1. There are 5 closed boxes on a table. Three of the boxes have good prizes inside and the other two don't. You open boxes one at a time until you find a prize. Let X be the number of boxes you open.

   (a). Find the pmf of X.

   Ans: X may take values $\in [1, 3]$. Let $f(X)$ be the pmf of X.

$$f(X) = \begin{cases} \frac{3}{5} & x = 1 \\ \frac{2}{5} \cdot \frac{3}{4} & x = 2 \\ \frac{2}{5} \cdot \frac{1}{4} \cdot 1 & x = 3 \\ 0 & x > 3 \end{cases}$$

   (b). Find $E(X)$.

   Ans:

$$E(X) = \sum_k kP(X = k) \tag{1}$$

   Alternate symbol of $E[X] = \mu$

   Thus

$$\begin{aligned} E(X) &= 1 \cdot \frac{2}{5} + 2\left(\frac{2}{5} \cdot \frac{3}{4}\right) + 3 \cdot \left(\frac{2}{5} \cdot \frac{1}{4}\right) \\ &= 1.3 \end{aligned}$$

   (c). Find Var(X).

   Ans:

$$\mathrm{Var}(X) = \sum_k (k - \mu)^2 P(X = k) \tag{2}$$

$$\begin{aligned} \mathrm{Var}(X) &= (1 - 1.3)^2 \cdot \frac{2}{5} + (2 - 1.3)^2\left(\frac{2}{5} \cdot \frac{3}{4}\right) + (3 - 1.3)^2\left(\frac{2}{5} \cdot \frac{1}{4}\right) \\ &= .4720 \end{aligned}$$

   (d). Suppose the good prize inside each of the three boxes is \$100, but each empty box you open costs you \$100. What is your expected gain or loss in the game? (Hint: Express the gain or loss as a function of X.)

Ans:

$$f(X) = \begin{cases} 100 & x = 1 \\ 0 & x = 2 \\ -100 & x = 3 \end{cases}$$

$$\begin{aligned} E[X] &= 100 \cdot \frac{3}{5} + 0 - 100 \cdot \left(\frac{2}{5} \cdot \frac{1}{4}\right) \\ &= 50 \end{aligned}$$

2. Let X be a random variable with density function

$$f(x) = \begin{cases} \frac{1}{4} & 1 < x < 2 \\ c & 3 < x < 5 \\ 0 & \text{otherwise} \end{cases}$$

(a). Find the value of c.

Ans: By the properties of a density function,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

which is only possible given c is equal to

$$\begin{aligned} c(5-3) &= 1 - \frac{1}{4}(2-1) \\ 2c &= .75 \\ c &= \frac{3}{8} \end{aligned}$$

(b). Find $P(1.5 < X < 4)$

Ans:

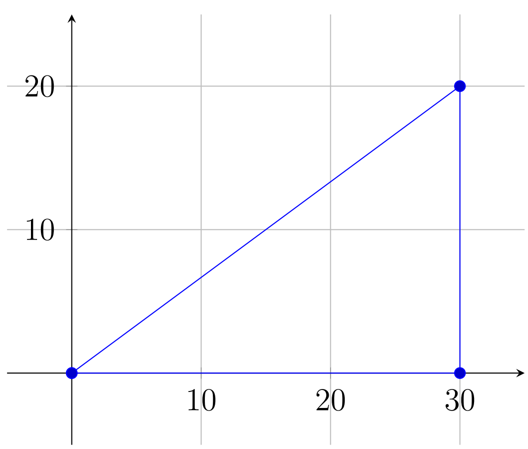
$$\begin{aligned} P(a < X < b) &= \int_b^a f(x)\,dx \\ P(1.5 < X < 4) &= \int_{1.5}^4 f(x)\,dx \\ &= \int_{1.5}^2 f(x)\,dx + \int_2^3 f(x)\,dx + \int_3^4 f(x)\,dx \\ &= \frac{1}{2}(2-1.5) + 0 + \frac{3}{8}(4-3) \\ &= .50 \end{aligned}$$

(c). Find the cdf of X.

Ans:

$$\begin{aligned} F(s) = P(X \le s) &= \int_{-\infty}^s f(x)\,dx \\ &= \begin{cases} 0 & s < 1 \\ \frac{s-1}{4} & 1 < s < 2 \\ \frac{3}{8}(s-3) + .25 & 3 < s < 5 \\ 1 & x > 5 \end{cases} \end{aligned}$$

3. Choose a point (X, Y) uniformly inside the triangle with vertices (0,0), (30,0) and (30,20).



(a). Find the cdf of X.

Ans: The area inside the triangle may be modeled by

$$\int_0^{30} \frac{2}{3}x\,dx = 300$$

as X may take values $\in [0, 30]$.

$$P(X \leq x) = \frac{1}{300}\int_0^{x} \frac{2}{3}x\,dx$$

Thus, we write the cdf as

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{300} \cdot \frac{1}{3}x^2 & 0 \leq x \leq 30 \\ 1 & x > 30 \end{cases}$$

(b). Using (a), find the density function of X.

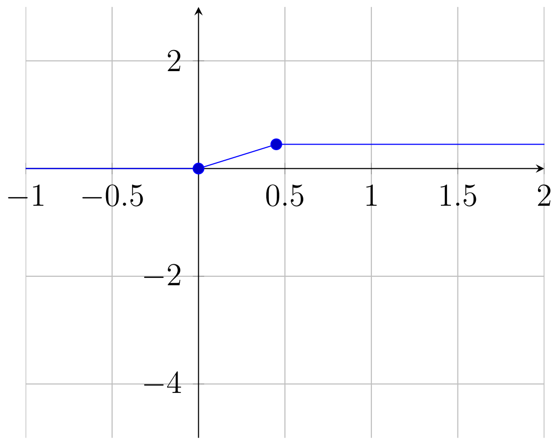
Ans:

$$f_X(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{450}x & x \in [0, 30] \\ 0 & x > 30 \end{cases}$$

4. Suppose that when you take the bus to school, it arrives after a uniformly distributed number of hours in the interval [0,1] after you get to the stop. However, if it does not arrive for 45 minutes, you take an uber. Let X be the number of hours you wait.

(a). Find the cdf of X.

Ans: The graph of X is as follows.



Let Y be the time which the bus arrives. $Y \sim U[0,1]$.