

L14Subspaces

1 Subspaces

So far have been working with vector spaces like $\mathbb{R}^2, \mathbb{R}^3$.

But there are more vector spaces. . .

Today we'll define a **subspace** and show how the concept helps us understand the nature of matrices and their linear transformations.

Definition. A *subspace* is any set H in \mathbb{R}^n that has three properties:

1. The zero vector is in H .
2. For each \mathbf{u} and \mathbf{v} in H , the sum $\mathbf{u} + \mathbf{v}$ is in H .
3. For each \mathbf{u} in H and each scalar c , the vector $c\mathbf{u}$ is in H .

Another way of stating properties 2 and 3 is that H is *closed* under addition and scalar multiplication.

Examples. Many of the vector sets we've discussed so far are subspaces.

For example, if \mathbf{v}_1 and \mathbf{v}_2 are in \mathbb{R}^n and $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, then H is a subspace of \mathbb{R}^n .

Let's check this:

- 1) The zero vector is in H

because $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2$ is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

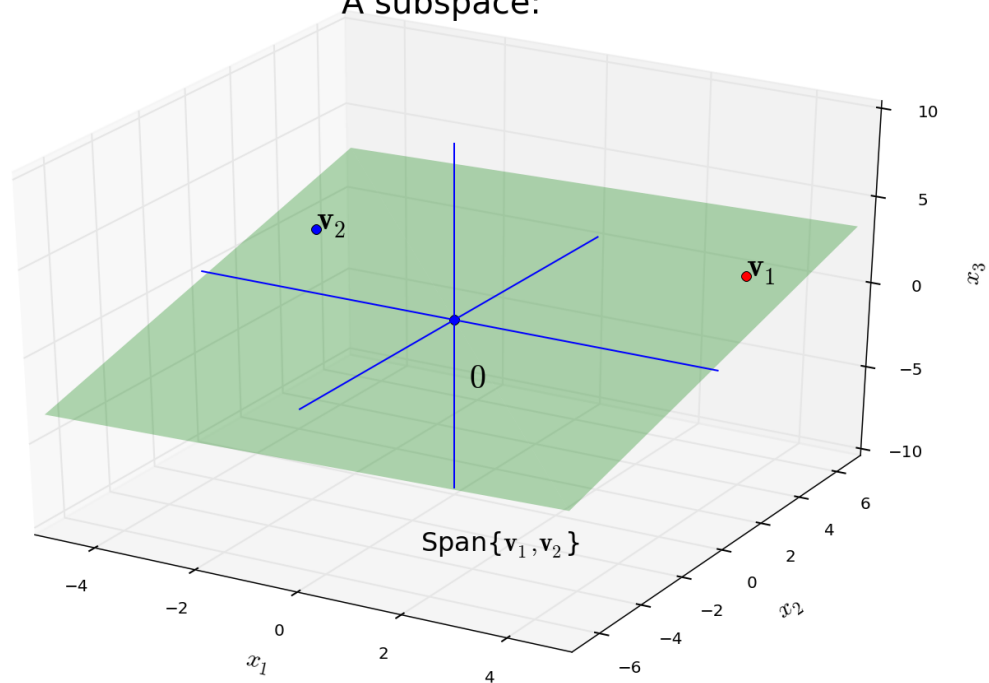
- 2) The sum of any two vectors in H is in H

In other words, if $\mathbf{u} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2$, and $\mathbf{v} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2$,
... their sum $\mathbf{u} + \mathbf{v}$ is $(s_1 + t_1)\mathbf{v}_1 + (s_2 + t_2)\mathbf{v}_2$,
... which is in H .

- 3) For any scalar c , $c\mathbf{u}$ is in H

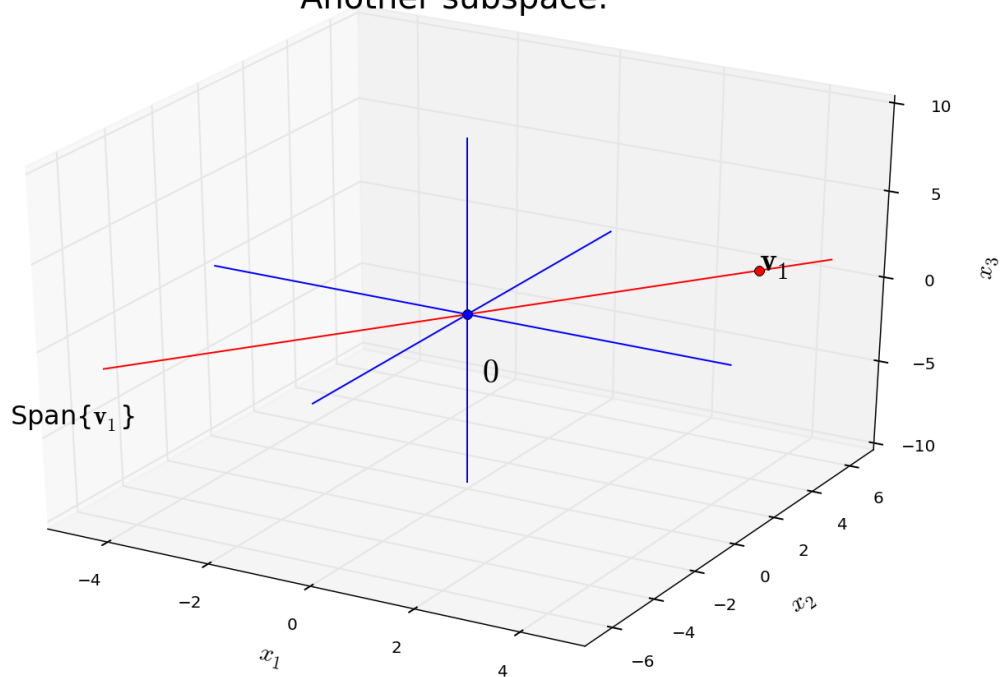
because $c\mathbf{u} = c(s_1\mathbf{v}_1 + s_2\mathbf{v}_2) = (cs_1\mathbf{v}_1 + cs_2\mathbf{v}_2)$.

A subspace:



Indeed, by similar arguments, *any* span of a vector set is a subspace. For example, $\text{Span}\{\mathbf{v}_1\}$, which is a line through the origin, is a subspace.

Another subspace:



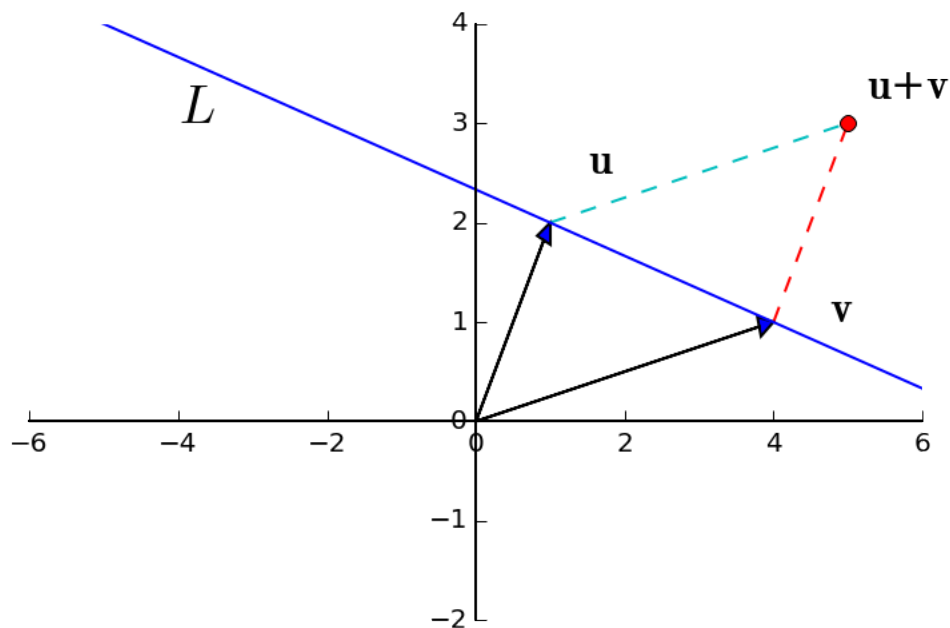
Because of this, we refer to $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ as **the subspace spanned by $\mathbf{v}_1, \dots, \mathbf{v}_p$** .

Is **any** line a subspace? What about a line that is not through the origin?

In fact, a line L not through the origin **fails all three** requirements for a subspace:

- 1) L does not contain the zero vector.
- 2) L is not closed under addition.
- 3) L is not closed under scalar multiplication.

Let's just look at 2):



1.1 Question Time! Q14.1

1.2 Column Space and Null Space of a Matrix

An important way to think about a matrix is in terms of two subspaces: **column space** and **null space**.

Definition. The **column space** of a matrix A is the set $\text{Col } A$ of all linear combinations of the columns of A .

If $A = [\mathbf{a}_1 \cdots \mathbf{a}_n]$, with columns in \mathbb{R}^m , then $\text{Col } A$ is the same as $\text{Span}\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$.

The column space of an $m \times n$ matrix is a subspace of \mathbb{R}^m .

In particular, note that $\text{Col } A$ equals \mathbb{R}^m only when the columns of A span \mathbb{R}^m . Otherwise, $\text{Col } A$ is only part of \mathbb{R}^m .

When a system of linear equations is written in the form $A\mathbf{x} = \mathbf{b}$, the column space of A is the set of all \mathbf{b} for which the system has a solution.

Equivalently, when we consider the linear operator $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ that is implemented by the matrix A , the column space of A is the **range** of T .

1.3 Question Time! Q14.2

Definition. The **null space** of a matrix A is the set $\text{Nul } A$ of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

When A has n columns, a solution of $A\mathbf{x} = \mathbf{0}$ is a vector in \mathbb{R}^n . So the null space of A is a subset of \mathbb{R}^n . In fact, $\text{Nul } A$ is a **subspace** of \mathbb{R}^n .

Theorem. The null space of an $m \times n$ matrix A is a subspace of \mathbb{R}^n .

Equivalently, the set of all solutions of a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of \mathbb{R}^n .

Proof.

1) The zero vector is in $\text{Nul } A$ because $A\mathbf{0} = \mathbf{0}$.

2) The sum of two vectors in $\text{Nul } A$ is in $\text{Nul } A$.

Take two vectors \mathbf{u} and \mathbf{v} that are in $\text{Nul } A$. By definition $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$.

Then $\mathbf{u} + \mathbf{v}$ is in $\text{Nul } A$ because $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$.

3) Any scalar multiple of a vector in $\text{Nul } A$ is in $\text{Nul } A$.

Take a vector \mathbf{v} that is in $\text{Nul } A$. Then $A(c\mathbf{v}) = cA\mathbf{v} = c\mathbf{0} = \mathbf{0}$.

Testing whether a vector \mathbf{v} is in $\text{Nul } A$ is easy: simply compute $A\mathbf{v}$ and see if the result is zero.

Comparing Col A and $\text{Nul } A$.

What is the relationship between these two subspaces that are defined using A ?

Actually, there is no particular connection (at this point in the course).

The important thing to note at present is that these two subspaces live in different “universes”. For an $m \times n$ matrix, the column space is a subset of \mathbb{R}^m (all its vectors have m components), while the null space is a subset of \mathbb{R}^n (all its vectors have n components).

(However: next lecture we will make a connection!)

1.4 Basis for a Subspace

A subspace usually contains an infinite number of vectors.

Often it is convenient to work with a small set of vectors that span the subspace. The smaller the set, the better.

It can be shown that the smallest possible spanning set must be linearly independent.

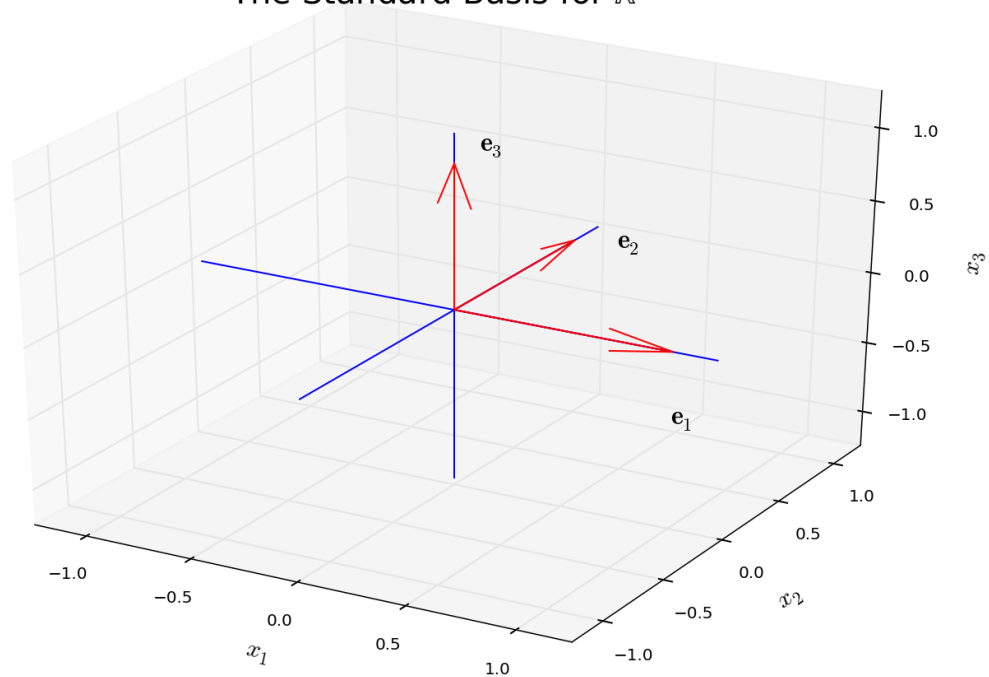
Definition. A **basis** for a subspace H of \mathbb{R}^n is a linearly independent set in H that spans H .

Example. The columns of **any** invertible $n \times n$ matrix form a basis for \mathbb{R}^n . This is because, by the Invertible Matrix Theorem, they are linearly independent, and they span \mathbb{R}^n .

So, for example, we could use the identity matrix, I . Its columns are $\mathbf{e}_1, \dots, \mathbf{e}_n$.

The set $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is called the **standard basis** for \mathbb{R}^n .

The Standard Basis for \mathbb{R}^3



1.5 Question Time! Q14.3

1.5.1 Finding a basis for the nullspace.

We will often want to find a basis for $\text{Col } A$ or for $\text{Nul } A$.

We'll start with finding a basis for the null space of a matrix.

Example. Find a basis for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

Solution. We would like to describe the set of all solutions of $A\mathbf{x} = \mathbf{0}$.

We start by writing the solution of $A\mathbf{x} = \mathbf{0}$ in parametric form:

$$[A \ \mathbf{0}] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{array}{rclcl} x_1 & -2x_2 & & -x_4 & +3x_5 & = & 0 \\ & & & x_3 & +2x_4 & -2x_5 & = & 0 \\ & & & & & 0 & = & 0 \end{array}$$

So x_1 and x_3 are basic, and x_2, x_4 , and x_5 are free.

So the general solution is:

$$\begin{aligned} x_1 &= 2x_2 + x_4 - 3x_5, \\ x_3 &= -2x_4 + 2x_5. \end{aligned}$$

Now, what we want to do is write the solution set as a weighted combination of vectors. The free variables will become the weights.

$$\begin{aligned}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} &= \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} \\
&= x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \\
&= x_2 \mathbf{u} + x_4 \mathbf{v} + x_5 \mathbf{w}.
\end{aligned}$$

Now what we have is an expression that describes the entire solution set of $A\mathbf{x} = \mathbf{0}$.

So $\text{Nul } A$ is the set of all linear combinations of \mathbf{u} , \mathbf{v} , and \mathbf{w} . That is, $\text{Nul } A$ is the subspace spanned by $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

Furthermore, this construction automatically makes \mathbf{u} , \mathbf{v} , and \mathbf{w} linearly independent.

Since each weight appears by itself in one position, the only way for the whole weighted sum to be zero is if every weight is zero – which is the definition of linear independence.

So $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a **basis** for $\text{Nul } A$.

Conclusion: by finding a parametric description of the solution of the equation $A\mathbf{x} = \mathbf{0}$, we can construct a basis for the nullspace of A .

1.5.2 Finding a basis for the column space.

Warmup. We start with a warmup example. Suppose we have a matrix B that happens to be in reduced echelon form:

$$B = \begin{bmatrix} 1 & 0 & -3 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Denote the columns of B by $\mathbf{b}_1, \dots, \mathbf{b}_5$ and note that $\mathbf{b}_3 = -3\mathbf{b}_1 + 2\mathbf{b}_2$ and $\mathbf{b}_4 = 5\mathbf{b}_1 - \mathbf{b}_2$.

So any combination of $\mathbf{b}_1, \dots, \mathbf{b}_5$ is actually just a combination of $\mathbf{b}_1, \mathbf{b}_2$, and \mathbf{b}_5 .

So $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_5\}$ spans $\text{Col } B$.

Also, $\mathbf{b}_1, \mathbf{b}_2$, and \mathbf{b}_5 are linearly independent, because they are columns from an identity matrix.

So: the pivot columns of B form a basis for $\text{Col } B$.

Note that this means: **there is no combination of columns 1, 2, and 5 that yields the zero vector.** (Other than the trivial combination of course.)

The general case. Now I'll show that the pivot columns of A form a basis for $\text{Col } A$ for any A .

Consider the case where $A\mathbf{x} = \mathbf{0}$ for some nonzero \mathbf{x} .

This says that there is a linear dependence relation between some of the columns of A .

If any of the entries in \mathbf{x} are zero, then those columns do not participate in the linear dependence relation.

When we row-reduce A to its reduced echelon form B , the columns are changed, but the equations $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same solution set.

So this means that the columns of A and the columns of B have exactly the same dependence relationships as the columns of B .

This means that any vector equation that is true for the columns of A is true for the columns of B .

In other words:

- 1) If some column of B can be written as a combination of other columns of B , then the same is true of the corresponding columns of A .

- 2) If no combination of certain columns of B yields the zero vector, then no combination of corresponding columns of A yields the zero vector.

So, if some columns of B are a basis for $\text{Col } B$, then the corresponding columns of A are a basis for $\text{Col } A$.

Example. Consider the matrix A :

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 & -9 \\ -2 & -2 & 2 & -8 & 2 \\ 2 & 3 & 0 & 7 & 1 \\ 3 & 4 & -1 & 11 & -8 \end{bmatrix}$$

It is row equivalent to the matrix B that we considered above. So to find its basis, we simply need to look at the basis for its reduced row echelon form. We already computed that a basis for $\text{Col } B$ was columns 1, 2, and 5.

Therefore we can immediately conclude that a basis for $\text{Col } A$ is A 's columns 1, 2, and 5.

So a basis for $\text{Col } A$ is:

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -9 \\ 2 \\ 1 \\ -8 \end{bmatrix} \right\}$$

Theorem. The pivot columns of a matrix A form a basis for the column space of A .

Be careful here – note that you compute the reduced row echelon form of A to find which columns are pivot columns, but you used the columns of A itself as the basis for $\text{Col } A$!