Section 7: Identification and MLE (solutions)

ARE 210

October 17, 2017

1) Let $Y \sim \text{Bern}(p)$, and suppose we observe $\{Y_i\}_{i=1}^n$ iid Y.

a) Construct the MLE of p, $\widehat{p}_{MLE}$.

$P_p[Y_1, \ldots, Y_n] = p^{\sum_i Y_i}(1-p)^{n-\sum_i Y_i}$. Taking logs yields $\log P_p[Y_1, \ldots, Y_n] = (\sum_i Y_i) \log p + (n - \sum_i Y_i) \log(1-p)$. Taking derivatives with respect to p, and setting equal to 0 to maximize yields $\hat{p}_{MLE} = \sum_{i=1}^n Y_i / n$.

b) What is the probability distribution of $\widehat{p}_{MLE}$? Note that this case is relatively unique in that the distribution of this estimator has a closed form solution.

$n\hat{p} \sim \text{Binomial}[n, p]$. Written alternatively, $P[\hat{p} = k/n] = \binom{n}{k} p^k (1-p)^{n-k}$.

c) Derive the asymptotic distribution of $\widehat{p}_{MLE}$.

$\sqrt{n}(\hat{p}_n - p) \to N(0, p(1-p))$.

2) (Problem set 4, Question 7) Consider the following model for an independently distributed sample $\{Y_i\}_{i=1}^n = \{Y_{i1}, Y_{i2}\}_{i=1}^n$ where each Y_{ij} is a binary valued random variable and $Y_{ij} \sim \text{Bernoulli}(\mu_{ij})$ for $i = 1, \ldots, n$. Suppose that (using the logit function, $\ell(\mu) = \log \frac{\mu}{1+\mu}$),

$$\ell(\mu_{ij}) = \beta_1 + \beta_2 z_{ij} + \theta_i$$

for a sequence of known constants $\{z_{i1}, z_{i2}\}_{i=1}^n$ and unknown constants $\{\theta_i\}_{i=1}^n$. Consider the statistic $T(Y_i) = Y_{i1} + Y_{i2}$ and show that the probability

$$P(Y_{i1} = a, Y_{i2} = b | T(Y_i) = 1)$$

does not depend upon θ_i (the θ_i are known as <u>fixed effects</u>).

First, note that $\mu_{ij} = \frac{\exp(\beta_1 + \beta_2 z_{ij} + \theta_i)}{1 + \exp(\beta_1 + \beta_2 z_{ij} + \theta_i)}$. Next, since $a = 1 - b$, we have $P(Y_{i1} = a, Y_{i2} = b | T(Y_i) = 1) = \frac{[\mu_{i1}(1-\mu_{i2})]^a [\mu_{i2}(1-\mu_{i1})]^b}{\mu_{i1}(1-\mu_{i2}) + \mu_{i2}(1-\mu_{i1})}$. Simplifying this yields $P(Y_{i1} = a, Y_{i2} = b | T(Y_i) = 1) = \frac{\exp(\beta_2 z_{i1})^a \exp(\beta_2 z_{i1})^b}{\exp(\beta_2 z_{i1}) + \exp(\beta_2 z_{i2})}$, which does not depend on θ_i.

3) (Midterm 2016, Question 7) Consider the following two-stage experiment. In the first stage we draw a random variable X with support $[0,1]$. In the second stage we toss a coin thrice independently with the probability of heads in each toss given by the observed value of X. To fix ideas, let the density function of X be given by

$$f(x) = 6x(1-x)\mathbf{1}\{x \in [0,1]\}$$

Let Y denote the number of heads in three independent tosses of a coin with probability x of heads (where $X = x$).

a) (4 points) What is the support of the vector of random variables (X, Y)? Does the joint distribution have any atoms?

$([0,1] \times \{0,1,2,3\}) \setminus \{(0,1),(0,2),(0,3),(1,0),(1,1),(1,2)\}$. The joint distribution has no atoms.

b) (6 points) What is the density (mass) function $p_Y(y)$ of Y (i.e. the marginal distribution of Y)? Hint: Note that

$$\int_0^1 x^{a-1}(1-x)^{b-1}dx = \frac{(a-1)!(b-1)!}{(a+b-1)!}$$

for $a, b \in \mathbf{N}_+$. Compute $p_Y(0)$.

$$\begin{aligned} p_Y(y) &= \int_0^1 \binom{3}{y} x^y (1-x)^{3-y} P[dx] \\ &= \binom{3}{y} \int_0^1 6x^{y+1}(1-x)^{4-y}dx \\ &= \frac{(y+1)(4-y)}{20} \end{aligned}$$

$p_Y(0) = \frac{1}{5}$.

c) (2 points) What is $\mathbf{E}(Y|X)$?

$\mathbf{E}[Y|X] = 3X$

d) (2 points) What is $\mathrm{Var}(Y|X)$?

$\text{Var}(Y|X) = 3X(1-X)$

e) (8 points) Use the variance equality

$$\text{Var}(Y) = \mathbf{E}(\text{Var}(Y|X)) + \text{Var}(\mathbf{E}(Y|X))$$

to compute the variance of Y.

First, $\mathbf{E}[X] = \frac{1}{2}$. Second, $\mathbf{E}[X^2] = \int_0^1 (6x^3 - 6x^4)dx = \frac{3}{10}$. This implies $\text{Var}[X] = \frac{1}{20}$, and that $\text{Var}[\mathbf{E}[Y|X]] = \frac{9}{20}$. Next, $\mathbf{E}[X(1-X)] = \int_0^1 6x^2(1-x)^2 dx = \frac{1}{5}$, which implies $\mathbf{E}[\text{Var}(Y|X)] = \frac{3}{5}$. Together this gives $\text{Var}[Y] = \frac{3}{5} + \frac{9}{20} = \frac{21}{20}$.