

HARMONIC OSCILLATIONS

Oscillatory motion is everywhere in nature. Any object which has both inertia and a restoring force will oscillate around an equilibrium position if displaced from that equilibrium. As we will see, the descriptions of essentially all oscillating systems are very similar - and hence we will look most closely at mass-spring systems and pendula, the most common oscillators in our daily experience.

Describing the motions of objects that oscillate about a point of equilibrium, as with the motion of any object, requires a solid understanding of Newton's laws. It is useful at this point to review the essential ideas contained in those laws. As we will then see, the descriptions of the motions of masses on springs, pendula, marbles oscillating about in the bottom of a bowl, and even the periodic motions of buildings or bridges or violin strings will follow from understanding the forces acting and then solving Newton's second law. So we will begin with a review of Newton's laws of motion.

NEWTON'S LAWS

While the kinematics equations describe the motions of objects, it is Newton's three laws that relate the motion to the *causes* of the motion. As simple as Newton's laws are at one level, it is difficult to overemphasize their importance. They represent the fundamental principles that govern how things move - the connection between how objects interact with each other and the changes in motion that result from that interaction. Solving Newton's law problems is often very difficult for many students. To be successful at it requires *internalizing* what is actually meant by Newton's laws - not just learning the statements or knowing the equation that expresses Newton's second law. When Newton's laws are understood, setting up problems to solve for the motion of some object simply becomes an exercise in identifying all the forces on the object and expressing Newton's second law in algebraic form specialized to each object in the specific problem being described.

Newton's 1st Law: The Law of Inertia

In the absence of a net force, an object either remains at rest or moves in a straight line at constant speed.

The first law just identifies that changes of motion occur because of forces - and so the state of motion remains unchanged either if no forces act or if all the forces that do act are balanced in such a way to add to zero. This is the basis of all equilibrium problems.

The *inertia* of an object is its tendency to remain in its current state of motion. The object's mass is a measure of its inertia. The significance of the first law is to state that because of the inertia of an object, an unbalanced force is required to change the object's motion.

Newton's 2nd Law: The Law of Motion

The result of a net force (or unbalanced forces) on an object is an acceleration in the direction of the net force. The acceleration will be directly proportional to the net force and inversely proportional to the mass of the object.

$$\mathbf{F}_{\text{net}} = \Sigma \mathbf{F} = m\mathbf{a} \quad \text{or} \quad \mathbf{a} = \mathbf{F}_{\text{net}}/m$$

This is the central idea of classical mechanics. If you can determine the forces that act on an object, you can also determine the resulting acceleration, from which the velocity and position as functions of time can be determined. That is, $\mathbf{F}=m\mathbf{a}$ lets you solve for the motion of the mass. Notice that the second law "contains" the first: If the net force is zero, so is the acceleration - and the vector velocity of the object is then a constant (ie, either the object remains at rest or it moves in a line at constant speed).

The significance of the second law is that it relates the forces that act on an object to the resulting acceleration. The second law says **HOW** the motion is changed due to the forces that act.

Newton's 3rd Law: The Law of Interaction

All forces are the result of an interaction between two objects. If an object exerts a force on some other object, there is an equal and opposite force being exerted on the first object by the second.

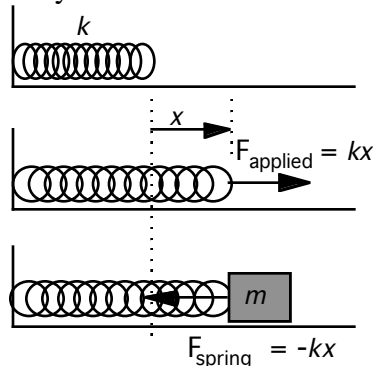
This may be the most important of the three laws because it speaks to what is meant by a force. It says that all forces are caused by some other object and always exist in equal and opposite pairs - that is, forces are mutual interactions *between* objects. Understanding the third law is essential to being able to solve problems using the second. In order to correctly set up any force problem, it is necessary to identify the forces that act on each object. Newton's third law can be summarized by writing the equation in the form $\mathbf{F}_{1 \rightarrow 2} = -\mathbf{F}_{2 \rightarrow 1}$, meaning the force exerted by object 1 on object 2 is equal in magnitude but opposite in direction to that exerted by 2 on 1. This important idea will ultimately explain why complex objects can be treated as if all forces act at the center of mass of the object and also leads to the important laws of conservation of linear momentum and conservation of energy.

Comments on Newton's Laws of Motion

- Forces are never the cause of motion, but rather the cause of *changes* in motion.
- Forces are always the result of an interaction. Forces on objects are always the result of some other object exerting the force. That point is not self-evident to most beginning students and is the very core of Newton's laws. Understand that point well, and setting up problems will be easier.
- Newton's 2nd law is for the resultant force or the net force. The equation $\mathbf{F} = m\mathbf{a}$ does not define force, but rather relates the change in motion of an object to all of the forces that act on it. Notice that a mass times an acceleration is not a force (though it has the units of force). Forces are always caused by some other object - the acceleration \mathbf{a} is just the response of the mass m to all the forces that act on it.
- Inertia* is the property of an object which describes its tendency to remain at rest or in uniform motion. The *mass* of the object is a measure of its inertia. The concept of inertia also surfaces in the discussion of any problem involving rotation about an axis with the quantity called the *moment of inertia* being a measure of how an object resists a change in its rotation.

SIMPLE HARMONIC MOTION

Simple harmonic motion is the motion of an object or system that is subject to a restoring force that is proportional to the displacement. The simplest example of such a system is a mass on a spring that obeys Hooke's law.



Consider a spring (with one end fixed) that is on a horizontal frictionless surface. According to Hooke's law, the force required to stretch or compress the spring is proportional to the amount the spring stretches or compresses. That is, $F_{\text{applied}} = kx$, where k is the spring constant and F_{applied} is the force necessary to displace the spring an amount x from its equilibrium. By Newton's third law, the *spring* exerts an equal and opposite force on whatever it is attached to. That is, $F_{\text{spring}} = -kx$, where the negative sign expresses that the spring force is opposite to the displacement.

If a mass m is attached to the end of the spring, displaced from equilibrium and released, the only unbalanced force acting on the mass is that exerted by the spring. Newton's second law can then be expressed as

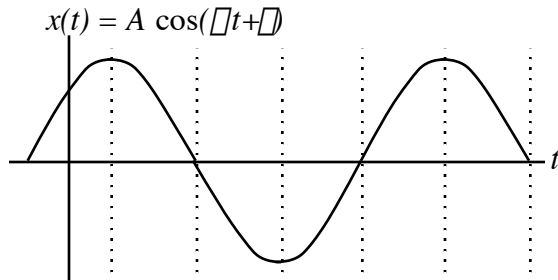
$$F_{\text{net}} = F_{\text{spring}} = -kx = ma$$

This, of course, can be expressed in differential form

$$m \frac{d^2x}{dt^2} = -kx$$

The solution to this second order differential equation is either a sine or a cosine function with time as the variable. In general, the solution can be written as a sine or cosine function with an arbitrary *phase* (the angle ϕ) in the argument.

$$x(t) = A \cos(\omega t + \phi)$$



Notice that the role of the phase ϕ is to identify where the object is with respect to the equilibrium position at time $t=0$. That is, if the mass is displaced an amount A and then released at $t=0$, the phase is just $\phi=0$ and $x(t)=A \cos \omega t$. If it is passing through the origin at $t=0$ then $\phi=\pi/2$ and the function can be written as $x(t)=A \sin \omega t$.

To show that $x(t) = A \cos(\omega t + \phi)$ is a solution to the differential equation of motion, simply differentiate $x(t)$ twice and substitute into the differential equation. When done,

which is an identity only when the angular frequency ω is related to the spring constant and mass by

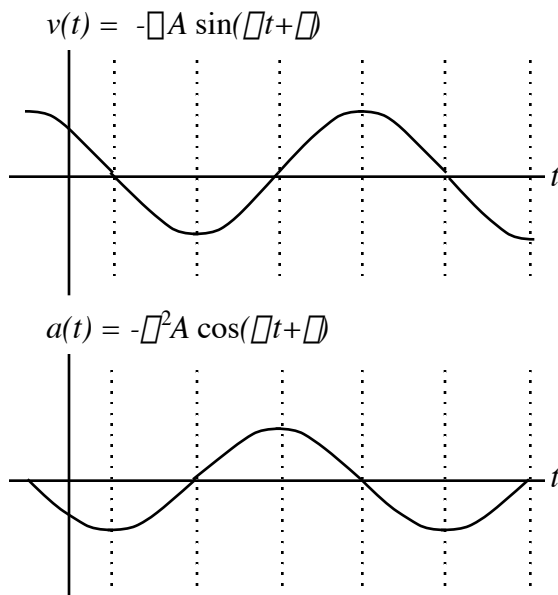
$$\omega^2 = k/m \quad \text{or} \quad \omega = \sqrt{k/m}$$

Notice that the stiffer the spring, the larger the oscillation angular frequency and the larger the mass, the smaller the frequency. The angular frequency relates to the period of the oscillation by $2\pi/T$, where T is the period of oscillation. That is:

$$\omega = 2\pi/T$$

If the displacement is described by $x(t) = A \cos(\omega t + \phi)$, then the velocity and acceleration, respectively, will be given by

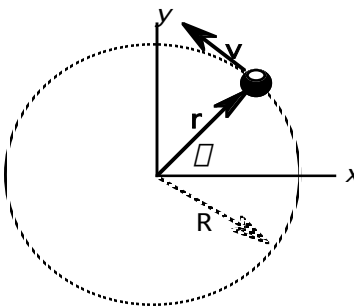
$$v(t) = -\omega A \sin(\omega t + \phi) \quad \text{and} \quad a(t) = -\omega^2 A \cos(\omega t + \phi)$$



Notice that both the velocity and acceleration are also sinusoidal functions when the motion itself is sinusoidal - but that the velocity and acceleration are $\pi/2$ and π out of phase, respectively, with respect to the displacement. In particular, notice that the acceleration is proportional to the displacement ($A \cos(\omega t + \phi)$ and always in the direction opposite to the displacement. That is an essential feature of harmonic motion. Also, from the velocity and acceleration expressions, the magnitude of the velocity (ie, its maximum value) can be seen to be just ωA and the magnitude of the acceleration (its maximum value) is $\omega^2 A$, where ω is the angular frequency of oscillation. These results will be extremely important in later discussions.

$$v_{max} = \omega A \quad \text{and} \quad a_{max} = \omega^2 A$$

The phase relationship between the displacement and the velocity and acceleration can be understood by recognizing that where the slope of the displacement is positive, so is the velocity. Where the slope is zero (ie, at the extrema), the velocity is also zero, etc. And similarly for the acceleration. It should also be recognized that the acceleration is simply opposite the displacement - where one is positive, the other is negative.



The description of simple harmonic motion is closely related to uniform circular motion. The connection can be seen by considering an object which moves in a circle at constant speed. It crosses the x axis at time $t=0$ and the angle ϕ increases at a constant rate - that is the angle can be expressed $\phi = \omega t$, where ω is a constant. [In this case, the quantity ω is called the angular velocity and is just the rate at which the angle changes - ie, $\omega = d\phi/dt$.]

When the object is at the angle ϕ with respect to the x axis, its coordinates are:

$$x = R \cos \phi = R \cos(\omega t) \quad \text{and} \quad y = R \sin \phi = R \sin(\omega t)$$

Notice that describing the motion of an object by the sinusoidal function $A \cos(\omega t + \phi)$ is essentially the same as just considering the x -component of the motion of an object in uniform circular motion with ϕ representing the angle with respect to the x -axis when $t=0$. The role of the angular velocity ω in the description of circular motion is identical to that of the angular frequency in the description of simple harmonic motion. That is, in both cases, ω is just 2π divided by the period of the rotation or oscillation, respectively.

$$\omega = 2\pi/T$$

It should be emphasized that even though this discussion focuses on a mass-spring system, the results must be the same for any system in which there is a restoring force that is proportional to the displacement. That is, the mathematical result of sinusoidal oscillation only depends on the acceleration and displacement terms being proportional. It is for that reason that a torsional pendulum or a simple pendulum subject to only small oscillations are also simple harmonic oscillators.

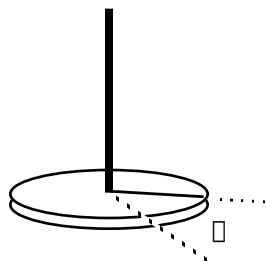
Rotational oscillators - Pendula

Newton's second law for rotational motion relates a net *torque* on an object about a rotation axis to the *angular acceleration* of the object about that axis. That is, in general,

$$\tau_{net} = I\alpha$$

where I is the moment of inertia of the object about the rotation axis and α is the angular acceleration.

Torsional pendulum

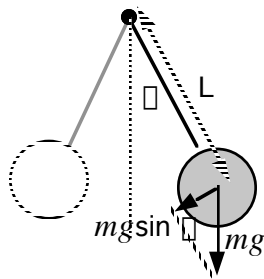


Consider a disk on the end of a torsion rod, the torque is proportional to the angular displacement and would be written $\tau_{net} = \kappa\phi$, where κ is a torsion constant (ie, a measure of how stiff the torsion rod is) and ϕ is the angular displacement. Since, by definition, the angular acceleration is just $\alpha = d^2\phi/dt^2$ the differential equation of motion is given by

$$I \frac{d^2\phi}{dt^2} = \kappa\phi$$

which has as its solution the same form as the mass on the spring $\phi(t) = A \cos(\omega t + \phi)$, where now the angular frequency is just $\omega = \sqrt{\kappa/I}$ where κ is the torsion constant of the rod.

Physical Pendula



For a more conventional pendulum, in which the restoring force is supplied by gravity rather than a mechanical restoring force like a torsion rod, the torque is given by $\tau = -mgL \sin \theta$, where L is the distance from the pivot to the center of mass of the pendulum. It is only for small angles that the numerical value of $\sin \theta$ is essentially equal to θ (when the angle is measured in radians) and the restoring torque is proportional to the angular displacement. That is, $\tau = -I\ddot{\theta}$ becomes (for angles sufficiently small that $\sin \theta \approx \theta$):

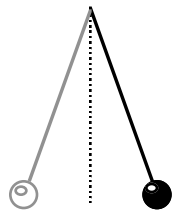
$$I \frac{d^2 \theta}{dt^2} = -(mgL) \theta.$$

By comparing the mathematical form to the that of the mass-spring system, it can be seen that the solution must again be $\theta(t) = A \cos(\omega t + \phi)$. Notice that the form is identical to both the mass-spring system and the torsional pendulum but with the mass replaced by the moment of inertia and the quantities (mgL) replacing the spring constant. In this case, the angular frequency is given by

$$\omega = \sqrt{\frac{mgL}{I}}$$

where I is the moment of inertia of the pendulum about point of rotation and L is the distance from the point of rotation to the center of mass of the pendulum. (In general, $I = mL^2 + I_{cm}$, where I_{cm} is the moment of inertia of the suspended object about its center of mass and L is the distance from the pivot to that center of mass.) It should be noticed that I is proportional to the mass of the pendulum - since I_{cm} is also - hence the mass will cancel and the period of the pendulum does not depend on the mass.

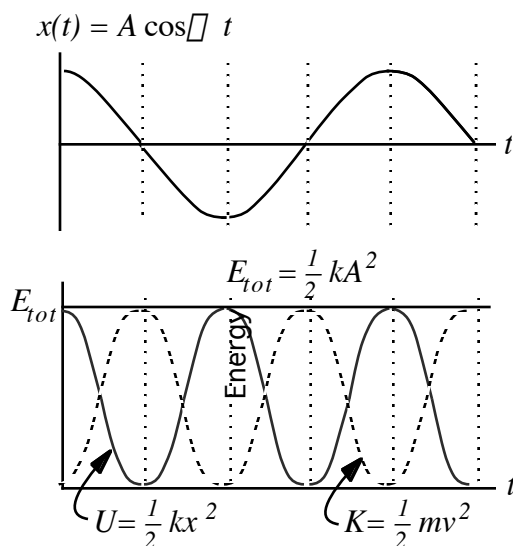
Simple Pendula



For a simple pendulum - a small object suspended from a long string - the moment of inertia would just be $I = mL^2$ (ie, we assume that I_{cm} is negligibly small compared to mL^2 since the radius of the mass itself is so small compared to the pendulum length) and the angular frequency and period expressions then reduce to the usual forms

$$\omega = \sqrt{\frac{g}{L}} \quad \text{and} \quad T = 2\pi \sqrt{\frac{L}{g}}$$

Energy of Oscillation



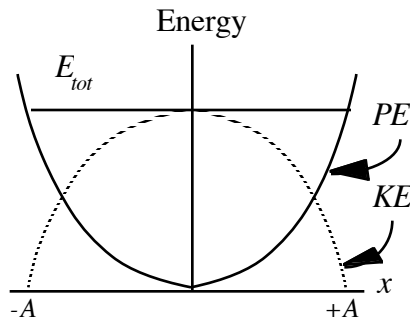
The energy associated with a harmonic oscillator is just the sum of the kinetic and potential energies. For a mass on a spring, that is just $E_{total} = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$. That is, the potential energy stored in a spring is just $\frac{1}{2}kx^2$ since the force is given by Hooke's law. But the displacement of a harmonic oscillator can be written $x(t) = A \cos(\omega t + \phi)$ and the velocity expression then becomes $v(t) = -A\omega \sin(\omega t + \phi)$, so the total energy can be written:

$$E_{total} = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi) + \frac{1}{2}kA^2 \cos^2(\omega t + \phi)$$

substituting $\omega^2 = k/m$ reduces the above to

$$E_{total} = \frac{1}{2}kA^2 (\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)) = \frac{1}{2}kA^2$$

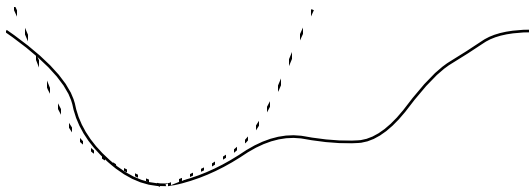
That is, the total energy of an ideal simple mass-spring harmonic oscillator depends only on the amplitude of oscillation and the spring constant. This makes sense, because at the endpoints of the oscillation, ie when $x=A$, the mass comes to rest and the total energy is in the form of potential stored in the spring.



If the energy is graphed as a function of displacement from equilibrium, the potential energy is just a parabola, ie, $\frac{1}{2}kx^2$. The total energy is then just a constant at a value equal to $\frac{1}{2}kA^2$. The kinetic energy is just the difference between the total and potential energies. Notice that the kinetic energy goes to zero when the potential energy is a maximum, and vice versa. The maximum values of both the potential and kinetic energies equals the total energy.

$$E_{total} = \frac{1}{2}kA^2$$

An object at rest at equilibrium would be at the origin of the graph. Any displacement from equilibrium would require an increase in potential energy - hence moving up the sides of the parabola on one side or the other. If the oscillator is displaced from equilibrium and released, it would lose potential energy while gaining kinetic energy, go through the origin and then lose kinetic while gaining potential energy. That motion would repeat itself - hence exhibit oscillatory motion. If the potential energy is a parabola, the restoring force is necessarily linear in the displacement - leading to simple harmonic motion.



A very interesting result follows from this energy approach to harmonic motion. For any potential energy function that has a local minimum - even some function that is not itself a parabola, the motion can still be shown to be simple harmonic. That is, the displacement is a sinusoidal function of time, if the displacement is sufficiently small.

To see this, consider some arbitrary potential energy function $U(x)$ written as a function of the displacement. For small enough displacements from equilibrium, the potential energy function can be approximated by a parabola that "fits" in the minimum of the potential well. Mathematically, it is just a Taylor series approximation. For small values of the displacement, only the quadratic term is significant regardless of the shape of the entire potential energy function. That is:

$$U(x) \approx U_0 + \left. \frac{dU}{dx} \right|_0 x + \frac{1}{2} \left. \frac{d^2U}{dx^2} \right|_0 x^2 + \dots \approx \frac{1}{2}kx^2$$

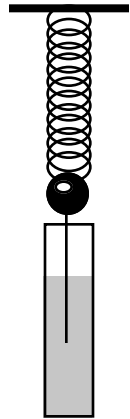
where k is just the second derivative of the potential energy evaluated at the equilibrium point. The first term is just the minimum in the potential energy - and that is a purely constant, since the potential energy can be assigned the value of zero at any defined point (like the equilibrium point). The next term is exactly zero - since the first derivative is the slope of the $U(x)$ function evaluated at x_0 . So the only position dependent term varies as the square of the displacement if the higher order terms can be ignored due to small displacements. If the potential energy can be approximated to be in the form $U(x) = \frac{1}{2}kx^2$, then the motion is the equivalent of that of a mass-spring system.

You might recall that potential energy is defined in terms of the work done by a conservative force (that is, $U = -\int F dx$) so that the force associated with a potential energy function is given by $F = -dU/dx$. But that means that a potential energy function which is quadratic would yield a restoring force that is proportional to the displacement itself - ie, $F(x) = -kx$. And this is just the condition that leads to simple harmonic motion. This leads to a very important generality:

Simple harmonic motion is a natural consequence of an oscillation about an equilibrium position.

Damped Oscillations

The description of simple harmonic motion assumes that the only forces acting on the system being described result in a restoring force that is proportional to the displacement from equilibrium. When that is true, of course, the resulting motion is purely sinusoidal with a constant amplitude - that is, the oscillator will continue to oscillate forever with undiminished energy. None of that can be correct, of course, because there will always be either some damping of the oscillation due to either imposed or inadvertent resistive forces such as air resistance, internal resistance in the spring itself, or some externally applied mechanism to damp the oscillations.



Consider a mass-spring system which has some damping mechanism which depends on the speed of the oscillator. An attached "fin" for example which moves in a viscous fluid would provide such a mechanism. The resistive force would depend on the speed of the motion as well as the size and shape of the fin and the viscosity of the fluid. The resistive force can be expressed as

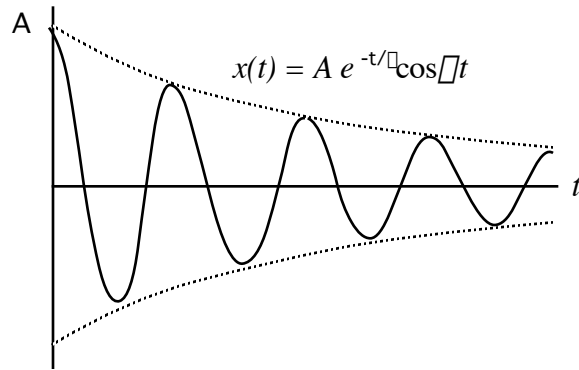
$$f = -bv$$

and that force would then be in addition to the restoring force of the spring. So Newton's second law for such an oscillator would be written:

$$m \frac{d^2x}{dt^2} = -kx - b \frac{dx}{dt}$$

The method for solving the above equation is not important to this discussion, but it IS important to consider the the *form* of the solution. That is, solving the differential equation yields

$$x(t) = A(t) \cos \omega t = A e^{-\frac{b}{2m}t} \cos \omega t$$



The primary effect of the damping force on the motion of the oscillator - assuming the damping is not so great as to prevent it from oscillating at all (a situation called "over-damping") - is to cause the amplitude of the oscillation to diminish in time. The time dependence of the amplitude is given by the decreasing exponential function - with the argument of the exponential given by $b/2m$. That implies that the larger the damping coefficient b in the damping force, the more quickly the oscillations diminish (which should not be surprising) and the larger the mass attached to the spring, the less quickly the oscillations

die out. The parameter $\tau = 2m/b$, with units of time, is called the *decay time* - and represents a relative measure of how quickly the oscillations "die out". The frequency of oscillation is only slightly affected by the damping forces - at least when the damping is not close to the "critical damping" that is required to prevent oscillations at all. The angular frequency is given by

$$\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

which is smaller than the undamped angular frequency ($\sqrt{k/m}$), but only slightly for small damping.

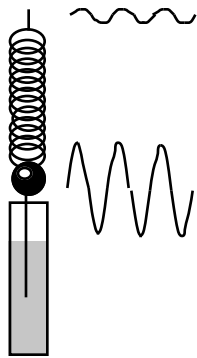
Since the energy of the oscillator is just $E_{tot} = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$ where A is the amplitude of the oscillation (and is time dependent when damped) the total energy of the oscillator decreases as a function of time according to $E_{tot} = \frac{1}{2}kA^2 e^{-\frac{b}{m}t}$ a result of the dissipation of the energy of oscillation due to the damping in the system.

Driven Oscillators and Resonance

Energy can be transferred to a system that has a natural oscillation frequency by a phenomenon called *resonance*. It is a common occurrence - a child on a swing can achieve ever higher oscillation amplitudes by being pushed at the same point on each successive swing, an out-of-balance automobile tire can be made to bounce wildly if driven at exactly the right speed, or running a wet finger around a wine glass can cause it to "ring". All of these are examples of resonance - a stimulated oscillation caused by "driving" a system at its natural oscillation frequency. To see why it occurs, consider a damped mass-spring system which also has an additional force that itself is sinusoidal - a driving force. That is, consider a system with three forces acting - the spring force, damping force, and driving force which is given by $F_o \sin \omega_d t$. The corresponding differential equation would be written:

$$m \frac{d^2 x}{dt^2} = -kx - b \frac{dx}{dt} + F_o \sin \omega_d t$$

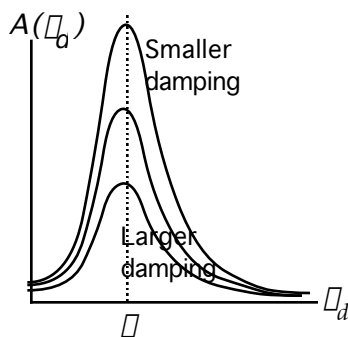
where F_o is the magnitude of the driving force and ω_d is the angular frequency of the driving mechanism. (Notice that the driving mechanism is the result of an outside agent acting on the oscillating system, so the driving frequency is not necessarily the same as the natural frequency of the system itself.)



Assume the mass spring system is being driven by shaking the top of the spring sinusoidally through some driving amplitude A_d , the magnitude of the driving force is then just the spring constant k times the amplitude at which the top of the spring is being driven - ie, $F_o = kA_d$. The solution to this differential equation is identical to that of the damped oscillator with the addition of a *steady state response term* (which mathematically is just the *particular solution* to the differential equation) and is given by

$$x(t) = Ae^{-\frac{b}{2m}t} \cos \omega t + A(\omega_d) \sin(\omega_d t + \phi)$$

The first term is called the *transient term* which is an oscillation at the natural frequency of the system - but which decays as time goes on (and becomes negligible when $t \gg 2m/b$). The second term - the *steady state term* - is an oscillation at the same frequency as the driving mechanism and has an amplitude $A(\omega_d)$ that depends on the driving frequency. It is this steady state term that is of interest. In fact, it is the *amplitude* of the steady state term that is of interest.



After the transient term dies out due to the damping, the steady state term remains. But the amplitude of the steady state term depends on the driving frequency (and how close it is to the natural frequency). At driving frequencies well away from the natural frequency - either much smaller or much larger - the steady state response term is small compared to the driving amplitude. That is, there is a relatively small response of the system to the additional force driving the system. But when the driving frequency is near or equal to the natural frequency, the steady state response amplitude can be many times larger than the driving amplitude. It is this effect that is called resonance.

When driven at resonance, the driving mechanism transfers a small "impulse" to the system that incrementally increases the energy of the oscillator on each successive oscillation. The result is that the system oscillates at an ever increasing amplitude until the damping forces dissipate energy at the same rate it is delivered to the system - at which time the system is said to be in steady state oscillation. In that case the motion of the system can be described by $x(t) = A(\omega_d) \sin(\omega_d t + \phi)$ - and that oscillation amplitude can be quite large. The limit to the response is due to the damping. And the larger the damping in the system, the smaller the maximum steady state response amplitude.