**Homogenous Linear Systems with Constant Coefficients**

$$x' = Ax$$

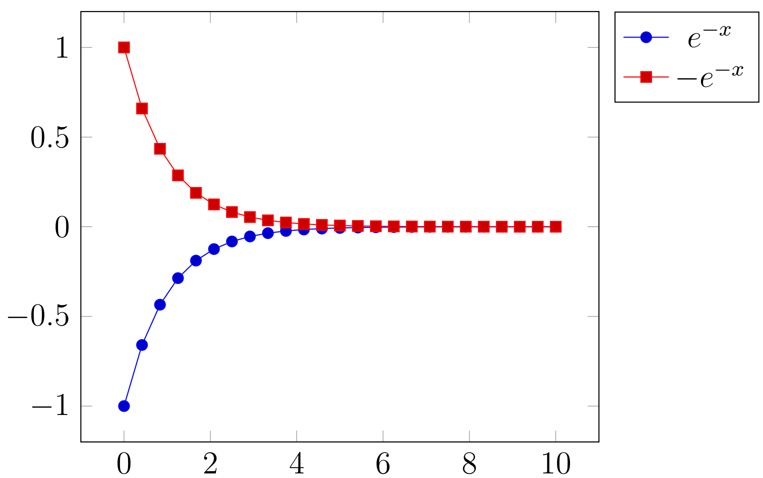
where $A \in \mathbb{R}^{n\times n}$

If $n = 1$, $\frac{dx}{dt} = ax$, where $a \neq 0$, and

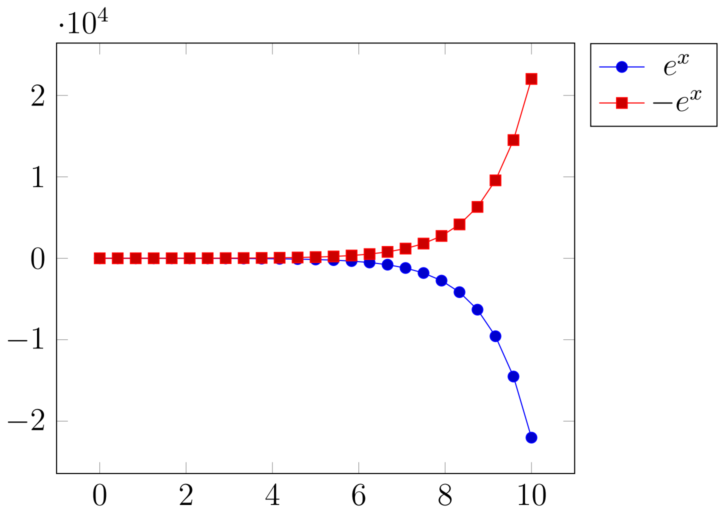
$$x(t) = Ke^{at}$$

If $a \neq 0, x(t) = 0$ is the only equilibrium solution ($\frac{dx}{dt} = 0$).

If $a < 0$, all nontrivial solutions approach $x(t) = 0$. $x(t) = 0$ is an asymptotically stable solution.




If $a > 0$, every solution except for the equilibrium solution moves further away from $x(t) = 0$. $x(t) = 0$ is an unstable equilibrium solution.




**Ex:** Find the general solution of

$$x' = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} x$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2x_1 \\ -3x_2 \end{bmatrix}$$

$$x_1 = C_1 e^{2t} \, x_2 = C_2 e^{-3t}$$

$$\begin{aligned} x &= \begin{bmatrix} C_1 e^{2t} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ C_2 e^{-3t} \end{bmatrix} \\ &= C_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-3t} \end{aligned}$$

$$\begin{aligned} W[x^{(1)} \; x^{(2)}] &= \begin{vmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{vmatrix} \\ &= e^{-t} \\ &\neq 0 \end{aligned}$$

Solution of $x' = Ax$ has form

$$x = \xi e^{rt}$$

where the exponent $r$ and the vector $\xi$ must be determined.

$$\begin{aligned} x' &= r\xi e^{rt} \\ r\xi e^{rt} &= A \cdot \xi e^{rt} \\ r \cdot \xi &= A \cdot \xi \\ r \cdot I \cdot \xi &= A \cdot \xi \\ A\xi - rI\xi &= 0 \\ (A - rI)\xi &= 0 \end{aligned}$$

**Ex:** Find the general solution of

$$x' = \begin{bmatrix} 1 & 1 \\ 4 & 6 \end{bmatrix} x$$

Assume $x = \xi e^{rt}$.

$$\begin{aligned} (A - rI)\xi &= 0 \\ \begin{bmatrix} 1-r & 1 \\ 4 & 1-r \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} (1-r)^2 - 4 &= 0 \\ 1 - 2r + r^2 - 4 &= 0 \\ (r-3)(r+1) &= 0 \end{aligned}$$

$r = 3$.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ 4 & -2 & 0 \end{array}\right]$$
$$\xi_1^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
$$\xi_2^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Thus,

$$x^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t},\ x^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$$

$$\begin{aligned} W[X^{(1)}\, X^{(2)}] &= \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{t} \end{vmatrix} \\ &= -2e^{2t} - 2e^{3t} \\ &= -4e^{2t} \\ &\neq 0 \end{aligned}$$

Our general solution is

$$\begin{aligned} x &= C_1 x^{(1)}(t) + C_2 x^{(2)}(t) \\ &= C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + C_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} C_1 e^{3t} + C_2 e^{-t} \\ 2C_1 e^{3t} - 2C_2 e^{-t} \end{bmatrix}$$

For large $t$ values, $x_1 \approx C_1 e^{2t}$, and $x_2 \approx 2C_1 e^{3t}$, thus,

$$\begin{aligned} \frac{x_1}{x_2} &= \frac{1}{2} \\ x_2 &= 2x_1 \end{aligned}$$

For large $t$, the first term of each $x_1, x_2$ become dominant, and the second term becomes negligible. $\rightarrow$ All the solutions with non-zero $C_1$ will asymptotically approach $x_2 = 2x_1$ as $t \rightarrow \infty$.