Exponential Distributions:

Exponential Distributions model the waiting time between arrivals.

- The time you wait from one earthquake to the next.
- The time between customer arrivals.

The exponential distribution is the **continuous counterpart of the geometric distribution.**

Definition: Let $0 < \lambda < \infty$. A random variable x has exponential distribution with parameter λ if x has the density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$x \sim \text{Exp}(x)$.

x is exponential with **rate** λ.

x has the c.d.f.

$$\begin{aligned} F(t) &= P(x \leq t) \\ &= \int_{-\infty}^{t} f(x)\, dx \\ &= 0 + \int_{0}^{t} \lambda e^{-\lambda}\, dx \\ &= \begin{cases} -e^{-\lambda x} \Big|_0^t & t \geq 0 \\ 0 & t < 0 \end{cases} \\ &= \begin{cases} 1 - e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases} \end{aligned}$$

$$\begin{aligned} P(\frac{x}{2} > t) &= P(x > 2t) \\ &= \int_{2t}^{\infty} \lambda e^{-\lambda x} \\ &= e^{-2\lambda t} \end{aligned}$$

Example: The length of a call, T is modeled by an exponential with a mean length of 10 minutes. What is $P(T > 8)$? What is $P(8 \leq T \leq 22)$.

Solution: We have $E(T) = 10$, then $\lambda = \frac{1}{10}$.

$$\begin{aligned} P(T > 8) &= \frac{1}{10} \int_{8}^{\infty} e^{\frac{x}{10}} dx \\ &= e^{-\frac{1}{10} \cdot 8} \\ &\approx 0.4493 \end{aligned}$$

$$\begin{aligned} P(8 \leq T \leq 22) &= P(T \geq 8) - P(T > 22) \\ &= e^{-\frac{8}{10}} - e^{-\frac{22}{10}} \\ &\approx 0.3385 \end{aligned}$$

Example: There is a certain protein in human cells which takes T time to be destroyed. The common assumption in biochemistry is that $T \sim \exp(\lambda n)$, where $\lambda > 0$.

Suppose n is the number of copies of this protein in a given cell. What is the value of n such that $P(T > 10^{-2}) < \frac{1}{10}$.

Example: An alarm clock is rigged to ring at some time T, such that $T \sim \exp(\frac{1}{3})$. You've waited for 7 hours, and it hasn't been set off yet. What's the probability that you will wait x more hours.

Solution: $P(T \leq 7 + x \,|\, T \geq 7)$. By memorylessness,

$$\begin{aligned} P(T \leq 7 + x \,|\, T \geq 7) &= P(T > x) \\ &= e^{-\frac{x}{3}} \end{aligned}$$

Example: Suppose $x \sim \exp(2)$. Find the real number $a < 1$ such that the events $\{x \in [0,1]\}$, and $\{x \in [a,2]\}$ are independent.

Solution: We want

$$P(x \in [0,1] \cap x \in [a,2]) = P(x \in [0,1]) \cdot P(x \in [a,2])$$

We know $P(x \in [0,1]) = 1 - e^{-2}$.

$$P(x \in [a,2]) = \begin{cases} e^{-2} - e^{-4} & a < 0 \\ e^{-2a} - e^{-4} & 0 \leq a < 1 \end{cases}$$

$$P(x \in [0,1] \cap x \in [a,2]) = \begin{cases} 1 - e^{-2} & a < 0 \\ e^{-2a} - e^{-2} & 0 \leq a < 1 \end{cases}$$

$$\begin{aligned} (1 - e^{-2})(e^{-2a} - e^{-4}) &= e^{-2a} - e^{-2} \\ e^{-2a} - e^{-2-2a} - e^{-4} + e^{-6} &= e^{-2a} - e^{-2} \\ e^{-2-2a} &= e^{-2} - e^{-4} + e^{-6} \\ e^{-2a} &= 1 - e^{-2} + e^{-4} \end{aligned}$$