

3. (11a)

From Eqn (8)

$$[V_T]_T^{n+1} = \left[\frac{\partial^2 S}{\partial} V_{SS} \right]_T^{n+1} + [r - \lambda(J-1)] S V_S]_T^{n+1} - [rV]_T^{n+1} + \lambda [V(JS_T, T^n)]_T^n - \lambda [V]_T^n$$

Eqn (7)

$$V_T^{n+1} [1 + \Delta t(r + \alpha_T + \beta_T)] = V_T^n + \Delta t \alpha_T V_{T-1}^{n+1} + \Delta t \beta_T V_{T+1}^{n+1}$$

divide Δt both side

$$\frac{V_T^{n+1}}{\Delta t} + V_T^{n+1} (r + \alpha_T + \beta_T) = \frac{V_T^n}{\Delta t} + \alpha_T V_{T-1}^{n+1} + \beta_T V_{T+1}^{n+1}$$

$$\frac{V_T^{n+1} - V_T^n}{\Delta t} = -(r + \alpha_T + \beta_T) V_T^{n+1} + \alpha_T V_{T-1}^{n+1} + \beta_T V_{T+1}^{n+1}$$

$$[V_T]_T^{n+1} = \underline{-(r + \alpha_T + \beta_T) V_T^{n+1} + \alpha_T V_{T-1}^{n+1} + \beta_T V_{T+1}^{n+1}}$$

By Eqn (6), we also know that

$$[V_T]_T^{n+1} = \underline{\left[\frac{\partial^2 S}{\partial} V_{SS} \right]_T^{n+1} + [r - \lambda(J-1)] S V_S]_T^{n+1} - [rV]_T^{n+1}}$$

Thus, in Eqn (8),

$$[V_T]_T^{n+1} = (-r + \alpha_T + \beta_T) V_T^{n+1} + \alpha_T V_{T-1}^{n+1} + \beta_T V_{T+1}^{n+1} + \lambda [V(JS_T, T^n)]_T^n - \lambda V_T^n$$

$$\frac{V_T^{n+1} - V_T^n}{\Delta t} = -(-r + \alpha_T + \beta_T) V_T^{n+1} + \alpha_T V_{T-1}^{n+1} + \beta_T V_{T+1}^{n+1} + \lambda [V(JS_T, T^n)]_T^n - \lambda V_T^n$$

$$[1 + \Delta t(r + \alpha_T + \beta_T)] V_T^{n+1} = (1 - \Delta t \lambda) V_T^n + \Delta t \alpha_T V_{T-1}^{n+1} + \Delta t \beta_T V_{T+1}^{n+1} + \Delta t \lambda [V(JS_T, T^n)]_T^n$$

To make T stable, $1 - \Delta t \lambda \geq 0$
 $\Delta t \leq \frac{1}{\lambda}$

3. (1) (b)

$$\begin{aligned} \frac{V_i^{n+1} - V_i^n}{\Delta t} &= -(r + \alpha_i + \beta_i) V_i^{n+1} + \alpha_i V_{i-1}^{n+1} + \beta_i V_{i+1}^{n+1} \\ &\quad + \lambda [V(JS_i, \tau^n)]_i^n - \lambda V_i^n \\ &= -(r + \alpha_i + \beta_i) V_i^{n+1} + \alpha_i V_{i-1}^{n+1} + \beta_i V_{i+1}^{n+1} + \\ &\quad \lambda (\omega_i V(JS_i, \tau^n) + (1 - \omega_i) V(JS_{i+1}, \tau^n)) - \lambda V_i^n. \end{aligned}$$

$$\Rightarrow V_i^{n+1} (1 + \Delta t (r + \alpha_i + \beta_i)) = V_i^n + \Delta t (\alpha_i V_{i-1}^{n+1} + \beta_i V_{i+1}^{n+1} + \Delta t \lambda (\omega_i V_k^n + (1 - \omega_i) V_{k+1}^n) - \lambda V_i^n).$$

noting that r, α_i, β_i are all non-negative,

$$\begin{aligned} |V_i^{n+1}| [1 + \Delta t (r + \alpha_i + \beta_i)] &\leq (1 - \lambda \Delta t) |V_i^n| + \Delta t \alpha_i |V_{i-1}^{n+1}| \\ &\quad + \Delta t \beta_i |V_{i+1}^{n+1}| + \Delta t \lambda \omega_i |V_k^n| + (1 - \omega_i) \Delta t \lambda |V_{k+1}^n| \\ \text{Let } \|V^n\| &= \max_i |V_i^n| \end{aligned}$$

$$\begin{aligned} |V_i^{n+1}| [1 + \Delta t (r + \alpha_i + \beta_i)] &\leq (1 - \lambda \Delta t) \|V^n\| + \Delta t (\alpha_i + \beta_i) \|V^{n+1}\| \\ &\quad + \Delta t \lambda \omega_i \|V^n\| + (1 - \omega_i) \Delta t \lambda \|V^n\| \end{aligned}$$

$$\text{Let } \|V^{n+1}\| = \max_i |V_i^{n+1}|$$

$$\begin{aligned} \|V^{n+1}\| [1 + \Delta t (r + \alpha_i + \beta_i)] &\leq (1 - \lambda \Delta t) \|V^n\| + \Delta t (\alpha_i + \beta_i) \|V^{n+1}\| \\ &\quad + \Delta t \lambda \omega_i \|V^n\| + (1 - \omega_i) \Delta t \lambda \|V^n\| \end{aligned}$$

We have assumed $1 - \lambda \Delta t \geq 0$.

$$\begin{aligned} \Rightarrow \|V^{n+1}\| [1 + \Delta t r] &\leq (1 - \lambda \Delta t + \lambda \Delta t) \|V^n\| \\ \|V^{n+1}\| &\leq \frac{\|V^n\|}{1 + \Delta t r} \leq \|V^n\| \end{aligned}$$

unconditionally converge if $\Delta t \leq \frac{1}{\lambda}$.

Thus, the stable condition is $\Delta t \leq \frac{1}{\lambda}$.

3.12)(a)

From Eqn (10)

$$[V_T]_T^{n+1} = \left[\frac{\partial \Sigma}{\partial V_{SS}} V_{SS} \right]_T^{n+1} + [(r - \lambda(J-1)) \Sigma V_S]_T^{n+1} - [rV]_T^{n+1} + \lambda [V(JS_T, T^n)]_T^n - \lambda [V]_T^{n+1}$$

From Eqn (7)

$$V_T^{n+1} [1 + \Delta t(r + d_T + \beta_T)] = V_T^n + \Delta t d_T V_{T-1}^{n+1} + \Delta t \beta_T V_{T+1}^{n+1} \\ \Rightarrow [V_T]_T^{n+1} = -(r + d_T + \beta_T) V_T^{n+1} + d_T V_{T-1}^{n+1} + \beta_T V_{T+1}^{n+1}$$

By Eqn (6), we also know that

$$[V_T]_T^{n+1} = \left[\frac{\partial \Sigma}{\partial V_{SS}} V_{SS} \right]_T^{n+1} + [(r - \lambda(J-1)) \Sigma V_S]_T^{n+1} - [rV]_T^{n+1}$$

Thus, in Eqn (8)

$$[V_T]_T^{n+1} = -(r + d_T + \beta_T) V_T^{n+1} + d_T V_{T-1}^{n+1} + \beta_T V_{T+1}^{n+1} + \lambda [V(JS_T, T^n)]_T^n - \lambda [V]_T^{n+1}$$

$$\frac{V_T^{n+1} - V_T^n}{\Delta t} = -(r + d_T + \beta_T) V_T^{n+1} + d_T V_{T-1}^{n+1} + \beta_T V_{T+1}^{n+1} + \lambda [V(JS_T, T^n)]_T^n - \lambda [V]_T^{n+1}$$

$$V_T^{n+1} (1 + \Delta t(r + d_T + \beta_T + \lambda)) = V_T^n + \Delta t d_T V_{T-1}^{n+1} + \Delta t \beta_T V_{T+1}^{n+1} + \lambda \Delta t [V(JS_T, T^n)]_T^n$$

3. (a)(b). It follows that $\alpha_T, \beta_T, \gamma$ are all nonnegative

$$|V_i^{n+1}| (1 + \Delta\tau(r + \alpha_T + \beta_T + \lambda)) \leq |V_i^n| + \alpha_T \Delta\tau |V_{i-1}^{n+1}| + \beta_T \Delta\tau |V_{i+1}^{n+1}| + \lambda \Delta\tau |w_T V_K^n + (1-w_T) V_{K+1}^n|$$

Let $\|V^n\| = \max_i |V_i^n|$

$$\Rightarrow |V_i^{n+1}| (1 + \Delta\tau(r + \alpha_T + \beta_T + \lambda)) \leq \|V^n\| + \Delta\tau(\alpha_T + \beta_T) \|V^{n+1}\| + \lambda \Delta\tau w_T \|V^n\| + (1-w_T) \lambda \Delta\tau \|V^n\|$$

Then let $|V_{i^*}^{n+1}| = \max_i |V_i^{n+1}|$

$$\|V^{n+1}\| (1 + \Delta\tau(r + \alpha_T + \beta_T + \lambda)) \leq \|V^n\| + \Delta\tau(\alpha_T + \beta_T) \|V^{n+1}\| + \lambda \Delta\tau w_T \|V^n\| + (1-w_T) \lambda \Delta\tau \|V^n\|$$

$$\|V^{n+1}\| (1 + \Delta\tau(r + \lambda)) \leq (1 + \lambda \Delta\tau) \|V^n\|$$

$$\|V^{n+1}\| \leq \frac{1 + \lambda \Delta\tau}{1 + \Delta\tau(r + \lambda)} \|V^n\|$$

$\frac{1 + \lambda \Delta\tau}{1 + \Delta\tau(r + \lambda)} \leq 1$ for sure since r is non-negative

\Rightarrow It is unconditionally stable.

$$\|V^{n+1}\| \leq \frac{1 + \lambda \Delta\tau}{1 + \Delta\tau(r + \lambda)} \|V^n\| \leq \|V^n\|$$