1. There are 5 closed boxes on a table. Three of the boxes have good prizes inside and the other two don't. You open boxes one at a time until you find a prize. Let X be the number of boxes you open.

   (a). Find the pmf of X.

   Ans: X may take values $\in [1, 3]$. Let $f(X)$ be the pmf of X.

$$f(X) = \begin{cases} \frac{3}{5} & x = 1 \\ \frac{2}{5} \cdot \frac{3}{4} & x = 2 \\ \frac{2}{5} \cdot \frac{1}{4} \cdot 1 & x = 3 \\ 0 & x > 3 \end{cases}$$

   (b). Find $E(X)$.

   Ans:

$$E(X) = \sum_k kP(X = k) \tag{1}$$

   Alternate symbol of $E[X] = \mu$

   Thus

$$\begin{aligned} E(X) &= 1 \cdot \frac{3}{5} + 2\left(\frac{2}{5} \cdot \frac{3}{4}\right) + 3 \cdot \left(\frac{2}{5} \cdot \frac{1}{4}\right) \\ &= 1.5 \end{aligned}$$

   (c). Find $\mathrm{Var}(X)$.

   Ans:

$$\mathrm{Var}(X) = \sum_k (k - \mu)^2 P(X = k) \tag{2}$$

$$\begin{aligned} \mathrm{Var}(X) &= (1 - 1.5)^2 \cdot \frac{3}{5} + (2 - 1.5)^2\left(\frac{2}{5} \cdot \frac{3}{4}\right) + (3 - 1.5)^2\left(\frac{2}{5} \cdot \frac{1}{4}\right) \\ &= .45 \end{aligned}$$

   (d). Suppose the good prize inside each of the three boxes is \$100, but each empty box you open costs you \$100. What is your expected gain or loss in the game? (Hint: Express the gain or loss as a function of X.)

Ans:

$$f(X) = \begin{cases} 100 & x = 1 \\ 0 & x = 2 \\ -100 & x = 3 \end{cases}$$

$$\begin{aligned} E[X] &= 100 \cdot \frac{3}{5} + 0 - 100 \cdot \left(\frac{2}{5} \cdot \frac{1}{4}\right) \\ &= 50 \end{aligned}$$

2. Let X be a random variable with density function

$$f(x) = \begin{cases} \frac{1}{4} & 1 < x < 2 \\ c & 3 < x < 5 \\ 0 & \text{otherwise} \end{cases}$$

(a). Find the value of c.

Ans: By the properties of a density function,

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

which is only possible given c is equal to

$$\begin{aligned} c(5-3) &= 1 - \frac{1}{4}(2-1) \\ 2c &= .75 \\ c &= \frac{3}{8} \end{aligned}$$

(b). Find $P(1.5 < X < 4)$

Ans:

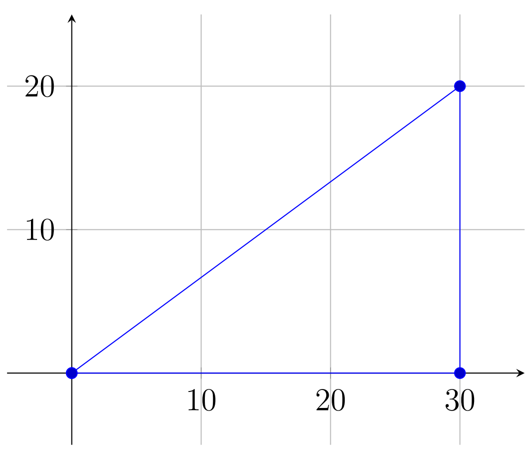
$$
\begin{aligned}
P(a < X < b) &= \int_b^a f(x)\,dx \\
P(1.5 < X < 4) &= \int_{1.5}^4 f(x)\,dx \\
&= \int_{1.5}^2 f(x)\,dx + \int_2^3 f(x)\,dx + \int_3^4 f(x)\,dx \\
&= \frac{1}{4}(2-1.5) + 0 + \frac{3}{8}(4-3) \\
&= .50
\end{aligned}
$$

(c). Find the cdf of X.

Ans:

$$
\begin{aligned}
F(s) = P(X \le s) &= \int_{-\infty}^s f(x)\,dx \\
&= \begin{cases} 0 & s < 1 \\ \frac{s-1}{4} & 1 < s < 2 \\ \frac{3}{8}(s-3) + .25 & 3 < s < 5 \\ 1 & x > 5 \end{cases}
\end{aligned}
$$

3. Choose a point (X, Y) uniformly inside the triangle with vertices (0,0), (30,0) and (30,20).



(a). Find the cdf of X.

Ans: The area inside the triangle may be modeled by

$$\int_0^{30} \frac{2}{3}x\,dx = 300$$

as X may take values $\in [0, 30]$.

$$P(X \leq x) = \frac{1}{300}\int_0^x \frac{2}{3}x\,dx$$

Thus, we write the cdf as

$$f_X(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{300} \cdot \frac{1}{3}x^2 & 0 \leq x \leq 30 \\ 1 & x > 30 \end{cases}$$

(b). Using (a), find the density function of X.

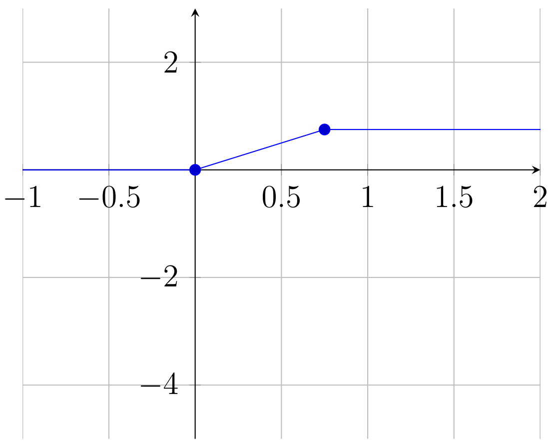
Ans:

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{450}x & x \in [0, 30] \\ 0 & x > 30 \end{cases}$$

4. Suppose that when you take the bus to school, it arrives after a uniformly distributed number of hours in the interval [0,1] after you get to the stop. However, if it does not arrive for 45 minutes, you take an uber. Let X be the number of hours you wait.

(a). Find the cdf of X.

Ans: The graph of X is as follows.



Let Y be the time which the bus arrives. $Y \sim U[0,1]$.

- $0 \leq Y \leq \frac{3}{4} \to X = Y$.
- $Y > \frac{3}{4} \to X = \frac{3}{4}$.

Thus, the cdf of X is

$$f_X(x) = \begin{cases} 0 & x < 0 \\ \frac{x-0}{1} & 0 \leq x \leq \frac{3}{4} \\ 1 & x > \frac{3}{4} \end{cases}$$

5.

6. Let $Z \sim \text{Bin}(10, \frac{1}{3})$. Find the value of its cdf at 2 and at 8.

Ans:

$$Z \sim \text{Bin}(10, \frac{1}{3}) \to P(Z = k) = \binom{10}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{10-k} \quad \text{for } k = 0, 1, \ldots, 10$$

$$P(Z \leq k) = \sum_{i=0}^{k} \binom{10}{i} \left(\frac{1}{3}\right)^i \left(\frac{2}{3}\right)^{10-i}$$

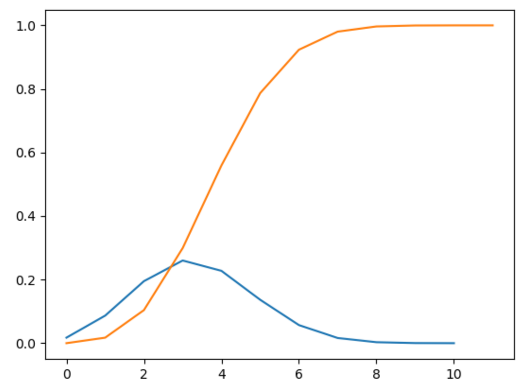


Figure 1: Matplotlib Figure of the cdf, pdf of Z

$k = 2$

$$\begin{aligned} P(Z \leq 2) &= \sum_{i=0}^{2} \binom{10}{i} \left(\frac{1}{3}\right)^{i} \left(\frac{2}{3}\right)^{10-i} \\ &= \binom{10}{0}\left(\frac{1}{3}\right)^{0}\left(\frac{2}{3}\right)^{10} + \binom{10}{1}\left(\frac{1}{3}\right)^{1}\left(\frac{2}{3}\right)^{9} + \binom{10}{2}\left(\frac{1}{3}\right)^{2}\left(\frac{2}{3}\right)^{8} \\ &= 0.2991 \end{aligned}$$

$k = 8$

$$\begin{aligned} P(Z \leq 8) &= \sum_{i=0}^{8} \binom{10}{i} \left(\frac{1}{3}\right)^{i} \left(\frac{2}{3}\right)^{10-i} \\ &= 0.9996443631560223 \end{aligned}$$

7. Let $c > 0$ and $X \sim \text{Unif}[0, c]$. Show that the random variable $Y = c - X$ has the same cdf as X and hence also the same density function.

Ans:

$$X \sim \text{Unif}[0, c] \to f(x) = \begin{cases} \frac{1}{c} & x \in [0, c] \\ 0 & \text{otherwise} \end{cases}$$

where $f(x)$ is the density function of X.

We want to find $P(Y \leq x)$.

$$\begin{aligned} P(Y \leq x) &= P(c - X \leq x) \\ &= P(X \geq c - x) \end{aligned}$$

$$P(c - x \leq X) = \begin{cases} 0 & x < c \\ \frac{c-(c-x)}{c} & 0 \leq x \leq c \\ 1 & x > c \end{cases}$$

If we take the derivative of $P(c - x \leq X)$ we get exactly $f(x)$.