Question 1

1(a) $dV = -\lambda(V-\bar{V})dt + \eta\sqrt{V}dZ_t$

$V_{n+1} - V_n = -\lambda(V_n - \bar{V})\Delta t + \eta\sqrt{V_n}(Z_{n+1} - Z_n)$

Z_n is a standard brownian motion.

$Z_{n+1} - Z_n \sim N(0, \Delta t)$.

$\Rightarrow Z_{n+1} - Z_n = \sqrt{\Delta t}\,\phi_t \qquad \phi_t \sim N(0,1)$

$V_{n+1} = V_n - \lambda(V_n - \bar{V})\Delta t + \eta\sqrt{V\cdot\Delta t}\,\phi_t$

By assumption $V_n = \bar{V}$

$V_{n+1} = \bar{V} - \lambda(\bar{V} - \bar{V})\Delta t + \eta\sqrt{\bar{V}\cdot\Delta t}\,\phi_t$

$= \bar{V} + \eta\sqrt{\bar{V}\cdot\Delta t}\,\phi_t$

Since $V_{n+1} = \bar{V} + \eta\sqrt{\bar{V}\cdot\Delta t}\,\phi_t < 0$

$\phi < \frac{-\bar{V}}{\eta\sqrt{\bar{V}\Delta t}} = -\frac{1}{\eta}\sqrt{\frac{\bar{V}}{\Delta t}}$

(b) $V_{n+1} = V_n - \lambda(V_n - \bar{V})\Delta t + \eta\sqrt{V_n}(Z_{n+1} - Z_n)$

$+ \frac{1}{2}\eta\sqrt{V_n}\cdot\eta\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{V_n}}((\Delta Z_n)^2 - \Delta t)$

$= V_n - \lambda(V_n - \bar{V})\Delta t + \eta\sqrt{V_n\Delta t}\,\phi_t + \frac{1}{4}\eta^2((\phi_t\sqrt{\Delta t})^2 - \Delta t)$

$= V_n - \lambda(V_n - \bar{V})\Delta t + \eta\sqrt{V_n\Delta t}\,\phi_t + \frac{1}{4}\eta^2\Delta t(\phi_t^2 - 1)$.

Follow the assumption $V_n = 0$

$V_{n+1} = 0 - \lambda(0 - \bar{V})\Delta t + \eta\cdot\sqrt{0\cdot\Delta t}\,\phi_t + \frac{1}{4}\eta^2\Delta t(\phi_t^2 - 1)$

$= \lambda\bar{V}\Delta t + \frac{1}{4}\eta^2\Delta t(\phi_t^2 - 1)$

By the condition

$V_{n+1} > 0 \Rightarrow \lambda\bar{V}\Delta t + \frac{1}{4}\eta^2\Delta t(\phi_t^2 - 1) > 0 \qquad$ since $\Delta t > 0$

$= \lambda\bar{V} + \frac{1}{4}\eta^2(\phi_t^2 - 1) > 0$