$$Ax = y$$

Eigenvectors: Using A, we may transform x into y, where

$$y = \lambda x$$

and

$$\begin{aligned} Ax &= \lambda x \\ &= \lambda I x \\ (A - \lambda I)x &= 0 \end{aligned}$$

There exists non-zero solutions to $(A - \lambda I)x = 0$ if and only if the characteristic polynomial

$$\det(A - \lambda I) = 0$$

Eigenvalues: λ of A can be real or complex, and each have corresponding eigenvectors.

Ex: Find eigenvalues, eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{vmatrix} \\ &= (3-\lambda)(-2-\lambda) + 4 \\ &= -6 - 3\lambda + 2\lambda + \lambda^2 + 4 \\ &= -2 - 1\lambda + \lambda^2 \\ &= (\lambda + 1)(\lambda - 2) \end{aligned}$$

$$[A - \lambda | 0] = \left[\begin{array}{cc|c} 4 & -1 & 0 \\ 4 & -1 & 0 \end{array}\right]$$

$R_2 - R_1$

$$\left[\begin{array}{cc|c} 4 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore

$$x = \begin{bmatrix} \frac{1}{4}t \\ t \end{bmatrix}$$

(We will only compute this eigenvector in this example)

Normalization: For some vector x, choose a constant k such that $\|k \cdot x\| = 1$

$$\frac{1}{\sqrt{17}} \cdot \begin{bmatrix} \frac{1}{4}t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{4\sqrt{17}}t \\ \frac{t}{\sqrt{17}} \end{bmatrix}$$

Multiplicity: If a given eigvenvalue appears m times as a root of the characteristic polynomial, then that eigenvalue is said to have an algebraic multiplicity of m.

If an eigenvalue has q linearly independent eigenvectors, then it is said to have geometric multiplicity of q.

Properties of Eigenvalues & Eigenvectors:

1. If each eigenvalue of A is simple (has algebraic multiplicity of 1), then each eigenvalue also has geometric multiplicity of 1.
2. If A has k unique eigenvalues $\lambda_1, \ldots, \lambda_k$, and their corresponding eigenvectors $x_1, \ldots, x_k$. Then $x_1, \ldots, x_k$ are linearly independent.
3. If $A \in \mathbb{R}^{nxn}$ has one or more repeated eigenvalues, there *may* be fewer than n linearly independent eigenvectors associated with A.

Ex: Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} \\ &= -\lambda(-\lambda \cdot -\lambda - 1) - (-\lambda - 1) + (1 + \lambda) \\ &= -\lambda^3 + \lambda + \lambda + 1 + 1 + \lambda \\ &= -\lambda^3 + 3\lambda + 2 \\ &= (-\lambda^2 + 2\lambda) + (\lambda + 2) \\ &= (\lambda + 1)^2(\lambda - 2) \end{aligned}$$

$\lambda = 2$

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

$$\vdots$$

$$\left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Let $t = 1$.

$$x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\lambda = -1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$\vdots$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x = \begin{bmatrix} -t - s \\ s \\ t \end{bmatrix}$$

$$x = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} s$$

We extract two eigenvectors. For x_1, let $t = 1$, $s = 0$, and for x_2, let $t = 0$, $s = 1$.

$$x_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},\ x_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Hermitian Matrix: For A Hermitian,

1. $A^* = A$.
2. All eigenvalues are real.
3. There exists n independent eigenvectors.
4. If x_1, x_2 are eigenvectors corresponding to λ_1, λ_2,

   $$(x_1, x_2) = 0$$

   Therefore, if all eigenvalues are simple, the associated eigenvectors form an orthogonal set of vectors.
5. It is possible to choose m eigenvectors that are mutually orthogonal which correspond to and eigenvalue of algebraic multiplicity m.