Suppose $S_n \sim Bin(n,p)$, then

$$\begin{aligned}\lim_{n\to\infty} P(a \leq \frac{S_n - np}{\sqrt{np(1-p)}} \leq b) &= \Phi(b) - \Phi(a) \\ &= P(a \leq Z \leq b)\end{aligned}$$

(Central Limit Theorem)

(Not covered) Continuity Correction: Suppose we wanted to estimate $P(k_1 \leq S_n \leq k_2)$.

$$\begin{aligned}P(k_1 \leq S_n \leq k_2) &= P\left(\frac{k_1 - np}{\sqrt{np(1-p)}} \leq \frac{S_n - np}{\sqrt{np(1-p)}} \leq \frac{k_2 - np}{\sqrt{np(1-p)}}\right) \\ &\approx \Phi\left(\frac{k_2 - np}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{k_1 - np}{\sqrt{np(1-p)}}\right)\end{aligned}$$

For a better approximation, we use

$$P(k_1 \leq S_n \leq k_2) \approx \Phi\left(\frac{(k_2 + \frac{1}{2}) - np}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{(k_1 - \frac{1}{2}) - np}{\sqrt{np(1-p)}}\right)$$

This approximation is more accurate if k_1, k_2 are close to one another or $np(1-p)$ is not large.

(Not covered)

Law of Large Numbers - [4.2]

Confidence Intervals - [4.3]

Maximum Likelihood Estimation

Random Walks

4.4: Poisson Approximation Let $\lambda > 0$. A random variable has the Poisson(x) distribution if x takes non-negative integer values with pmf:

$$P(x = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

for $k \in \{0, 1, 2, 3, \ldots\}$

Recall the Taylor series expansion of

$$\begin{aligned}
e^{\lambda} &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \\
&= \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \\
&= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \\
&= 1
\end{aligned}$$

Let $x \sim \text{Poisson}(x)$, then $E(x) = \lambda$, and $\text{Var}(x) = \lambda$.

Proof.

$$\begin{aligned}
E(x) &= \sum_{k=0}^{\infty} k \cdot \frac{e^{-\lambda}\lambda^k}{k!} \\
&= \sum_{k=1}^{\infty} k \cdot \frac{e^{-\lambda}\lambda^k}{k!} \\
&= \sum_{k=1}^{\infty} \frac{e^{-\lambda}\lambda^k}{(k-1)!}
\end{aligned}$$

Let $t = k - 1$.

$$\begin{aligned} &= \sum_{t=0}^{\infty} \frac{e^{-\lambda}\lambda^{t+1}}{t!} \\ &= \lambda \sum_{t=0}^{\infty} \frac{e^{-\lambda}\lambda^{t}}{t!} \\ &= \lambda \end{aligned}$$

$\square$

Proof. **WTS**: $\mathrm{Var}(x) = \lambda$.

Recall $\mathrm{Var}(x) = E(x^2) - (E(x))^2$.

Consider

$$\begin{aligned} E(x(x-1)) &= E(x^2 - x) \\ &= E(x^2) - E(x) \end{aligned}$$

$$\begin{aligned} E(x(x-1)) &= \sum_{k=0}^{\infty} k(k-1)\frac{e^{-\lambda}\lambda^{k}}{k!} \\ &= \sum_{k=2}^{\infty} k(k-1)\frac{e^{-\lambda}\lambda^{k}}{k!} \\ &= \sum_{k=2}^{\infty} \frac{e^{-\lambda}\lambda^{k}}{(k-2)!} \end{aligned}$$

Let $t = k - 2$.

$$\begin{aligned} &= \sum_{t=0}^{\infty} \frac{e^{-\lambda}\lambda^{t+2}}{t!} \\ &= \lambda^2 \sum_{t=0}^{\infty} \frac{e^{-\lambda}\lambda^{t}}{t!} \\ &= \lambda^2 \end{aligned}$$

$$\begin{aligned} E(x^2) &= E(x(x-1)) + E(x) \\ &= \lambda^2 + \lambda \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(x) &= E(x^2) - (E(x))^2 \\ &= \lambda^2 + \lambda - \lambda^2 \\ &= \lambda \end{aligned}$$

$\square$

Interesting things to do with the Poisson distribution:

1. Poisson approximation to the binomial.

   If p is "very small," in particular, $np = c$ (expected number of successes for $S_n \sim \mathrm{Bin}(np)$), then, S_n is well approximated by a Poisson(np).

   Example: Toss a coin n times, where the chances of heads is $\frac{5}{n}$.

n	p	np
10	.5	5
100	.05	5
1000	.005	5

   Good instance to use Poisson approximation.

   Theorem: Let $\lambda > 0$ and consider positive integers n for which $\frac{\lambda}{n} < 1$.
   Let $S_n \sim B_n(n, \frac{\lambda}{n})$, then

   $$\lim_{n\to\infty} P(S_n = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

   for $k \in \{1, 2, 3, \ldots\}$.

Example: $S_n \sim \text{Bin}(n,p)$. Then

$$\begin{aligned}
P(S_n = k) &= \binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k} \\
&= \frac{n!}{(n-k)!k!}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k} \\
&= \frac{n(n-1)(n-2)\dots(n-k+1)}{k!}\frac{\lambda^k}{n^k}\frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k} \\
&= \frac{\lambda^k}{k!}(1-\frac{\lambda}{n})^n\left[\frac{n(n-1)(n-2)\dots(n-k+1)}{n\cdot n\cdot n\dots n}\right]\cdot\frac{1}{(1-\frac{\lambda}{n})^k} \\
&= \frac{\lambda^k}{k!}(1-\frac{\lambda}{n})^n\left[1\cdot(1-\frac{1}{n})\cdot(1-\frac{2}{n})\dots(1-\frac{k}{n})\right]\cdot\frac{1}{(1-\frac{\lambda}{n})^k}
\end{aligned}$$

Since k is a fixed integer,

$$\lim_{n\to\infty}(1-\frac{1}{n})\cdot\ldots\cdot(1-\frac{k}{n}) = 1$$

$$\lim_{n\to\infty}\frac{1}{(1-\frac{\lambda}{n})^k}1$$

Thus,

$$\begin{aligned}
\lim_{n\to\infty}\frac{\lambda^k}{k!}(1-\frac{\lambda}{n})^n\left[1\cdot(1-\frac{1}{n})\cdot(1-\frac{2}{n})\dots(1-\frac{k}{n})\right]\cdot\frac{1}{(1-\frac{\lambda}{n})^k} &= \frac{\lambda^k}{k!}e^{-\lambda} \\
&= P(\text{Poisson}(\lambda) = k)
\end{aligned}$$

Theorem: Let $x \sim \text{Bin}(n,p)$, $y \sim \text{Poisson}(np)$, then for any $A \subset \{0,1,2,\dots\}$,

$$|P(x\in A) - P(y \in A)| \le np^2$$

for any $k \in \{0,1,2,3,\dots\}$.