7.4: Basic Theory of Systems of 1^{st} Order Linear D.Es

$$\begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} p_{11}(t) & \dots & p_{1n}(t) \\ \vdots & \ddots & \vdots \\ p_{n1} & \dots & p_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{bmatrix}$$

$$x' = P(t)x + g(t) \tag{1}$$

Theorem 7.4.1: Principle of Superposition If the vector functions x_1, x_2 are solutions of (1), then the linear combination

$$c_1x_1 + c_2x_2$$

is also a solution of (1) for any constants c_1, c_2.

If x_1, …, x_k are solutions of (1), then

$$x = c_1x_1(t) + c_2x_2(t) + \ldots + c_kx_k(t)$$

is also a solution of (1) for any constants c_1, …, c_k.

If (1) has n solutions, x_1, …, x_k, and the solutions form a matrix

$$\begin{bmatrix} x_{11}(t) & \dots & x_{1n}(t) \\ \vdots & \ddots & \vdots \\ x_{n1}(t) & \dots & x_{nn}(t) \end{bmatrix},$$

the solutions $x_1, \ldots, x_n$ are linearly independent if

$$W[x_1, \ldots, x_n] \neq 0$$
$$W[x_1, \ldots, x_n] = \det(x(t))$$

Theorem 7.4.2 If the vector functions $x_1, \ldots, x_n$ are linearly independent solutions of the system (1) for each point in the interval $\alpha < t < \beta$, then each solution $x = x(t)$ of (1) may be expressed as a linear combination of $x_1, \ldots, x_n$

$$x(t) = c_1 x_1 + \ldots + c_n x_n \tag{2}$$

in exactly one way.

(2) is the general solution of (1). Any set of solutions $\{x_1, \ldots, x_n\}$ of (1) that is linearly independent at each point in $\alpha < t < \beta$ is said to be a fundamental set of solutions for the interval.

Theorem 7.4.5 Consider system (1), where each element of P is a real-valued continuous function. If $x = u(t) + iv(t)$ is a complex valued solution of (1), then $u(t)$ and $v(t)$ are also solutions of (1).