Ex: Find the general solution of

$$x' \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} x$$

$$x = \xi e^{rt}$$

$$(A - rI)\xi = 0$$

$$\begin{bmatrix} -3 - r & \sqrt{2} \\ \sqrt{2} & -2 - r \end{bmatrix} \xi = 0$$

$$\begin{aligned} (-3-r)(-2-r) - 2 &= 0 \\ 6 + 3r + 2r + r^2 - 2 &= 0 \\ r^2 + 5r + 4 &= 0 \\ (r+4)(4+1) &= 0 \end{aligned}$$

$r = -1$

$$\left[\begin{array}{cc|c} -2 & \sqrt{2} & 0 \\ \sqrt{2} & -1 & 0 \end{array}\right]$$

$R_2 + \frac{\sqrt{2}}{2} R_1$

$$\left[\begin{array}{cc|c} -2 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\xi_1 = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$$

$$\vdots$$

$$\xi = \begin{cases} x_1 = C_1 e^{-t} - C_2 \sqrt{2} e^{-4t} \\ x_2 = C_1 \sqrt{2} e^{-t} + C_2 e^{-4t} \end{cases}$$

To solve nonlinear systems, we need to find eigenvalues $r_1, r_2, \ldots, r_n$, by solving characteristic polynomial $(A - rI) = 0$, and finding the associated eigenvectors.

There are a couple cases

1. All eigenvalues are real and different
2. Some eigenvectors occur in complex conjugate pairs
3. Case 2 + and real or complex are repeated.

Case 1: Eigenvalues $r_1, r_2, \ldots, r_n$ are real and different. Then the associated eigenvectors $\xi_1, \ldots, \xi_n$ are linearly dependent.

The corresponding solution of $x' = Ax$ is

$$x^{(1)} = \xi^{(1)} \cdot e^{r_1 t}, \ldots, x^{(1)}(t) = \xi^{(n)} \cdot e^{r_n t}$$

The general solution of $x' = Ax$ is

$$x = c_1 \xi^{(1)} e^{(r_1 t)} + \ldots + c_n \xi^{(n)} e^{r_n t} \tag{1}$$

If A is real and symmetric,

1. All eigenvalues $r_1, \ldots, r_n$ are real.
2. There is a set of n eigenvectors $\xi^{(1)}, \ldots, \xi^{(n)}$ that are linearly independent.
3. The solution will also be written as (1).

Ex: Find general solution of

$$x' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Eigenvalues, Eigenvectors:

$\lambda = -1$

$$x_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \, x_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Fundamental Set of Solutions

$$x^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t}, \, x^{(2)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t}, \, x^{(3)} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t}$$

General Solution:

$$x = C_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t}$$

Ex 1: Find a fundamental set of real-valued solutions of the system

$$x' = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & \frac{1}{2} \end{bmatrix}$$

$$(-\frac{1}{2} - r)(\frac{1}{2} - r) + 1 = 0$$