`Section 3.5 p. 141] #2, 3, 5, 6, 7, 12, 13, 14`

`Section 3.6 p. 146] #4, 6`

`Section 3.7 p. 157] # 1, 2, 3, 4`

(2). $y'' - y' - 2y = -2t + 4t^2$

Ans: We know that our solution is the sum of the complementary and particular solution. Solving the corresponding homogenous problem

$$y'' - y' - 2y = 0$$

Using $y = e^{rt}$.

$$\begin{aligned}(r^2 - r - 2)e^{rt} &= 0\\ (r^2 - r - 2) &= 0\\ (r+1)(r-2) &= 0\end{aligned}$$

Therefore, we have

$$y = C_1 e^{2t} + C_2 e^{-t}$$

We let our potential solution equal

$$y = A + Bt + Ct^2$$

such that our ODE equals

$$\begin{aligned}(A + Bt + Ct^2)'' - (A + Bt + Ct^2)' - 2(A + Bt + Ct^2) &= -2t + 4t^2\\ 2C - B - 2Ct - 2A - 2Bt - 2Ct^2 &= -2t + 4t^2\\ (-2A - B + 2C) + (-2B - 2C)t + (-2C)t^2 &= -2t + 4t^2\end{aligned}$$

Giving us the system

$$\begin{bmatrix} -2 & -1 & 2\\ 0 & -2 & -2\\ 0 & 0 & -2 \end{bmatrix} \cdot \phi = \begin{bmatrix} 0\\ -2\\ 4 \end{bmatrix}$$

where

$$\phi = \begin{bmatrix} -3.5\\ 3\\ -2 \end{bmatrix}$$

Leaving us with our complete solution of

$$y = C_1 e^{2t} + C_2 e^{-t} - 3.5 + 3t - 2t^2$$

(3). $y'' + y' - 6y = 12e^{3t} + 12e^{-2t}$

As we did in (2). we solve the corresponding homogenous problem

$$y'' + y' - 6y = 0$$

$$\begin{aligned} r^2 + r - 6 &= 0 \\ (r-3)(r+2) &= 0 \end{aligned}$$

Leaving us with

$$y = C_1 e^{3t} + C_2 e^{-2t}$$

We assume then our particular solution must take the form

$$y = Ae^{3t} + Be^{-2t}$$

Which means our ODE equals

$$\begin{aligned} (Ae^{3t} + Be^{-2t})'' + (Ae^{3t} + Be^{-2t})' - 6(Ae^{3t} + Be^{-2t}) &= 12e^{3t} + 12e^{-2t} \\ 9Ae^{3t} + 4Be^{-2t} + 3Ae^{3t} - 2Be^{-2t} - 6Ae^{3t} - 6Be^{-2t} &= 12e^{3t} + 12e^{-2t} \\ 6Ae^{3t} - 4Be^{-2t} &= 12e^{3t} + 12e^{-2t} \end{aligned}$$

Which is true if $A = 2$, and $B = -3$.

Leaving us with the complete solution

$$y = C_1 e^{3t} + C_2 e^{-2t} + 2e^{3t} - 3e^{-2t}$$

(5) $y'' + 2y' = 3 + 4\sin(2t)$

Solving corresponding homogenous problem

$$\begin{aligned} r^2 + 2r &= 0 \\ r(r+2) &= 0 \end{aligned}$$

Leaving us with the complementary solution

$$y = C_1 + C_2 e^{-2t}$$

Because our nonhomogenous problem includes a sin function and a constant, we let our particular solution equal

$$y = A\cos(2t) + B\sin(2t) + Ct$$

Thus

$$\begin{aligned}(A\cos(2t) + B\sin(2t) + Ct)'' + 2(A\cos(2t) + B\sin(2t) + Ct)' &= 3 + 4\sin(2t)\\ -4A\cos(2t) - 4B\sin(2t) - 4A\sin(2t) + 4B\cos(2t) + 2C &= 3 + 4\sin(2t)\\ (-4A + 4B)\cos(2t) + (-4A - 4B)\sin(2t) + 2C &= 3 + 4\sin(2t)\end{aligned}$$

We conclude that $C = \frac{3}{2}$, and that the following system must hold

$$\begin{bmatrix} -4 & 4 \\ -4 & -4 \end{bmatrix} \cdot \phi = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

Which we solve to produce

$$\phi = \begin{bmatrix} -.5 \\ -.5 \end{bmatrix}$$

Thus, our complete solution is

$$y = C_1 + C_2 e^{-2t} - \frac{1}{2}\cos(2t) - \frac{1}{2}\sin(2t) + \frac{3}{2}t$$

(6). $y'' + 2y' + y = 2e^{-t}$

Solving corresponding homogenous problem

$$\begin{aligned} r^2 + 2r + 1 &= 0\\ (r+1)(r+1) &= 0\end{aligned}$$

In the case of repeated roots, we let our solution equal

$$y = C_1 e^{-t} + C_2 t e^{-t}$$

Also in the case of repeated roots, we let our particular solution be

$$y = At^2 e^{-t}$$

Therefore

$$\begin{aligned}(At^2e^{-t})'' + 2(At^2e^{-t})' + (At^2e^{-t}) &= 2e^{-t}\\ A(t^2e^{-t} - 2te^{-t} + 2(-te^{-t} + e^{-t})) + 2A(-t^2e^{-t} + 2te^{-t}) + (At^2e^{-t}) &= 2e^{-t}\\ At^2e^{-t} - 2Ate^{-t} - 2Ate^{-t} + 2Ae^{-t} - 2At^2e^{-t} + 4Ate^{-t} + At^2e^{-t} &= 2e^{-t}\\ 2Ate^{-t} &= 2e^{-t}\end{aligned}$$

Leaving us with $A = 1$, and our complete solution being

$$y = C_1 e^{-t} + C_2 t e^{-t} + t^2 e^{-t}$$

(7). $y'' + y = 3\sin(2t) + t\cos(2t)$

Solving corresponding homogenous problem

$$r^2 + 1 = 0$$

Using Euler's formula

$$e^{i\phi} = \cos(\phi) + i\sin(\phi)$$

We find the complementary solution

$$y = C_1 \cos(t) + C_2 \sin(t)$$

We find that the cos function in our nonhomogenous problem has a linear coefficient. Thus, we choose the following as our particular solution

$$y = A\sin(2t) + Bt\cos(2t) + C\sin(2t)$$

Making the left hand sum of our ODE equal to

$$\begin{aligned}
(A\sin 2t + Bt\cos 2t + C\sin 2t)'' + (A\sin 2t + Bt\cos 2t + C\sin 2t) \\
= (2A\cos 2t + B(-2t\sin 2t + \cos 2t) + 2C\cos 2t)' \\
+(A\sin 2t + Bt\cos 2t + C\sin 2t) \\
= (-4A\sin 2t + B(-4t\cos 2t - 2\sin 2t - 2\sin 2t) - 4C\sin 2t) \\
+(A\sin 2t + Bt\cos 2t + C\sin 2t) \\
= (-4A\sin 2t + -4Bt\cos 2t - 4B\sin 2t - 4C\sin 2t) \\
+(A\sin 2t + Bt\cos 2t + C\sin 2t) \\
= (-3A - 4B - 3C)\sin 2t - 3Bt\cos 2t
\end{aligned}$$

Giving us

$$(-3A - 4B - 3C)\sin 2t - 3Bt\cos 2t = 3\sin 2t + t\cos 2t$$

and the following system

$$\begin{bmatrix} -3 & -4 & -3 \\ 0 & -3 & 0 \end{bmatrix} \cdot \phi = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Using Octave, we find

$$\phi = \begin{bmatrix} -0.2778 \\ -0.3333 \\ -0.2778 \end{bmatrix}$$

and a complete solution of

$$y = C_1 \cos(t) + C_2 \sin(t) - \frac{5}{9}\sin(2t) - \frac{1}{3} t\cos(2t)$$

(12). $y'' + 4y = t^2 + 3e^t$, $y(0) = 0, y'(0) = 2$

$$r^2 + 4 = 0$$

$$r = \pm 2i$$

Complementary Solution:

$$y = C_1 \cos 2t + C_2 \sin 2t$$

We choose the following function as our potential particular solution:

$$y = (A + Bt + Ct^2) + De^t$$

Which makes our ODE equal to

$$\begin{aligned} ((A + Bt + Ct^2) + De^t)'' + 4((A + Bt + Ct^2) + De^t) &= t^2 + 3e^t \\ 2C + De^t + 4A + 4Bt + 4Ct^2 &= t^2 + 3e^t \\ (4A + 2C) + 4Bt + 4Ct^2 + 4De^t &= t^2 + 3e^t \end{aligned}$$

And the following system

$$\begin{bmatrix} 4 & 0 & 2 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \cdot \phi = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}$$

Using Octave

$$\phi = \begin{bmatrix} -0.1250 \\ 0 \\ 0.2500 \\ 0.6000 \end{bmatrix}$$

Therefore

$$y = C_1 \cos 2t + C_2 \sin 2t - \frac{1}{8} + \frac{1}{4}t^2 + \frac{3}{5}e^t$$

and

$$y' = -2C_1 \sin 2t + 2C_2 \cos 2t + \frac{1}{2}t + \frac{3}{5}e^t$$

using our initial conditions

$$\begin{aligned} 0 &= C_1 - \frac{1}{8} + \frac{3}{5} \\ -\frac{19}{40} &= C_1 \\ 2 &= 2C_2 + \frac{3}{5} \\ \frac{7}{10} &= C_2 \end{aligned}$$

and the solution to our IVP is

$$y = -\frac{19}{40} \cos 2t + \frac{7}{10} \sin 2t - \frac{1}{8} + \frac{1}{4}t^2 + \frac{3}{5}e^t$$

(13). $y'' - 2y' + y = te^t + 4,\ y(0) = 1, y'(0) = 1$

$$\begin{aligned} r^2 - 2r + 1 &= 0 \\ (r-1)(r-1) &= 0 \end{aligned}$$

Complementary Solution:

$$y = C_1 e^t + C_2 t e^t$$

Due to our repeated roots, we let our potential particular solution be

$$y = At^2e^2 + Bt^3e^t + C$$

And our ODE becomes equal to

$$\begin{aligned} (At^2e^2 + Bt^3e^t + C)'' - 2(At^2e^2 + Bt^3e^t + C)' + (At^2e^2 + Bt^3e^t + C) \\ = te^t + 4 \\ \text{(using special mathway and derivative calculator properties)} \\ \vdots \\ 2Ae^t + 6Bte^t + C = te^t + 4 \end{aligned}$$

Giving us the following system

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \phi \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$$

$$\phi = \begin{bmatrix} 0 \\ 0.1666 \\ 4 \end{bmatrix}$$

Therefore, our general solution is

$$y = C_1 e^t + C_2 t e^t + \frac{1}{6} t^3 e^t + 4$$

and

$$y' = C_1 e^t + C_2 t e^t + C_2 e^t + \frac{1}{6} t^3 e^t + \frac{1}{2} t^2 e^t$$

Applying initial conditions

$$1 = C_1 + 4$$
$$-3 = C_1$$

$$1 = -3 + C_2$$
$$4 = C_2$$

The solution to our IVP is

$$y = -3e^t + 4te^t + \frac{1}{6} t^3 e^t + 4$$

(14). $y'' + 4y = 3\sin(2t),\ y(0) = 2, y'(0) = -1$

$$r^2 + 4 = 0$$

$$r = \pm 2i$$

Complementary Solution:

$$y = C_1 \cos 2t + C_2 \sin 2t$$

Let our potential particular solution be

$$y = A \sin 2t + B \cos 2t$$

These are already part of the homogenous solution, so let's multiply by a factor of t.

$$y = At\sin 2t + Bt\cos 2t$$

Which makes our ODE equal to

$$\begin{aligned} (At\sin 2t + Bt\cos 2t)'' + 4(At\sin 2t + Bt\cos 2t) &= 3\sin 2t \\ (2At\cos 2t + A\sin 2t - 2Bt\sin 2t + B\cos 2t)' & \\ +4At\sin 2t + 4Bt\cos 2t &= 3\sin 2t \\ -4At\sin 2t + 2A\cos 2t + 2A\cos 2t - 4Bt\cos 2t - 2B\sin 2t - 2B\sin 2t & \\ +4At\sin 2t + 4Bt\cos 2t &= 3\sin 2t \\ 4A\cos 2t - 4B\sin 2t &= 3\sin 2t \end{aligned}$$

Giving us $A = 0$, and $B = -\frac{3}{4}$, and a general solution

$$y = C_1\cos 2t + C_2\sin 2t - \frac{3}{4}t\cos 2t$$

and

$$y' = -2C_1\sin 2t + 2C_2\cos 2t + \frac{3}{2}\sin 2t + \frac{3}{4}\cos 2t$$

Applying initial conditions

$$2 = C_1$$

$$\begin{aligned} -1 &= 2C_2 + \frac{3}{4} \\ \frac{1}{8} &= C_2 \end{aligned}$$

Thus, the solution to our IVP is

$$y = 2\cos 2t - \frac{1}{8}\sin 2t - \frac{3}{4}t\cos 2t$$

(4.) $y'' + y = \tan t, 0 < t < \frac{\pi}{2}$

Homogenous solution:

$$y = C_1\cos t + C_2\sin t$$

Using variation of parameters

$$
\begin{aligned}
(C_1(t)\cos t + C_2(t)\sin t)'' + (C_1(t)\cos t + C_2(t)\sin t) &= \tan t \\
&\vdots \\
C_1''(t)\cos t - 2C_1'(t)\sin t + C_2''(t)\sin t + 2C_2'(t)\cos t &= \tan t
\end{aligned}
$$

I don't have time tonight to solve the rest, but I promise to go to the library tomorrow and finish this :D