Section 4: Vector random variables (solutions)

ARE 210

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1) Derive the characteristic function for a multivariate normal random variable X with mean $\mu_{n\times 1}$ and variance $V_{n\times n}$

We're interested in $\Psi_X(r^{\mathrm{T}}) \equiv \mathbf{E}[\exp(ir^{\mathrm{T}}X)]$. Since X is MVN, $r^{\mathrm{T}}X \sim N(r^{\mathrm{T}}\mu, r^{\mathrm{T}}Vr)$. Further, note that, for some random variable Y, $\Psi_{aY+b}(t) = \exp(ibt)\Psi_Y(at)$. Therefore

$$\begin{aligned}
\Psi_X(r^{\mathrm{T}}) &= \Psi_{r^{\mathrm{T}}X}(1) \\
&= \exp(ir^{\mathrm{T}}\mu)\Psi_{r^{\mathrm{T}}X-r^{\mathrm{T}}\mu}(1) \\
&= \exp(ir^{\mathrm{T}}\mu)\Psi_{\frac{r^{\mathrm{T}}X-r^{\mathrm{T}}\mu}{\sqrt{r^{\mathrm{T}}Vr}}}(\sqrt{r^{\mathrm{T}}Vr})
\end{aligned}$$

However, $\frac{r^{\mathrm{T}}X-r^{\mathrm{T}}\mu}{\sqrt{r^{\mathrm{T}}Vr}} \sim N(0,1)$. Looking up the characteristic function of the standard normal ($\exp(-t^2/2)$), this yields $\Psi_X(r^T) = \exp(ir^{\mathrm{T}}\mu - r^{\mathrm{T}}Vr/2)$.

2) **Bayesian updating**: Suppose I have a prior belief that some population parameter $\Theta \sim N(0,1)$. Suppose I observe an unbiased signal of Θ, T, with normally distributed noise with variance σ_T^2. What are $f_{T|\Theta}(t|\theta)$ and $f_\Theta(\theta)$? Derive $f_T(t)$ and $f_{\Theta|T}(\theta|t)$. Interpret $f_{\Theta|T}(\theta|t)$.

First, $f(T|\Theta) = \frac{1}{\sigma_T}\phi\left(\frac{t-\theta}{\sigma_T}\right)$. Second, $f_\Theta(\theta) = \phi(\theta)$. Next, note that $f_{T,\Theta}(t,\theta) = \frac{1}{\sigma_T}\phi\left(\frac{t-\theta}{\sigma_T}\right)\phi(\theta)$.

Integrating, $f_T(t) = \int_{-\infty}^{\infty} f_{T,\Theta}(t,\theta)d\theta = \frac{1}{\sqrt{1+\sigma_T^2}}\phi\left(\frac{t}{\sqrt{1+\sigma_T^2}}\right)$. Applying the continuous equivalent of Bayes' Rule, $f_{\Theta|T}(\theta|t) = \frac{f_{T,\Theta}(t,\theta)}{f_T(t)} = \frac{1}{\sigma_t/\sqrt{1+\sigma_t^2}}\phi\left(\frac{\theta-\frac{t}{1+\sigma_t^2}}{\sigma_t/\sqrt{1+\sigma_t^2}}\right)$. Note that $\Theta|T \sim N\left(\frac{t}{1+\sigma_t^2}, \frac{\sigma_t^2}{1+\sigma_t^2}\right)$.

This is the posterior distribution. Its mean is a weighted average of the old mean (0) and the signal (t), where the weights are the inverses of the variances. The new variance is just the variance of this weighted average.