1. There are 5 closed boxes on a table. Three of the boxes have good prizes inside and the other two don't. You open boxes one at a time until you find a prize. Let X be the number of boxes you open.

(a). Find the pmf of X.

Ans: X may take values $\in [1, 3]$. Let $f(X)$ be the pmf of X.

$$f(X) = \begin{cases} \frac{3}{5} & x = 1 \\ \frac{2}{5} \cdot \frac{3}{4} & x = 2 \\ \frac{2}{5} \cdot \frac{1}{4} \cdot 1 & x = 3 \\ 0 & x > 3 \end{cases}$$

(b). Find $E(X)$.

Ans:

$$E(X) = \sum_k kP(X = k) \tag{1}$$

Alternate symbol of $E[X] = \mu$

Thus

$$\begin{aligned} E(X) &= 1 \cdot \frac{3}{5} + 2\left(\frac{2}{5} \cdot \frac{3}{4}\right) + 3 \cdot \left(\frac{2}{5} \cdot \frac{1}{4}\right) \\ &= 1.5 \end{aligned}$$

(c). Find $\mathrm{Var}(X)$.

Ans:

$$\mathrm{Var}(X) = \sum_k (k - \mu)^2 P(X = k) \tag{2}$$

$$\begin{aligned} \mathrm{Var}(X) &= (1 - 1.5)^2 \cdot \frac{3}{5} + (2 - 1.5)^2\left(\frac{2}{5} \cdot \frac{3}{4}\right) + (3 - 1.5)^2\left(\frac{2}{5} \cdot \frac{1}{4}\right) \\ &= .45 \end{aligned}$$

(d). Suppose the good prize inside each of the three boxes is \$100, but each empty box you open costs you \$100. What is your expected gain or loss in the game? (Hint: Express the gain or loss as a function of X.)

Ans:

$$f(X) = \begin{cases} 100 & x = 1 \\ 0 & x = 2 \\ -100 & x = 3 \end{cases}$$

$$\begin{aligned} E[X] &= 100 \cdot \frac{3}{5} + 0 - 100 \cdot \left(\frac{2}{5} \cdot \frac{1}{4}\right) \\ &= 50 \end{aligned}$$

2. Let X be a random variable with density function

$$f(x) = \begin{cases} \frac{1}{4} & 1 < x < 2 \\ c & 3 < x < 5 \\ 0 & \text{otherwise} \end{cases}$$

(a). Find the value of c.

Ans: By the properties of a density function,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

which is only possible given c is equal to

$$\begin{aligned} c(5-3) &= 1 - \frac{1}{4}(2-1) \\ 2c &= .75 \\ c &= \frac{3}{8} \end{aligned}$$

(b). Find $P(1.5 < X < 4)$

Ans:

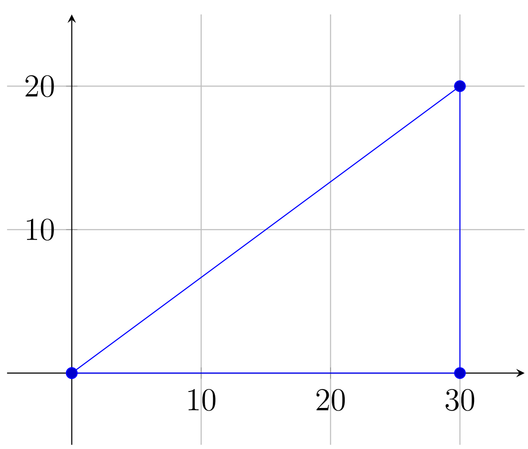
$$\begin{aligned}
P(a < X < b) &= \int_b^a f(x)\,dx \\
P(1.5 < X < 4) &= \int_{1.5}^4 f(x)\,dx \\
&= \int_{1.5}^2 f(x)\,dx + \int_2^3 f(x)\,dx + \int_3^4 f(x)\,dx \\
&= \frac{1}{4}(2-1.5) + 0 + \frac{3}{8}(4-3) \\
&= .50
\end{aligned}$$

(c). Find the cdf of X.

Ans:

$$\begin{aligned}
F(s) = P(X \le s) &= \int_{-\infty}^s f(x)\,dx \\
&= \begin{cases} 0 & s < 1 \\ \frac{s-1}{4} & 1 < s < 2 \\ \frac{3}{8}(s-3) + .25 & 3 < s < 5 \\ 1 & x > 5 \end{cases}
\end{aligned}$$

3. Choose a point (X, Y) uniformly inside the triangle with vertices (0,0), (30,0) and (30,20).



(a). Find the cdf of X.

Ans: The area inside the triangle may be modeled by

$$\int_0^{30} \frac{2}{3} x \, dx = 300$$

as X may take values $\in [0, 30]$.

$$P(X \leq x) = \frac{1}{300} \int_0^x \frac{2}{3} x \, dx$$

Thus, we write the cdf as

$$f_X(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{300} \cdot \frac{1}{3} x^2 & 0 \leq x \leq 30 \\ 1 & x > 30 \end{cases}$$

(b). Using (a), find the density function of X.

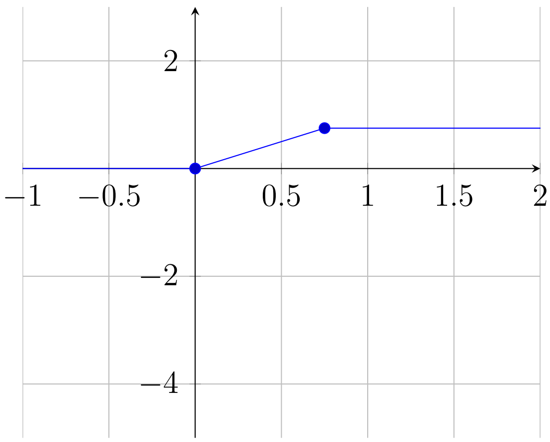
Ans:

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{450} x & x \in [0, 30] \\ 0 & x > 30 \end{cases}$$

4. Suppose that when you take the bus to school, it arrives after a uniformly distributed number of hours in the interval [0,1] after you get to the stop. However, if it does not arrive for 45 minutes, you take an uber. Let X be the number of hours you wait.

   (a). Find the cdf of X.

Ans: The graph of X is as follows.



Let Y be the time which the bus arrives. $Y \sim U[0, 1]$.

- $0 \leq Y \leq \frac{3}{4} \to X = Y$.
- $Y > \frac{3}{4} \to X = \frac{3}{4}$.

Thus, the cdf of X is

$$f_X(x) = \begin{cases} 0 & x < 0 \\ \frac{x-0}{1} & 0 \leq x \leq \frac{3}{4} \\ 1 & x > \frac{3}{4} \end{cases}$$

5. Let X be a normal random variable with mean 3 and variance 4. Find

   (a). $P(2 < X < 6)$

   Ans: A normal random variable has density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

   Thus X has the density function

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi \cdot 4}} e^{-\frac{(x-3)^2}{2\cdot 4}} \\ &= \frac{1}{\sqrt{8\pi}} e^{-\frac{(x-3)^2}{8}} \end{aligned}$$

$$\begin{aligned} P(2 < X < 6) &= P(\frac{6-3}{2}) - P(\frac{2-3}{2}) \\ &= \Phi(\frac{3}{2}) - \Phi(-\frac{1}{2}) \\ &= .24173 \end{aligned}$$

(b). The value c such that $P(X > c) = 0.33$.

Ans:

$$\begin{aligned} P(X > c) &= .33 \\ P(Z > \frac{c-3}{2}) &= .33 \\ \frac{c-3}{2} &= .04 \\ c &= 3.88 \end{aligned}$$

(c). $E(X^2)$. Hint: You can integrate with the density function but it is easier to relate to the mean and variance of X.

Ans:

$$\begin{aligned} \mathrm{Var}(X) &= E(X^2) - (E(X))^2 \\ E(X^2) &= \mathrm{Var(X)} + E(X)^2 \\ &= 4 + 9 \\ &= 13 \end{aligned}$$

6. Let $Z \sim \mathrm{Bin}(10, \frac{1}{3})$. Find the value of its cdf at 2 and at 8.

Ans:

$$Z \sim \mathrm{Bin}(10, \frac{1}{3}) \to P(Z = k) = \binom{10}{k}\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^{10-k} \text{ for } k = 0, 1, \ldots, 10$$

$$P(Z \leq k) = \sum_{i=0}^{k}\binom{10}{i}\left(\frac{1}{3}\right)^i\left(\frac{2}{3}\right)^{10-i}$$

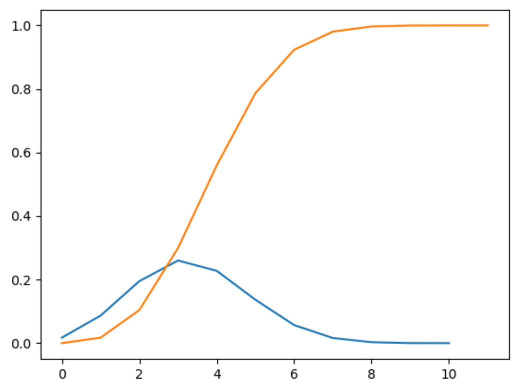


Figure 1: Matplotlib Figure of the cdf, pdf of Z

$k = 2$

$$\begin{aligned}P(Z \leq 2) &= \sum_{i=0}^{2} \binom{10}{i} \left(\frac{1}{3}\right)^i \left(\frac{2}{3}\right)^{10-i} \\ &= \binom{10}{0} \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{10} + \binom{10}{1} \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^{9} + \binom{10}{2} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^{8} \\ &= 0.2991\end{aligned}$$

$k = 8$

$$\begin{aligned}P(Z \leq 8) &= \sum_{i=0}^{8} \binom{10}{i} \left(\frac{1}{3}\right)^i \left(\frac{2}{3}\right)^{10-i} \\ &= 0.9996443631560223\end{aligned}$$

7. Let $c > 0$ and $X \sim \text{Unif}[0, c]$. Show that the random variable $Y = c - X$ has the same cdf as X and hence also the same density function.

Ans:

$$X \sim \text{Unif}[0, c] \to f(x) = \begin{cases} \frac{1}{c} & x \in [0, c] \\ 0 & \text{otherwise} \end{cases}$$

where $f(x)$ is the density function of X.

We want to find $P(Y \leq x)$.

$$\begin{aligned} P(Y \leq x) &= P(c - X \leq x) \\ &= P(X \geq c - x) \end{aligned}$$

$$P(c - x \leq X) = \begin{cases} 0 & x < c \\ \frac{c-(c-x)}{c} & 0 \leq x \leq c \\ 1 & x > c \end{cases}$$

If we take the derivative of $P(c - x \leq X)$ we get exactly $f(x)$.

8. My bus is scheduled to depart at noon. However, in reality, the departure time varies randomly with average departure time being 12 noon with a standard deviation on 6 minutes. Assume the departure time is normally distributed. If I get to the stop at 5 minutes past noon, what is the chance the bus has not yet departed?

   Ans: Let X be a random variable $X \sim N(720, 36)$, representing the departure time of the bus in minutes.

   We are looking to find $1 - P(X \leq 725)$, the probability the bus has *not* left before minute 725.

$$\begin{aligned} 1 - P(X \leq 725) &= 1 - P(Z \leq \frac{725 - 720}{6}) \\ &= 1 - \Phi(\frac{5}{6}) \\ &\approx 1 - .79673 \\ &\approx .20327 \end{aligned}$$

9. In an axe throwing competition, you are blindfolded and spun around 9 times. You then try to hit the target, which is a point in the middle of a horizontal log with your axe. You receive:

   - 15 points if you hit within 3cm of the target.

- 10 points if you hit between 3cm and 10cm of the target.
- 5 points if you hit between 10cm and 20cm of the target.
- 0 points if you hit more than 20cm off the target.

Let Y be the position of the hit so that $Y = y > 0$ corresponds to missing the target on the right by y cm and $Y = -y < 0$ corresponds to missing the target on the left by y cm. Suppose Y is normally distributed with mean $\mu = 0$cm and variance 100cm^2. Find the expected number of points that you win.

Ans: We can start solving this problem by finding the probabilities of scoring each point discretization.

- 15 points: $\Phi(\frac{3}{10}) - \Phi(-\frac{3}{10}) = .2358$.
- 10 points: $(\Phi(1) - \Phi(\frac{3}{10})) + (\Phi(-\frac{3}{10}) - \Phi(-1)) = .4468$.
- 5 points: $(\Phi(2) - \Phi(1)) + (\Phi(-1) - \Phi(-2)) = .2718$.
- 0 points: $2(1 - \Phi(2)) = .0456$.

(As a double check, these probabilities cumulatively sum to 1).

Then

$$\begin{aligned} E(\text{points}) &= 15(.2358) + 10(.4468) + 5(.2718) + 0(.0456) \\ &= 9.364 \end{aligned}$$