Expectation of Random Variables

Let g be a real valued function defined on the range of a random variable x.

If x is discrete such that it has some probability mass function, f,

$$E(g(x)) = \sum_k g(k) \cdot P(x = k)$$

If x is continuous with density f,

$$E(g(x)) = \int_{-\infty}^{\infty} g(x) f(x) \, dx$$

Example: A stick of length l is broken at a uniformly chosen point. What is the expected length of the longer piece?

Sol: We define a variable L to be the length of the longer piece when we break the stick at a point x. $x \sim U[0, l]$.

$$L = \begin{cases} l - x & x < \frac{l}{2} \\ x & x \geq \frac{l}{2} \end{cases}$$

$$\begin{aligned}
E(L(x)) &= \int_{-\infty}^{\infty} L(x) f(x) \, dx \\
&= \int_0^{\frac{l}{2}} L(x) f(x) \, dx + \int_{\frac{l}{2}}^{l} L(x) f(x) \, dx \\
&= \int_0^{\frac{l}{2}} (l - x) \frac{1}{l} \, dx + \int_{\frac{l}{2}}^{l} x \frac{1}{l} \, dx \\
&= \frac{(l - x)^2}{2l} \bigg|_{\frac{l}{2}}^{0} + \frac{x^2}{2l} \bigg|_{\frac{l}{2}}^{l} \\
&= \frac{3l}{4}
\end{aligned}$$

Ex: C is the amount of money paid for an accident. The cost of the accident $Y \sim U[100, 1500]$.

$$C = g(Y)$$
$$g(Y) = min(Y, 500)$$
$$g(Y) = \begin{cases} y & y < 500 \\ 500 & y \geq 500 \end{cases}$$

What is $E(C) = E(g(Y))$?

Sol:

$$\begin{aligned} E(g(Y)) &= \int_{-\infty}^{\infty} g(Y) f(Y)\, dY \\ &= \frac{1}{1400} \int_{100}^{1500} g(Y)\, dY \\ &= \frac{1}{1400} \left(\int_{100}^{500} Y\, dY + \int_{500}^{1500} 500\, dY \right) \\ &= \frac{1}{1400} \left(\frac{250000 - 10000}{2} + 500 \cdot 1000 \right) \\ &= \$442.86 \end{aligned}$$

Variance, Standard Deviation

$$SD = \sqrt{\frac{\sum_{i=1}^{n} (s_i - \hat{s})^2}{n}}$$

Let x be a random variable with mean $E(x) = \mu$. The variance of x is given by

$$E(x - \mu)^2$$

— also denoted as σ^2

Discrete Variables

$$E((x-\mu)^2) = \sum_k (k-\mu)^2 \cdot P(x=k)$$

Ex: $x \sim \text{Ber}(p)$. What is $\text{Var}(x)$?

Sol:

$$\begin{aligned}
\text{Var}(x) &= \sum_{k=0,1} (k-p)^2 \cdot P(x=k) \\
&= (0-p)^2(1-p) + (1-p)^2(p) \\
&= p(1-p)(p+1-p) \\
&= p(1-p)
\end{aligned}$$

Note

$$\begin{aligned}
\text{Var}(x) &= E[(x-E(x))^2] \\
&= E(x^2) - (E(x))^2
\end{aligned}$$

Linear Functions, Properties of Variance

Let g be a linear function $g(x) = ax + b$.

$$\begin{aligned}
E[g(x)] &= E[ax+b] \\
&= a \cdot E(x) + b \\
&= E[ax] + E[b]
\end{aligned}$$

We also see that $E[cx] = c \cdot E[x]$, $c \in \mathbb{R}$, and

$$\text{Var}(ax+b) = a^2 \text{Var}(x)$$

To help model this, suppose we had a dataset of midterm scores S. Suppose we added two points to each midterm score.

- The average, μ of our scores shifts forward by two points.
- The variance does not move.

So

$$\mathrm{Var}(x + b) = \mathrm{Var}(x).$$

Suppose instead we multiply all the scores by 5,

- Then all our scores get stretched by five fold, with equal probability measure.
- Our average shifts to 5 times what it was previously.
- If our distances have stretched by five fold, then our variance has increased by 25.
- Our standard deviation has shifted by five fold.