

L21OrthogonalSets

1 Orthogonal Sets

Today we'll study the properties of **sets** of orthogonal vectors. These can be very useful.

A set of vectors $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ in \mathbb{R}^n is said to be an **orthogonal set** if each pair of distinct vectors from the set is orthogonal, i.e.,

$$\mathbf{u}_i^T \mathbf{u}_j = 0 \text{ whenever } i \neq j.$$

Example. Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set, where

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}.$$

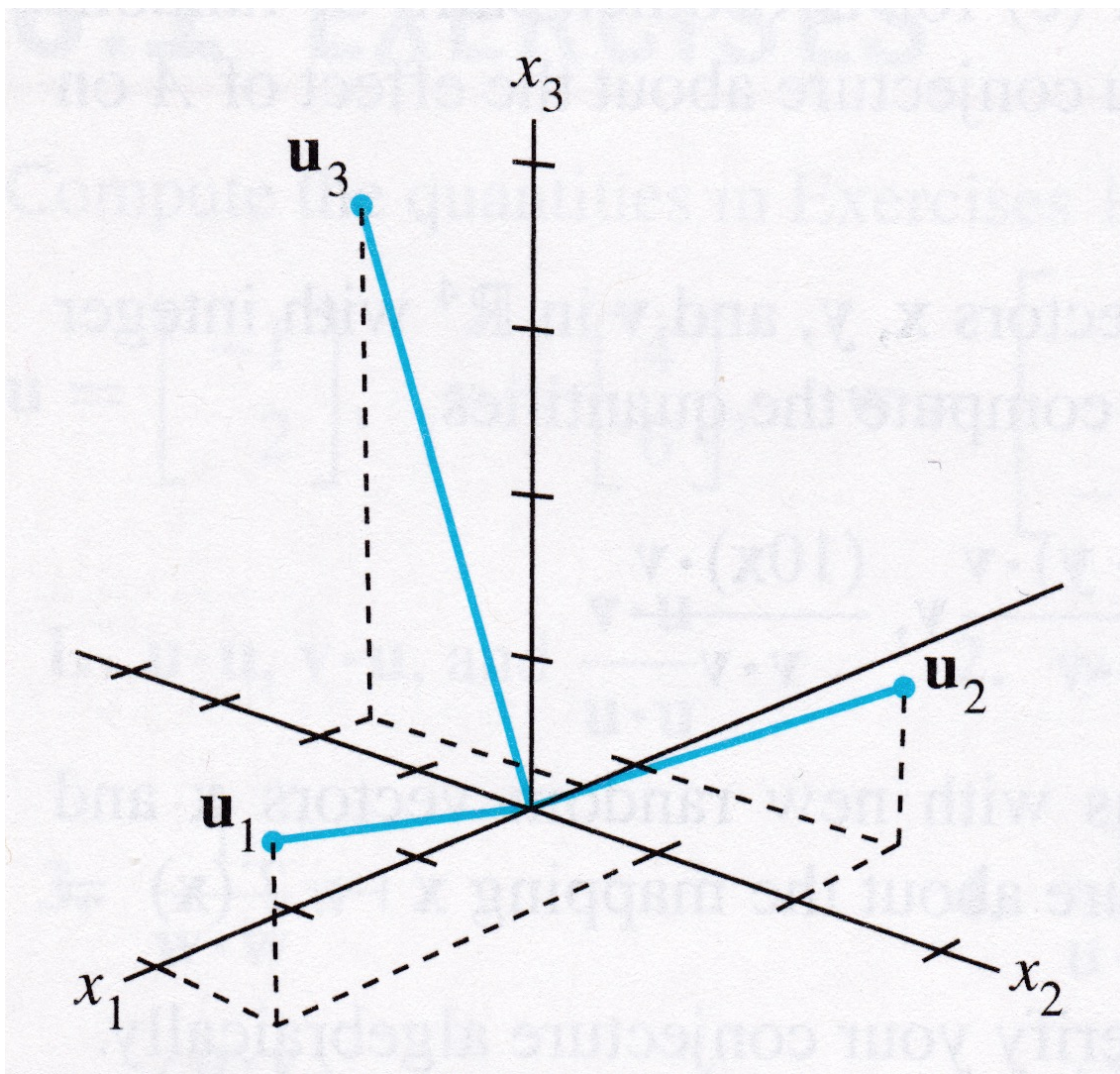
Solution. Consider the three possible pairs of distinct vectors, namely, $\{\mathbf{u}_1, \mathbf{u}_2\}$, $\{\mathbf{u}_1, \mathbf{u}_3\}$, and $\{\mathbf{u}_2, \mathbf{u}_3\}$.

$$\mathbf{u}_1^T \mathbf{u}_2 = 3(-1) + 1(2) + 1(1) = 0$$

$$\mathbf{u}_1^T \mathbf{u}_3 = 3(-1/2) + 1(-2) + 1(7/2) = 0$$

$$\mathbf{u}_2^T \mathbf{u}_3 = -1(-1/2) + 2(-2) + 1(7/2) = 0$$

Each pair of distinct vectors is orthogonal, and so $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set. In three space they describe three lines that are mutually perpendicular.



1.1 Question Time! Q21.1

Theorem. If $S = \{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in \mathbb{R}^n , then S is linearly independent and hence is a basis for the subspace spanned by S .

Proof. We will prove that there is no linear combination of the vectors in S with nonzero coefficients that yields the zero vector.

Assume $\mathbf{0} = c_1\mathbf{u}_1 + \dots + c_p\mathbf{u}_p$ for some scalars c_1, \dots, c_p . Then:

$$\begin{aligned} \mathbf{0} &= c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \dots + c_p\mathbf{u}_p \\ 0 &= (c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \dots + c_p\mathbf{u}_p)^T \mathbf{u}_1 \\ &= (c_1\mathbf{u}_1)^T \mathbf{u}_1 + (c_2\mathbf{u}_2)^T \mathbf{u}_1 + \dots + (c_p\mathbf{u}_p)^T \mathbf{u}_1 \\ &= c_1(\mathbf{u}_1^T \mathbf{u}_1) + c_2(\mathbf{u}_2^T \mathbf{u}_1) + \dots + c_p(\mathbf{u}_p^T \mathbf{u}_1) \end{aligned}$$

Because \mathbf{u}_1 is orthogonal to $\mathbf{u}_2, \dots, \mathbf{u}_p$:

$$= c_1(\mathbf{u}_1^T \mathbf{u}_1)$$

Since \mathbf{u}_1 is nonzero, $\mathbf{u}_1^T \mathbf{u}_1$ is not zero and so $c_1 = 0$.

We can use the same kind of reasoning to show that, c_2, \dots, c_p must be zero.

In other words, there is no nonzero combination of \mathbf{u}_i 's that yields the zero vector –so S is linearly independent.

Definition. An **orthogonal basis** for a subspace W of \mathbb{R}^n is a basis for W that is also an orthogonal set.

We have seen that for any subspace, there may be many different sets of vectors that can serve as a basis for W .

However an orthogonal basis is a particularly nice basis, because the weights (coordinates) of any point can be computed easily.

Theorem. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of \mathbb{R}^n . For each \mathbf{y} in W , the weights of the linear combination

$$\mathbf{y} = c_1 \mathbf{u}_1 + \dots + c_p \mathbf{u}_p$$

are given by

$$c_j = \frac{\mathbf{y}^T \mathbf{u}_j}{\mathbf{u}_j^T \mathbf{u}_j} \quad j = 1, \dots, p$$

Proof. As we saw in the last proof, the orthogonality of $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ means that

$$\mathbf{y}^T \mathbf{u}_1 = (c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \dots + c_p \mathbf{u}_p)^T \mathbf{u}_1$$

$$= c_1(\mathbf{u}_1^T \mathbf{u}_1)$$

Since $\mathbf{u}_1^T \mathbf{u}_1$ is not zero, the equation above can be solved for c_1 . To find any other c_j , compute $\mathbf{y}^T \mathbf{u}_j$ and solve for c_j .

Example. The set S which we saw earlier, ie,

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix},$$

is an orthogonal basis for \mathbb{R}^3 .

Then, express the vector $\mathbf{y} = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$ as a linear combination of the vectors in S (ie, in the basis S or in the coordinate system S).

Solution. Compute

$$\mathbf{y}^T \mathbf{u}_1 = 11, \quad \mathbf{y}^T \mathbf{u}_2 = -12, \quad \mathbf{y}^T \mathbf{u}_3 = -33,$$

$$\mathbf{u}_1^T \mathbf{u}_1 = 11, \quad \mathbf{u}_2^T \mathbf{u}_2 = 6, \quad \mathbf{u}_3^T \mathbf{u}_3 = 33/2$$

So

$$\begin{aligned} \mathbf{y} &= \frac{\mathbf{y}^T \mathbf{u}_1}{\mathbf{u}_1^T \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{y}^T \mathbf{u}_2}{\mathbf{u}_2^T \mathbf{u}_2} \mathbf{u}_2 + \frac{\mathbf{y}^T \mathbf{u}_3}{\mathbf{u}_3^T \mathbf{u}_3} \mathbf{u}_3 \\ &= \frac{11}{11} \mathbf{u}_1 + \frac{-12}{6} \mathbf{u}_2 + \frac{-33}{33/2} \mathbf{u}_3 \end{aligned}$$

$$= \mathbf{u}_1 - 2\mathbf{u}_2 - 2\mathbf{u}_3.$$

Let's stop for a moment and think about how we would have done this if we had not known that the vectors \mathbf{u}_1 , \mathbf{u}_2 , and \mathbf{u}_3 form an orthogonal set.

We would have been looking for

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{y}$$

The way we would find c_1, c_2, c_3 in that case would be to solve the linear system

$$[\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{u}_3] \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \mathbf{y}$$

which would have been much more trouble than what we did.

Instead, because the basis is an orthogonal basis, each coefficient c_i can be found separately, and simply.

1.2 An Orthogonal Projection

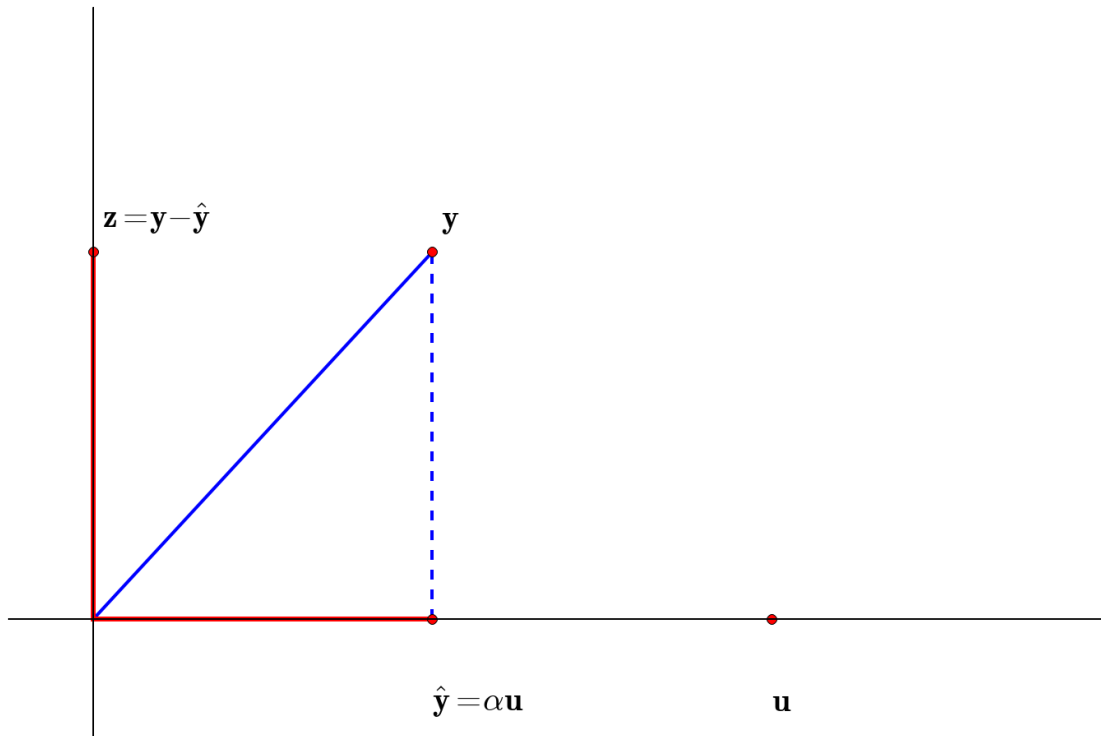
Given a nonzero vector \mathbf{u} in \mathbb{R}^n , consider the problem of decomposing a vector \mathbf{y} in \mathbb{R}^n into the sum of two vectors:

- one that is a multiple of \mathbf{u} , and
- one that is orthogonal to \mathbf{u} .

In other words, we wish to write:

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

where $\hat{\mathbf{y}} = \alpha\mathbf{u}$ for some scalar α and \mathbf{z} is some vector orthogonal to \mathbf{u} .



That is, we are given \mathbf{y} and \mathbf{u} , and asked to compute \mathbf{z} and $\hat{\mathbf{y}}$.

To solve this, assume that we have some α , and with it we compute $\mathbf{y} - \alpha\mathbf{u} = \mathbf{y} - \hat{\mathbf{y}} = \mathbf{z}$.

We want \mathbf{z} to be orthogonal to \mathbf{u} .

Now $\mathbf{z} = \mathbf{y} - \alpha\mathbf{u}$ is orthogonal to \mathbf{u} if and only if

$$0 = (\mathbf{y} - \alpha\mathbf{u})^T \mathbf{u}$$

$$= \mathbf{y}^T \mathbf{u} - (\alpha\mathbf{u})^T \mathbf{u}$$

$$= \mathbf{y}^T \mathbf{u} - \alpha(\mathbf{u}^T \mathbf{u})$$

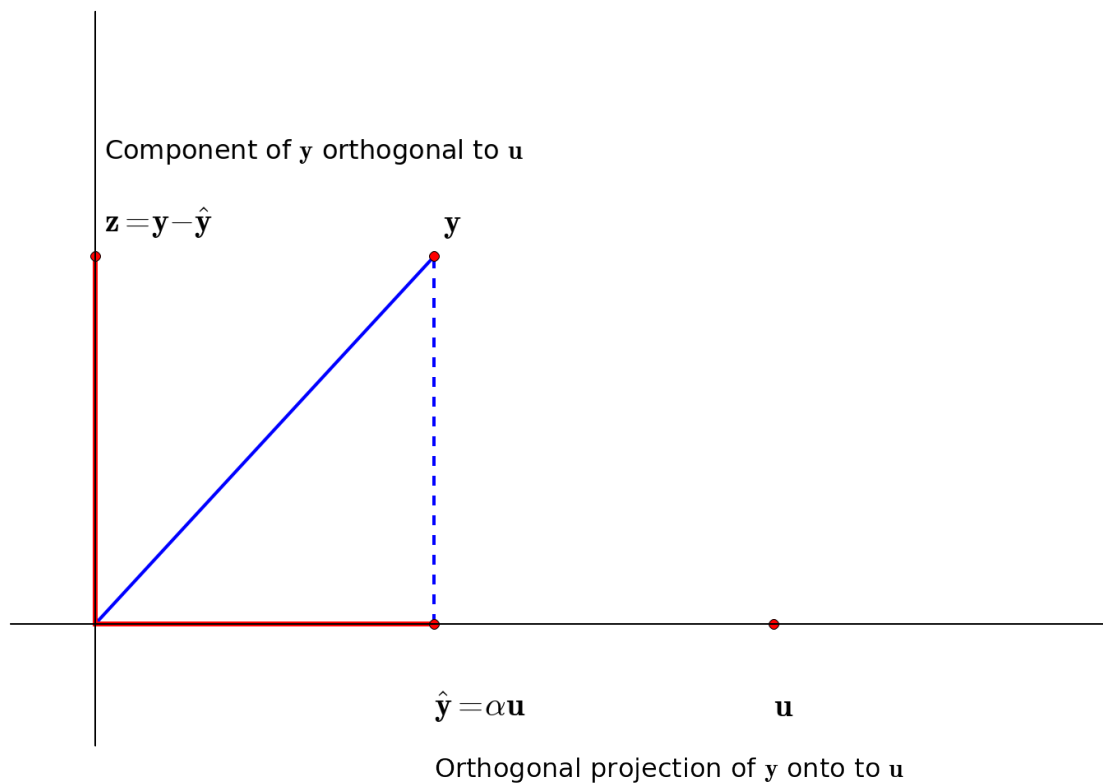
That is, the solution in which \mathbf{z} is orthogonal to \mathbf{u} happens if and only if

$$\alpha = \frac{\mathbf{y}^T \mathbf{u}}{\mathbf{u}^T \mathbf{u}}$$

and since $\hat{\mathbf{y}} = \alpha\mathbf{u}$,

$$\hat{\mathbf{y}} = \frac{\mathbf{y}^T \mathbf{u}}{\mathbf{u}^T \mathbf{u}} \mathbf{u}.$$

The vector $\hat{\mathbf{y}}$ is called the **orthogonal projection of \mathbf{y} onto \mathbf{u}** , and the vector \mathbf{z} is called the **component of \mathbf{y} orthogonal to \mathbf{u}** .



Now, note that if we had scaled \mathbf{u} by any amount (ie, moved it to the right or the left), we would not have changed the location of $\hat{\mathbf{y}}$.

This can be seen as well by replacing \mathbf{u} with $c\mathbf{u}$ and recomputing $\hat{\mathbf{y}}$:

$$\hat{\mathbf{y}} = \frac{\mathbf{y}^T c\mathbf{u}}{c\mathbf{u}^T c\mathbf{u}} c\mathbf{u} = \frac{\mathbf{y}^T \mathbf{u}}{\mathbf{u}^T \mathbf{u}} \mathbf{u}.$$

Thus, the projection of \mathbf{y} is determined by the *subspace* L that is spanned by \mathbf{u} – in other words, the line through \mathbf{u} and the origin.

Hence sometimes $\hat{\mathbf{y}}$ is denoted by $\text{proj}_L \mathbf{y}$ and is called the **orthogonal projection of \mathbf{y} onto L** . Specifically:

$$\mathbf{y} = \text{proj}_L \mathbf{y} = \frac{\mathbf{y}^T \mathbf{u}}{\mathbf{u}^T \mathbf{u}} \mathbf{u}$$

Example. Let $\mathbf{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

Find the orthogonal projection of \mathbf{y} onto \mathbf{u} . Then write \mathbf{y} as the sum of two orthogonal vectors, one in $\text{Span}\{\mathbf{u}\}$, and one orthogonal to \mathbf{u} .

Solution. Compute

$$\mathbf{y}^T \mathbf{u} = \begin{bmatrix} 7 & 6 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 40$$

$$\mathbf{u}^T \mathbf{u} = \begin{bmatrix} 4 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 20$$

The orthogonal projection of \mathbf{y} onto \mathbf{u} is

$$\begin{aligned} \hat{\mathbf{y}} &= \frac{\mathbf{y}^T \mathbf{u}}{\mathbf{u}^T \mathbf{u}} \mathbf{u} \\ &= \frac{40}{20} \mathbf{u} = 2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \end{aligned}$$

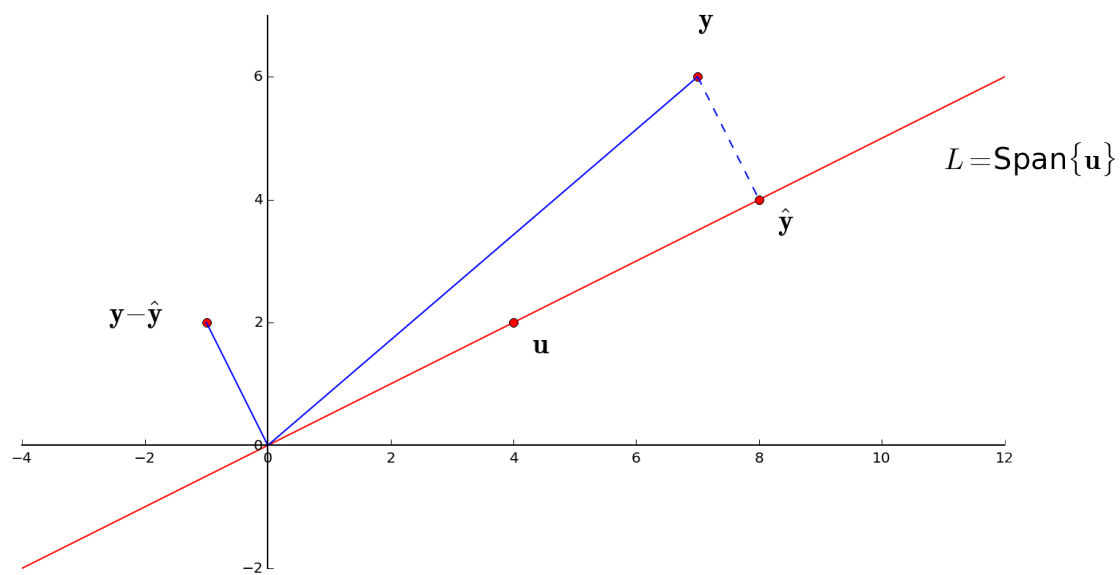
And the component of \mathbf{y} orthogonal to \mathbf{u} is

$$\mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

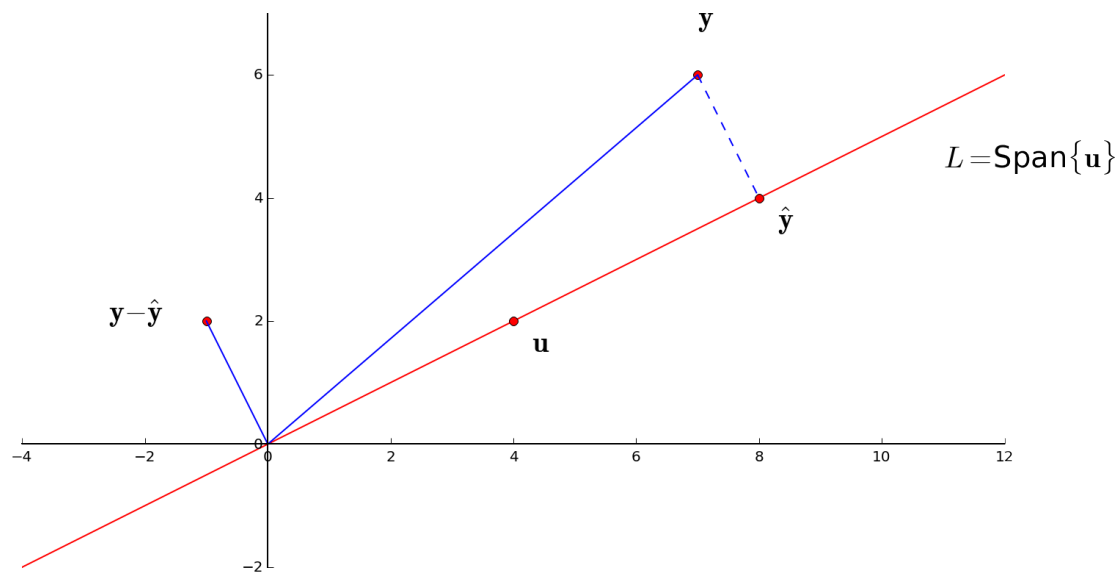
So

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

$$\begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$



2 Question Time! Q21.2



The closest point.

Recall from geometry that given a line and a point P , the closest point on the line to P is given by the perpendicular from P to the line.

So this gives an important interpretation of $\hat{\mathbf{y}}$: it is **the closest point to \mathbf{y} in the subspace L** .

The distance from \mathbf{y} to L

The distance from \mathbf{y} to L is the length of the perpendicular from \mathbf{y} to its orthogonal projection on L , namely $\hat{\mathbf{y}}$.

This distance equals the length of $\mathbf{y} - \hat{\mathbf{y}}$.

In this example, the distance is

$$\|\mathbf{y} - \hat{\mathbf{y}}\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}.$$

2.1 A Geometric Interpretation

Earlier today, we saw that when we decompose a vector \mathbf{y} into a linear combination of vectors $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ in an orthogonal set, we have

$$\mathbf{y} = c_1 \mathbf{u}_1 + \dots + c_p \mathbf{u}_p$$

where

$$c_j = \frac{\mathbf{y}^T \mathbf{u}_j}{\mathbf{u}_j^T \mathbf{u}_j}$$

And just now we have seen that the projection of \mathbf{y} onto the subspace spanned by \mathbf{u} is

$$\text{proj}_L \mathbf{y} = \frac{\mathbf{y}^T \mathbf{u}}{\mathbf{u}^T \mathbf{u}} \mathbf{u}.$$

So a decomposition like $\mathbf{y} = c_1 \mathbf{u}_1 + \dots + c_p \mathbf{u}_p$ is really decomposing \mathbf{y} into **a sum of orthogonal projections onto one-dimensional subspaces**.

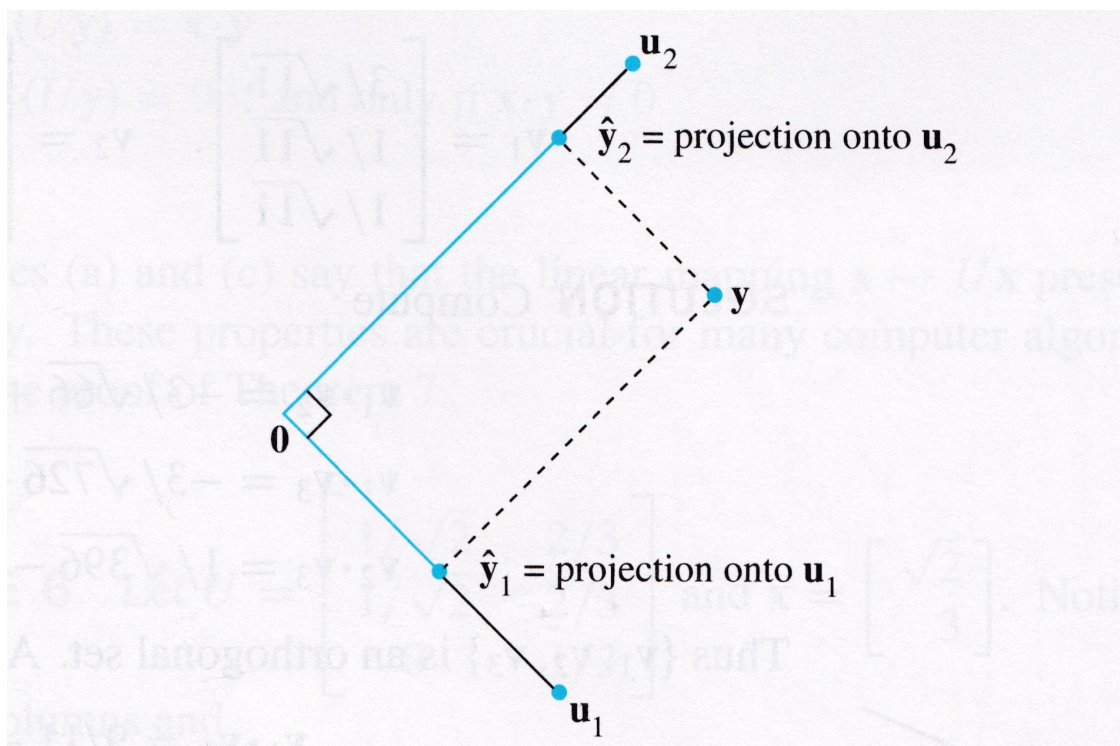
For example, let's take the case where $\mathbf{y} \in \mathbb{R}^2$. Let's say we are given $\mathbf{u}_1, \mathbf{u}_2$ such that \mathbf{u}_1 is orthogonal to \mathbf{u}_2 , and so together they span \mathbb{R}^2 .

Then \mathbf{y} can be written in the form

$$\mathbf{y} = \frac{\mathbf{y}^T \mathbf{u}_1}{\mathbf{u}_1^T \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{y}^T \mathbf{u}_2}{\mathbf{u}_2^T \mathbf{u}_2} \mathbf{u}_2.$$

The first term is the projection of \mathbf{y} onto the subspace spanned by \mathbf{u}_1 and the second term is the projection of \mathbf{y} onto the subspace spanned by \mathbf{u}_2 .

So this equation expresses \mathbf{y} as the sum of its projections onto the (orthogonal) axes determined by \mathbf{u}_1 and \mathbf{u}_2 .



2.2 Question Time! Q21.3

2.3 Orthonormal Sets

A set $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an **orthonormal set** if it is an orthogonal set of **unit** vectors.

If W is the subspace spanned by such as a set, then $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an **orthonormal basis** for W since the set is automatically linearly independent.

The simplest example of an orthonormal set is the standard basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ for \mathbb{R}^n . Any nonempty subset of $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is orthonormal as well.

Pro tip: keep the terms clear in your head:

- **orthogonal** is (just) perpendicular, while
- **orthonormal** is perpendicular *and* unit length.

(You can see the word “normalized” inside “orthonormal”).

Matrices with orthonormal columns are particularly important.

Theorem. A $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I$.

Proof. Let us suppose that U has only three columns which are each vectors in \mathbb{R}^m (but the proof will generalize to n columns).

Let $U = [\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{u}_3]$. Then:

$$\begin{aligned} U^T U &= \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \mathbf{u}_3^T \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{u}_1^T \mathbf{u}_1 & \mathbf{u}_1^T \mathbf{u}_2 & \mathbf{u}_1^T \mathbf{u}_3 \\ \mathbf{u}_2^T \mathbf{u}_1 & \mathbf{u}_2^T \mathbf{u}_2 & \mathbf{u}_2^T \mathbf{u}_3 \\ \mathbf{u}_3^T \mathbf{u}_1 & \mathbf{u}_3^T \mathbf{u}_2 & \mathbf{u}_3^T \mathbf{u}_3 \end{bmatrix} \end{aligned}$$

The columns of U are orthogonal if and only if

$$\mathbf{u}_1^T \mathbf{u}_2 = \mathbf{u}_2^T \mathbf{u}_1 = 0, \quad \mathbf{u}_1^T \mathbf{u}_3 = \mathbf{u}_3^T \mathbf{u}_1 = 0, \quad \mathbf{u}_2^T \mathbf{u}_3 = \mathbf{u}_3^T \mathbf{u}_2 = 0$$

The columns of U all have unit length if and only if

$$\mathbf{u}_1^T \mathbf{u}_1 = 1, \quad \mathbf{u}_2^T \mathbf{u}_2 = 1, \quad \mathbf{u}_3^T \mathbf{u}_3 = 1.$$

So $U^T U = I$.

Theorem. Let U be an $m \times n$ matrix with orthonormal columns, and let \mathbf{x} and \mathbf{y} be in \mathbb{R}^n . Then:

1. $\|U\mathbf{x}\| = \|\mathbf{x}\|$.
2. $(U\mathbf{x})^T(U\mathbf{y}) = \mathbf{x}^T\mathbf{y}$.
3. $(U\mathbf{x})^T(U\mathbf{y}) = 0$ if and only if $\mathbf{x}^T\mathbf{y} = 0$.

Properties 1. and 3. say that the linear mapping $\mathbf{x} \mapsto U\mathbf{x}$ preserves lengths and orthogonality.

So, viewed as a linear operator, an orthonormal matrix is very special: the lengths of vectors, and therefore the **distances between points** is not changed by the action of U .

Example. Let $U = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}$. Notice that U has orthonormal columns, and

$$U^T U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let's verify that $\|Ux\| = \|x\|$.

$$U\mathbf{x} = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

$$\|U\mathbf{x}\| = \sqrt{9 + 1 + 1} = \sqrt{11}.$$

$$\|\mathbf{x}\| = \sqrt{2 + 9} = \sqrt{11}.$$

Orthonormal Square Matrices. Consider the case when U is square, and has orthonormal columns. Then the fact that $U^T U = I$ implies that $U^{-1} = U^T$.

Then U is called an **orthogonal** matrix.

(Note that this terminology could be confusing; the columns of U are not just orthogonal but actually orthonormal.)