Linear Systems of Differential Equations: A system of differential equations is called linear if each of $F_1, F_2, \ldots, F_n$ is a linear function of $x_1, x_2, \ldots, x_n$. Otherwise, it is called non-linear.

We are only considering linearity of the variable x.

$$x_1' = P_{11}(t)x_1 + p_{12}x_2(t) + \ldots + p_{1n}(t)x_n + g_1(t)$$
$$x_2' = P_{21}(t)x_1 + p_{22}x_2(t) + \ldots + p_{2n}(t)x_n + g_1(t)$$
$$\vdots$$
$$x_n' = P_{n1}(t)x_1 + p_{n2}x_2(t) + \ldots + p_{nn}(t)x_n + g_1(t)$$

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} p_{11}(t) & p_{12}(t) & \ldots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \ldots & p_{2n}(t) \\ \vdots & & \ddots & \vdots \\ p_{n1}(t) & p_{n2}(t) & \ldots & p_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Theorem 7.1.2: If the functions p_{11}, p_{12}, $\ldots$, p_{nn}, g_1, g_2, $\ldots$, g_n are continuous on an open interval I, $\alpha < t < \beta$, then there exists an unique solution $x_1 = \phi_1(t), \ldots, x_n = \phi_n(t)$ of the system that satisfies the initial condition problem where t_0 is any point in I and $x_1^{[0]}, \ldots, x_n^{[0]}$ are any prescribed numbers.

Then the solution exists throughout the interval I.

Linear Algebra Review

AB

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -1 \\ 2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 0 \\ 2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 \\ 0 & -1 & -1 \\ 7 & 0 & -1 \end{bmatrix}$$

BA

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -1 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 0 \\ 1 & -4 & 2 \\ 4 & -5 & 4 \end{bmatrix}$$

$AB \neq BA$

Orthogonality: Two vectors x, y, are orthogonal if and only if $(x,y) = 0$.

Ex:

$$z = \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}$$

$$z^T z = 1 + i^2 = 0$$

$$(z, z) = \begin{bmatrix} 1 & 0 & i \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -i \end{bmatrix} = 2$$

Identity Matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Inverse: $A \in \mathbb{R}^{nxn}$ is called nonsingular or invertible if there exists another matrix B such that $AB = I$, and $BA = I$.

B is the inverse of A, also written $B = A^{-1}$.

$$AA^{-1} = A^{-1}A = I$$

Determinant of a Matrix: Let $A \in \mathbb{R}^{nxn}$.

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

$$\det(A) = c_{11} + c_{12} + \dots + c_{1n}$$

where c_{ij} is the cofactor.

Ex:

$$\begin{aligned} \begin{vmatrix} 1 & 2 & -4 \\ 2 & 1 & 1 \\ -1 & 3 & -11 \end{vmatrix} &= 1(-11-3) - 2(-22+1) - 4(6+1) \\ &= -14 + 42 - 28 \\ &= 0 \end{aligned}$$

Note: A matrix is called singular if and only if $\det(A) = 0$.

Computing A^{-1}:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$R_2 - 3R_1,\ R_3 - 2R_1$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 5 & -3 & 1 & 0 \\ 0 & 4 & 5 & -2 & 0 & 1 \end{array}\right]$$

$R_1 + \frac{1}{2}R_2,\ R_3 - 2R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{2} & \frac{-1}{2} & \frac{1}{2} & 0 \\ 0 & 2 & 5 & -3 & 1 & 0 \\ 0 & 0 & -5 & 4 & -2 & 1 \end{array}\right]$$

$R_1 + \frac{3}{10}R_3,\ R_2 + R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{10} & \frac{-1}{10} & \frac{3}{10} \\ 0 & 2 & 0 & 1 & -1 & 1 \\ 0 & 0 & -5 & 4 & -2 & 1 \end{array}\right]$$

$\frac{1}{2}R_2,\ \frac{-1}{5}R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & .7 & -.1 & .3 \\ 0 & 1 & 0 & .5 & -.5 & .5 \\ 0 & 0 & 1 & -.8 & .4 & -.2 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} .7 & -.1 & .3 \\ .5 & -.5 & .5 \\ -.8 & .4 & -.2 \end{bmatrix}$$