Locally Linear Systems

Theorem 9.3.1 The critical point $x = 0$ of the linear system

$$x' = Ax. \tag{1}$$

1. The system is asymptotically stable if the eigenvalues r_1, r_2 are real and negative, or have negative real parts.

2. The system is stable but not asymptotically stable if r_1, r_2 are purely imaginary

3. The system is unstable if r_1, r_2 are real and exclusively one is positive or if they both have real positive parts.

Consider a nonlinear two-dimensional system

$$x' = f(x)$$

If $x = \begin{bmatrix} x \\ y \end{bmatrix}$, and $f(x) = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$, then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$$

Suppose that $x' = Ax + g(x)$ and that $x = 0$ is its isolated critical point (There is some circle about the origin within which there are no other critical points).

Additionally, assume that the $\det(A) \neq 0$, which implies $x = 0$ is an isolated critical point of the system $x' = Ax$.

For the nonlinear system to be "close" to the linear system, we assume that $g(x)$ is small, i.e. $g(x)$ satisfies the following:

1. The components of $g(x)$ have continuous first partial derivatives.

2. It satisfies the following limit condition

$$\frac{\|g(x)\|}{\|x\|} \to 0$$

Such a system is caalled a locally linear system in the neighborhood of the critical point $x = 0$.

If $x = \begin{bmatrix} x \\ y \end{bmatrix}$, $\|x\| = \sqrt{x^2 + y^2} = r$, and

$g(x) = \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \end{bmatrix}$, $\|g(x)\| = \sqrt{(g_1(x,y))^2 + (g_2(x,y))^2}$, then our conditions are satisfied if and only if

$$\frac{g_1(r\cos(\theta), r\sin(\theta))}{r} \to 0, \frac{g_2(r\cos(\theta), r\sin(\theta))}{r} \to 0,$$

as $r \to 0$ for all $0 \leq \theta \leq 2\pi$.

Example 1: Determine whether the system

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -x^2 - xy \\ -0.75x - 0.25x^2 \end{bmatrix}$$

is locally linear in the neighborhood of (0,0).

Solution: (0,0) is a critical point of our system. $\det(A) \neq 0$.

The other critical points are (0,2), (1,0), $(\frac{1}{2}, \frac{1}{2}) \to (0,0)$ is an isolated critical point.

$$\frac{g_1(r\cos\theta, r\sin\theta)}{r} = -r(\cos^2\theta + \cos\theta\sin\theta)$$

This approaches 0 as r approaches 0. We can use the same argument for $\frac{g_2(x,y)}{r}$, therefore, our system is locally linear near the origin.

Theorem 9.3.2 Nonlinear systems are locally linear in the neighborhood of a critical point (x_0, y_0) whenever the functions F, G have continuous partial derivatives up to the second order.