Section 2: Random variables (solutions)

ARE 210

September 5, 2017

1) **Probability of complement**: Show that $P(A^c) = 1 - P(A)$.

By the law of total probability, $P(A^c) + P(A) = 1$. Subtracting yields $P(A^c) = 1 - P(A)$.

2) **Conditional Bayes' Theorem**: Show that $P(A|B,C) = \frac{P(A,B|C)}{P(B|C)}$

$$\begin{aligned} P(A|B,C) &= \frac{P(A,B,C)}{P(B,C)} \\ &= \frac{P(A,B|C)P(C)}{P(B|C)P(C)} \\ &= \frac{P(A,B|C)}{P(B|C)} \end{aligned}$$

3) **Mixed random variables**: Let (X_1, X_2) be mixed random variables with joint distribution $F = \alpha F^C + (1-\alpha)F^D$, and joint density $f = \alpha f^C + (1-\alpha) f^D$, where F^C is a continuous CDF, F^D is a discrete CDF, f^C is the density for a continuous RV, and f^D is the density for a discrete RV (i.e. a sum of dirac functions). How would you calculate the CDF of X_1? the density of X_1?

$$F_1(x_1) = \lim_{x_2 \to \infty} F(x_1, x_2)$$

$$f_1(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2) dx_2$$

4) **Independence and densities**: Let X_1, X_2 be independent continuous random variables. Show that their joint density is equal to the product of their marginal densities, and assume for simplicity that each has a continuous marginal density.

First, their joint distribution $F(x_1,x_2)=F_1(x_1)F_2(x_2)$. Substituting for the definitions of the CDF in terms of the density and merging the integration yields $F(x_1,x_2)=\int_{-\infty}^{x_1}\int_{-\infty}^{x_2} f_1(\widetilde{x_1})f_2(\widetilde{x_2})d\widetilde{x_2}d\widetilde{x_1}$, which implies that $f(x_1,x_2)=f_1(x_1)f_2(x_2)$.

5) **Distribution practice**: Let X_1, X_2 be standard normal random variables, each with density $f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$.
a) What is the joint distribution F of X_1 and X_2? What is the joint density f?

$$F(x_1,x_2)=\Phi(x_1)\Phi(x_2)=\frac{1}{2\pi}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\exp\left\{-\frac{\widetilde{x_1}^2+\widetilde{x_2}^2}{2}\right\}d\widetilde{x_2}d\widetilde{x_1}$$

$$f(x_1,x_2)=\phi(x_1)\phi(x_2)=\frac{1}{2\pi}\exp\left\{-\frac{\widetilde{x_1}^2+\widetilde{x_2}^2}{2}\right\}$$

b) What is the distribution F_Y of $Y=X_1+X_2$? What is the density f_Y?

Let ϕ be the standard normal density and Φ be the standard normal CDF.

$$\begin{aligned}F_Y(y)&=\int_{-\infty}^{\infty}\int_{-\infty}^{y-x_1}\phi(x_1)\phi(x_2)dx_2dx_1\\&=\int_{-\infty}^{\infty}\phi(x_1)\Phi(y-x_1)dx_1\end{aligned}$$

Differentiating with respect to y yields

$$\begin{aligned}f_Y(y)&=\int_{-\infty}^{\infty}\phi(x_1)\phi(y-x_1)dx_1\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left\{-\frac{x_1^2}{2}\right\}\exp\left\{-\frac{(y-x_1)^2}{2}\right\}dx_1\\&=\frac{1}{2\sqrt{\pi}}\exp\left\{-\frac{y^2}{4}\right\}\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}\exp\left\{-\left(x_1-\frac{y}{2}\right)^2\right\}dx_1\\&=\frac{1}{2\sqrt{\pi}}\exp\left\{-\frac{y^2}{4}\right\}\end{aligned}$$

Integrating again yields

$$F_Y(y) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{y} \exp\left\{-\frac{\widetilde{y}^2}{4}\right\} d\widetilde{y}$$