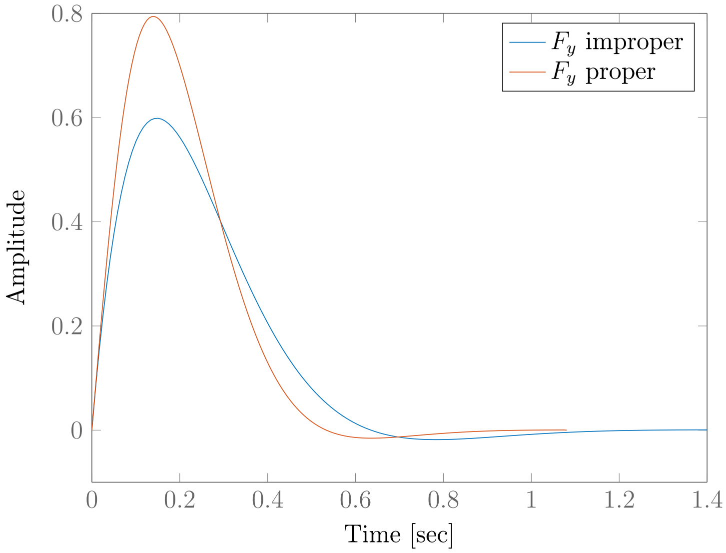

Figure 10: The response of the transfer function from d to y for a step in the disturbance when F_y is proper (`red`) and improper (`blue`). $\omega_I = 0.5\omega_c$, $p_1 = p_2 = 5\omega_c$.

2.3 Exercises 3 and 4

In this subsection we try to bridge all the gaps and meet all specifications. In order to do so, first we add a lead component to the controller, so that F_y is now given by

$$F_y(s) = \frac{s+\omega_I}{s} \cdot K\frac{\tau_D s+1}{\beta\tau_D s+1} \cdot \frac{p_1 \cdot p_2}{(s+p_1)(s+p_2)} \cdot G^{-1}G_d$$

Since there is no specification for the phase margin, we can set it to a sensible 30° at a crossover frequency larger than the actual one, so that ω_c is well inside the control region, $\omega_c + 6 \approx 15$ rad/s, for which $\beta = 0.6565, \tau_D = 0.0826$ and $K = 1.2557$.

As for the prefilter, it is given by F_r:

$$F_r = \frac{1}{\tau s+1}$$

Through a process of trial and error, if $\tau = 0.1$, then all requirements are met. Specifically, controllers F_y, F_r, with coefficients given in table 3 make the system behave as follows:

ω_c	ω_I	K	τ_D	β	$p_1 = p_2$	τ
9.9473	$0.5\omega_c$	1.2557	0.0826	0.6565	$5\omega_c$	0.1

Table 3: The values of coefficients for $F_y(s)$ and $F_r(s)$.

- ☐ The rise time for a step change in the reference signal is $0.1561 < 0.2$s
- ☐ The overshoot is $6.5716 < 10\%$
- ☐ For a step in the disturbance, $|y(t)| \leq 1\ \forall t$ and $|y(t)| \leq 0.1$ for $t > 0.445$s
- ☐ The control signal obeys $|u(t)| \leq 0.511 \leq 1\ \forall t$

Figure 11 shows the frequency response of the sensitivity S and complimentary sensitivity T transfer functions, figure 12 illustrates the step response of the closed-loop system and figure 13 shows depicts the response of the transfer function from d to y for a step disturbance.

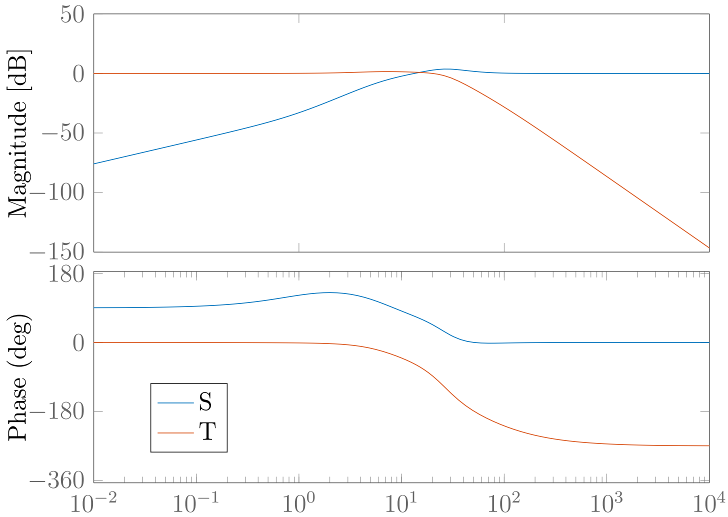

Figure 11: The frequency responses of the sensitivity (`blue`) and complimentary sensitivity function (`red`).

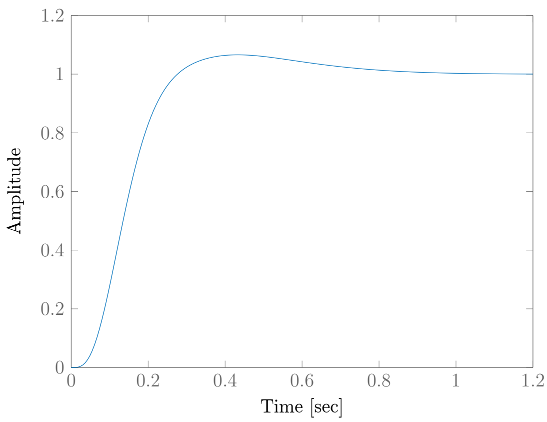

Figure 12: The step response of the system.

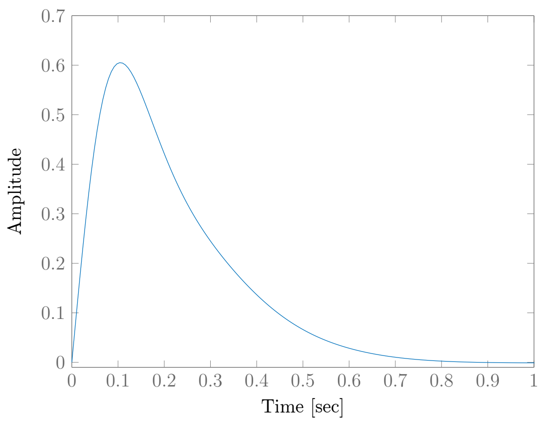

Figure 13: The response of the transfer function from d to y for a step in the disturbance.

3 Conclusion

In summary, we had to implement controllers that either track the reference signal or attenuate disturbances. This meant an act of balancing between coefficients, zeros' and poles' placement so that the specified requirements were met.