Assignment 9 - DATA 605

John Mazon

Q1 Pilsdorf Beer Company Stock Price

Let the stock price on day n of the year be Y_n. The differences, $X_n = Y_{n+1} - Y_n$ are observed to be independent and normally distributed $N(\mu = 0, \sigma^2 = 1/4)$. If $Y_1 = 100$, predict the likelihood of $Y_{365} \geq 100$. Repeat for values of 110 and 120.

I begin by noting that $X_2 + X_1 = Y_3 - Y_1$. By extension:

$$X_{364} + X_{363} + \cdots + X_1 = Y_{365} - Y_1 = Y_{365} - 100$$

Thus, Y_{365} can be expressed as:

$$Y_{365} = X_{364} + X_{363} + \cdots + X_1 + 100$$

The expected value of Y_{365} follows from:

$$\mathbb{E}(Y_{365}) = \mathbb{E}\left[\sum_{i=1}^{364} X_i\right] + 100$$

$$\mathbb{E}(Y_{365}) = \sum_{i=1}^{364} \mathbb{E}\left[X_i\right] + 100$$

$$\mathbb{E}(Y_{365}) = 100$$

Because the X_i are independent, the variance of Y_{365} is the sum of variances of the X_i:

$$\mathbb{E}\left[(Y_{365} - 100)^2\right] = \frac{364}{4} = 91$$

Hence Y_{365} is normally distributed with $\mu = 100$ and $\sigma^2 = 91$.

By definition, $P\left(Y_{365} \geq 100\right) = 0.5$. For values of 110 and 120, we use R’s pnorm function.

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pge110 <- round(1 - pnorm(110, 100, 91), 4)
pge120 <- round(1 - pnorm(120, 100, 91), 4)
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We find $P\left(Y_{365} \geq 110\right) =$ 0.4562 and $P\left(Y_{365} \geq 120\right) =$ 0.413.

Q2

Find the mean and variance of the binomial distribution from it’s moment-generating function.

The MGF for the binomial distribution is:

$$M(t) = (q + pe^t)^n$$

To get the mean and variance, we compute the first two derivatives of $M(t)$:

$$M'(t) = npe^t(q + pe^t)^{n-1}$$

$$M''(t) = n(n-1)p^2 e^{2t}(q + pe^t)^{n-2} + npe^t(q + pe^t)^{n-1}$$

The mean (first non-centered moment) is obtained by evaluating $M'(0)$:

$$\mu = M'(0) = np(q + p)^{n-1} = np(1 - p + p)^{n-1} = np$$

The variance is obtained from:

$$\sigma^2 = M''(0) - \mu^2 = n(n-1)p^2 + np - n^2p^2$$

$$\sigma^2 = -np^2 + np = np(1-p) = npq$$

Q3

Find the mean and variance of the exponential distribution from it’s moment-generating function.

The MGF for the exponential distribution is:

$$M(t) = \frac{\lambda}{\lambda - t} \text{ for } t < \lambda$$

Taking the first and second derivatives yields:

$$M'(t) = \frac{\lambda}{(\lambda - t)^2}$$

$$M''(t) = \frac{2\lambda}{(\lambda - t)^3}$$

Thus the mean is:

$$\mu = M'(0) = \frac{1}{\lambda}$$

and the variance:

$$\sigma^2 = M''(0) - \mu^2 = \frac{2\lambda}{(\lambda)^3} - \frac{1}{\lambda} = \frac{1}{\lambda^2}$$

$$\sigma^2 = \frac{1}{\lambda^2}$$