Control Theory and Practice Advanced Course

Laboratory experiment: THE FOUR-TANK PROCESS

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Abstract

This laboratory experiment investigates a coupled four-tank process from a control systems view. It derives a linearized model of the system and uses it to design suitable controllers. The report present manual, decentralized and Glover-McFarlane robust controllers for two different settings of the model. The controllers are tested on a simulation of the system and results of multiple tests are presented, such as step-response and disturbance attenuation . The report conclude that depending on the settings of the model adding a Glover-McFarlane robustification can be necessary for acceptable stability or be an unnecessary implementation cost when applied on an already stable system.

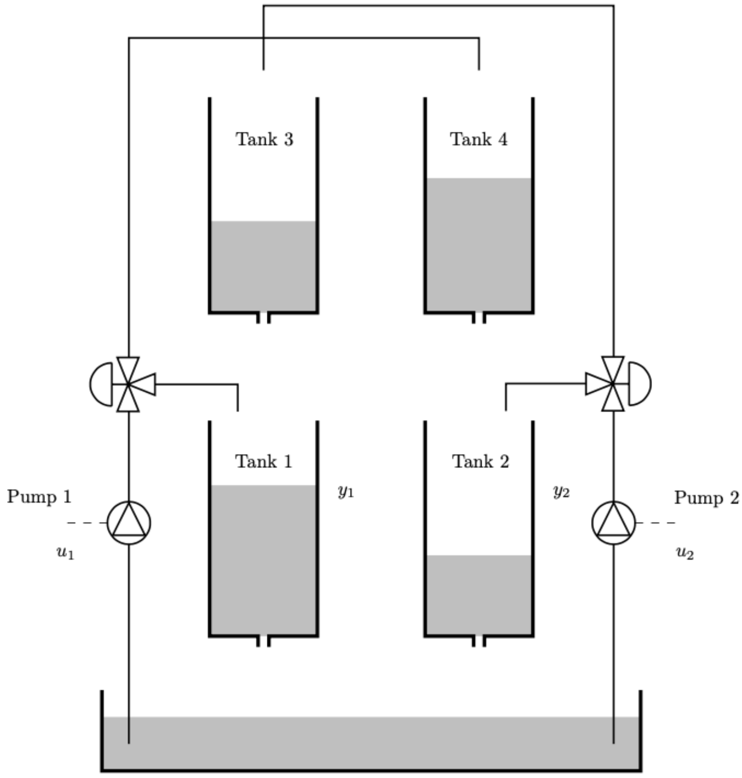

Figure 1: The four-tank system. Image from [3].

1 Introduction

In the modern world of the 21st century automation of complex systems have become increasingly important. We rely on these systems for our electricity, our water supply, our internet and many other fundamental infrastructures of our society. At the heart of many of these complex automated systems are often carefully designed control systems managing the control of the system in order to achieve the performance and stability desired.

This laboratory experiment and investigation aims to analyse the characteristics, specifications and limitation of such a control system in a mathematically simple situation. Thereto find several different controllers with different characteristics to control and stabilize the system at hand. The controllers to be analysed are manual control, decentralized PI-control and a robustified control using the Glover-Mcfarlane method.

The system used in this experiment is shown in Figure 1. It shows a dynamically complex situation due to the inevitable coupling of the different tanks resulting in a difficult control problem in some cases.

2 Models and characteristics of the system

Modeling the system can be divided in multiple parts. Firstly the physical models need to be derived. Then a linearization can be preformed to enable the then relevant control theory characterization of the system.

2.1 Physical models

Using the trivial arguments of mass conservation presented in the laboratory instructions [3] one arrive at the fact that all tanks fulfill

$$A\frac{dh}{dt} = q_{in} - q_{out}, \tag{1}$$

where A is the cross sectional area of the tank, h is the water height of the tank and q_{in} and q_{out} are the flow rates of water into and out of the tank respectively. Furthermore the laboratory instructions give us $q_{out} = a\sqrt{2gh}$ with a being the outlet hole area and g being the gravitational acceleration (approx. $981cm/s^2$).

It is also stated in the laboratory instructions that the flow rate of the pump, q, can be described as being proportional to the applied voltage u according to $q = ku$. The constant k will be determined experimentally in at a later stage. Lastly we have splitters splitting the water of the pumps between one of the upper tanks and one of the lower tanks (see Figure 1). The relation between the flow rates into the upper and the lower tank are given by

$$q_L = \gamma ku, \qquad q_U = (1-\gamma)ku \tag{2}$$

with $\gamma \in [0,1]$ and indices $_U$ and $_L$ denote the upper and the lower tank respectively.

With this information it is now possible to derive the differential equation governing the dynamics of the system. Using the schematics of Figure 1 we see that for tanks 3 and 4 $q_{in} = (1-\gamma)ku$ for their respective γ, k and u. For tanks 1 and 2 the flow into the tank is given as a sum of the flow from the pump and from the outlet of the tank above, giving $q_{in} = \gamma ku + a\sqrt{2gh}$. Substituting q_{in} and q_{out} of Equation 1 for each tank and indexing each area, height and constant according to the numbering of Figure 1 we have the following system of differential equations.

$$\begin{aligned}
\frac{dh_1}{dt} &= -\frac{a_1}{A_1}\sqrt{2gh_1} + \frac{a_3}{A_1}\sqrt{2gh_3} + \frac{\gamma_1 k_1}{A_1}u_1 \\
\frac{dh_2}{dt} &= -\frac{a_2}{A_2}\sqrt{2gh_2} + \frac{a_4}{A_2}\sqrt{2gh_4} + \frac{\gamma_2 k_2}{A_2}u_2 \\
\frac{dh_3}{dt} &= -\frac{a_3}{A_3}\sqrt{2gh_3} + \frac{(1-\gamma_2)k_2}{A_3}u_2 \\
\frac{dh_4}{dt} &= -\frac{a_4}{A_4}\sqrt{2gh_4} + \frac{(1-\gamma_1)k_1}{A_4}u_1
\end{aligned} \tag{3}$$

As well as the equations of Equation 3 we have the outputs y_1 and y_2 given by

$$y_i = k_c h_i, \tag{4}$$

where k_c is a constant of the measurement equipment.

Using Equation 3 we can derive the equilibrium of the system. At the equilibrium the derivative of tank height, and in turn the outputs y_1 and y_2 are 0. Assuming $u_1^0, u_2^0, h_1^0, h_2^0, h_3^0, h_4^0, y_1^0$ and y_2^0 are the operation points of the equilibrium we have, using the fact that $\frac{dh_i}{dt} = 0$ and with some simple algebra, that

$$
\begin{aligned}
0 &= -a_1\sqrt{2gh_1^0} + a_3\sqrt{2gh_3^0} + \gamma_1 k_1 u_1^0, \\
0 &= -a_2\sqrt{2gh_2^0} + a_4\sqrt{2gh_4^0} + \gamma_2 k_2 u_2^0, \\
0 &= -a_3\sqrt{2gh_3^0} + (1-\gamma_2) k_2 u_2^0, \\
0 &= -a_4\sqrt{2gh_4^0} + (1-\gamma_1) k_1 u_1^0,
\end{aligned} \tag{5}
$$

and

$$
\begin{aligned}
y_1^0 &= k_c h_1^0, \\
y_2^0 &= k_c h_2^0.
\end{aligned} \tag{6}
$$

2.2 Linearization

To be able to analyse the system using basic control theory we need to linearize the system given by Equation 3. Introducing $\Delta u_i = u_i - u_i^0$, $\Delta h_i = h_i - h_i^0$ and $\Delta y_i = y_i - y_i^0$, where u_i^0, h_i^0 and y_i^0 are equilibrium points we can linearize the system around the equilibrium point. Firstly we have that

$$
\frac{dh_i}{dt} = \frac{d}{dt}(\Delta h_i + h_i^0) = \frac{d\Delta h_i}{dt}. \tag{7}
$$

Using Equation 7 and Taylor expansion of the first degree around the equilibrium point we get the expressions

$$
\begin{aligned}
\frac{d\Delta h_1}{dt} &= -\frac{a_1\sqrt{2g}}{A_1}\left(\sqrt{h_1^0} + \frac{\Delta h_1}{2\sqrt{h_1^0}}\right) + \frac{a_3\sqrt{2g}}{A_1}\left(\sqrt{h_3^0} + \frac{\Delta h_3}{2\sqrt{h_3^0}}\right) + \frac{\gamma_1 k_1}{A_1}\left(u_1^0 + \Delta u_1\right), \\
\frac{d\Delta h_2}{dt} &= -\frac{a_2\sqrt{2g}}{A_2}\left(\sqrt{h_2^0} + \frac{\Delta h_2}{2\sqrt{h_2^0}}\right) + \frac{a_4\sqrt{2g}}{A_2}\left(\sqrt{h_4^0} + \frac{\Delta h_4}{2\sqrt{h_4^0}}\right) + \frac{\gamma_2 k_2}{A_2}\left(u_2^0 + \Delta u_2\right), \\
\frac{d\Delta h_3}{dt} &= -\frac{a_3\sqrt{2g}}{A_3}\left(\sqrt{h_3^0} + \frac{\Delta h_3}{2\sqrt{h_3^0}}\right) + \frac{(1-\gamma_2)k_2}{A_3}\left(u_2^0 + \Delta u_2\right), \\
\frac{d\Delta h_4}{dt} &= -\frac{a_4\sqrt{2g}}{A_4}\left(\sqrt{h_4^0} + \frac{\Delta h_4}{2\sqrt{h_4^0}}\right) + \frac{(1-\gamma_1)k_1}{A_4}\left(u_1^0 + \Delta u_1\right), \\
\Delta y_1 + y_1^0 &= k_c\left(\Delta h_1 + h_1^0\right), \\
\Delta y_2 + y_2^0 &= k_c\left(\Delta h_2 + h_2^0\right).
\end{aligned} \tag{8}
$$

Using the expressions in Equation 5 and Equation 6 we can simplify expressions in Equation 8 to

$$
\begin{aligned}
\frac{d\Delta h_1}{dt} &= -\frac{a_1}{A_1}\sqrt{\frac{g}{2h_1^0}}\Delta h_1 + \frac{a_3}{A_1}\sqrt{\frac{g}{2h_3^0}}\Delta h_3 + \frac{\gamma_1 k_1}{A_1}\Delta u_1,\\
\frac{d\Delta h_2}{dt} &= -\frac{a_2}{A_2}\sqrt{\frac{g}{2h_2^0}}\Delta h_2 + \frac{a_4}{A_2}\sqrt{\frac{g}{2h_4^0}}\Delta h_4 + \frac{\gamma_2 k_2}{A_2}\left(u_2^0 + \Delta u_2\right),\\
\frac{d\Delta h_3}{dt} &= -\frac{a_3}{A_3}\sqrt{\frac{g}{2h_3^0}}\Delta h_3 + \frac{(1-\gamma_2)k_2}{A_3}\Delta u_2,\\
\frac{d\Delta h_4}{dt} &= -\frac{a_4}{A_4}\sqrt{\frac{g}{2h_4^0}}\Delta h_4 + \frac{(1-\gamma_1)k_1}{A_4}\Delta u_1,\\
\Delta y_1 &= k_c \Delta h_1,\\
\Delta y_2 &= k_c \Delta h_2.
\end{aligned}
\tag{9}
$$

Introducing the vectors

$$
u = \begin{bmatrix}\Delta u_1\\ \Delta u_2\end{bmatrix}, \qquad h = \begin{bmatrix}\Delta h_1\\ \Delta h_2\\ \Delta h_3\\ \Delta h_4\end{bmatrix}, \qquad y = \begin{bmatrix}\Delta y_1\\ \Delta y_2\end{bmatrix}
$$

and the abbreviation

$$
T_i = \frac{A_i}{a_i}\sqrt{\frac{2h_i^0}{g}}
$$

we can now write the linearized system of Equation 9 as

$$
\frac{dx}{dt} = \begin{bmatrix} -\frac{1}{T_1} & 0 & \frac{A_3}{A_1 T_3} & 0\\ 0 & -\frac{1}{T_2} & 0 & \frac{A_4}{A_2 T_4}\\ 0 & 0 & -\frac{1}{T_3} & 0\\ 0 & 0 & 0 & -\frac{1}{T_4} \end{bmatrix} x + \begin{bmatrix} \frac{\gamma_1 k_1}{A_1} & 0\\ 0 & \frac{\gamma_2 k_2}{A_2}\\ 0 & \frac{(1-\gamma_2)k_2}{A_3}\\ \frac{(1-\gamma_1)k_1}{A_4} & 0 \end{bmatrix} u \tag{10}
$$

$$
y = \begin{bmatrix} k_c & 0 & 0 & 0\\ 0 & k_c & 0 & 0 \end{bmatrix} x \tag{11}
$$

2.3 Control theory analysis

2.3.1 Transfer matrix

Now we have derived the linearized model of the system and described it in a state-space form as seen in Equation 10. To analyse the system using control theory it is useful to have the transfer matrix of the system as this would allow us to use common control theroy methods. As explained by Glad and Ljung [1] we have that the transfer matrix of a state space model

$$\frac{dx}{dt} = Ax + Bu \tag{12}$$

$$y = Cx + Du \tag{13}$$

is given by

$$G(s) = C(sI - A)^{-1}B + D. \tag{14}$$

In our case our model neglects the disturbance D and results in the transfer matrix

$$\begin{aligned} G(s) &= \begin{bmatrix} k_c & 0 & 0 & 0 \\ 0 & k_c & 0 & 0 \end{bmatrix} \begin{bmatrix} s + \frac{1}{T_1} & 0 & \frac{A_3}{A_1 T_3} & 0 \\ 0 & s + \frac{1}{T_2} & 0 & \frac{A_4}{A_2 T_4} \\ 0 & 0 & s + \frac{1}{T_3} & 0 \\ 0 & 0 & 0 & s + \frac{1}{T_4} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\gamma_1 k_1}{A_1} & 0 \\ 0 & \frac{\gamma_2 k_2}{A_2} \\ 0 & \frac{(1-\gamma_2) k_2}{A_3} \\ \frac{(1-\gamma_1) k_1}{A_4} & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{\gamma_1 k_1 c_1}{1+sT_1} & \frac{(1-\gamma_2) k_2 c_1}{(1+sT_3)(1+sT_1)} \\ \frac{(1-\gamma_1) k_1 c_2}{(1+sT_4)(1+sT_2)} & \frac{\gamma_2 k_2 c_2}{1+sT_2} \end{bmatrix}, \end{aligned} \tag{15}$$

where $c_i = \frac{T_i k_c}{A_i}$.

2.3.2 System zeros

Future it is interesting to analyse the stability of the system by examining the natural placement of zeros. Using what is given in the laboratory instructions [3] we have that the zeros of $G(s)$ are given by zeros of

$$detG(s) = \frac{\gamma_1 \gamma_2 k_1 k_2 c_1 c_2}{\prod_{i=1}^{4}(1 + sT_i)} \left[(1 + sT_3)(1 + sT_4) - \frac{(1 - \gamma_1)(1 - \gamma_2)}{\gamma_1 \gamma_2} \right]. \tag{16}$$

The fraction in front of the bracket in Equation 16 will never be zero. We can therefore get the zeros by solving the second order equation

$$(1 + sT_3)(1 + sT_4) - \frac{(1 - \gamma_1)(1 - \gamma_2)}{\gamma_1 \gamma_2} = 0. \tag{17}$$

Solving for s gives

$$s = -\frac{T_3 + T_4}{2T_3 T_4} \pm \sqrt{\left(\frac{T_3 + T_4}{2T_3 T_4} \right)^2 + \frac{(1 - \gamma_1)(1 - \gamma_2)}{\gamma_1 \gamma_2 T_3 T_4} - \frac{1}{T_3 T_4}}. \tag{18}$$

We have that for $s < 0 \forall s$

$$\frac{T_3 + T_4}{2T_3T_4} < \sqrt{\left(\frac{T_3 + T_4}{2T_3T_4}\right)^2 + \frac{(1-\gamma_1)(1-\gamma_2)}{\gamma_1\gamma_2T_3T_4} - \frac{1}{T_3T_4}} \tag{19}$$

must be fulfilled. Using the fact that $T_3, T_4, \gamma_1, \gamma_2 > 0$ we have with some calculation that

$$\frac{(1-\gamma_1)(1-\gamma_2)}{\gamma_1\gamma_2} > 1 \tag{20}$$

and in turn that

$$1 > \gamma_1 + \gamma_2. \tag{21}$$

Since $\gamma_1, \gamma_2 \in [0, 1]$ this gives a condition on γ_1, γ_2 for the system to be minimum or non-minimum phase, with $0 < \gamma_1 + \gamma_2 < 1$ giving minimum phase and $1 \leq \gamma_1 + \gamma_2 \leq 2$ giving non-minimum phase.

2.3.3 Relative gain array

Studying the Relative gain array, RGA, we can observe the best strategy for decoupling the system. We therefore calculate the RGA at $s = 0$ analytically using the transfer matrix. Glad and Ljung [1] state that the RGA is given by $RGA = G \circ (G^{-1})^T$, with $\circ$ being the element-wise multiplication of matrices. For an arbitrary $2x2$ matrix

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{22}$$

we have

$$X \circ (X^{-1})^T = \begin{bmatrix} \frac{ad}{ad-bc} & -\frac{bc}{ad-bc} \\ -\frac{bc}{ad-bc} & \frac{ad}{ad-bc} \end{bmatrix} = \begin{bmatrix} \lambda & 1-\lambda \\ 1-\lambda & \lambda \end{bmatrix}, \tag{23}$$

with $\lambda = \frac{ad}{ad-bc}$. Using the static gain matrix

$$G(0) = \begin{bmatrix} \gamma_1k_1c_1 & (1-\gamma_2)k_2c_1 \\ (1-\gamma_1)k_1c_2 & \gamma_2k_2c_2 \end{bmatrix}. \tag{24}$$

we have

$$\lambda = \frac{\gamma_1\gamma_2k_1k_2c_1c_2}{\gamma_1\gamma_2k_1k_2c_1c_2 - (1-\gamma_1)(1-\gamma_2)k_1k_2c_1c_2} = \frac{\gamma_1\gamma_2}{\gamma_1\gamma_2 - (1-\gamma_1)(1-\gamma_2)} = \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2-1}. \tag{25}$$

The laboratory instructions [3] give $\gamma_1 = \gamma_2 = 0.625$ in the minimum phase case and $\gamma_1 = \gamma_2 = 0.375$ in the non-minimum phase case, giving the relative gain arrays

$$RGA_{min}(0) = \begin{bmatrix} 1.5625 & -0.5625 \\ -0.5625 & 1.5625 \end{bmatrix}, \tag{26}$$

$$RGA_{non-min}(0) = \begin{bmatrix} -0.5625 & 1.5625 \\ 1.5625 & -0.5625 \end{bmatrix}. \tag{27}$$

2.4 Determining constants

Some constants are given by the laboratory instructions [3]. These are presented together with the experimentally derived constants in Table 1.

We will start by finding values of k_1 and k_2 describing the linearity of change in voltage to change in flow-rate of the two pumps. These can be found by

1. Empty all tanks.
2. Close the outlet holes of tanks 3 and 4.
3. Set full voltage on the pumps (15 V).
4. Observe the rate of change of the water height, $\frac{dh_3}{dt}$ and $\frac{dh_4}{dt}$.
5. Calculate the constants according to

$$k_1 = \frac{A_4 \frac{dh_4}{dt}}{(1-\gamma_1)u_1}, \tag{28}$$

$$k_2 = \frac{A_3 \frac{dh_3}{dt}}{(1-\gamma_2)u_2}, \tag{29}$$

   using the constants of Table 1.

Now having the constants k_1 and k_2 we seek the hole areas of the tanks' outlet holes. These can be found by

1. Set half voltage (7.5 V) on both pumps.
2. Wait for the system to stabilize to an equilibrium.
3. Using the expressions of Equation 5 and Equation 6 and the constants of Table 1 we can calculate all the output hole areas using the equilibrium point of the system.

Table 1: Constants of the system. Some given by [3] and others derived according to experiments described in the text.

Constant	**Value**
A_i	$15.52\ cm^2$
k_c	$0.2\ \frac{V}{cm}$
$\gamma_{i,min}$	0.625
$\gamma_{i,nonmin}$	0.375
k_1	$4.32\ \frac{cm^3}{Vs}$
k_2	$3.74\ \frac{cm^3}{Vs}$
a_1	$0.1661\ cm^2$
a_2	$0.1702\ cm^2$
$a_{3,min}$	$0.0751\ cm^2$
$a_{3,nonmin}$	$0.1651\ cm^2$
$a_{4,min}$	$0.0708\ cm^2$
$a_{4,nonmin}$	$0.1829\ cm^2$

3 Control theory

3.1 Manual control

The problems were solved for both the minimum phase and the non-minimum phase, were the pumps were set at 50% voltage in the four-tank simulation process. The equilibrium heights were recorded for all four tanks and the results are compared in the table below between the analytical values and the measured values.

Table 2: Analytically calculated and measured values of the tanks given a manually set static control.

	u_1^0	u_2^0	h_1^0	h_2^0	h_3^0	h_4^0
Analytical Min-phase	7.5	7.5	17.5	15.5	10.0	15.0
Analytical Non-Min-phase	7.5	7.5	16.5	16.65	5.75	6.25
Measured Min-phase	7.37	7.59	17.0	15.75	10.25	14.5
Measured Non-Min-phase	8.15	7.78	16.5	14.25	5.00	5.13

The levels of voltage and heights for all cases are quite similar with higher deviations being in the non-minimum phase, since this case takes a lot of time to stabilize.

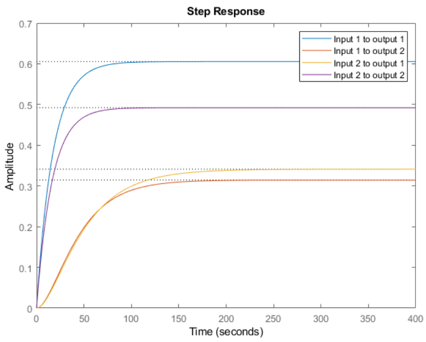

Figure 2: Step response for minimum phase

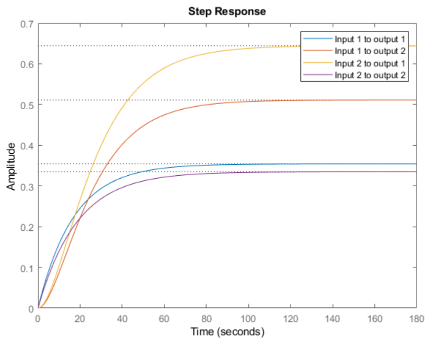

Figure 3: Step response for Non-minimum phase

The system seems to be coupled for minimum phase according to the RGA in matrices 26 and 30 from the analytical and measured cases. The diagonal elements are the same for both cases in the non-minimum phase cases according to 27 and 31,but a slight difference appears cross diagonal elements and this shows in the step response figure 3 where the step responses are inverted. The transient time for the minimum phase case is 112 seconds and for the non-minimum phase 84 seconds.

$$RGA_{min}(0) = \begin{bmatrix} 1.5625 & -0.5625 \\ -0.5625 & 1.5625 \end{bmatrix}, \tag{30}$$

$$RGA_{nonmin}(0) = \begin{bmatrix} -0.5625 & 1.2389 \\ 1.9706 & -0.5625 \end{bmatrix} \tag{31}$$

3.2 Decentralized control

The transfer function G(s) is designed using the RGA properties for the decentralised controller, where it should be stable in each loop and the pairings should be designed by firstly making the diagonal elements of $RGA(G(i\omega_c))$ very close to 1 in the complex plane and also avoiding the negative diagonal elements for $RGA(G(0))$.

Minimum phase

The controller from input to output for the minimum phase system is:

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}. \tag{32}$$

where,

$$F_{11} = \frac{1.579s^4 + 0.3771s^3 + 0.01835s^2 + 0.0002509s + 9.767 \times 10^{-22}}{s^4 + 0.06002s^3 + 0.0008886s^2 + 4.336 \times 10^{-22}s}, \tag{33}$$

$$F_{12} = \frac{-0.02904s^3 - 0.006167s^2 - 0.0001739s - 7.594 \times 10^{-22}}{s^4 + 0.06002s^3 + 0.0008886s^2 + 4.336 \times 10^{-22}s}, \tag{34}$$

$$F_{21} = \frac{-0.03063s^3 - 0.006665s^2 - 0.0001553s + 2.147 \times 10^{-21}}{s^4 + 0.06002s^3 + 0.0008886s^2 + 4.336 \times 10^{-22}s}, \tag{35}$$

$$F_{22} = \frac{1.761s^4 + 0.4422s^3 + 0.02176s^2 + 0.000299s + 4.76 \times 10^{-22}}{s^4 + 0.06002s^3 + 0.0008886s^2 + 4.336 \times 10^{-22}s} \tag{36}$$

Non-minimum phase

The controller from input to output for the non-minimum system is:

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}. \tag{37}$$

where,

$$F_{11} = \frac{-1.129s^2 - 0.1728s - 0.006183}{s^2 + 0.2s}, \tag{38}$$

$$F_{12} = \frac{0.2089s + 0.0119}{s^2 + 0.2s}, \tag{39}$$

$$F_{21} = \frac{0.1789s + 0.00944}{s^2 + 0.2s}, \tag{40}$$

$$F_{22} = \frac{-1.309s^2 - 0.1931s - 0.006543}{s^2 + 0.2s} \tag{41}$$

3.3 Robust control

The controllers for the robust control are designed according to computer exercise 4 [2]. Using Glover-McFarlane method, the robustness magnitude lies powerfully in the measurement of γ. The controllers will not be presented explicitly because the degrees in the elements vary for each computer and are quite high but the method is explained below.

The nominal loop gain is designed with the following equation,

$$L_0(s) = G(s)W_1(s)\tilde{F}(s) \tag{42}$$

The method adds a link $F_r(s)$ to the loop gain such that the controller is

$$F(s) = W_1(s)\tilde{F}(s)F_r(s) \tag{43}$$

where the weight W_1 is the pre-compensation designed such that the correct cross-over frequency ω_c is achieved as well as better decoupling and closeness of the singular values around the crossover.

$$W_1 = \begin{bmatrix} 1 & -g_{21}(s)/g_{22}(s) \\ -g_{12}(s)/g_{11}(s) & 1 \end{bmatrix} \tag{44}$$

and the controller $\tilde{F}$ is given by :

$$\tilde{F} = \begin{bmatrix} \tilde{f}_1 & 0 \\ 0 & \tilde{f}_2 \end{bmatrix} \tag{45}$$

The final controller is designed with the Glover-McFarlane link F_r is seen in equation 43, which is derived in MATLAB according to Computer Exercise 4 [2].

4 Results

The results were produced by a four tank simulation tool in MATLAB. The simulation tool represents the four water tanks as seen in Figure 1. The tool provides the options for setting reference levels when using the computed controllers as feedback loops for the system, but also comes with functions that can simulate various disturbances. The tests in the simulation tool investigates the stability and response of the controllers when they get affected by a step response and disturbances by adding or removing water from the tank.

4.1 Decentralized Control

Minimum phase

For the dynamical decoupled controls as presented in section 3.2, the step responses for each input of the minimum phase system, $u1$ and $u2$, can be seen in Figure (4) and (5). The disturbances induced on the system can be seen in Figure (6)(7)(8)(9).

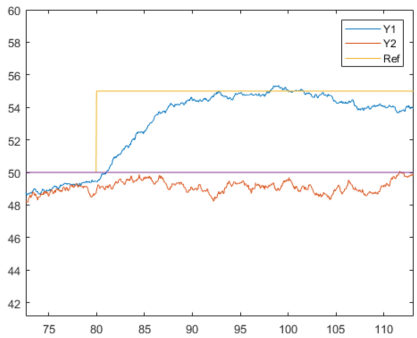

Figure 4: Step response on u1.

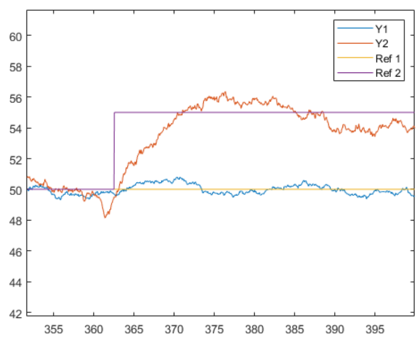

Figure 5: Step response on u2.

	Value
Overshoot Step in u_1 minimum phase	10%
Overshoot Step in u_2 minimum phase	30%
Rise time Step in u_1 minimum phase	8.5s
Rise time Step in u_2 minimum phase	6.6s

Table 3: Overshoot and rise time of the dynamical decoupled system for the minimum phase.

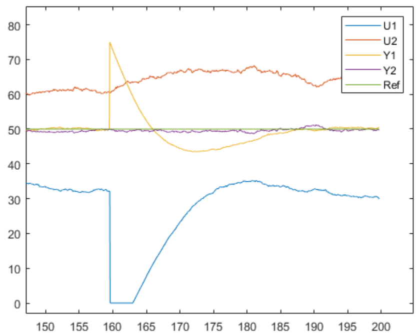

Figure 6: Adding water to tank 1.

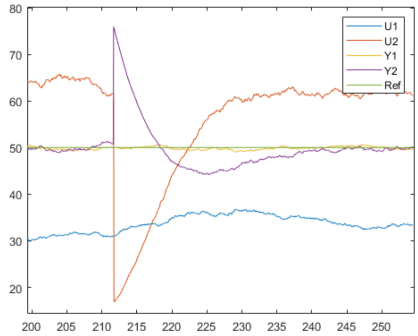

Figure 7: Adding water to tank 2.

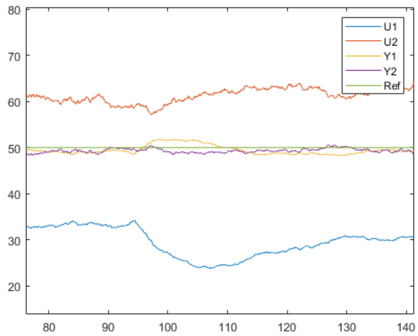

Figure 8: Opening outlet of tank 3.

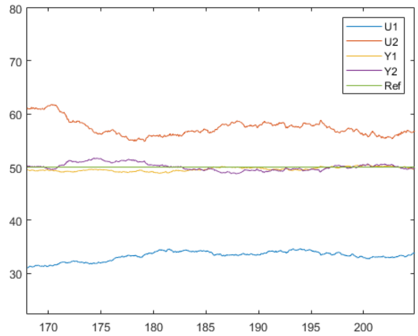

Figure 9: Opening outlet of tank 4.

Looking at the results from the disturbance tests of the minimum case, it can be observed that pouring water into the tanks causes the controller to turn of the pump momentarily and increase the pump output corresponding to the slope of the water-level in the tank. This keeps up until the tank reached the reference again. Opening any of the tank outlets on the top tanks, to let more water flow out into the bottom ones, results in a slight decrease of corresponding bottom tank pumps.

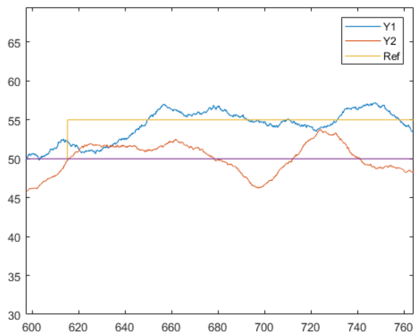

Figure 10: Step response on u1.

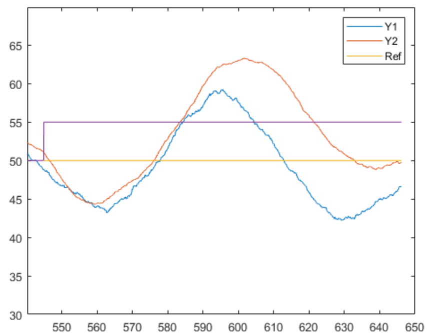

Figure 11: Step response on u2.

Non-minimum phase

For the non-minimum phase, the step responses for each input, $u1$ and $u2$, can be seen in Figure (10) and (11). The disturbances induced on the system can be seen in Figure (12)(13)(14)(15).

Constant	**Value**
Overshoot Step in u_1 non-minimum phase	44%
Overshoot Step in u_2 non-minimum phase	165%
Rise time Step in u_1 non-minimum phase	13s
Rise time Step in u_2 non-minimum phase	5s

Table 4: Overshoot and rise time of the dynamical decoupled system for the non-minimum phase.

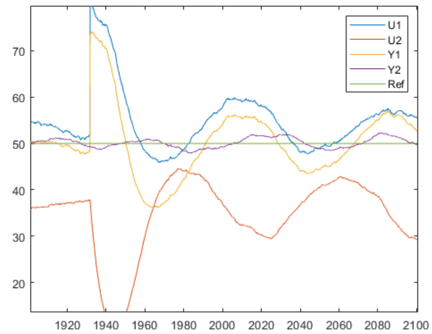

Figure 12: Adding water to tank 1.

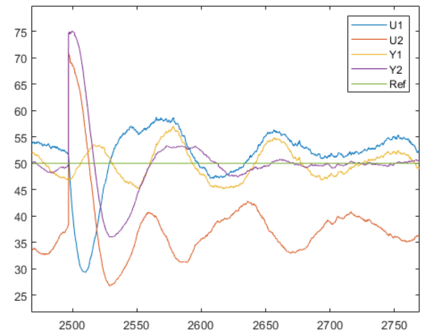

Figure 13: Adding water to tank 2.

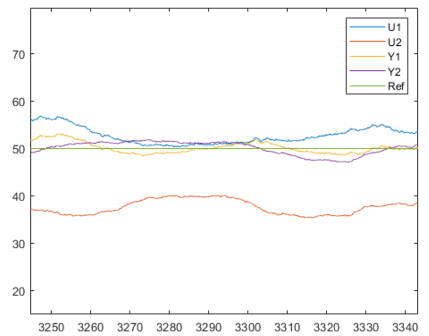

Figure 14: Opening outlet of tank 3.

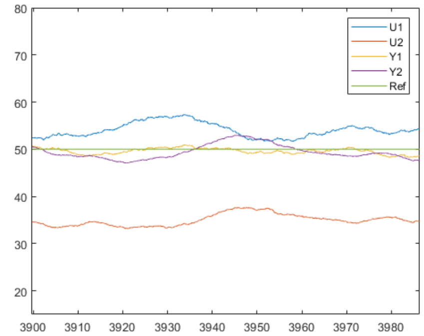

Figure 15: Opening outlet of tank 4.

Looking at the results from the disturbance tests of the non-minimum case, it can be observed that pouring water into the tanks causes the controller to turn of the pump momentarily and increase the pump output corresponding to the slope of the water-level in the tank. This is the same behaviour as what we observed for the minimum phase. However, this time the controller takes much longer time to stabilise the system as it starts to oscillate. Opening any of the tank outlets on the top tanks, to let more water flow out into the bottom ones, does not produce any clear results, however some observations can be made. The inputs to one of the pumps seem to increase output while the other decreases. The water level in the tanks barely see any change.

4.2 Robust Control

Minimum phase

For the dynamical decoupled controls as presented in section 3.3, the step responses for each input of the minimum phase system, $u1$ and $u2$, can be seen in Figure (16) and (17). The disturbances induced on the system can be seen in Figure (18)(19)(20)(21).

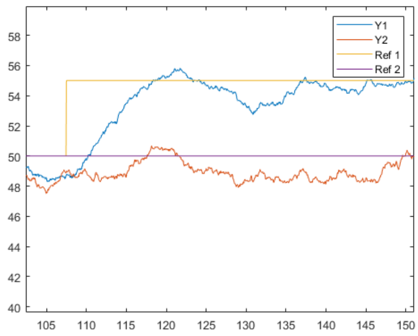

Figure 16: Step response on u1.

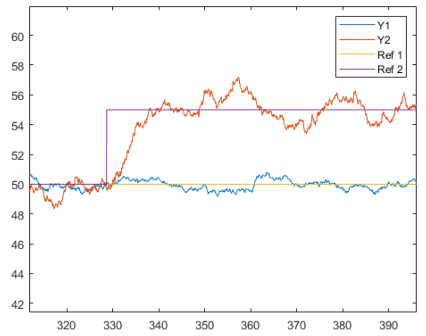

Figure 17: Step response on u2.

Constant	Value
Overshoot Step in u_1 minimum phase	16%
Overshoot Step in u_2 minimum phase	45%
Rise time Step in u_1 minimum phase	7s
Rise time Step in u_2 minimum phase	7.5s

Table 5: Overshoot and risetime of the Glover-Mcfarlane Robust controlled system for the minimum phase.

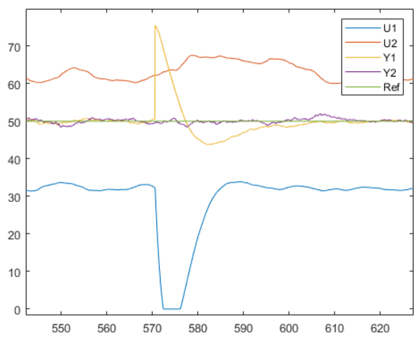

Figure 18: Adding water to tank 1.

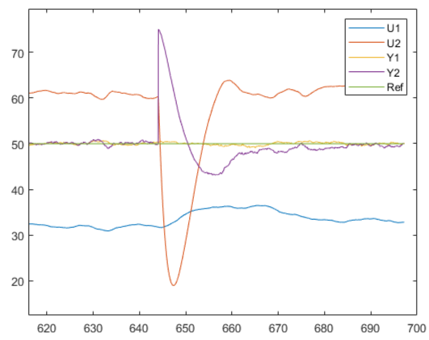

Figure 19: Adding water to tank 2.

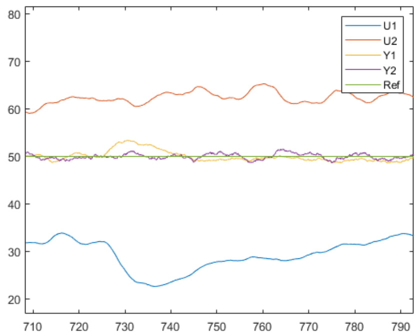

Figure 20: Opening outlet of tank 3.

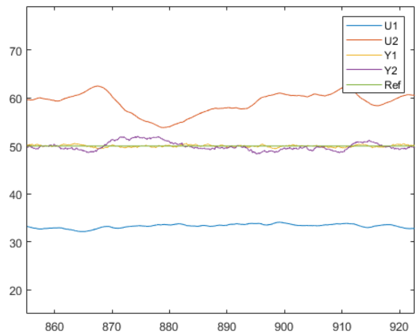

Figure 21: Opening outlet of tank 4.

Looking at the results from the disturbance tests of the minimum case, it can be observed that pouring water into the tanks causes the controller to turn of the pump momentarily and increase the pump output corresponding to the slope of the water-level in the tank. This keeps up until the tank reached the reference again. Opening any of the tank outlets on the top tanks, to let more water flow out into the bottom ones, results in a slight decrease of corresponding bottom tank pumps. This is the same results as the ones that was observed in the dynamical decoupled case.

Non-minimum phase

For the non-minimum phase, the step responses for each input, $u1$ and $u2$, can be seen in Figure (22) and (23). The disturbances induced on the system can be seen in Figure (24)(25)(26)(27).

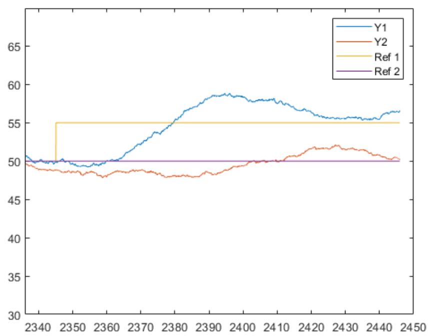

Figure 22: Step response on u1.

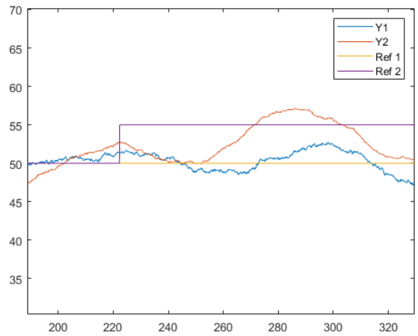

Figure 23: Step response on u2.

Constant	Value
Overshoot Step in u_1 nonminimum phase	80%
Overshoot Step in u_2 nonminimum phase	60%
Risetime Step in u_1 nonminimum phase	14s
Risetime Step in u_2 nonminimum phase	15s

Table 6: Overshoot and risetime of the Glover-Mcfarlane Robust controlled system for the non-minimum phase.

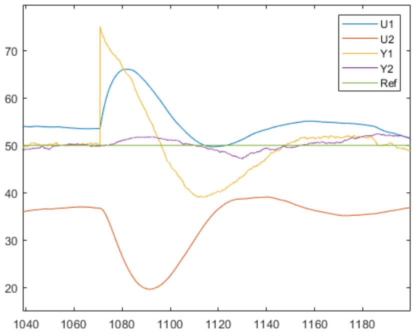

Figure 24: Adding water to tank 1.

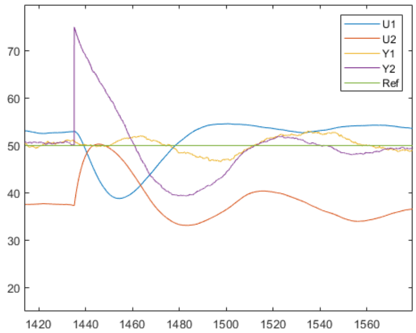

Figure 25: Adding water to tank 2.

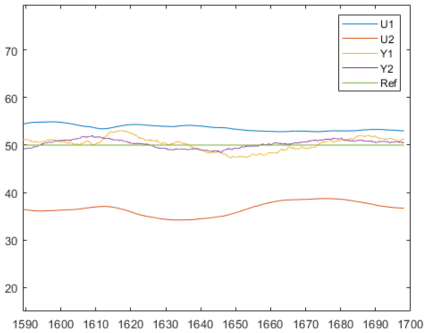

Figure 26: Opening outlet of tank 3.

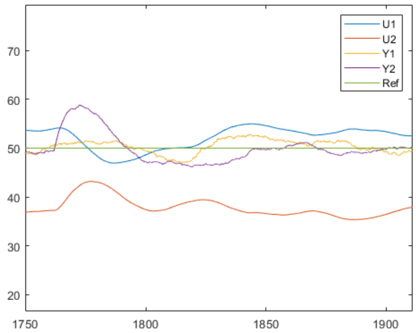

Figure 27: Opening outlet of tank 4.

Looking at the results from the disturbance tests of the non-minimum case, it can be observed that pouring water into one of the tanks causes the corresponding pump to decrease the output, but at the same time the other pump, increases the output. Opening the tanks in the non-minimum phase case, causes both pumps to increase it's output.

5 Discussion

In the manual control, the major differences between the two phases are that the step responses are inverted in the non-minimum phase case. The non-minimum phase gives larger deviations in the heights between the measured and analytical values than the minimum phase. This behaviour is expected and can be understood by the fact that when γ_1 and γ_2 is small an error in the hole-sizes, a_i, does affect the error of the heights more as seen when analyzing Equation 5.

The controllers behave differently depending on whether the controller is decoupled or if robust Glover-McFarlane control is added. The results of the system also varies if the system is a minimum or a non-minimum phase. In this case we argue that it is not entirely feasible to compare the systems of the minimum and the non-minimum phase. There are big differences in the overshoot between the minimum and the non-minimum phase. The overshoot and rise time for the minimum phase is good, with an overshoot of max 30% on the u_2 step response, as shown in Table 3. However, in the non-minimum phase case the overshoot increases dramatically to 165%, as shown in Table 4. However, due to the oscillations of the non-minimum phase it is hard to tell where from the rise time should be calculated.

Adding robust control to the minimum phase does not affect the system significantly. We were expecting to see a decrease in overshoot, but the values of the overshoot is similar between the decoupled and the robust controller. When it comes to the non-minimum phase however, adding the robust Glover-McFarlane controller decreases the overshoot of the system, but at the same time increases the rise time, which is in accordance with our expectations.

When adding water to the tanks we get the same response from the decoupled controller and the robust controller. However, the way the controllers act are different between the minimum phase and the non-minimum phase. In the minimum phase, when adding water to tank 1, it causes pump one to go down to zero and increase from that. The same goes for tank 2 and pump 2. When adding water in the non-minimum phase system, the reaction of the pumps is of opposite sign. This behavior could be understood by the different RGA matrices presented in Equations 26 and 27. We see that in the minimum phase case the diagonal elements are positive and thus the inputs are connected to their respective outputs. In the non-minimum phase case the RGA matrix has a negative diagonal and each input will therefore mainly react inversely to their respective output, as shown in Figures 12 and 24. Albeit the difference in behaviour in both the minimum phase case and the non-minimum phase case the controller managed to cope with the disturbance, but in the non-robust case of the non-minphase that stabilization is very slow and the disturbance is a source of a lengthy process of oscillations.

When introducing disturbances by opening the outlets of the top tanks, the system compensates by reducing the input for the pump providing the tank below with water. This happens because the pump tries to compensate for the increase water flow to the lower tanks. This behaviour is the same for both the minimum and the non-minimum phase case with both controllers, with the exception for the outlet of tank 4 when dealing with the non-minimum phase robust controller as shown in Figure 27. Here the input for u_1 decreases and the input for u_2 increases.

The disturbances eliminates quite fast in the minimum phase case, but in the non-minimum phase the system becomes very oscillatory and takes very long time to properly stabilize.

6 Conclusion

The best controller for the different cases has to be put into consideration depending on weather they are used with our minimum phase or non-minimum phase system. For the minimum phase case, the differences in system response, disturbance attenuation and robustness are almost as good with just the decoupled controller as with the robust controller. Because of this, it would be better to just run with the decoupled controller to reduce implementation cost of the controller. However, for the non-minimum phase, the added robustness with the Glover-McFarlane controller performs better than the decoupled. The overshoot is less and the disturbance attention is faster and more stable. In this case the robust controller would be best to implement.

References

[1] Torkel Glad. *Reglerteori : flervariabla och olinjära metoder*. Studentlitteratur, Lund, 2. uppl.. edition, 2003.

[2] Anders Hansson, Alf Isaksson, and Magnus Jansson. Computer exercise: Decoupling & glover-mcfarlane robustloop-shaping, 1999.

[3] Anders Hansson, Ola Markusson, and Magnus Åkerblad. Laboratory experiment: The four-tank process, 1999. Last revision by Y. Wang (2020).