Normal Variables

Suppose $Z \sim N(0,1)$. Let $\mu \in \mathbb{R}$, $\sigma > 0$.

Let $X = \sigma Z + \mu$.

$$\begin{aligned} E(x) &= E(\sigma Z + \mu) \\ &= \sigma E(z) + \mu \\ &= \mu \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(x) &= \mathrm{Var}(\sigma Z + \mu) \\ &= \sigma^2 \mathrm{Var}(z) \\ &= \sigma^2 \end{aligned}$$

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= P(\sigma Z + \mu \leq x) \\ &= P(\sigma Z \leq x - \mu) \\ &= P(Z \leq \frac{x-\mu}{\sigma}) \\ &= \Phi(\frac{x-\mu}{\sigma}) \end{aligned}$$

$$\begin{aligned} f_X(x) &= \frac{d}{dx} F_X(x) \\ &= \frac{d}{dx} \Phi(\frac{x-\mu}{\sigma}) \\ &= \frac{1}{\sigma} \phi(\frac{x-\mu}{\sigma}) \\ &= \frac{1}{\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{(\frac{x-\mu}{\sigma})^2}{2}} \end{aligned}$$

Def: A random variable X has $N(\mu, \sigma^2)$ distribution if its density is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\frac{x-\mu}{\sigma})^2}{2}}$$

X has mean μ and variance σ^2.

A similar argument can be used to show that if $X \sim N(\mu, \sigma^2)$, then $X - \frac{\mu}{\sigma} \sim N(0, 1)$.

$$X = \sigma Z + \mu$$
$$\rightarrow z = \frac{x - \mu}{\sigma}$$