5). A mass of 20 g stretches a spring 5 cm. Suppose that the mass is also attached to a viscous damper with a damping constant of 400dyn $\cdot$ s/cm. If the mass is pulled down an additional 2 cm and then released, find its position u at any time t. Determine the quasi-frequency and the quasi-period. Determine the ratio of the quasiperiod to the period of the corresponding undamped motion.

Note:

$$1N = 1\frac{\text{kg}}{s^2}$$

Ans: Apply the following formulas.

$$mg = kL$$

where L is the elongation of the spring, k is the spring constant, m is the mass of the object, and g is acceleration due to gravity.

We know $L = 5$ cm, $g = 9.8m/s^2$

$$\begin{aligned} 20g \cdot \frac{1kg}{1000g} \cdot \frac{9.8m}{s^2} &= k \cdot .05m \\ .196N &= k \cdot .05m \\ 3.92\frac{N}{\text{m}} &= k \end{aligned}$$

We use the following D.E. to model this spring's movement.

$$mu''(t) + \gamma u'(t) + ku(t) = F(t)$$

where γ is the damping constant, m is the mass of the object, and k is the spring constant.

We convert γ to a usable unit in the following manner:

$$400 \text{ dyn} \cdot \frac{\text{s}}{\text{cm}} \cdot \frac{1N}{10^5 \text{ dyn}} \cdot 100\frac{\text{cm}}{1 \text{ m}} = 0.4\frac{N \cdot s}{\text{m}}$$

Plugging in constants:

$$0.02\text{kg } u''(t) + 0.40\frac{N \cdot s}{\text{m}}u'(t) + 3.92\frac{N}{\text{m}}u(t) = F(t)$$

Using the following equation discussed in class,

$$\begin{aligned}\gamma^2 - 4k\text{m} &= .40^2 - 4 \cdot 3.92 \cdot .02 \\ &= -0.1536 \\ &< 0\end{aligned}$$

Thus, we have the following roots,

$$\begin{aligned}r &= \left\{\frac{-\gamma - i\sqrt{4km - \gamma^2}}{2m}, \frac{-\gamma + i\sqrt{4km - \gamma^2}}{2m}\right\} \\ &= \left\{-\frac{\gamma}{2m} - i\mu, -\frac{\gamma}{2m} + i\mu\right\}\end{aligned}$$

and our homogenous solution takes the form

$$u = e^{-\frac{\gamma}{2m}t}(A\cos\mu t + \beta\sin\mu t)$$

Our particular solution takes the form

$$mu''(t) + \gamma u'(t) + ku(t) = mg$$

The inhomogenous term mg is constant, therefore, the particular solution is also constant, let

$$u(t) = x.$$

Therefore

$$\begin{aligned}m \cdot 0 + \gamma \cdot 0 + kx &= mg \\ x &= \frac{mg}{k}\end{aligned}$$

Our complete solution is

$$u = e^{-\frac{\gamma}{2m}t}A\cos\mu t + e^{-\frac{\gamma}{2m}t}\beta\sin\mu t + \frac{mg}{k}$$

and

$$\begin{aligned}u' = &-A\frac{\gamma}{2m}e^{-\gamma t/2m}\cos\mu t - A\mu e^{-\gamma t/2m}\sin\mu t - \\ &\beta\frac{\gamma}{2m}e^{-\gamma t/2m}\sin\mu t + B\mu e^{-\gamma t/2m}\cos\mu t\end{aligned}$$

Applying initial conditions:

$$\begin{aligned} u(0) &= 5\text{cm} + 2\text{cm} \\ &= .07m \end{aligned}$$

$$u'(0) = \frac{0m}{s}$$

Leaving us with

$$0.07 = A + \frac{mg}{k}$$

$$0 = -A\frac{\gamma}{2m} + B\mu$$

which gives us

$$u = e^{-10t}.02\cos 9.8t + e^{-10t}.02\sin 9.8t + .05$$

Quasi-Frequency:

$$\mu = \frac{\sqrt{4mk - \gamma^2}}{2m} \approx 9.81\frac{rad}{s}$$

Quasi-Period:

$$T_d = \frac{2\pi}{\mu} \approx 0.641$$

Ratio:

$$\begin{aligned} \frac{T_d}{T} &= \frac{\omega}{\mu} \\ &\approx 1.43 \end{aligned}$$

6). A spring is stretched 10 cm by a force of 3 N. A mass of 2 kg is hung from the spring and is also attached to a viscous damper that exerts a force of 3 N when the velocity of the mass is 5 m/s. If the mass is pulled down 5 cm below its equilibrium position and given an initial downward velocity of 10 cm/s, determine its position u at any time t. Find the quasi-frequency μ and the ratio of μ to the natural frequency of the corresponding undamped motion.

7). A spring is stretched 6 in by a mass that weighs 8 lb. The mass is attached to a dashpot mechanism that has a damping constant of 1 4 lb·s/ft and is acted on by an external force of 4 cos(2t) lb.

a. Determine the steady-state response of this system.

6). A mass that weighs 8lb stretches a spring 6 in. The system is acted on by an external force of $8\sin(8t)$ lb. If the mass is pulled down 3 in and then released, determine the position of the mass at any time. Determine the first four times at which the velocity of the mass is zero.

3). $u^{(4)} - u = 0$

5). $u'' + p(t)u' + q(t)u = g(t), \quad u(0) = u_0, \quad u'(0) = u_0'$