CS 476/676: Assignment 2

Winter 2020

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Lecture Times :	MW 2:30-3:50	E2 1732
Yuying Li, OH :	Thursday 10:00-11:00	DC3623
Chendi Ni, OH	chendi.ni@uwaterloo.ca	DC3594
OH (Chendi,DC3594):	Feb 18, 4-6 pm	Feb 25, 4-6pm
OH (Chendi,DC3594):	Feb 28, 5-6 pm	

Course Web Site: http://www.student.cs.uwaterloo.ca/~cs476

Due Mar 2,2020

IMPORTANT: In this and future assignments, you will be asked to write pseudo code. **Pseudo-code** is a detailed, high level description of an algorithm, when given to a first year CS student who does not know anything about computational finance, can still code this up in any programming language. It is important, therefore to specify how variables are initialized, etc. Note further that marks for programming questions are allocated for explanations of algorithms and discussions of results. If all you hand in is the listing of the "Raw Code" or "Raw Output" by itself, you will get poor marks. The TA will take off marks for poor documentation. Finally, **vectorize your code**.

1. **(10 marks)** (Numerical Scheme for SDE)

   The variance in the stochastic volatility of the Heston model

   $$dv = -\lambda(v - \bar{v})dt + \eta\sqrt{v}dZ$$

   is a mean reverting Cox-Ingersoll-Ross (CIR) process.

   Let the time step Δt be given. Assume that the variance v_n at time t_n is given.

   (a) Write down the Euler-Maruyama formula for computing the variance value v_{n+1} at t_{n+1}. Assume $v_n = \bar{v}$. Write down the condition on the standard normal sample which leads to $v_{n+1} < 0$.

   (b) Given v_n, write down the Milstein method for computing v_{n+1}. Assume $v_n = 0$. Provide a condition which guarantees that v_{n+1} is always positive. Hint: the condition is an inequality using parameters $\lambda, \bar{v}, \eta$.

2. **(10 marks)** (Monte Carlo Pseudo Code)

   Consider the following interest rate model for the short rate r and the variance ξ (under the risk neutral measure)

   $$\begin{aligned} dr &= a(\bar{r} - r)\ dt + \sqrt{\xi r}\ dZ_1 \\ d\xi &= b(\bar{\xi} - \xi)\ dt + c\sqrt{\xi}\ dZ_2 \\ & \quad a, \bar{r}, b, \bar{\xi}, c \text{ positive constants} \\ & \quad E[dZ_1 dZ_2] = \rho\ dt \quad . \end{aligned} \tag{1}$$

   Give precise pseudo code for Monte Carlo valuation of a zero coupon bond, given $r(t=0), \xi(t=0)$. Use forward Euler timestepping. Assume the existence of a function which returns uncorrelated random variables ϕ, with $\phi \sim N(0,1)$.

   The price of a zero coupon bond is

   $$Value \quad = \quad E^Q\left[\exp\left[\int_0^T -r(s)\ ds\right]\right] \tag{2}$$

   Recall that the Choleski factor of a 2×2 correlation matrix is

   $$\begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} 1 & \rho \\ 0 & \sqrt{1-\rho^2} \end{bmatrix} \quad = \quad \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \tag{3}$$

Table 1: Data for Hedging Simulations

σ	.25
r	.03
μ	0.18
Time to expiry	1.0 years
Strike Price K	$S(t_0)$
Initial asset price $S(t_0)$	$90
N	250

3. (14 marks) (Delta Hedging Analysis for a European Option).

Assume that the underlying (no dividend) stock price follows

$$\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t \tag{4}$$

where Z_t is a standard Brownian, μ and σ are constants. You are to conduct hedging effectiveness analysis using MC simulations of (4) for the European straddle, whose payoff equals the sum of the call payoff and put payoff with the same strike K, i.e., $\max(S-K,0)+\max(K-S,0)$. Simulate asset price paths from (4) so that there is no time discretization error in the simulated prices. Using the parameter values given in Table 1, determine the hedging error using discrete delta hedging for the straddle. You will need to produce *efficient* Matlab code to carry out this simulation. Assume that the initial option price is computed from **blsprice**.

(a). Using the data in Table 1, matlab function **blsprice**, write a matlab script to compute hedging positions at time $t = t_{N-1}$ for a European straddle, which has the payoff function $\max(S-K,0)+\max(K-S,0)$ using the parameters in Table 1. Plot the computed hedge position δ against S, using $S = \text{linspace}(70, 110, 100)$.

(b) For the straddle considered, plot the histogram of the relative hedging error P&L, for no hedging, hedging position rebalanced at $n = 0$ (once), $n = 1:1:N$ (daily), $n = 5:5:N$ (weekly), $n = 20:20:N$ (monthly) respectively, assuming hedging positions are computed from the Matlab function **blsdelta**. The Matlab functions *histc*, *bar* might prove useful. Use at least 50 bins in your histogram. Comment on your observations.

(c) Write a Matlab function [var,cvar]=dVaRCVaR(P&L, β) which returns VaR and CVaR for a discrete P&L distribution with M independent samples using the procedure described. In Matlab, ordering can be done using **sort**.

(d) Compute the standard deviation, VaR (98%) and CVaR (98%). For the straddle, report in a table, mean, standard deviation, VAR (98%), CVAR (98%), of $P\&L$ for no hedging, rebalancing monthly, weekly, and daily. Discuss how hedging performance changes with the rebalancing frequency.

(e) What changes need to be made for hedging analysis of an American option? Explain.

Note: you should be using *at least* 10,000 simulations to get a reasonable histogram for the probability density. Vectorize your code.

4. (10 marks) Assume that the interest rate $r = 0$ and it is possible to trade **continuously** in time. Suppose that a writer has sold one European call option with strike K and expiry T. Assume that $S_0 < K$, i.e. the initial asset price is less than K. Assume that we can implement the following simple hypothetic strategy (assuming continuously trading is possible): As soon as $S_t > K$, the writer borrows from cash account and buys the stock at t. As soon as $S_t < K$, the writer sells the stock at t.

Assume that the interest $r = 0$. Let $\delta_t S_t + B_t$ denotes the value of the hedging portfolio of the underlying and bond and C_t denote the call option value at t.

(a) Write down a mathematical expression for the definition of this strategy δ_t, $0 \leq t < T$.

(b) Under the above continuous trading assumption, explain why the value of hedging portfolio value $\delta_t S_t + B_t$ is either $S_t - K + C_0$ or C_0 for any $0 < t < T$.

(c) What is the relative hedging error $(-C_T + \delta_T S_T + B_T)/C_0$?

(d) The answers to the above questions seem to suggest the existence of an arbitrage (or a perfect hedge with zero initial cost $\delta_0 S_0 + B_0$). Perform the following computation to investigate whether this is true. Simulate this strategy for a European call option with $S_0 = 99$, $K = 100$ and other parameters in Table 1. Determine the relative hedging error (final discounted $P\&L$, divided by the initial Black Scholes option price. Assume that the hedger receives the Black-Scholes price in cash initially. Use about $80,000$ simulations. Use $100, 200, 400, 800$ rebalancing times. Show a table of results (mean, standard deviation, 95% VAR and CVAR) versus number of rebalancing times. Generate plot of the probability density of the discounted hedging error for the case of 800 rebalancing times.

Comment on the statistical properties of the hedging error.

(f) **Bonus**. Explain why the computed hedging error is different from what is thought to be.

5. (12 marks) (Jump Model)

Assume that Z_t is a standard Brownian motion. Let q_t denotes a Poisson counting process where the change dq_t is defined as

$$dq_t = \begin{cases} 0 & \text{with probability } 1 - \lambda dt \\ 1 & \text{with probability } \lambda dt, \end{cases}$$

where $\lambda > 0$ denotes the jump intensity. Consider the following risk neutral jump-diffusion-stochastic-volatility process

$$\frac{dS(t)}{S(t^-)} = (r - \lambda\kappa)dt + \sigma dZ + (J-1)dq_t \tag{5}$$

where the value $S(t^-)$ is the price just before jump. Note that, if $dq_t = 1$, then after the jump

$$S(t) = S(t^-)J$$

Hence J corresponds to the random jump amplitude. Let $\kappa = E[J-1]$. Merton's jump model assumes that $\log J$ has a normal distribution, see 9, 9.1, 9.2, 9.4, 9.5 in the course notes for discussions on the Merton's jump diffusion model and its implementation.

A random variable X with an exponential distribution in $[0, +\infty)$ has a probability density function

$$f_X(x) = \begin{cases} \frac{1}{\mu} e^{-\frac{x}{\mu}} & \text{if } x \in [0, +\infty) \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

where $\mu > 0$ is a parameter. The matlab function **exprnd** generates a random sample from the distribution of X.

Assume that, when jump occurs, the up-jump occurs with a probability p_u and down-jump occurs with a probability $p_{down} = 1 - p_{up}$. Assume further that, when the up-jump occurs, $\log J$ has an exponential distribution (6) with a parameter μ_u.

(a) (1 mark) Suppose that when a down-jump occurs, $-\log J$ has an exponential distribution (6) with a parameter μ_d. Write down the probability density function for $\log J$ for the down-jump. Determine $\kappa = E(J-1)$ under these assumptions.

(b) (6 marks) Assume that the model parameters are provided in Table 2.

Write a Matlab code to compute the fair value of a European put with strike K and expiry T assuming the underlying follows a stochastic volatility jump diffusion process (5), using Euler time stepping and Monte Carlo method. Using the data in Table 2, carry out a sequence of tests with the number of (timesteps, simulations) = $(800, 25000), (1600, 100000), (3200, 400000)$. Show the results in a table.

Table 2: Model Parameter Data

σ	.15
r	.03
Time to expiry	1 years
Strike Price	\$90
Initial asset price S^0	\$90
Payoff	European put
μ_u	0.32
μ_d	0.3
p_{up}	0.4
Intensity of Poisson Process λ	.1
Δt	1/1000

(c) (4 marks) Use Matlab function **blsimpv** to plot the implied volatility of the put option price computed under the jump model (5) with (timesteps, simulations) = $(800, 25000)$ against the strike, e.g., K=linspace$(70, 120, 20)$ and $T = 1$. Discuss observed characteristics of the implied volatility from the assumed jump model.

6. (Graduate Student Question) (10 marks)

Practitioner's BS delta hedging consists of a portfolio $\Pi(t)$ with the following components at $t = t_n$: an option position $-V_n$, δ_n shares at price S_n, and the dollar amount B_n in a risk-free bank account, where

$$\delta_n = \frac{\partial V_{BS}}{\partial S}(S_n, t_n; \breve{\sigma}_n)$$

and $\frac{\partial V_{BS}}{\partial S}(S_n, t_n; \breve{\sigma}_n)$ is the BS delta at t_n using the implied volatility $\breve{\sigma}_n$ computed from the market option price V_n at the rebalancing time t_n. In [Hull and White(2017)] [1], the mean variance delta, δ_{MV} is modeled in a quadratic form, see (5) in [Hull and White(2017)].

You are given the S&P 500 index option data, in the file RawData.mat, from 01-Jan-2009 to 31-Dec-2010 as the training data set and the data from 01-Jan - 2011 to 31-Dec-2011 as the testing data set. In RawData.mat, values of following additional variables are also provided for you to use for training and testing:

- CSTrain: Daily change of the market index price for the training set.
- CVTrain: Daily change of the market index option price for the training set.
- CSTest: Daily change of the market index price for the testing set.
- CVTest: Daily change of the market option price for the testing set.
- DeltaTrain: BS delta δ_{BS} computed using implied volatility σ_{imp} for the training set.
- DeltaTest: BS delta δ_{BS} computed using implied volatility σ_{imp} for the testing set.
- VegaTrain: BS vega $vega_{BS}$ computed using implied volatility σ_{imp} for the training set.
- VegaTest: BS vega $vega_{BS}$ computed using implied volatility σ_{imp} for the testing set.
- STrain: market index price for the training set.
- STest: market index price for the testing set.
- TauTrain: time to expiry for the training set.
- TauTest: time to expiry for the testing set.

[1] *Optimal delta hedging for options*, Journal of Banking and Finance, 82, 180-190, 2017.

(a) Assume the hedging position function δ has the following parametric form

$$\delta = c_0 + c_1 \times \delta_{\mathrm{BS}} + c_2 \times \delta_{\mathrm{BS}}^2 + c_3 \times \mathrm{VEGA}_{\mathrm{BS}} + \mathbf{c_4} \times \mathrm{VEGA}_{\mathrm{BS}}^{\mathbf{2}} + \mathbf{c_5} \times \mathrm{VEGA}_{\mathrm{BS}}\delta_{\mathrm{BS}}$$

where δ_{BS} denotes the BS delta and $\mathrm{VEGA}_{\mathrm{BS}}$ denotes the BS vega. Learn this parametric function for the δ position by minimizing the MV error defined in [Hull and White(2017)].

(b) Assess the hedging performance of the estimated model for the daily hedging error $\Delta V - \delta\Delta S$, where $\Delta V = V(S(t_{n+1}), t_{n+1}) - V(S(t_n), t_n)$, $\Delta S = S(t_{n+1}) - S(t_n)$, δ is the position of the underlying at t_n, which is learned from (a). Plot the histogram of the hedging error $\Delta V - \delta\Delta S$, tabulate its mean, standard deviation, VaR and CVaR and compare these performance measures with those from hedging with $\delta = \delta_{BS}$ using the practitioner's BS delta hedging.

(c) Now learn the MV hedging model (5) in [Hull and White(2017)] from the provided option market prices. Tabulate the same performance measures for the MV hedging model. Repeat computation in (b) with the position in the underlying $\delta = \delta_{MV}$ for the MV hedging model (5). Compare results from (c) with the cubic parameterizations in (a), which indicates importance of choosing a parametric model.

You can use Matlab function **blsprice**, **blsimpv**, **blsdelta**.