Section 12: Hypothesis testing

ARE 210

November 21, 2017

1. Assume $\{X_i\}_{i=1}^n$ iid $P \in \mathbf{P} = \{P : V_P[X_i] = \sigma^2\}$. Let $\mathbf{P}_H = \{P : V_P[X_i] = \sigma^2, \mathbf{E}_P[X_i] = \mu\}$, representing the null hypothesis that the mean of X_i is μ.

   (a) Calculate $V[\overline{X}]$, where $\overline{X} = \sum_{i=1}^n X_i/n$.

   $V[\overline{X}] = \frac{\sigma^2}{n}$

   (b) Assume $P \in \mathbf{P}_H$. Place a lower bound on $P\left[\left|(\overline{X} - \mu)/\sqrt{V[\overline{X}]}\right| > t\right]$.

   Applying Chebyshev, $P\left[\left|(\overline{X} - \mu)/(\sigma/\sqrt{n})\right| > t\right] \leq t^2$

   (c) Let $\delta_\alpha = \mathbf{1}\left\{\left|(\overline{X} - \mu)/\sqrt{V[\overline{X}]}\right| > t_{1-\alpha/2}\right\}$. Derive the maximum threshold $t_{1-\alpha/2}$ such that δ_α is size α. Compare this threshold to $Z_{1-\alpha/2}$.[1]

   By the above result, $\mathbf{E}_P[\delta_\alpha] \leq (t_{1-\alpha/2})^{-2}$. Solving $\mathbf{E}_P[\delta_\alpha] = \alpha$ yields $t_{1-\alpha/2} = \alpha^{-1/2}$. As a comparison, for $\alpha = .05$, this yields $Z_{1-\alpha/2} \approx 2$ and $t_{1-\alpha/2} \approx 4.5$.

   (d) δ_α is not unbiased, provide a counter example. What intuition does this give you about the set of possible unbiased tests in this context? Think about what bdry($\mathbf{P}_H$) looks like.

   No. For intuition, note that bdry($\mathbf{P}_H$) $= \mathbf{P}_H$, so a continuous unbiased test would have to have power .05 for every possible distribution in $\mathbf{P}_H$, but many of these distributions are impossible to distinguish in any finite sample from a distribution just outside $\mathbf{P}_H$.
   For a counter example, consider a random variable X with pmf $P[X = \mu] = 1 - \epsilon$ and $P[X = \mu + k_\epsilon] = \epsilon$, where k_ϵ solves $V[X] = k_\epsilon^2\epsilon(1-\epsilon) = \sigma^2$. $P[\overline{X} \neq \mu] = 1 - (1-\epsilon)^n \leq n\epsilon$. Set $\epsilon = \alpha/(2n)$. Then $\mathbf{E}_P[\delta_\alpha] =\leq \alpha/2 < \alpha$. Since Chebyshev's inequality holds with equality for a discrete random variable that's $\mu + k$ with probability 0.5 and $\mu - k$ with probability 0.5, δ_α is size α. So δ_α is not unbiased.

[1] Recall $Z_{.84} \approx 1$, $Z_{.97} \approx 2$, $Z_{.998} \approx 3$.

(e) Construct useful bounds on $\mathbf{E}_P[\delta_\alpha]$, for $P \in \mathbf{P}_{\mu+k} \equiv \{P : \mathbf{E}_P[X_i] = \mu + k\}$.

First, we apply the proof to Chebyshev to get an upper bound.

$$\begin{aligned} P\left[\left|\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\right| > t_{1-\alpha/2}\right] &\leq \frac{\mathbf{E}[(\overline{X}-\mu)^2]}{(\sigma^2/n)(t_{1-\alpha/2})^2} \\ &= \alpha\frac{\sigma^2/n + k^2}{\sigma^2/n} \\ &= \alpha(1 + \frac{k^2}{\sigma^2/n}) \end{aligned}$$

Second... without stronger assumptions (e.g. common bounded support, finite higher moments, ...), I don't think a lower bound exists (besides trivially 0).