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A Geometrical Approach to Calculating Determinants of Wiener-Hopf Operators

J.P. MacCormick and B.S. Pavlov

Department of Mathematics, University of Auckland, Private Bag 92019, Auckland, New Zealand

Summary. Using a geometric idea from semigroup theory, we derive a formula for the determinants of certain Wiener-Hopf operators whose symbols are bounded and analytic in the upper half plane. For rational symbols, we propose a method for calculating the asymptotics of the eigenvalues of these operators.

1. Introduction

In [6] the general approach for asymptotic calculation of classical Szegö-Kac determinants [14, 9, 1] was proposed. This approach permits investigation of the asymptotics of the Fredholm determinants of the Wiener-Hopf operator T, defined on the finite interval $(0, a)$ by

$$\begin{array}{rccc} T \equiv T_a(g): & L_2(0,a) & \longrightarrow & L_2(0,a) \\ & u(x) & \longmapsto & u(x) + \int_0^a g(x-s)u(s)\,ds. \end{array}$$

More specifically (see [4]), the asymptotics as $a \to \infty$ can be calculated provided the symbol $\sigma = 1 + \int e^{iks} g(s) ds$ possesses real roots. In [3], an elegant description is given for the oscillating terms in the asymptotics, in the special case that the symbol has only two real roots. Recovering similar asymptotics in the more general case of matrix integral operators (see for instance [15, 16]), requires a more direct and general approach to the problem, which is supplied by the Lax-Phillips version of analytic semigroup theory suggested in [10] for scattering problems. This approach is equivalent [2], to the construction of functional models of dissipative operators [7].

In this paper we consider Wiener-Hopf operators on the finite interval $(0, a)$ whose symbol σ is analytic in the upper half plane. We also consider the case where σ is a real rational function.

Let T be the above Wiener-Hopf operator, and set $\rho = \sigma - 1 = \mathcal{F}^* g$, where $\mathcal{F}^*$ is the inverse Fourier transform. (The operator of multiplication by σ will also be denoted by σ — the meaning is always clear from the context). Our approach to the calculation of the determinant of T is based on the fact that T can be approximated by some other operators W_β whose eigenvalues are known exactly.

Recall that the inverse Fourier transform $\mathcal{F}^*$ maps $L_2[0, \infty]$ unitarily to the Hardy space H_2^+ of the upper half plane. Further, $\mathcal{F}^*(L_2[0, a]) = H_2^+ \ominus e^{ika} H_2^+$, and of course the convolution operator becomes the multiplication operator. So denoting $H_2^+ \ominus e^{ika} H_2^+$ by K_a, and writing P_a for orthogonal projection $H_2^+ \to K_a$, we see that T is unitarily equivalent to $W \equiv W_a(\sigma) = P_a \sigma P_a$. In other words, we have reduced our original problem to the calculation of

$$(*) \qquad \det (P_a \sigma P_a).$$

The exponential e^{ika} is a singular function, so we can find a sequence of Blaschke products tending uniformly to it on the upper half plane. A good choice turns out to be

$$\Pi_\beta(k) = \frac{e^{ika} - e^{-\beta}}{1 - e^{ika}e^{-\beta}},$$

which does indeed tend uniformly (on the upper half plane) to e^{ika} as $\beta \to \infty$. It's convenient to note here that the zeroes of $\Pi_\beta(k)$ occur at the points $k_l = 2\pi l/a + i\beta/a$, $l \in \mathbb{Z}$.

Let $K_\beta = H_2^+ \ominus \Pi_\beta H_2^+$, and $P_\beta =$ orthogonal projection $H_2^+ \to K_\beta$. We consider the operator

$$W_\beta = P_\beta \sigma P_\beta$$

as an approximation for W, because in a sense to be made precise later, $K_\beta \to K_a$ and $P_\beta \to P_a$ as $\beta \to \infty$. The idea is that instead of (*), we can use

$$(\dagger) \qquad \lim_{\beta \to \infty} \det (P_\beta \sigma P_\beta),$$

provided that the W_β approximate W well enough. But the whole point of this approach is that the operator $P_\beta \sigma P_\beta$ turns out to have a remarkably simple form, provided the function σ is *analytic in the upper half plane*: its eigenvectors form a complete set (and even a Riesz basis) in K_β, and the eigenvalues are just $\sigma(k_l)$. This fact immediately gives an explicit expression for (†).

A similar approach is applicable to the case of a rational symbol, including the case of zeroes on the real axis. We show that the spectral analysis of such Wiener-Hopf operators can be reduced to the investigation of a finite matrix, and a procedure for deriving the asymptotics of the eigenvalues for fixed a is suggested.

The straightforward plan outlined here meets some minor obstacles, such as the fact that the operators W and W_β are close in operator norm but not in trace norm. Therefore we need an intermediate operator, which is similar[1] to W_β (and therefore has the same determinant), but close to W in trace norm. This intermediate operator will be constructed as the image of W_β under the multiplication operator of an entire function f_β, which is bounded and invertible as an operator in $L_2(\mathbb{R})$. In summary, the plan is realised as the following chain of statements, which sketch the way of using Semigroup Theory (or the functional calculus for shift operators) for calculating Szegö-Kac determinants.

[1] In the technical sense, i.e. equal to $A W_\beta A^{-1}$ for some A.

2. Proof of the Main Theorem

In all the following results, $a > 0$ and $\beta > 1$. We first state a well-known result from the theory of semigroups.

Proposition 1 *Let Π_β be the family of Blaschke products*

$$\Pi_\beta(k) = \frac{e^{ika} - e^{-\beta}}{1 - e^{ika}e^{-\beta}},$$

approaching the singular function $e^{ika} \equiv \theta_a$ uniformly in upper half plane as $\beta \to \infty$. Consider the generators B_β of the contracting semigroup

$$Z_\beta(t) = P_\beta e^{ikt} P_\beta \equiv e^{iB_\beta t}, \ t > 0,$$

which arises as a compression of the shift group onto the coinvariant subspaces

$$K_\beta = H_+^2 \ominus \Pi_\beta H_+^2.$$

Then the B_β are simple dissipative operators, with eigenfunctions given by

$$\psi_l(k) = \frac{\Pi_\beta(k)}{k - k_l}, \ l \in \mathbb{Z}$$

and corresponding eigenvalues $k_l = 2\pi l/a + i\beta/a$.

For a proof, see for example [11].

A similar statement is valid for the systems of eigenvectors of the adjoint operators B_β^*. Actually in this case the eigenvectors conveniently coincide with the H_2^+ reproducing kernels: $\varphi_l(k) = \frac{1}{k - \bar{k}_l}$. In other words, we have,

Proposition 2 *The eigenvectors of B_β^* are*

$$\varphi_l(k) = \frac{1}{k - \bar{k}_l}, \ l \in \mathbb{Z}$$

with eigenvalues $\bar{k}_l = 2\pi l - i\beta$.

Proof: We need only check that $\{\varphi_l\}$ and $\{\psi_l\}$ are biorthogonal sets. The details are in [11], for example. ■

We also have the following fact which will be crucial for our proofs later on.

Proposition 3 *The sets $\{\varphi_l\}$ and $\{\psi_l\}$ both form Riesz bases[2] for the subspace K_β.*

[2] By a Riesz basis, we mean a basis obtained from an orthonormal basis by an invertible, bounded, linear transformation.

Proof: This follows from [13], and full details are given in [11]. The interesting point of the proof is that this problem was solved for us many years ago by Carleson [5], in the context of interpolation by analytic functions. The well-known Carleson condition states that the family $\{\varphi_l\}$ is a Riesz basis[3] if

$$\inf_m \prod_{l \neq m} \left| \frac{k_m - k_l}{k_m - \bar{k}_l} \right| > 0,$$

and a quick calculation shows that this condition holds for our set $\{\varphi_l\}$. ∎

In the next theorem we describe an important automorphism of $L_2(\mathbb{R})$ which maps K_β to K_a.

Theorem 4 *Write $\theta(k) = e^{ika}$, and let $f_\beta(k)$ be the entire function of exponential type defined by*

$$f_\beta(k) = 1 - e^{-\beta} e^{ika}.$$

Then the multiplication operator $u \mapsto f_\beta u$ is a bounded and invertible operator on $L_2(\mathbb{R})$, transforming the orthogonal sum

$$L_2(\mathbb{R}) = H^2_- \oplus K_\beta \oplus \Pi_\beta H^2_+$$

into the direct sum

$$\overline{\Pi}_\beta \theta H^2_- + K_a + \Pi_\beta H^2_+,$$

where $K_a = H^2_+ \ominus \theta H^2_+$ is a coinvariant subspace of the shift group corresponding to the singular function θ. The entire functions

$$\Phi_l = f_\beta \, \varphi_l$$

form a Riesz basis in K_a for each $\beta > 1$.

Proof: The full proof is in [11]. Multiplication by f_β is clearly bounded and invertible since $1 - e^{-\beta} \leq |f_\beta| \leq 1 + e^{-\beta}$ on the real axis, and the proof proceeds by checking the transformation of each orthogonal subspace of $L_2(\mathbb{R})$ separately. Of course we exploit our knowledge from Propositions 1 and 2 to examine the map $f_\beta : K_\beta \longrightarrow K_a$. The final claim that the $\{\Phi_l\}$ form a Riesz basis for K_a follows immediately from Proposition 3, as they were obtained from the Riesz basis $\{\varphi_l\}$ of K_β by a bounded invertible linear transformation. ∎

It is interesting to note here that in fact the functions Φ_l are Fourier images of the projections of exponentials $e^{-i\bar{k}_l x}$ in $L_2(\mathbb{R})$ onto $L_2(0, a)$.

As explained in the introduction, it will turn out that the operators $W_\beta \equiv P_\beta \sigma P_\beta$ do not approximate the operator $W_a \equiv P_a \sigma P_a$ well enough for our purposes. Therefore in the next theorem we introduce the operators $\mathcal{W}_\beta$ which have the same determinants as the W_β. Then in Theorem 6 we show that $\mathcal{W}_\beta$ is a good enough approximation to W_a.

[3] Actually, the Carleson condition guarantees only that we have a so-called *unconditional basis.* A Riesz basis must also satisfy $\inf \|\varphi_l\| > 0$ and $\sup \|\varphi_l\| < \infty$, but these conditions are clearly fulfilled here.

Theorem 5 *The operator*

$$\mathcal{P}_a^\beta = f_\beta P_\beta f_\beta^{-1}$$

is a skew (i.e. nonorthogonal) projection onto K_a parallel to the sum of subspaces

$$\overline{\Pi}_\beta \theta H_-^2 + \Pi_\beta H_+^2.$$

For each essentially bounded function σ defined on the real axis, the operator $W_\beta \equiv P_\beta \sigma P_\beta$ is bounded and is similar to the operator $\mathcal{W}_\beta \equiv \mathcal{P}_a^\beta \sigma \mathcal{P}_a^\beta$ acting in the subspace K_a; in particular, $\mathcal{W}_\beta$ and W_β have the same determinants.

Proof: The effect of f_β described in Theorem 4 means precisely that $\mathcal{P}_a^\beta$ is zero on $\Pi_\beta \theta H_2^- + \Pi_\beta H_2^+$, and the identity on K_a; this is the definition of a skew projection so the first statement is proved.

Now it is not generally true for infinite-dimensional determinants that

$$\det(PQP^{-1}) = \det Q,$$

but this formula does hold if P maps some Riesz basis of dom Q to a Riesz basis of dom PQP^{-1}. But we proved in Theorem 4 that f_β maps the Riesz basis $\{\varphi_l\}$ of K_β to the Riesz basis $\{\Phi_l\}$ of K_a. By the definition of $\mathcal{W}_\beta$ we have

$$\mathcal{W}_\beta|_{K_a} = f_\beta P_\beta \sigma P_\beta f_\beta^{-1}|_{K_a} = f_\beta W_a(\sigma) f_\beta^{-1}|K_a,$$

so the above remarks tell us that $\det \mathcal{W}_\beta = \det W_\beta$. ∎

Our next task is to estimate the norms of the various operators defined so far. Define

$$\epsilon_\beta = \sup_{k\in\mathbb{R}} (\Pi_\beta(k) - \theta(k)) = \sup_{k\in\mathbb{R}} \left(1 - \overline{\Pi}_\beta(k)\theta(k)\right).$$

(It is easy to see these suprema are equal, and in fact that $\epsilon_\beta \le \frac{2e^{-\beta}}{1-e^{-\beta}}$. In particular, we see that $\epsilon_\beta \to 0$ as $\beta \to \infty$.)

The next theorem states that the intermediate operator $\mathcal{W}_\beta$ is actually close to W in trace norm.

Theorem 6 *Let σ be a bounded analytic function in the upper half plane, set $\rho = \sigma - 1$ and suppose ρ can be expressed as the product of 3 functions, each in the intersection of the Hardy classes $H_\infty^+ \cap H_2^+$:*

(1) $$\rho = \rho_1\rho_2\rho_3, \quad \rho_j \in H_\infty^+ \cap H_2^+$$

Then

$$\left\| P_a \rho P_a - \mathcal{P}_a^\beta \rho \mathcal{P}_a^\beta \right\|_{Trace} \le \frac{\sqrt{2}\epsilon_\beta}{\left(1 - \sqrt{2}\epsilon_\beta\right)^3} \, const,$$

where the constant depends only on the L_2 and L_∞ norms of the factors ρ_j. In particular, $\mathcal{P}_a^\beta \rho \mathcal{P}_a^\beta \to P_a \rho P_a$ in trace norm as $\beta \to \infty$.

Proof: Full details are in [11]. The proof involves carefully exploiting the relationship between operator, Hilbert-Schmidt, and trace norms, together with some straightforward estimation. ■

At last we are in a position to prove our main theorem, obtaining a formula for $\det W$.

Theorem 7 *Suppose σ is an analytic function in the upper half plane, and $\sigma - 1$ satisfies the 3-factor condition (1) of the previous theorem. Then*

$$\det W_a(\sigma) = \lim_{\beta\to\infty} \prod_{l\in\mathbb{Z}} \sigma(k_l), \tag{2}$$

where $k_l = 2\pi l/a + i\beta/a$.

Proof: We apply our previous results to carry out the plan outlined in the introduction. Note that in the first line we will use the fact that det is continuous with respect to the trace norm. (This is proved in [4], for example). We have

$$\begin{aligned}
\det W_a(\sigma) &= \lim_{\beta\to\infty} \det \mathcal{W}_\beta, && \text{by Theorem 6}\\
&= \lim_{\beta\to\infty} \det W_\beta, && \text{by Theorem 5}\\
&= \lim_{\beta\to\infty} \prod(\text{evals of } W_\beta), && \text{by definition of det}\\
&= \lim_{\beta\to\infty} \prod_{l\in\mathbb{Z}} \sigma(k_l), && \text{by Proposition 1}
\end{aligned}$$

■

3. An Example

As an example, suppose $\sigma(z) = 1 + (z+i)^{-n}$. (This is a natural example since symbols of this form arise from kernels of W proportional to $x^{n-1}e^{-x}$). If $n \geq 3$, the 3-factor condition (1) is satisfied, so we can apply (2) to see

$$\begin{aligned}
\det W &= \lim_{\beta\to\infty} \prod_{l=-\infty}^{\infty} \left\{ 1 + \left[\frac{2\pi l}{a} + i\left(1 + \frac{2\pi\beta}{a}\right)\right]^{-n} \right\}\\
&= 1. \qquad (3)
\end{aligned}$$

(To calculate the limit, use the fact that $1+a_1+\ldots+a_n \leq (1+a_1)\ldots(1+a_n) \leq e^{a_1+\ldots+a_n}$, provided each $a_n > 0$.)

The Achiezer-Kac formula for the asymptotics of $\det W$ states that as $a \to \infty$,

$$\begin{aligned}
\det W \sim exp\left(\frac{a}{2\pi}\int_{-\infty}^{\infty} \log\sigma\, dz\right) \times \\
exp\left(\frac{1}{4\pi^2}\int_0^{\infty} z\,\mathcal{F}(\log\sigma)(z)\,\sigma(\log\sigma)(-z)\,dz\right),
\end{aligned} \tag{4}$$

and of course it should agree with (3). It is not hard to check this agreement directly; using contour integration on each factor of (4) shows that both are equal to 1.

Note that when $n = 2$, these calculations still work even though σ no longer satisfies (1). This provides some evidence that the condition (1) can be weakened without affecting formula (2).

If we are prepared to abandon rigour for a moment, we can make a direct connection between (2) and the Achiezer-Kac formula, in the case where σ is an outer function of the upper half plane. In this case we know that up to a constant of modulus 1, σ can be represented as

$$\sigma(z) = exp\left(\frac{1}{i\pi}\int_{-\infty}^{\infty}\frac{\log|\sigma(t)|}{t-z}\,dt\right).$$

After substituting this expression into (2), we obtain a $\sum_{l\in\mathbb{Z}}$ which is not absolutely convergent; nevertheless if we make the most obvious choice for ordering this sum — in order of increasing $|l|$ so that terms for $+l$ and $-l$ have a cancelling effect on each other — the result is absolutely convergent, and we can simplify the result using the Poisson Summation Formula, eventually obtaining

$$\begin{aligned} \det W &\stackrel{?}{=} exp\left(\frac{a}{2\pi}\int_{-\infty}^{\infty}\log|\sigma(t)|\,dt\right) \\ &= exp\left(\frac{a}{2\pi}\int_{-\infty}^{\infty}\log\sigma(t)\,dt\right). \end{aligned} \tag{5}$$

The last line follows since it can be shown $\arg\sigma$ is an odd function[4], and hence $\int\log|\sigma| = \int\log\sigma$. When σ is an outer function, the second factor in (4) turns out to be 1, so (5) agrees with the asymptotic behaviour predicted by the Achiezer-Kac formula.

4. The Rational Symbol Case

In [6] the case of a rational symbol is carefully investigated. Now we demonstrate how to generalise our approach to real rational symbols. Calculating the eigenvalues of the corresponding Wiener-Hopf operators will be reduced to the investigation of a finite dimensional matrix, whose determinant vanishes at the eigenvalues of the original Wiener-Hopf operator.

So, suppose the symbol σ is real on the real axis and thus can be represented the in form of a finite sum of Cauchy kernels with poles at prescribed complex points. In what follows we deal only with the case that all the poles

[4] That $\arg\sigma$ is odd follows from the fact that $1-\sigma$ is the Fourier Transform of a real-valued function.

are simple; generalising to multiple poles is fairly straightforward but makes the notation unnecessarily complicated. Thus σ has the form

$$\sigma(k) = 1 + \sum_{l=1}^{L} \frac{\alpha_l}{k - k_l} + \frac{\bar{\alpha}_l}{k - \bar{k}_l} \equiv 1 + \varphi(k), \quad \Im k_l > 0. \tag{6}$$

It is easy to see that the corresponding Wiener-Hopf operator $P_a \varphi|_{K_a}$ is compact. Let us denote by $z_n(\lambda)$, $n = -L, \dots -2, -1, 1, 2, \dots L$ the roots of the auxiliary equation $\varphi(k) = \lambda$ (counting multiplicity).

Theorem 8 *The eigenvalues of the Wiener-Hopf operator in the coinvariant subspace $K_a = H_2^+ \ominus \theta_a H_2^+$, $\theta_a(k) = e^{ika}$ coincide with the zeroes of the determinant of a finite square matrix:*

$$\det \begin{pmatrix} \dots & \dots & \dots & \dots \\ \dots & \frac{\alpha_l}{z_n(\lambda) - k_l} & \frac{\bar{\alpha}_l}{z_n(\lambda) - \bar{k}_l} e^{i z_n(\lambda) a} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

Here $n = -L, \dots -2, -1, 1, 2, \dots L$, and $l = 1, \dots L$.

Sketch of the proof. The proof is based on the following well-known statement (see [7]).

Proposition 9 *The following representations are true for the resolvents of the generators of compressions shifts and adjoint shifts semigroups onto the coinvariant subspace $K_a = H_2^+ \ominus \theta_a H_2^+$:*

$$P_a \frac{1}{k - p} u(k) = \frac{u - u(p)}{k - p}, \ \Im p > 0, \ u \in K_a$$
$$P_a \frac{1}{k - \bar{p}} u(k) = \frac{u - \theta(k)\bar{\theta}(\bar{p}) u(\bar{p})}{k - \bar{p}}, \ \Im \bar{p} < 0, \ u \in K_a.$$

So for φ of the form (6) and $u \in K_a$ we have

$$P_a(\phi u) = \phi(k)\ u - \sum_{l=1}^{L} \left(\alpha_l \frac{u(k_l)}{k - k_l} + \theta(k) \bar{\alpha}_l \frac{\bar{\theta}(\bar{k}_l) u(\bar{k}_l)}{k - \bar{k}_l} \right).$$

In particular, if u is an eigenfunction of $P_a \phi|_{K_a}$ with eigenvalue λ, we must have

$$\sum_{l=1}^{L} \left(\alpha_l \frac{u(k_l)}{k - k_l} + \theta(k) \bar{\alpha}_l \frac{\bar{\theta}(\bar{k}_l) u(\bar{k}_l)}{k - \bar{k}_l} \right) = 0 \tag{7}$$

whenever $k = z_n(\lambda)$ for some n.

Fix λ, and consider $u(k_l) \equiv u_l$ and $\bar{\theta}(\bar{k}_l)u(\bar{k}_l) \equiv v_l$ as the $2L$ unknowns of the linear system (7). (Note that (7) really does consist of $2L$ equations in these unknowns as there are $2L$ possible values of k.) Since the system has a solution, the determinant of its coefficients must be zero. It is clear from our definitions that these coefficients have the form $\frac{\alpha_l}{z_n(\lambda)-k_l}$ or $\frac{\bar{\alpha}_l\theta(z_n(\lambda))}{z_n(\lambda)-\bar{k}_l}$, exactly as stated in the theorem.

Conversely, suppose the determinant in the theorem is zero and choose $u_1, \ldots u_L, v_1, \ldots v_L$ to be a solution of (7). Define $u(k)$ by

$$u(k) = \frac{1}{\varphi(k)-\lambda}\sum_{l=1}^{L}\left(\frac{\alpha_l}{k-k_l}u_l^r + \frac{\theta(k)\bar{\alpha}_l}{k-k_l}v_l^r\right).$$

Then $u(k)$ is in K_a, since the zeroes of the denominator are compensated by the roots of the sum counting multiplicity, and it is easy to see u is an eigenfunction of the original Wiener-Hopf operator, and λ is the corresponding eigenvalue. ∎

The determinant of Theorem 8 can be simplified for small λ by using a Taylor series. If $z_n(0)$ is a simple zero of φ, then

$$(8) \qquad \begin{aligned} z_n(\lambda) &= \frac{1}{2\pi i}\oint_\gamma \frac{s\dot{\varphi}(s)}{\varphi(s)-\lambda}ds \\ &= z_n(0) + \sum_{j=1}^{\infty}\lambda^j\frac{1}{2\pi i}\oint_\gamma \frac{s\dot{\varphi}(s)}{\varphi^{j+1}(s)}ds, \end{aligned}$$

where the integral is taken over a small contour γ, encircling the simple root z_n once in the positive direction.

As remarked earlier, all of the above can be generalised to the case where φ has multiple poles; the details are in [11]. Instead of (8) we can use the generalised Taylor series, with a properly chosen factor λ^δ, δ defined by the multiplicity of the root of the symbol. Then the asymptotic behaviour of the determinant for small λ^δ is described by a quasipolynomial in $\lambda^\delta, \exp(d/\lambda^\delta)$. The asymptotics of the zeroes of these quasipolynomials for small λ can be derived using a construction involving Newton polygons ([8, 12]).

It is obvious that the determinant of the finite matrix under consideration coincides with the determinant or the regularized determinant of the operator $1 - \frac{1}{\lambda}P_a\varphi|_{K_a}$ up to some analytic factor which has no zeroes on the complex plane. It is a challenging problem to describe this factor in explicit form.

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