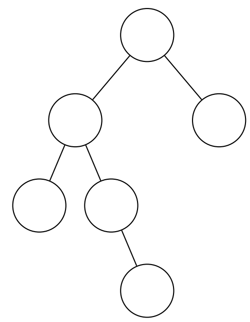
(1). (5 points) **Argue** that in a *binary* tree of N nodes, there are $N + 1$ null-pointer links representing children. Note that each node has a left link and a right link pointing to the (possibly empty) left child and the (possibly empty) right child, respectively.



Proof.

Theorem: A tree of N vertices has $N - 1$ edges.

By the above theorem, and by properties of trees, we know that we have $N - 1$ child nodes. For N nodes, we have $2N$ links. We know that $N - 1$ of these are *not* null.

$$2N - (N - 1) = N + 1$$

By this result, $N + 1$ of these links must be null. □

(2). (9 points) **Prove by induction** that the maximum number of nodes in a binary tree of height h is $2^{h+1} - 1$.

Proof. **Height of a Binary Tree:** The height of a binary tree is the number of edges between its root and its furthest leaf node.

Consider the tree which maximizes the number of nodes in a given binary tree of height h — this tree must be a perfect binary tree.

Then, for a perfect binary tree of height h, the number of nodes for height h is equal to

$$\sum_{i=0}^{h} 2^i$$

as each level must have double the number of nodes as the previous level.

Let $h = 0$.

$$\begin{aligned}\sum_{i=0}^{0} 2^i &= 2^{0+1} - 1 \\ 1 &= 1\end{aligned}$$

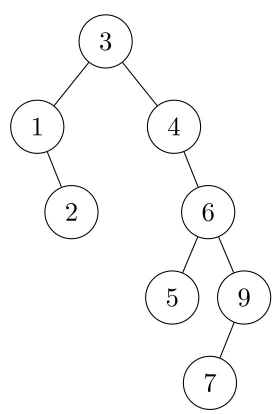
Then assume $h = k$.

$$\begin{aligned}\sum_{i=0}^{k} 2^i &= 2^{k+1} - 1 \\ \sum_{i=0}^{k+1} 2^i &= 2^{(k+1)+1} - 1 \\ \left(\sum_{i=0}^{k} 2^i\right) + 2^{k+1} &= 2^{k+2} - 1 \\ 2^{k+1} - 1 + 2^{k+1} &= 2^{k+2} - 1 \\ 2^{k+2} - 1 &= 2^{k+2} - 1\end{aligned}$$

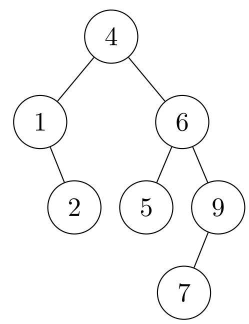
$\square$

(3). (10 points) (a). Show the result of inserting 3, 1, 4, 6, 9, 2, 5, 7 into an initially empty binary search tree. (b). Show the result of deleting the root.

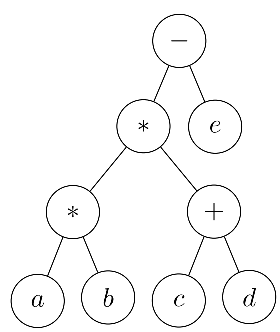
(a).



(b).



(4). (6 points) Give the prefix, infix, and postfix expressions corresponding to the following tree.



Prefix:

$$- * * ab + cde$$

Infix:

$$((a \cdot b) \cdot (c + d)) - e$$

Postfix:

$$ab * cd + * e-$$