1 Distributions

If the outcome is $F^{k-1} \cdot S$, then $x = k$. Therefore, $P(x = K) = (1-p)^{k-1} \cdot p$, for $k \geq 1$.

Def: Let $0 \ \leq p \leq 1$. A random variable x has geometric distribution with parameter P if x satisfies : $x \in \{1, 2, \ldots\}$, and $P(x = K) = (1-p)^{k-1} \cdot p$

Bernoulli: Ber(P) =

$$f(x) = \begin{cases} 0 & F \\ 1 & S \end{cases}$$

Binomial: Bin(n,p) = S_n, the number of successes in n independent trials.

Geometric: Geom(p) = X = number of trials for first success.

$$\begin{aligned} \sum_{k=1}^{\infty} P(x = k) &= \sum_{k=1}^{\infty} (1-p)^{k-1} \cdot p \\ &= P \sum_{k=1}^{\infty} (1-p)^{k-1} \end{aligned}$$

Exercise: Show that if x $\sim$ Geom(P), $P(x = n + k | x > n) = P(x = k)$.

Soln: This property is called memorylessness. We perform independent trials until we encounter a success. The event that $x > n$ means that we haven't had success by the time n arrives. Or, every trial including n has been a failure.

What is the chance that $x = n + k$? Or in other words, after $n + k$ trials, we have our first success.

Claim: The geometric distribution has no memory of previous trials.

It's sort of clear that this is the case. These are independent trials.

Proof.

$$\begin{aligned} P(x = n + k | x > n) &= \frac{P(x = n + k \cap x > n)}{P(x > n)} \\ &= \frac{P(x = n + k)}{(1 - p)^n} \\ &= \frac{(1 - p)^{n+k-1} \cdot p}{(1 - p)^n} \\ &= (1 - p)^{k-1} \cdot p \\ &= P(x = k) \end{aligned}$$

□

2 Continuous Random Variables

A random variable is discrete if it takes values in a discrete or countably infinite set.

$$\sum_{k=0} P(x = k) = 1 \tag{1}$$

Ex: x = position of a dart $\in [0, 1]$.

The histogram equivalent for continuous random variables is a probability density function.

Def: Let x be a random variable. If a function f satisfies

$$P(x \leq b) = \int_{-\infty}^{b} f(x)\, dx \tag{2}$$

for all $b \in \mathbb{R}$, then f is the probability density function of x.

Returning to our dart example . . .

$$f(x) = \begin{cases} 0 & b < 0 \\ b & 0 \leq b \leq 1 \\ 1 & b > 1 \end{cases}$$

We want to find f such that (2) is satisfied.

We can focus on the interval [0,1].

Let's define $f(x)$ as

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \in [0,1] \\ 0 & x > 1 \end{cases}$$

then

$$\int_{-\infty}^{b} f(x)\,dx = \begin{cases} 0 & x < 0 \\ \int_{-\infty}^{b} 1\,dx & x \in [0,1] \\ \int_{-\infty}^{b} 0\,dx + \int_{0}^{1} 1\,dx + \int_{1}^{b} 0\,dx & x > 1 \end{cases}$$

therefore, we have found the density function for the random variable x.

Property: x is a random variable with density function f.

$$P(x \leq b) = \int_{-\infty}^{b} f(x)\,dx \tag{3}$$

$$\begin{aligned} P(x \leq b) - P(x < a) &= P(x \leq b) - (P(x \leq a) - P(x = a)) \\ &= P(x \leq b) - P(x \leq a) + P(x = a) \\ &= \int_{-\infty}^{b} f(x)\,dx - \int_{-\infty}^{a} f(x)\,dx \\ &= \int_{a}^{b} f(x)\,dx \end{aligned}$$