For a continuous variable X with density function f

$$f(x) \neq P(X = x)$$

If the density is continuous at a point $a \in \mathbb{R}$ then for small constants $\epsilon > 0$,

$$P(a < X < a + \epsilon) \approx f(a) \cdot \epsilon$$

which implies

$$f(a) \approx \frac{P(a < X < a + \epsilon)}{\epsilon}$$

which we interpret as the probability that X is close to a.

Ex: Suppose a random variable X has density given by

$$f(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Approximate $P(0.50 < X < 0.51)$.

Soln: Let $a = .50$, and $\epsilon = .01$. $f(x)$ *is* continuous at $f(a)$, as $3x^2$ is continuous on [0,1].

Therefore, $P(a < X < a + \epsilon) = \frac{3}{4} \cdot .01 = .0075$

Calculating the exact probability:

$$\begin{aligned} \int_{.5}^{.51} 3x^2 \, dx &= x^3 \Big|_{.5}^{.51} \\ &= .007651 \end{aligned}$$

$$\begin{aligned} \mathcal{R} &= \frac{|.007651 - .0075|}{.007651} \\ &\approx 2\% \end{aligned}$$

Ex: We have a dart board of 9 in. radius. Suppose a dart hits the board at a uniformly random point (every point is equally likely). We measure the distance R of the dart to the bull's eye (center).

Find a function f_R such that f_R is a density for f.

Soln: This function should satisfy

$$P(R \leq r) = \int_{-\infty}^{r} f_R(s)\, ds$$

We can fix a number $r \in [0, 9]$ — we expect any value < 0 to take density 0, and any value > 9 to take value 1.

We know $P(r < R < r + \epsilon) \approx f_R(r) \cdot \epsilon$.

and

$$\begin{aligned} P(r < R < r + \epsilon) &= \frac{\pi(r+\epsilon)^2 - \pi r^2}{9^2\pi} \\ &= \epsilon \cdot \left(\frac{2r+\epsilon}{9^2}\right) \\ &\approx \epsilon f_R(r) \end{aligned}$$

This tells us

$$f_R(r) = \frac{2r}{9^2}$$

This result coming from

$$f_R(r) = \lim_{\epsilon \to 0} \frac{P(r < R < r + \epsilon)}{\epsilon}$$

Therefore

$$f_R(r) = \begin{cases} 0 & r < 0 \\ \frac{2r}{9^2} & 0 \leq r \leq 9\,, \epsilon > 0 \\ 0 & r > 9 \end{cases}$$

Cumulative Distribution Functions: For any random variable X,

$$F_x(x) = P(X \leq x).$$

For a continuous random variable X, with density $f(x)$,

$$F_x(x) = \int_{-\infty}^{x} f(t)\, dt$$

Ex: $t \sim U[a, b]$.

$$
\begin{aligned}
F_t(x) &= \int_{-\infty}^{x} f(t)\,dt \\
&= \begin{cases} 0 & x < 0 \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}
\end{aligned}
$$

Given the density of X, one can find the cdf, $F_x(x)$ by integration. Conversely, one can find the density from a given cdf through derivation.

Let a random variable X have cdf $F(x)$.

(a). Suppose $F$is piecewise constant. Then X is a discrete random variable. The possible values values of X are the locations where F has jumps. If x is such a point, then

$$P(X = x) = \text{magnitude of the jump at } x$$

(b). Suppose F is continuous and the derivative $F'(x)$ exists everywhere on the real line, except possibly at finitely many points. Then X is a continuous random variable, and its density is

$$f(x) = F'(x)$$

If f is not differentiable at x, then the value of f(x) can be set arbitrarily.

Ex: We throw a dart at the same disk as the earlier example. Find $f_R(r)$

$$
\begin{aligned}
F_R(r) &= P(R \le r) \\
&= \begin{cases} 0 & r < 0 \\ \frac{r^2}{9^2} & 0 \le r \le 9 \\ 1 & r > 9 \end{cases}
\end{aligned}
$$

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$$
f_R(r) = \begin{cases} 0 & r < 0 \\ \frac{2r}{9^2} & 0 \le r \le 9 \\ 0 & r > 2 \end{cases}
$$

Ex: Suppose you had an insurance policy on a car, and your deductible is \$500, i.e. you pay 100% of repair costs up to \$500, and the insurance company pays the rest.

Suppose the cost of repairs $C \sim U[100, 1500]$. Let X be the amount of money you pay, find the cdf of X.

Soln: $F_X(x) = P(X \leq x)$.

- If $100 \leq C \leq 500 \rightarrow X = C$
- If $C > 500 \rightarrow X = 500$

So X takes values $\in [100, 500]$, and

$$F_X(x) = \begin{cases} 0 & x < 100 \\ \frac{x-100}{1400} & 100 \leq x \leq 500 \\ 1 & x > 500 \end{cases}$$