Section 2: Random variables (solutions)

ARE 210

September 5, 2017

1) **Probability of complement**: Show that $P(A^c) = 1 - P(A)$.

By the law of total probability, $P(A^c) + P(A) = 1$. Subtracting yields $P(A^c) = 1 - P(A)$.

2) **Conditional Bayes' Theorem**: Show that $P(A|B, C) = \frac{P(A,B|C)}{P(B|C)}$

$$\begin{aligned} P(A|B,C) &= \frac{P(A,B,C)}{P(B,C)} \\ &= \frac{P(A,B|C)P(C)}{P(B|C)P(C)} \\ &= \frac{P(A,B|C)}{P(B|C)} \end{aligned}$$

3) **Mixed random variables**: Let (X_1, X_2) be random variables with joint distribution F. How would you calculate the marginal distribution of X_1?

$$F_1(x_1) = \lim_{x_2 \to \infty} F(x_1, x_2)$$

$$f_1(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2) dx_2$$

4) **Independence and densities**: Let X_1, X_2 be independent continuous random variables. Show that their joint density is equal to the product of their marginal densities, and assume for simplicity that each has a continuous marginal density.

First, their joint distribution $F(x_1, x_2) = F_1(x_1)F_2(x_2)$. Substituting for the definitions of the CDF in terms of the density and merging the integration yields $F(x_1, x_2) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f_1(\widetilde{x_1}) f_2(\widetilde{x_2}) d\widetilde{x_2} d\widetilde{x_1}$, which implies that $f(x_1, x_2) = f_1(x_1)f_2(x_2)$.

5) **Distribution practice**: Let X_1, X_2 be standard normal random variables, each with density $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.
a) What is the joint distribution F of X_1 and X_2? What is the joint density f?

$$F(x_1, x_2) = \Phi(x_1)\Phi(x_2) = \frac{1}{2\pi} \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \exp\left\{-\frac{\widetilde{x_1}^2 + \widetilde{x_2}^2}{2}\right\} d\widetilde{x_2} d\widetilde{x_1}$$

$$f(x_1, x_2) = \phi(x_1)\phi(x_2) = \frac{1}{2\pi} \exp\left\{-\frac{\widetilde{x_1}^2 + \widetilde{x_2}^2}{2}\right\}$$

b) What is the distribution F_Y of $Y = X_1 + X_2$? What is the density f_Y?

Let ϕ be the standard normal density and Φ be the standard normal CDF.

$$\begin{aligned} F_Y(y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{y - x_1} \phi(x_1)\phi(x_2) dx_2 dx_1 \\ &= \int_{-\infty}^{\infty} \phi(x_1)\Phi(y - x_1) dx_1 \end{aligned}$$

Differentiating with respect to y yields

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} \phi(x_1)\phi(y - x_1) dx_1 \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left\{-\frac{x_1^2}{2}\right\} \exp\left\{-\frac{(y - x_1)^2}{2}\right\} dx_1 \\ &= \frac{1}{2\sqrt{\pi}} \exp\left\{-\frac{y^2}{4}\right\} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} \exp\left\{-\left(x_1 - \frac{y}{2}\right)^2\right\} dx_1 \\ &= \frac{1}{2\sqrt{\pi}} \exp\left\{-\frac{y^2}{4}\right\} \end{aligned}$$

Integrating again yields

$$F_Y(y) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{y} \exp\left\{-\frac{\widetilde{y}^2}{4}\right\} d\widetilde{y}$$