Expectation

1. Suppose a random student is chosen. What is their expected score on a test?

   Ans: Let X be the score of a randomly chosen student. X takes values $s_1, s_2, \ldots, s_n$.

$$\begin{aligned} E(X) &= s_1 \cdot \frac{1}{n} + s_2 \cdot \frac{1}{n} + \ldots + s_n \cdot \frac{1}{n} \\ &= \text{Average Score} \end{aligned}$$

2. Suppose 5 fair coins are tossed. How many heads are expected?

3. Toss a coin 6 times, with $P(H) = \frac{1}{3}$. How many heads are expected?

   Ans: Let X be the number of heads in 6 tosses, where $P(H) = \frac{1}{3}$.

$$\begin{aligned} E(X) &= \sum_{k=0}^{6} k \cdot \binom{6}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{6-k} \\ &\vdots \\ &= 2 \end{aligned}$$

4. Roll a die 96 times. How many 6s are expected?

   Ans:

$$\frac{1}{6} \cdot 96 = 16$$

5. Roll a die. Let X be the outcome.

   Ans:

$$\begin{aligned} E(X) &= \sum_{k=1}^{6} k \cdot \frac{1}{6} \\ &= 3.5 \end{aligned}$$

Def: Suppose X is a discrete random variable.

$$E(X) = \sum_{k:P(X=k)>0} k \cdot P(X=k)$$

Bernoulli:

$$X \sim \mathrm{Ber}(p)$$

.

$$f(x) = \begin{cases} 0 & p \\ 1 & (1-p) \end{cases}$$

$$\begin{aligned} E(X) &= 1 \cdot P(X=1) + 0 \cdot P(X=0) \\ &= p \end{aligned}$$

Binomial:

$$S_n \sim \mathrm{Bin}(n,p).$$

Toss n coins, $S_n =$ Number of Heads, $P(n) = p$.

$$S_n = X_1 + \ldots + X_n$$

where X_i are n independent Bernoulli variables.

$$\begin{aligned} E(S_n) &= E(X_1 + \ldots + X_n) \\ &= E(X_1) + E(X_2) + \ldots + E(X_n) \\ &= np \end{aligned}$$

Proof.

$$\begin{aligned}
E(S_n) &= \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{k=1}^{n} \frac{k \cdot n!}{(n-k)!\, k!}\, p^k\, (1-p)^{n-k} \\
&= \sum_{k=1}^{n} \frac{n!}{(n-k!)\, (k-1)!}\, p^k\, (1-p)^{n-k} \\
&= np \sum_{k=1}^{n} \frac{(n-1)!}{(n-k!)\, (k-1)!}\, p^{k-1}\, (1-p)^{n-k} \\
&= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1}\, (1-p)^{n-k}
\end{aligned}$$

Let $t = k - 1$.

$$E(S_n) = np \sum_{t=0}^{n-1} \binom{n-1}{t} p^t\, (1-p)^{n-1-t}$$

$$\sum_{t=0}^{n-1} \binom{n-1}{t} p^t\, (1-p)^{n-1-t}$$

is the sum of the binomial probabilities $\mathrm{Bin}(n-1, p)$.

$$\begin{aligned}
E(S_n) &= np \cdot 1 \\
&= np
\end{aligned}$$

$\square$

Geometric: *Calculation in book.*

$$X \sim \mathrm{Geom}(p)$$

.

$$E(x) = \frac{1}{p}$$

Expectation of a Continuous Random Variable

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

Expectation is also known as the mean, and the first moment.

Ex: Let $X \sim U[a, b]$.

$$\begin{aligned} E(x) &= \int_{\infty}^{\infty} x \cdot f(x)\, dx \\ &= \int_{b}^{a} x \cdot \frac{1}{b-a}\, dx \\ &= \frac{x^2}{2(b-a)}\Bigg|_a^b \\ &= \frac{b^2 - a^2}{2(b-a)} \\ &= \frac{b+a}{2} \end{aligned}$$

This is precisely the center of our uniform density.

Ex: Roll a fair die. Let W be our winnings in dollars, and x be the number we roll.

$$W = \begin{cases} -1 & x \in \{1, 2, 3\} \\ 1 & x = 5 \\ 3 & x \in \{5, 6\} \end{cases}$$

$$\begin{aligned} E(W) &= -1 \cdot P(W = -1) + 1 \cdot P(W = 1) + 3 \cdot P(W = 3) \\ &= -1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{6} + 3 \cdot \frac{1}{3} \end{aligned}$$

Alternatively, let $x = i\,, 1 \leq i \leq 6$.

$$W = \begin{cases} -1 & x = 1, 2, 3 \\ 1 & x = 4 \\ 3 & x = 5, 6 \end{cases}$$

$W = g(x)$.

$$\begin{aligned} &P(W = -1) = P(X = 1) + P(X = 2) + P(X = 3) \\ &P(W = 1) = P(X = 4) \\ &P(W = 3) = P(X = 5) + P(x = 6) \end{aligned}$$

Then,

$$\begin{aligned} E(W) &= g(1) \cdot P(X = 1) + g(2) \cdot P(X = 2) + g(3) \cdot P(X = 3) \\ &+ g(4) \cdot P(X = 4) \\ &+ g(5) \cdot P(X = 5) + g(6) \cdot P(X = 6) \\ &= E(g(X)) \end{aligned}$$

Summarily,

$$E(g(x)) = \sum_{k=1}^{6} g(k)\, P(x = k)$$