$x = r\cos\theta$, $y = r\sin\theta$, , $\tan\theta = \frac{y}{x}$

$$r^2 = x^2 + y^2$$

$$\begin{aligned}
2r \cdot \frac{dr}{dt} &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\
r\frac{dr}{dt} &= x\frac{dx}{dt} + y\frac{dy}{dt} \\
&= x(x + y - x(x^2 + y^2)) + y(-x + y - y(x^2 + y^2)) \\
&= (x^2 + y^2)(1 - (x^2 + y^2)) \\
r\frac{dr}{dt} &= r^2(1 - r^2)
\end{aligned}$$

The critical points are (0,0): $r = 0.$, and $r = 1$.

$r = 1$ corresponds to the unit circle in the phase plane.

If $r > 1$, $\frac{dr}{dt}$ is negative, which means that outside the unit circle, the trajectories are directed inwards.

If $r < 1$, $\frac{dr}{dt}$ is postivie, and the trajectories inside the unit circle are directed outwards.

The goal is to find $\frac{d\theta}{dt}$

$$\begin{aligned}
\frac{dx}{dt} &= \frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt} \\
\frac{dy}{dt} &= \frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}
\end{aligned}$$

$$\begin{aligned}
x\frac{dy}{dt} &= r\cos\theta\left(\frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}\right) \\
&= r\frac{dr}{dt}\cos\theta \cdot \sin\theta + r^2\cos^2\theta\frac{d\theta}{dt}
\end{aligned}$$

$$
\begin{aligned}
y\frac{dx}{dt} &= r\sin\theta\left(\frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}\right) \\
&= r\frac{dr}{dt}\cos\theta\cdot\sin\theta - r^2\sin^2\theta\frac{d\theta}{dt}
\end{aligned}
$$

$$
\begin{aligned}
x\frac{dy}{dt} - y\frac{dx}{dy} &= r^2\frac{d\theta}{dt}(\cos^2\theta + \sin^2\theta) \\
&= r^2\frac{d\theta}{dt} \\
&= x\cdot(-x + y - y(x^2+y^2)) - y(x + y - x(x^2+y^2)) \\
&= -x^2 - y^2 \\
&= -(x^2+y^2) \\
&= -r^2 \\
-r^2 &= r^2\frac{d\theta}{dt} \\
\frac{d\theta}{dt} &= -1
\end{aligned}
$$

The system we have found is:

$$
\begin{cases}
r\frac{dr}{dt} = r^2(1-r^2) \\
\frac{d\theta}{dt} = -1
\end{cases}
$$

One of our solutions is:

$$
\begin{cases}
r = 1 \\
\theta = -t + t_0
\end{cases}
$$

Points satisfying this solution move around the unit circle clockwise.

Thus, the autonomous system has a periodic solution.

If $r \neq 0$ and $r \neq 1$, we can solve our equation as follows:

$$
\begin{aligned}
r\frac{dr}{dt} &= r^2(1-r^2) \\
\frac{dr}{dt} &= r(1^2-r^2) \\
&= r(1-r)(1+r) \\
&= \frac{1}{r(1+r)(1-r)} \\
&= dt \\
&= \frac{A}{r} + B(1+r) + C(1-r) \\
&= \frac{A(1+r)(1-r)}{r(1+r)(1-r)} + \frac{B(1-r)(r)}{r(1+r)(1-r)} + \frac{C(1+r)(r)}{r(1+r)(1-r)} \\
&= \frac{(-A-B+C)r^2 + (B+C)r + A}{r(1-r)(1+r)}
\end{aligned}
$$

$$
\left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right]
$$

$r_1 + r_2$

$$
\left[\begin{array}{ccc|c} 0 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right]
$$

$r_1 + r_2$

$$
\left[\begin{array}{ccc|c} 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right]
$$

$r_2 - \frac{1}{2}r_1$

$$
\left[\begin{array}{ccc|c} 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 1 & 0 & 0 & 1 \end{array}\right]
$$

$A = 1, B = -\frac{1}{2}, C = \frac{1}{2}$.

$$\begin{aligned}
\frac{1}{r(1-r)(1+r)} &= \frac{1}{r} - \frac{1}{2}\cdot\frac{1}{1+r} + \frac{1}{2}\cdot\frac{1}{1-r} \\
\int 2dt &= \int (\frac{2}{r} - \frac{1}{1+r} + \frac{1}{1-r})dr \\
&= 2\log r - \log|1+r| - \log|1-r| + C \\
&= \log\frac{r^2}{(1+r)(1-r)} \\
&= \log e^{2t+C} \\
&= \log e^{2t}\cdot e^{C} \\
\frac{r^2}{(1+r)|1-r|} &= e^C \cdot e^{2t} \\
\frac{r^2}{(1+r)(1-r)} &= \pm e^C \cdot e^{2t} \\
r^2 &= C_0 e^2 t(1-r^2) \\
&= 1 + C_0 e^{2t} \\
r^2(1+C_0e^{2t}) &= C_0 e^{2t} \\
r^2 &= \frac{C_0e^{2t}}{1+C_0e^{2t}} \\
&= \frac{1}{C_1e^{-2t}+1}
\end{aligned}$$

$$\begin{aligned}
r &= \frac{1}{\sqrt{C_1e^{-2t}+1}} \\
\theta &= -t + t_0
\end{aligned}$$