L08MatrixofLinearTranformation

June 1, 2018

1 The Matrix of a Linear Transformation

Not all linear transformations are matrix-vector multiplications. But, **every linear transformation from vectors to vectors is a matrix multiplication.**

Theorem. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. There there is a unique matrix A such that:

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e_j})$, where $\mathbf{e_j}$ is the $j$the column of the identity matrix in $\mathbb{R}^n$:

$$A = \left[T(\mathbf{e_1}) \dots T(\mathbf{e_n})\right].$$

A is called the *standard matrix* of T.

Proof. Write

$$\mathbf{x} = I\mathbf{x} = [\mathbf{e_1} \dots \mathbf{e_n}]\,\mathbf{x}$$

$$= x_1\mathbf{e_1} + \cdots + x_n\mathbf{e_n}.$$

Because T is linear, we have:

$$T(\mathbf{x}) = T(x_1\mathbf{e_1} + \cdots + x_n\mathbf{e_n})$$

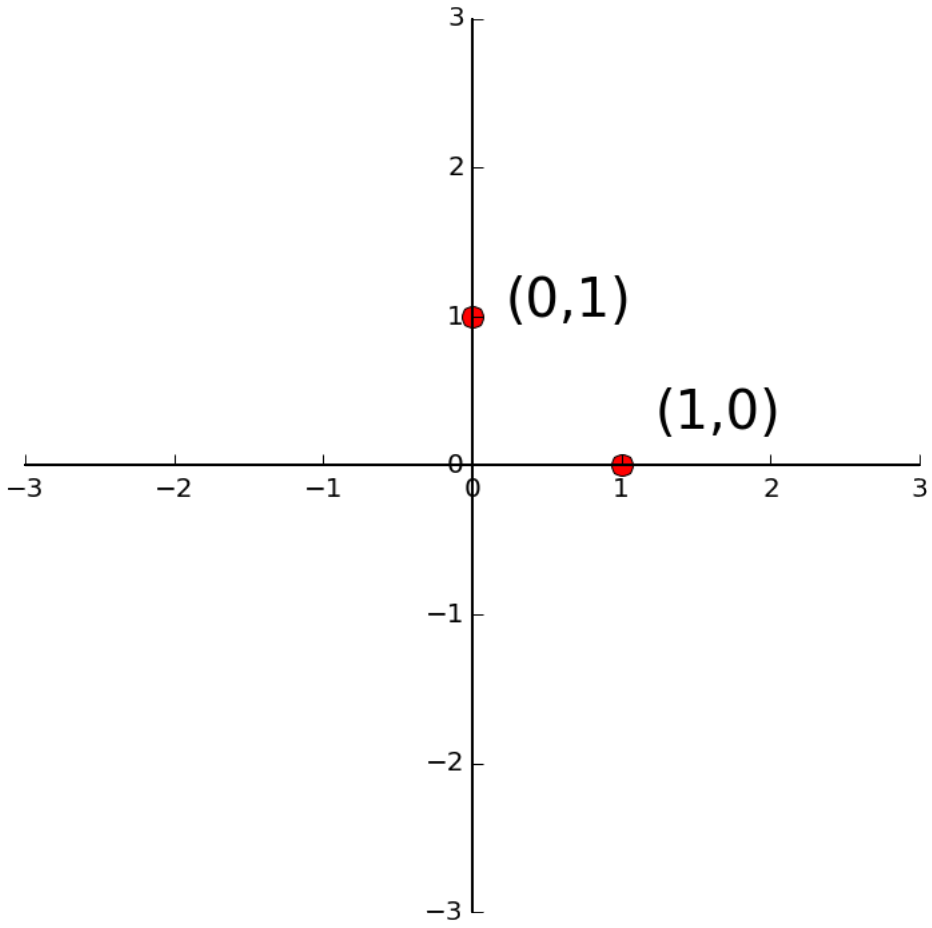
$$= x_1T(\mathbf{e_1}) + \cdots + x_nT(\mathbf{e_n})$$

$$= [T(\mathbf{e_1}) \dots T(\mathbf{e_n})] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A\mathbf{x}.$$

Hence we see that every linear transformation from vectors to vectors can be implemented as a matrix multiplication, and vice versa.

The term *linear transformation* focuses on a **property** of the mapping, while the term *matrix multiplication* focuses on how such a mapping is **implemented.**

For example, we find the standard matrix of a linear tranformation of $\mathbb{R}^2$ by asking what the transformation does to two particular points:



This is a **hugely** powerful tool.

Let's say we start from some given linear transformation; we can use this idea to find the matrix that implements that linear transformation.

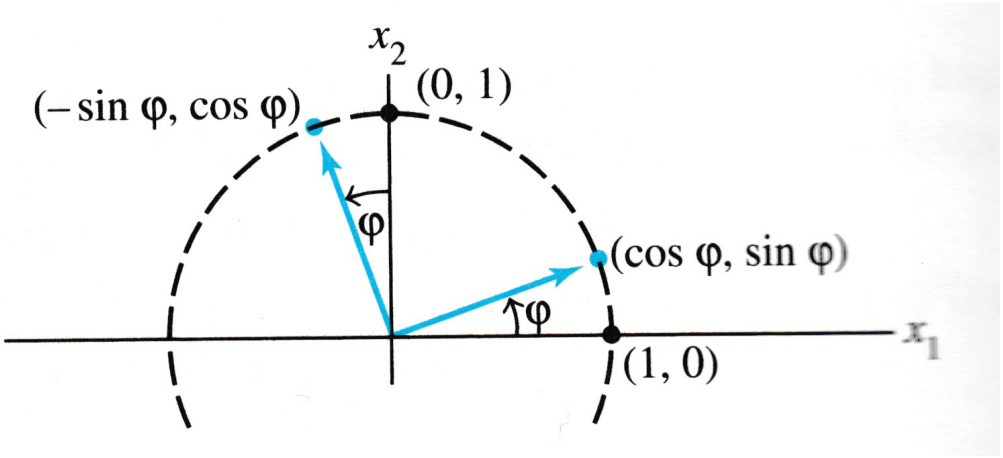
For example, let's see how to compute the linear transformation that is a rotation.

(By the way, can you see that a rotation is a linear transformation?)

Example. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle φ, with counterclockwise rotation for a positive angle. Find the standard matrix A of this transformation.

Solution. The columns of I are $\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Referring to the diagram below, we can see that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ rotates into $\begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ rotates into $\begin{bmatrix} -\sin\varphi \\ \cos\varphi \end{bmatrix}$.

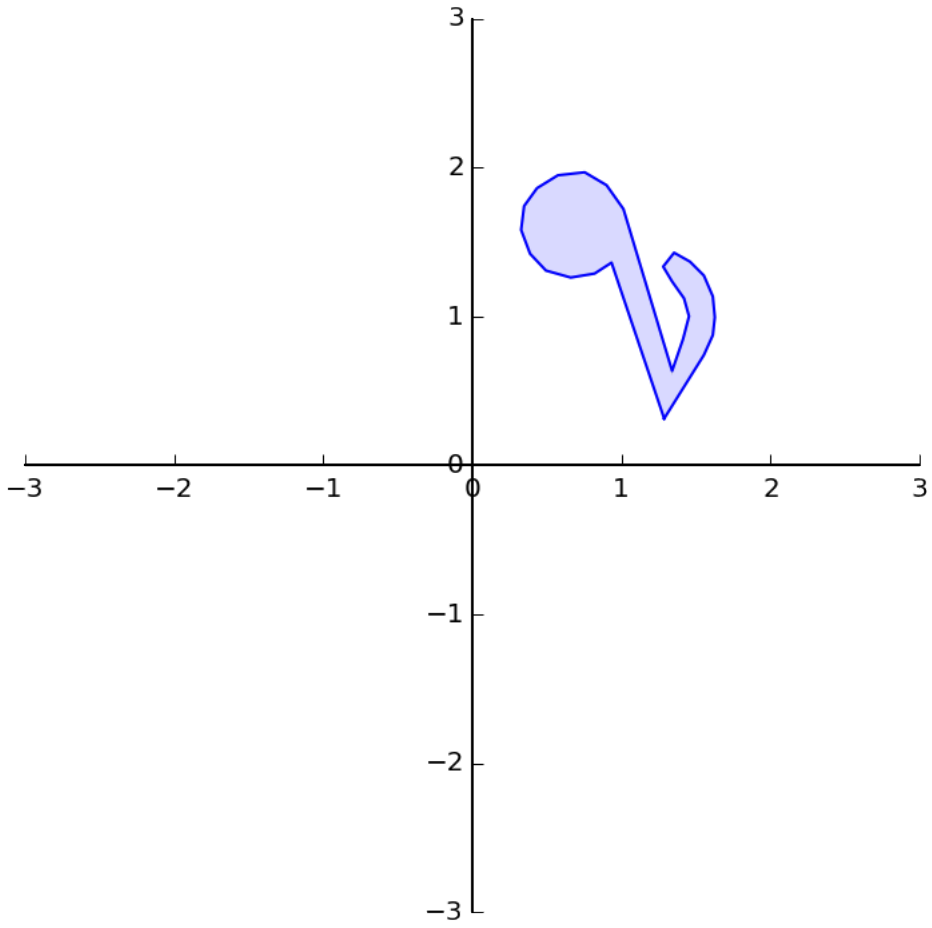


So by the Theorem above,

$$A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}.$$

To demonstrate the use of a rotation matrix, let's rotate the following shape:

In [6]:
```
dm.plotSetup()
note = dm.mnote()
dm.plotShape(note)
```



The variable `note` is a array of 26 vectors in $\mathbb{R}^2$ that define its shape.

In other words, it is a 2×26 matrix.

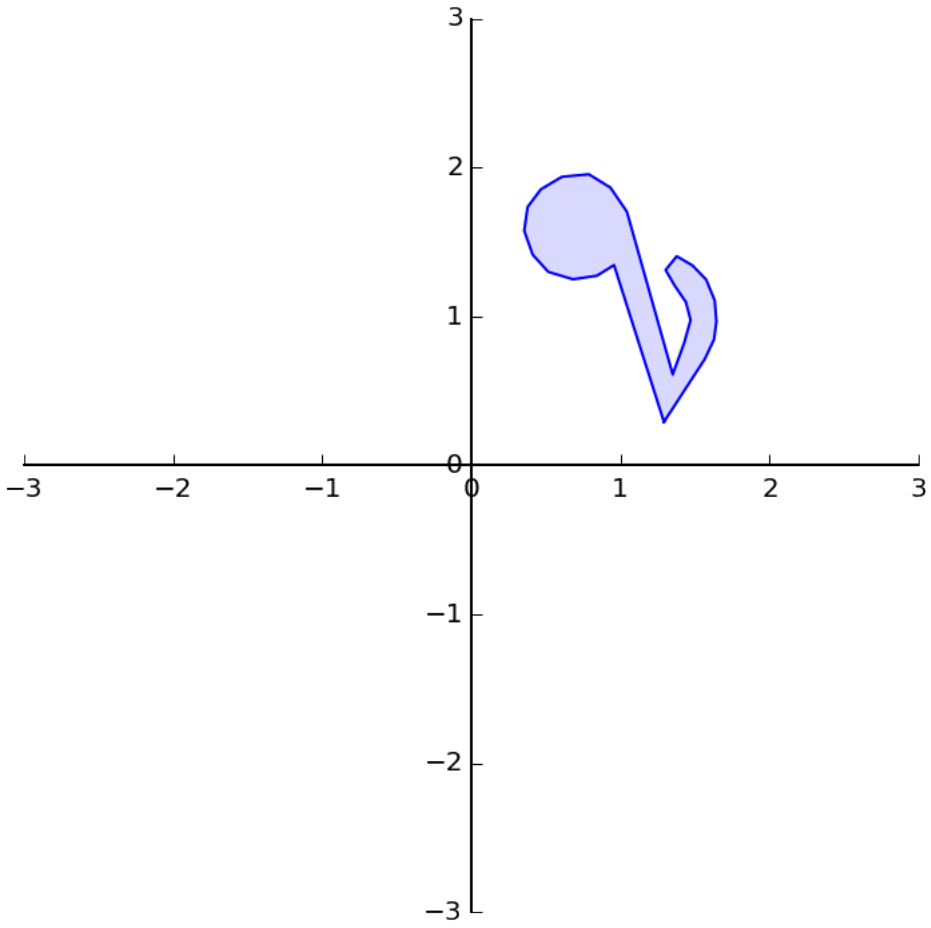
To rotate `note` we need to multiply each column of `note` by the rotation matrix A.

In `numpy`, every matrix has a `.dot()` method. This method multiplies the matrix by every column of its argument, returing the results as a matrix. That is:

`A.dot(B)`

where `A` and `B` are matrices, will multiply `A` by every column of `B`, and the resulting vectors will be formed into a matrix.

```
In [38]: dm.plotSetup()
         angle = 359.0
         phi = (angle/180) * np.pi
         A = np.array(
             [[np.cos(phi), -np.sin(phi)],
              [np.sin(phi), np.cos(phi)]])
         rnote = A.dot(note)
         dm.plotShape(rnote)
```



1.1 Question Time! Q8.1

1.2 Geometric Linear Transformations of $\mathbb{R}^2$

Let's use our understanding of how to constuct linear transformations to look at some specific linear transformations of $\mathbb{R}^2$ to $\mathbb{R}^2$.

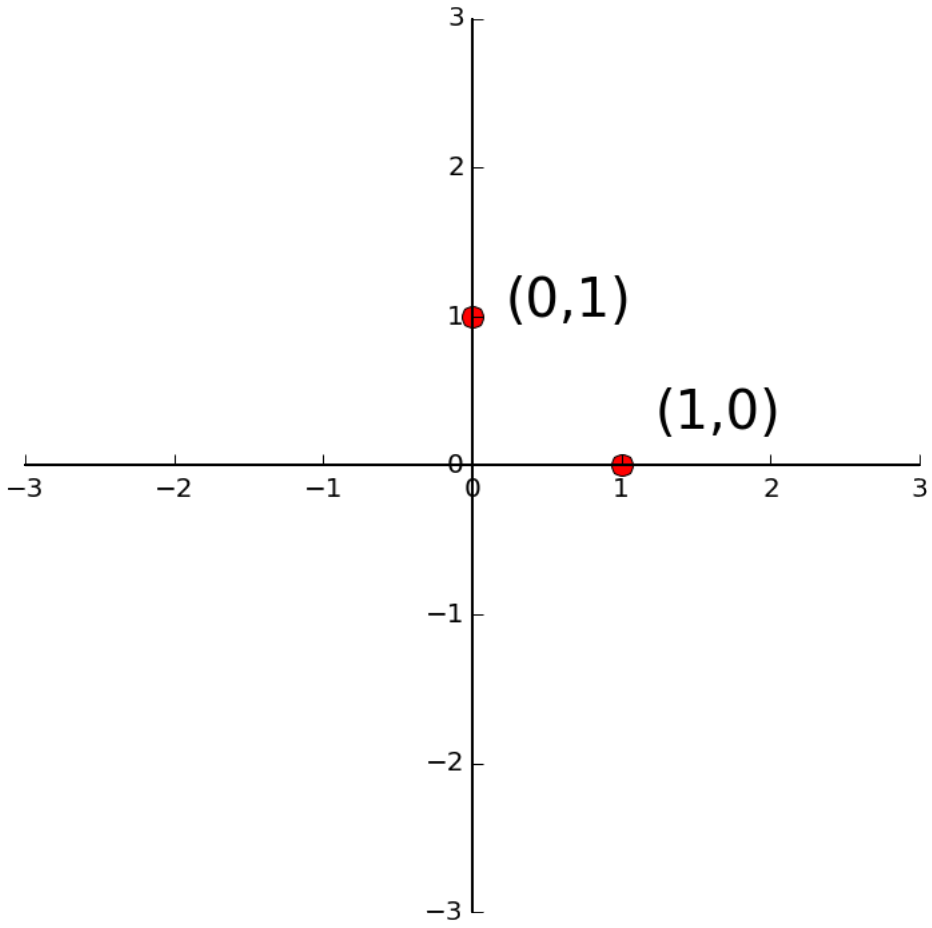
First, let's recall the linear transformation

$$T(\mathbf{x}) = r\mathbf{x}.$$

With $r > 1$, this is a dilation. It moves every vector further from the origin.

Let's say the dilation is by a factor of 2.5.

To construct the matrix A that implements this transformation, we ask: where do $\mathbf{e_1}$ and $\mathbf{e_2}$ go?

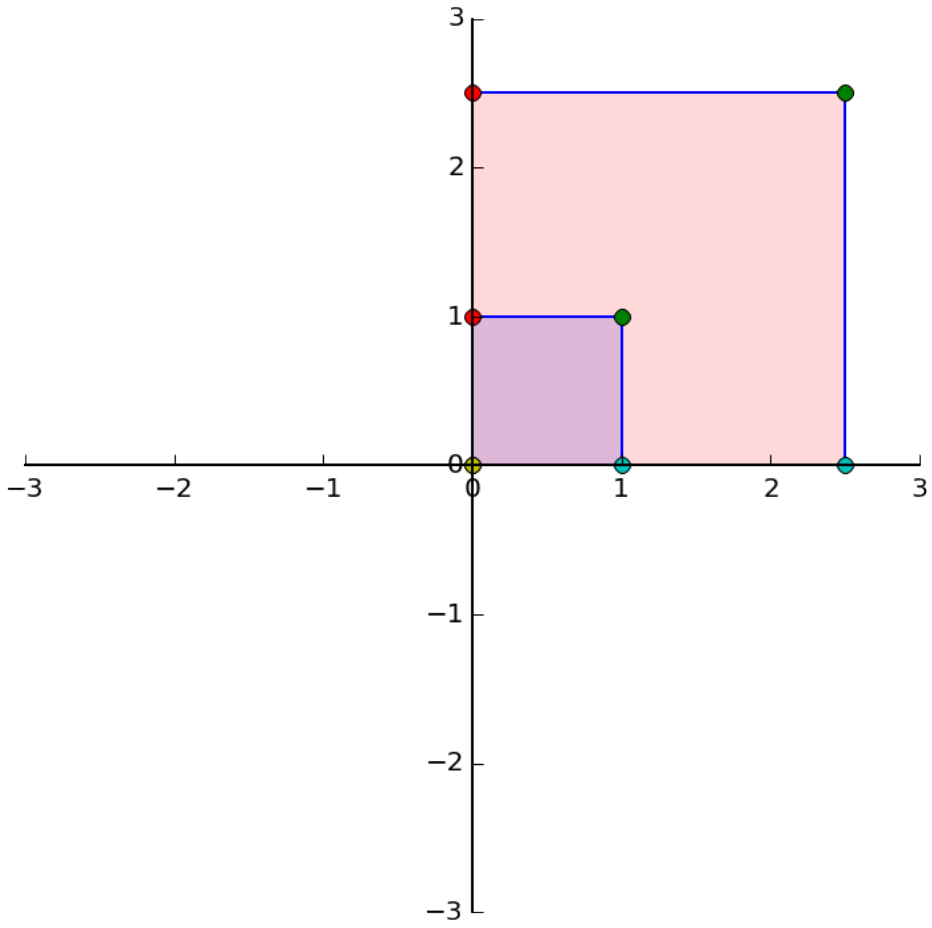


Under the action of A, $\mathbf{e}_1$ goes to $\begin{bmatrix} 2.5 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2$ goes to $\begin{bmatrix} 0 \\ 2.5 \end{bmatrix}$.

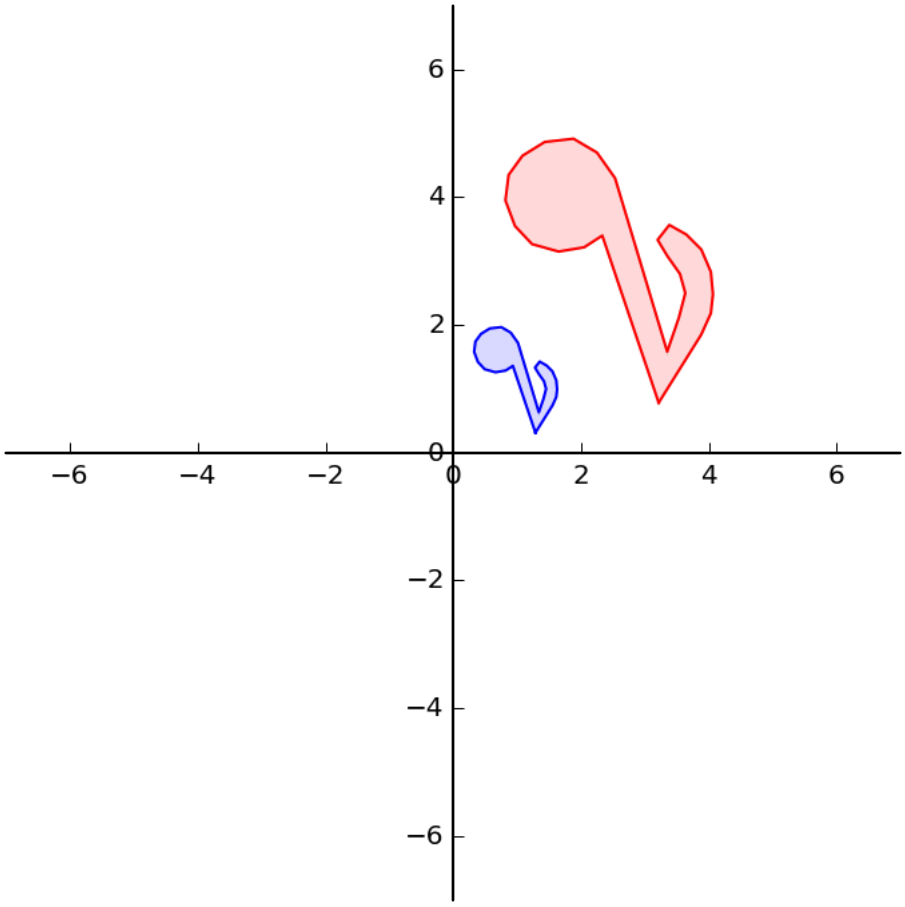
So the matrix A must be $\begin{bmatrix} 2.5 & 0 \\ 0 & 2.5 \end{bmatrix}$.

```
In [9]: square = np.array(
            [[0,1,1,0],
             [1,1,0,0]])
        A = np.array(
            [[2.5, 0],
             [0, 2.5]])
        print('A = \n',A)
        dm.plotSetup()
        dm.plotSquare(square)
        dm.plotSquare(A.dot(square),'r')
```

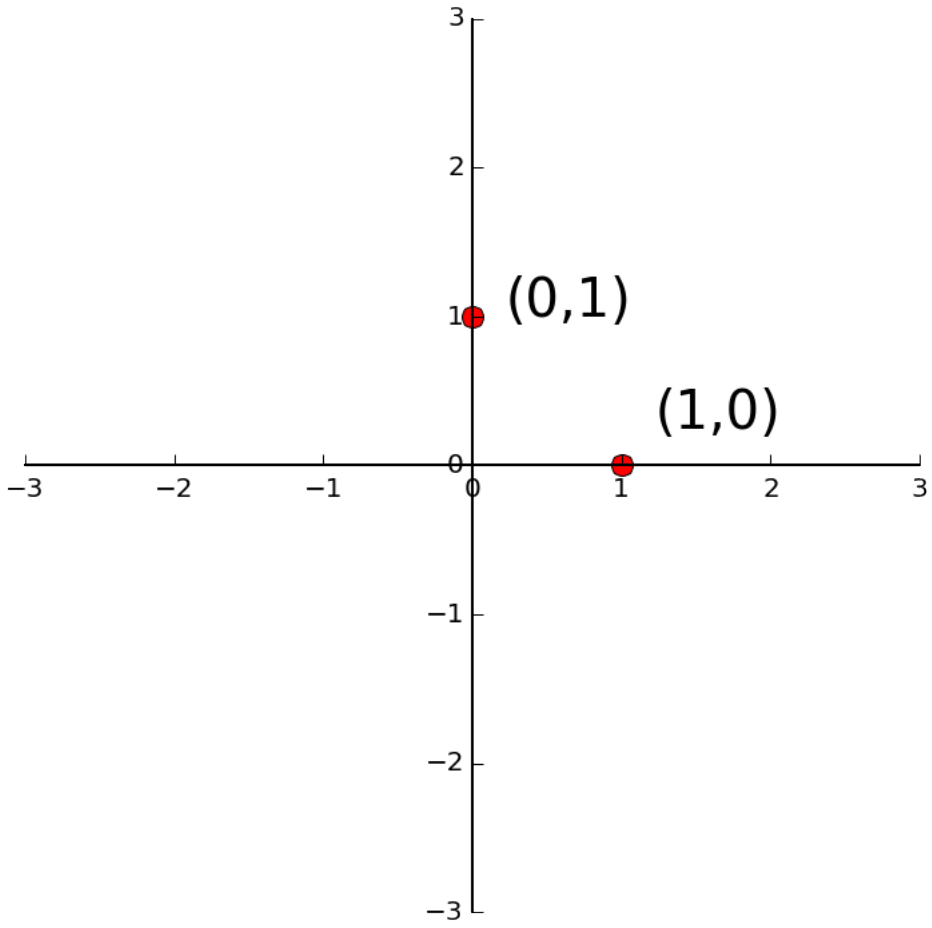
```
A =
 [[ 2.5  0. ]
 [ 0.   2.5]]
```




```
In [10]: dm.plotSetup(-7,7,-7, 7)
         dm.plotShape(note)
         dm.plotShape(A.dot(note),'r')
```



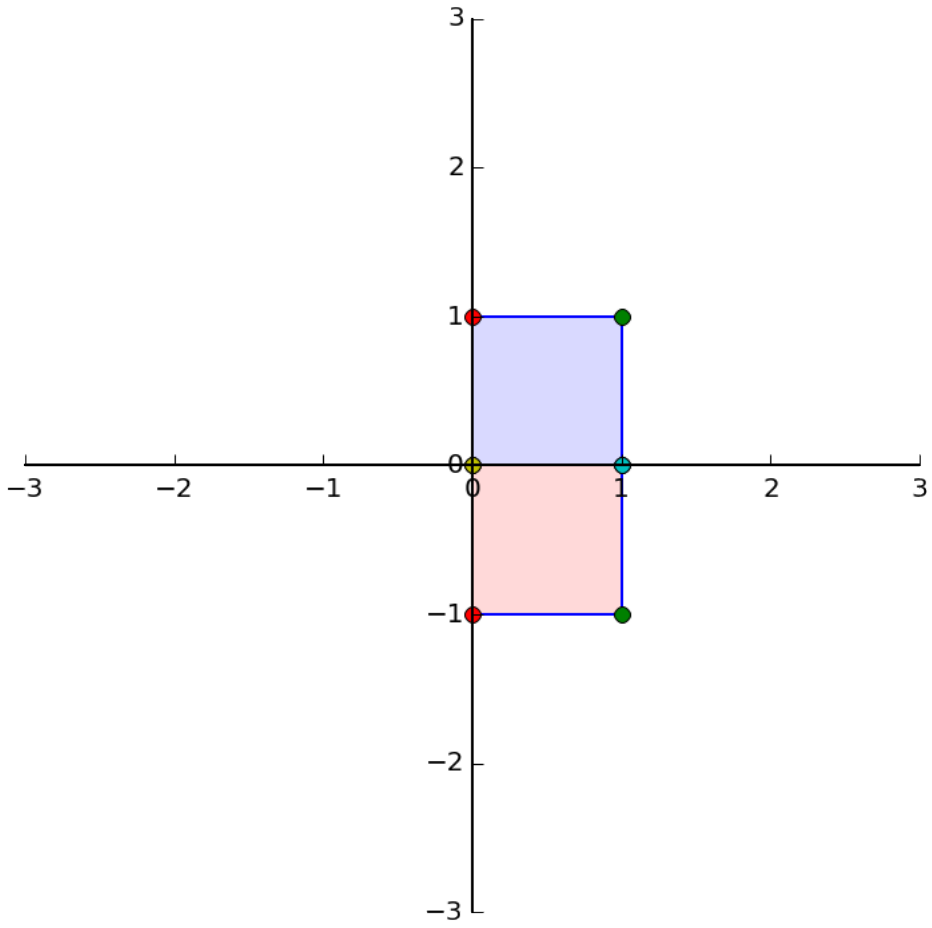
OK, now let's reflect through the x_1 axis. Where do $\mathbf{e_1}$ and $\mathbf{e_2}$ go?



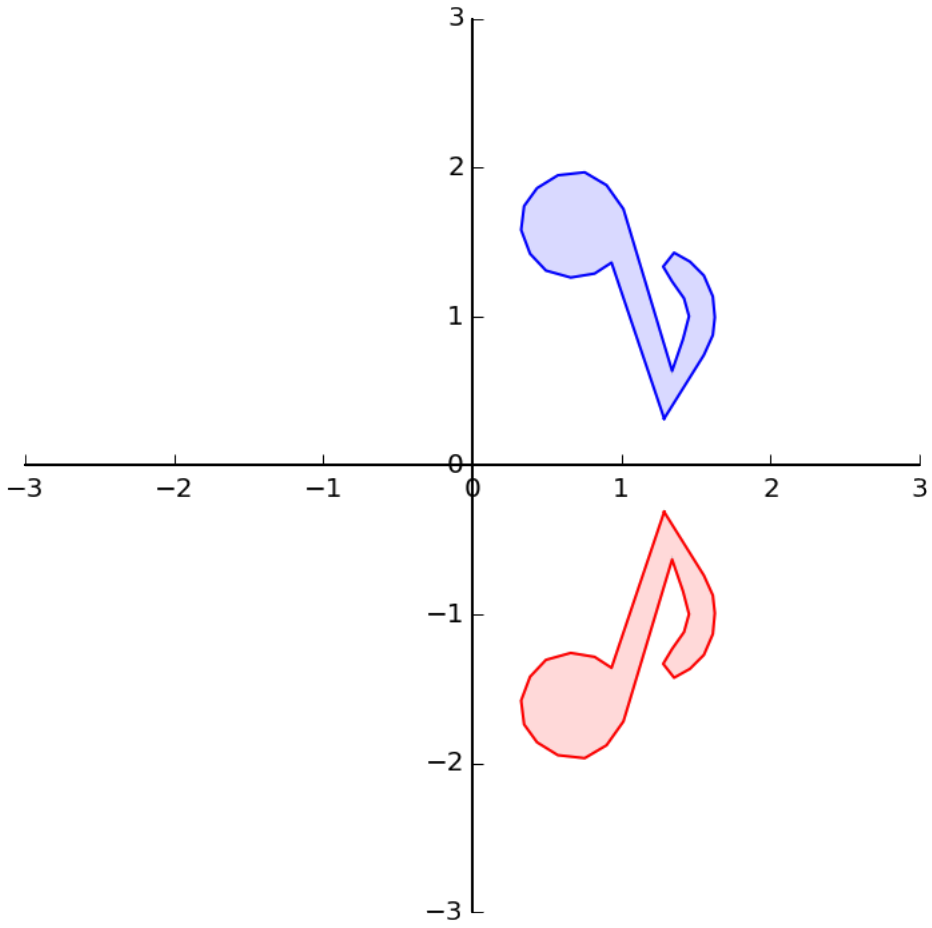

```
In [12]: A = np.array(
             [[1, 0],
              [0,-1]])
         dm.plotSetup()
         dm.plotSquare(square)
         dm.plotSquare(A.dot(square),'r')
         Latex(r'Reflection through the $x_1$ axis')
```

Out[12]:

Reflection through the x_1 axis




```
In [13]: dm.plotSetup()
         dm.plotShape(note)
         dm.plotShape(A.dot(note),'r')
```

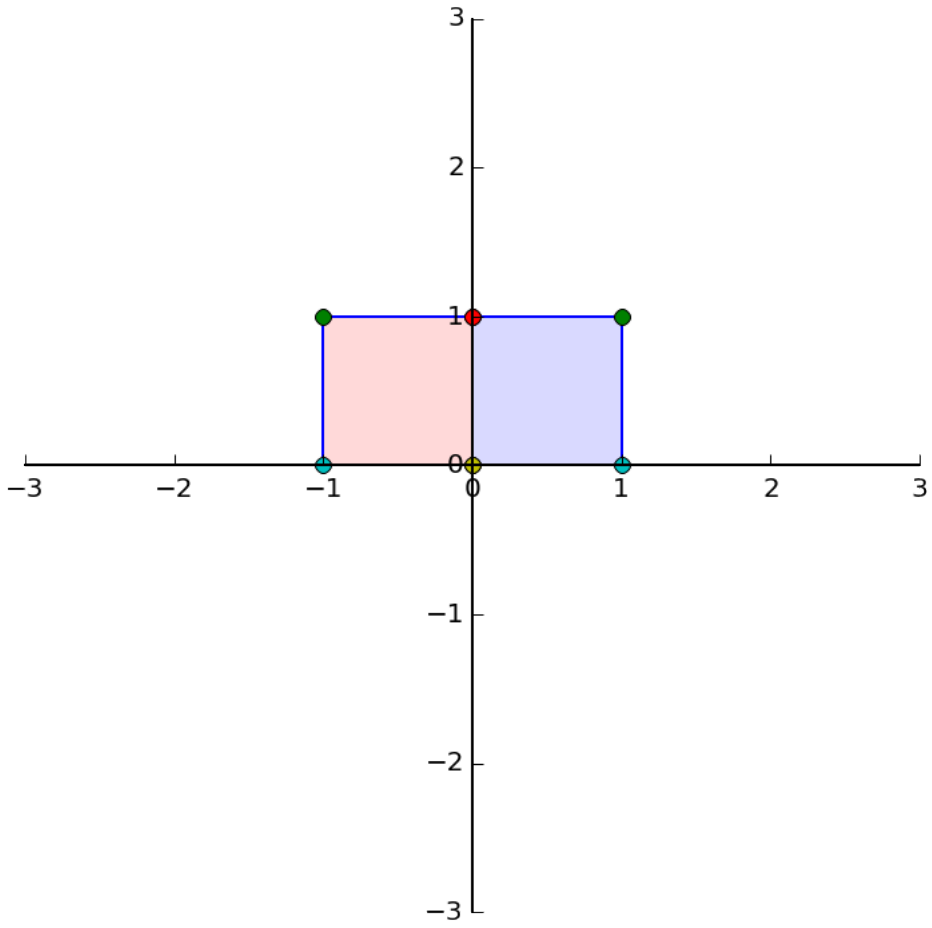


What about reflection through the x_2 axis?

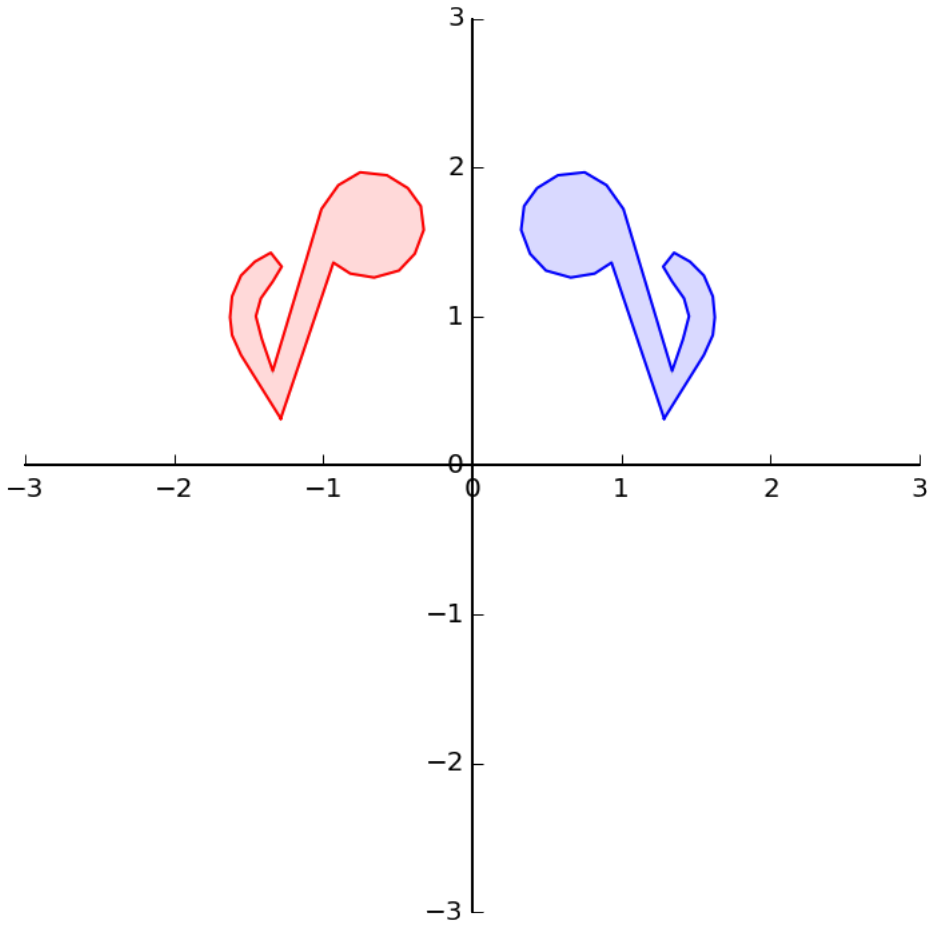
```
In [14]: A = np.array(
             [[-1,0],
              [0, 1]])
         dm.plotSetup()
         dm.plotSquare(square)
         dm.plotSquare(A.dot(square),'r')
         Latex(r'Reflection through the $x_2$ axis')
```

Out[14]:

Reflection through the x_2 axis




```
In [15]: dm.plotSetup()
         dm.plotShape(note)
         dm.plotShape(A.dot(note),'r')
```

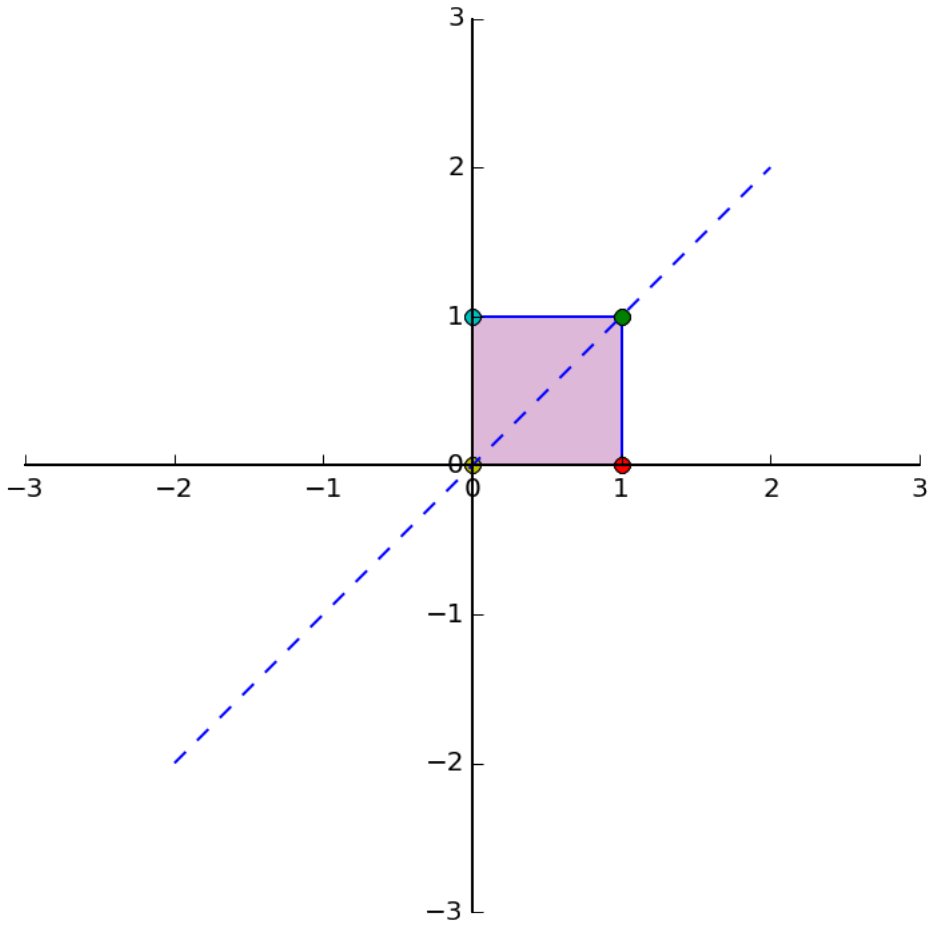


What about reflection through the line $x_1 = x_2$?

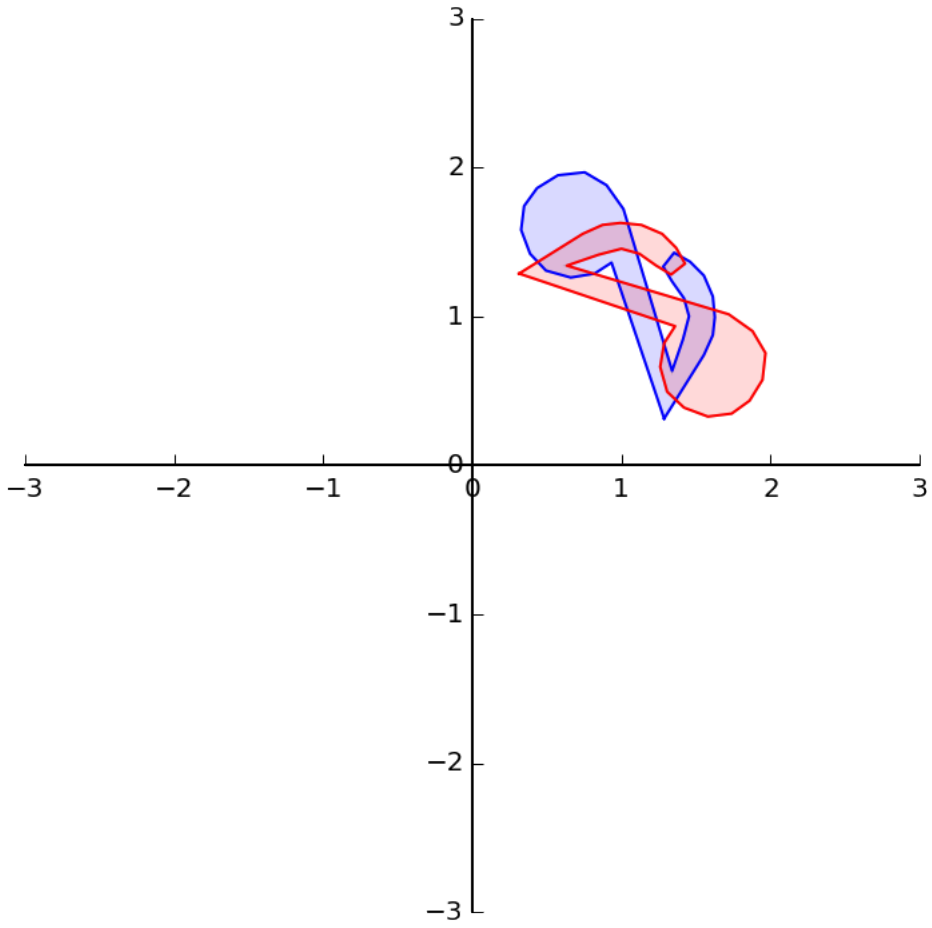
In [16]:
```
A = np.array(
    [[0,1],
     [1,0]])
dm.plotSetup()
dm.plotSquare(square)
dm.plotSquare(A.dot(square),'r')
plt.plot([-2,2],[-2,2],'b--')
Latex(r'Reflection through the line $x_1 = x_2$')
```

Out[16]:

Reflection through the line $x_1 = x_2$




```
In [17]: dm.plotSetup()
         dm.plotShape(note)
         dm.plotShape(A.dot(note),'r')
```

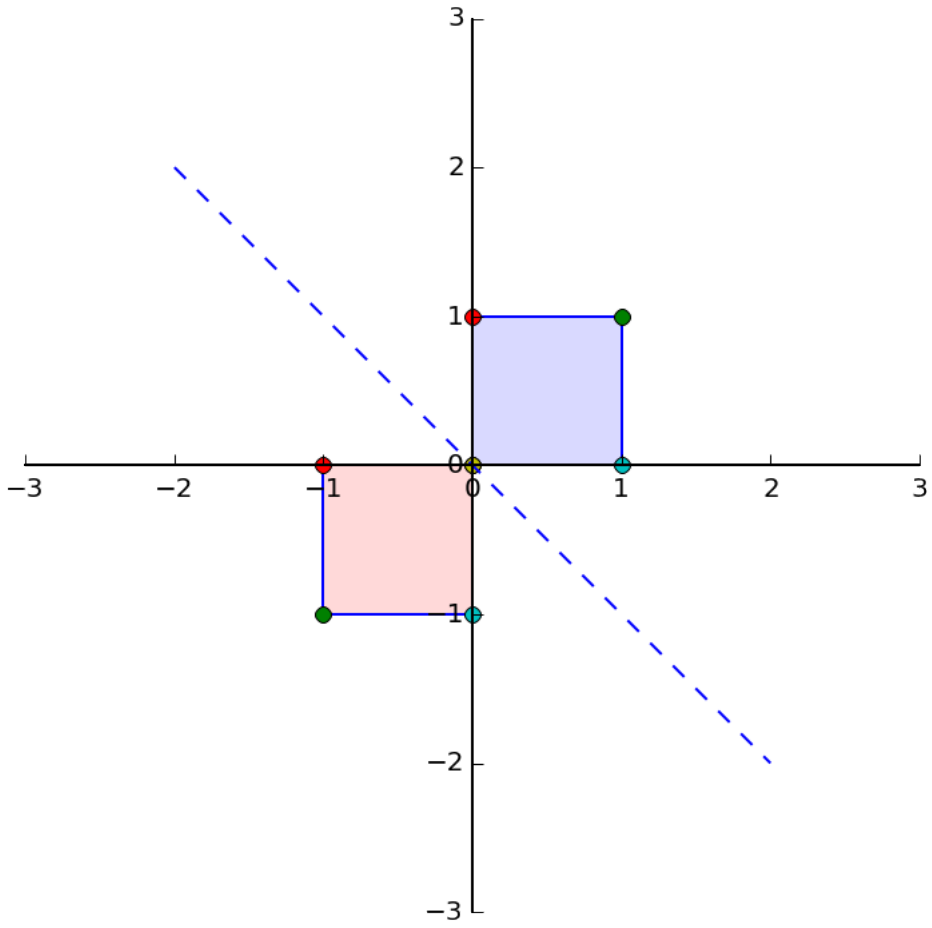


What about reflection through the line $x_1 = -x_2$?

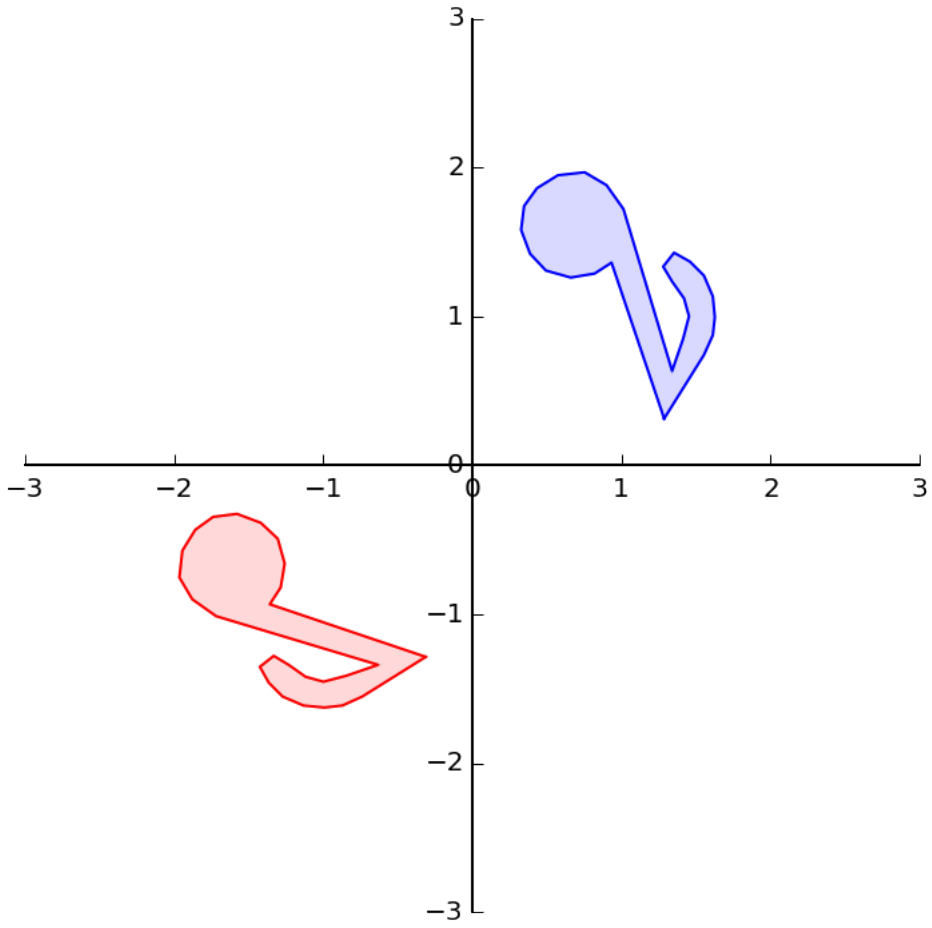
In [18]:
```
A = np.array(
    [[ 0,-1],
     [-1, 0]])
dm.plotSetup()
dm.plotSquare(square)
dm.plotSquare(A.dot(square),'r')
plt.plot([-2,2],[2,-2],'b--')
Latex(r'Reflection through the line $x_1 = -x_2$')
```

Out[18]:

Reflection through the line $x_1 = -x_2$




```
In [19]: dm.plotSetup()
         dm.plotShape(note)
         dm.plotShape(A.dot(note),'r')
```

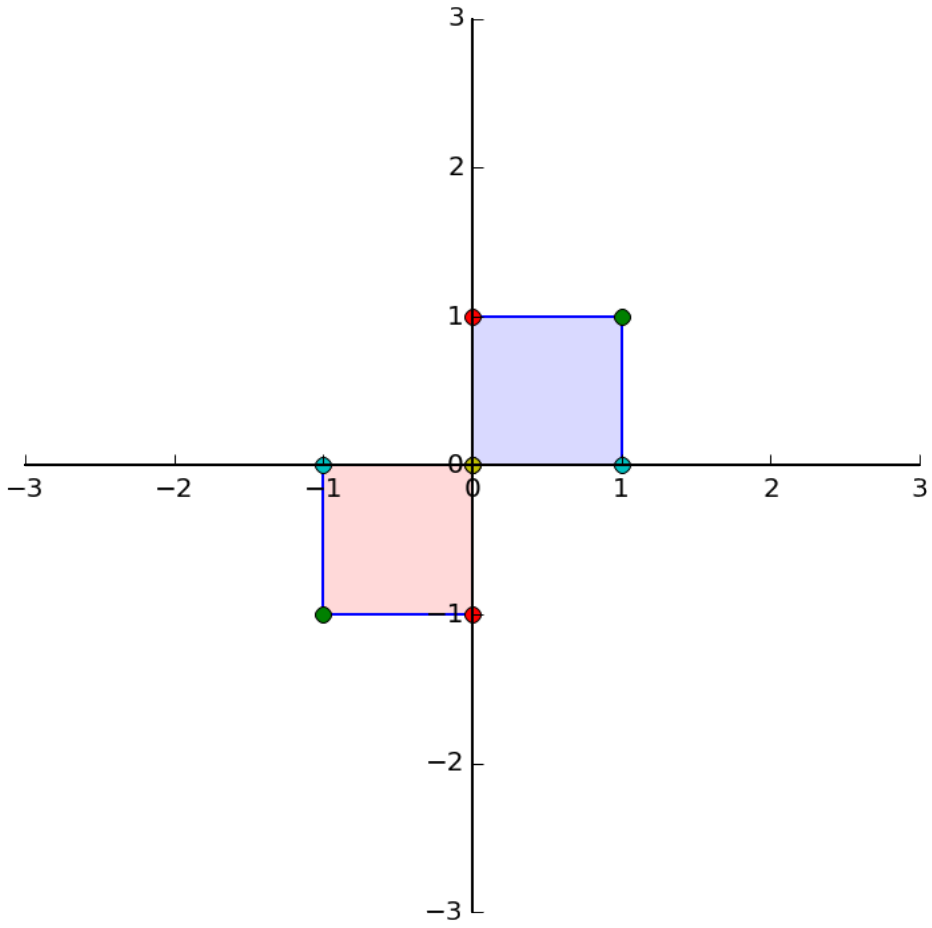


What about reflection through the origin?

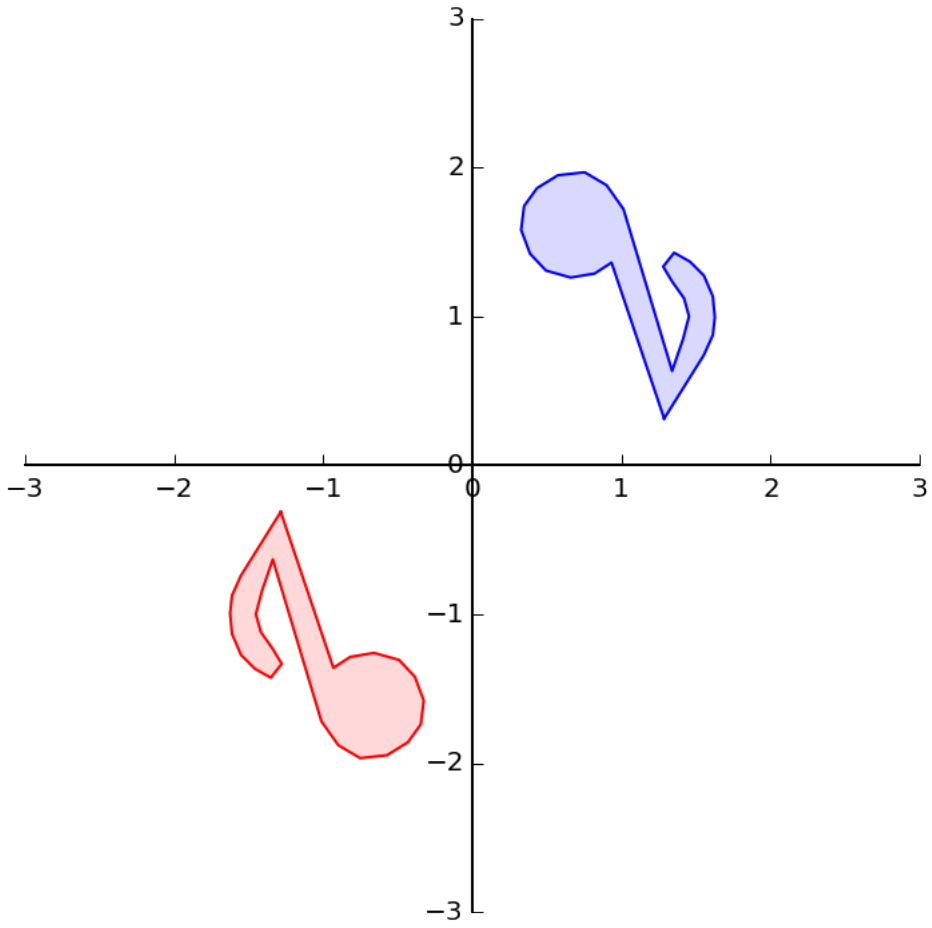
```
In [20]: A = np.array(
             [[-1, 0],
              [ 0,-1]])
         ax = dm.plotSetup()
         dm.plotSquare(square)
         dm.plotSquare(A.dot(square),'r')
         Latex(r'Reflection through the origin')
```

Out[20]:

Reflection through the origin



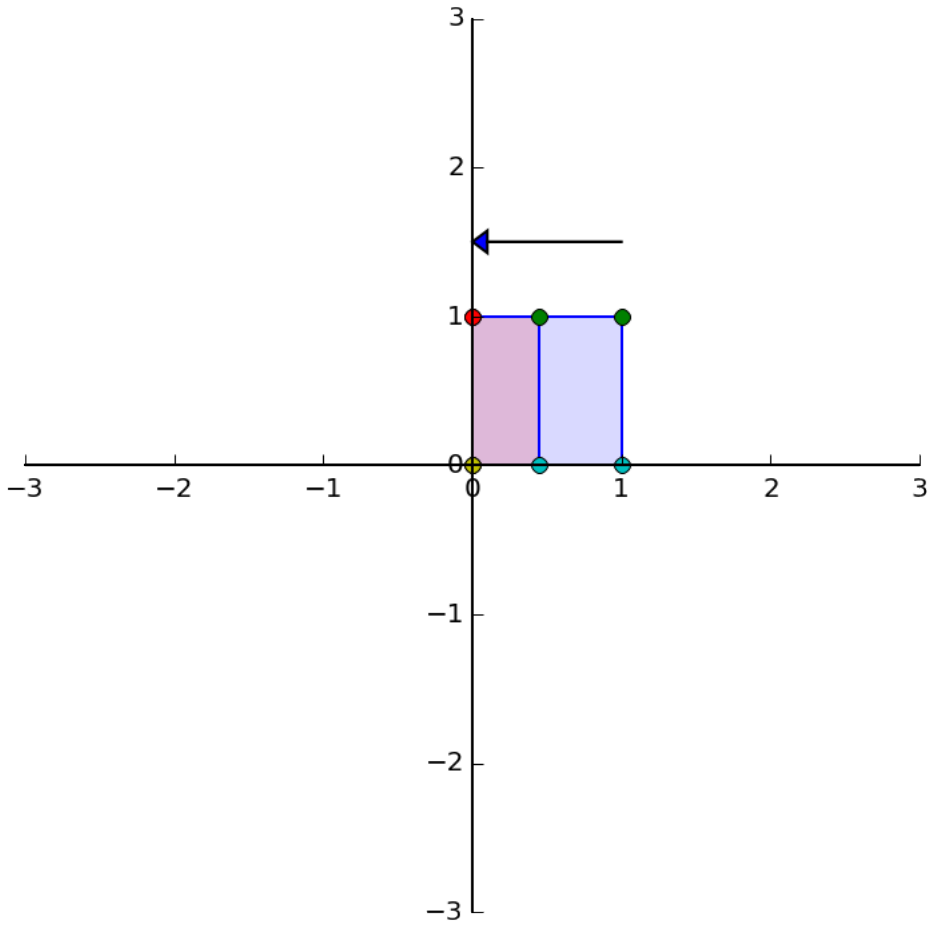

```
In [21]: dm.plotSetup()
         dm.plotShape(note)
         dm.plotShape(A.dot(note),'r')
```



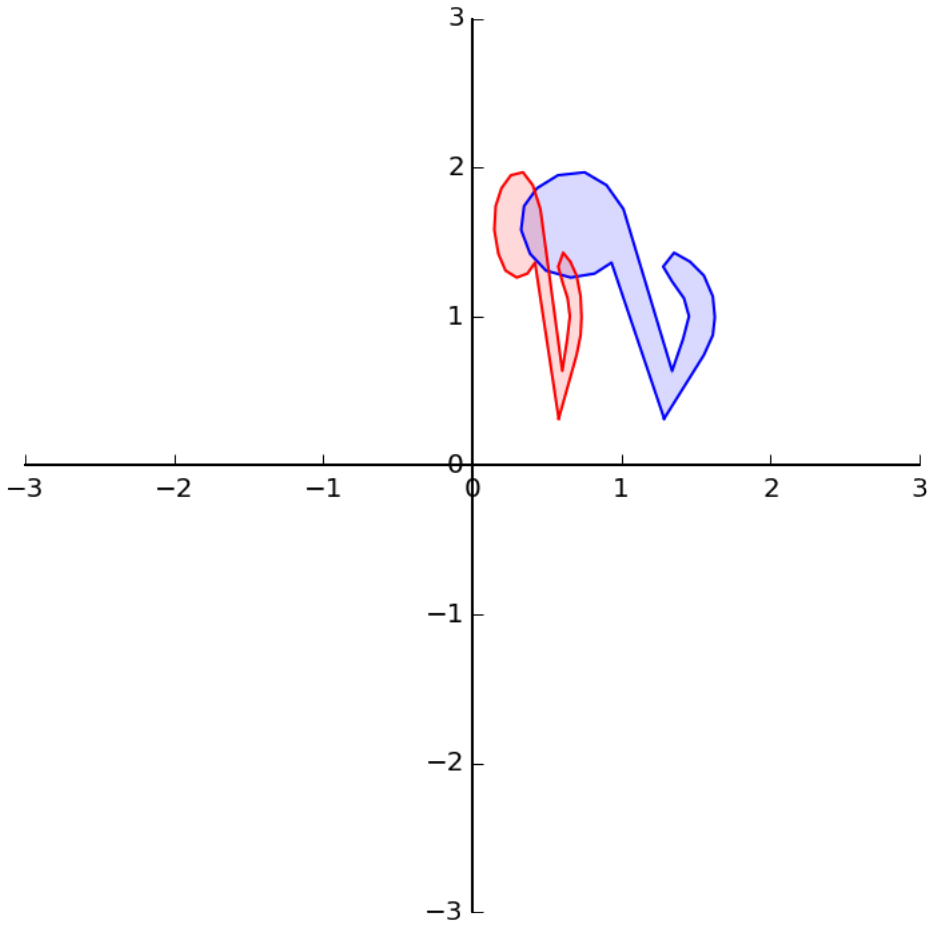

```
In [22]: A = np.array(
             [[0.45, 0],
              [0,    1]])
         ax = dm.plotSetup()
         dm.plotSquare(square)
         dm.plotSquare(A.dot(square),'r')
         ax.arrow(1.0,1.5,-1.0,0,head_width=0.15, head_length=0.1, length_includes_head=True)
         Latex(r'Horizontal Contraction')
```

Out[22]:

Horizontal Contraction



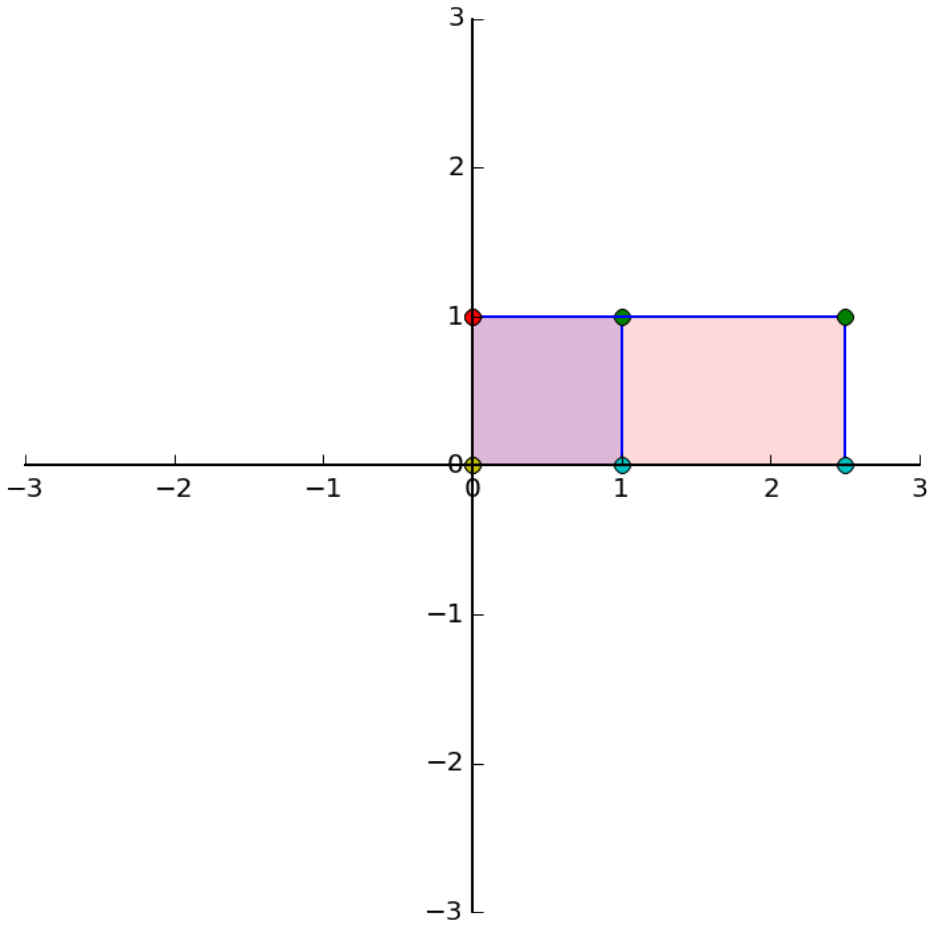

```
In [23]: dm.plotSetup()
         dm.plotShape(note)
         dm.plotShape(A.dot(note),'r')
```



In [24]:
```
A = np.array(
        [[2.5,0],
         [0,  1]])
dm.plotSetup()
dm.plotSquare(square)
dm.plotSquare(A.dot(square),'r')
Latex(r'Horizontal Expansion')
```

Out[24]:

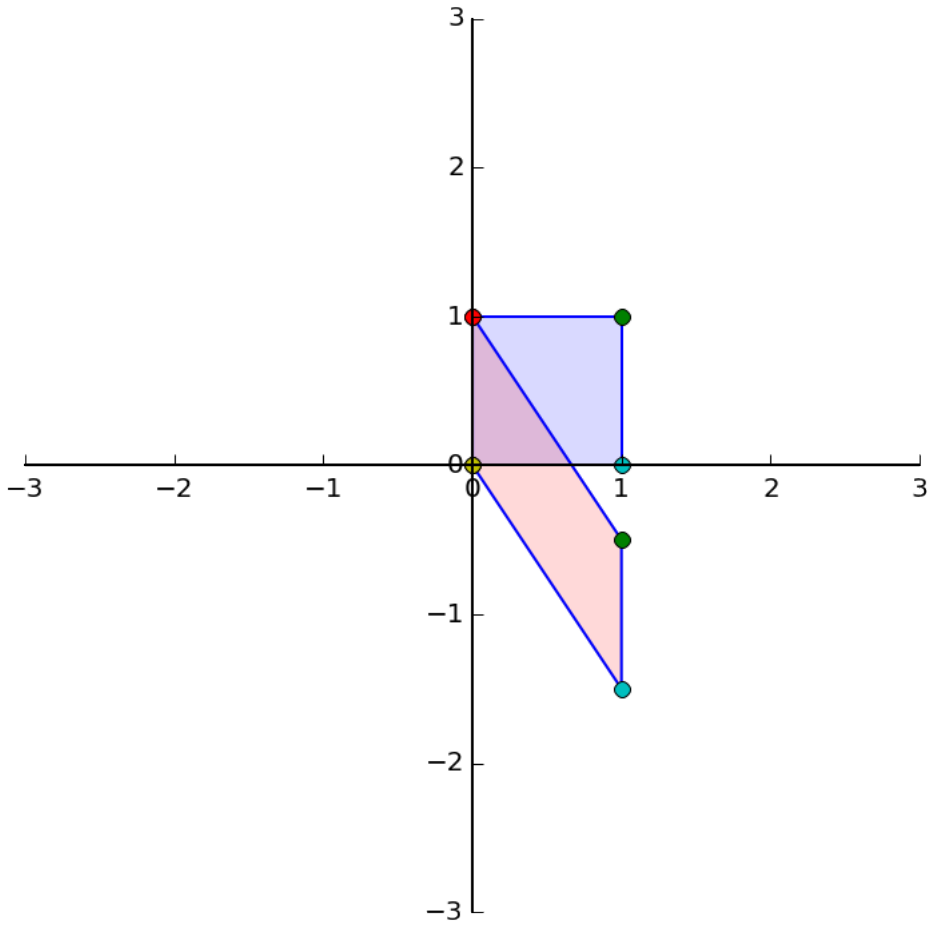
Horizontal Expansion



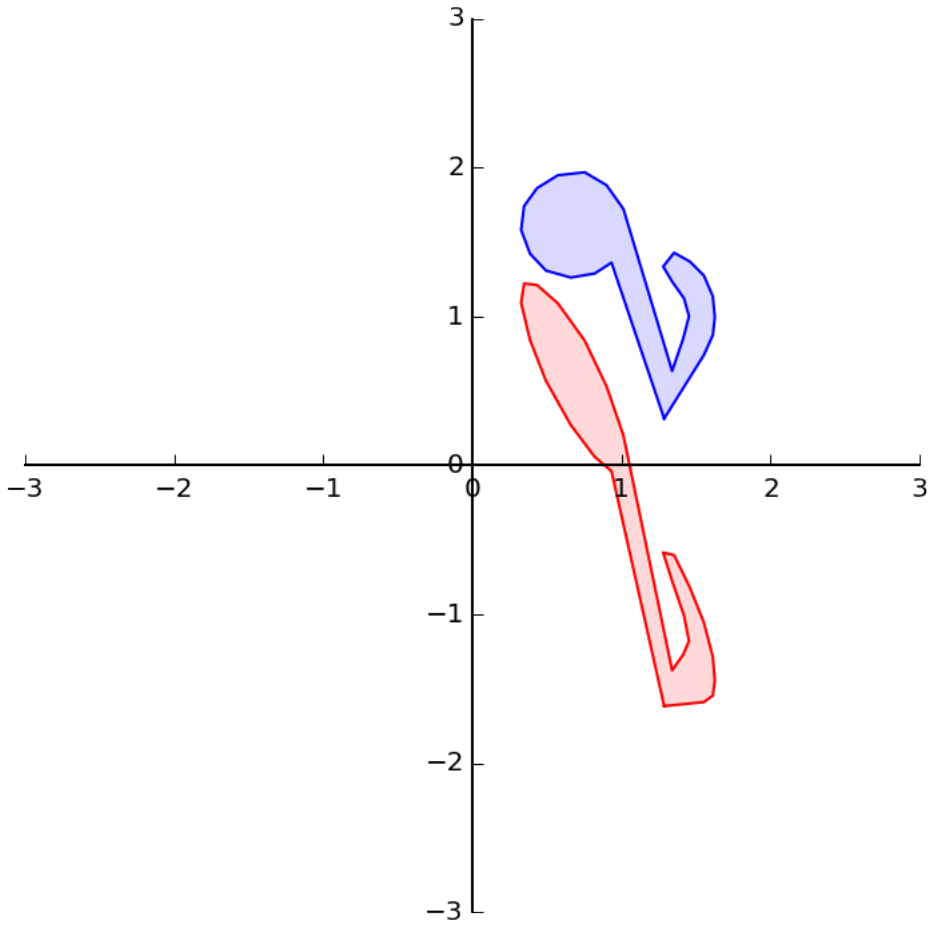
In [25]:
```
A = np.array(
        [[   1, 0],
         [-1.5, 1]])
dm.plotSetup()
dm.plotSquare(square)
dm.plotSquare(A.dot(square),'r')
Latex(r'Vertical Shear')
```

Out[25]:

Vertical Shear



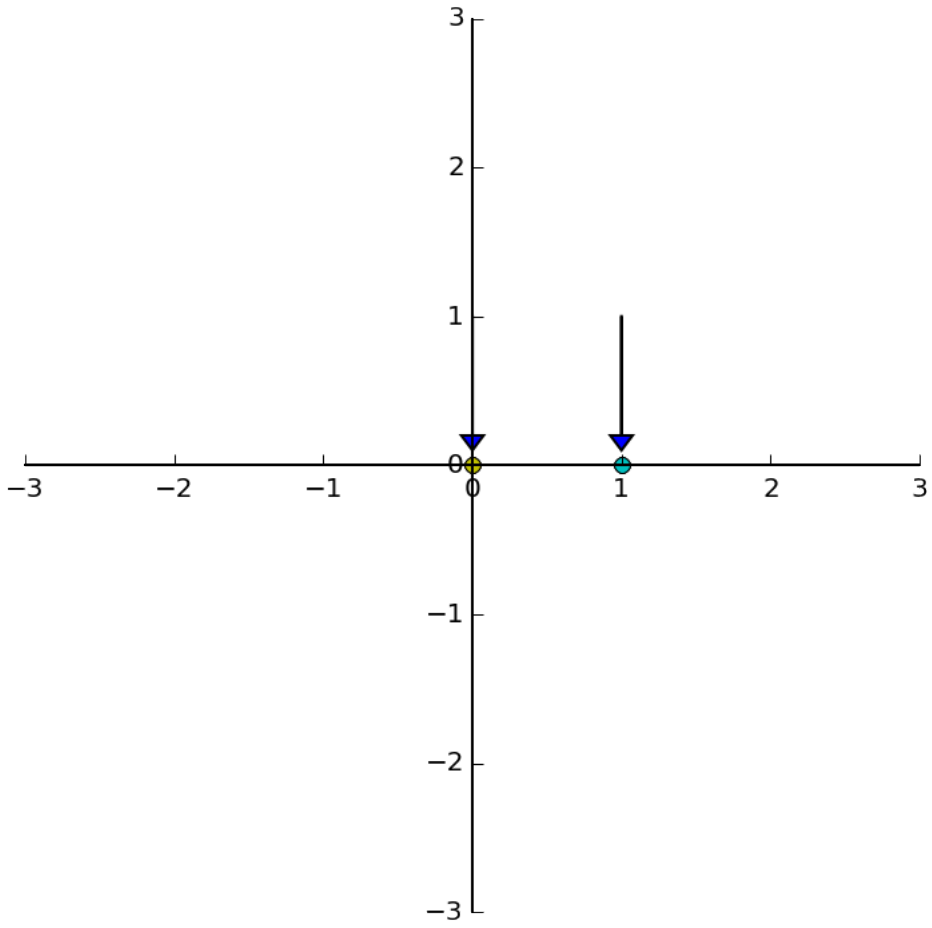
In [26]:
```
dm.plotSetup()
dm.plotShape(note)
dm.plotShape(A.dot(note),'r')
```



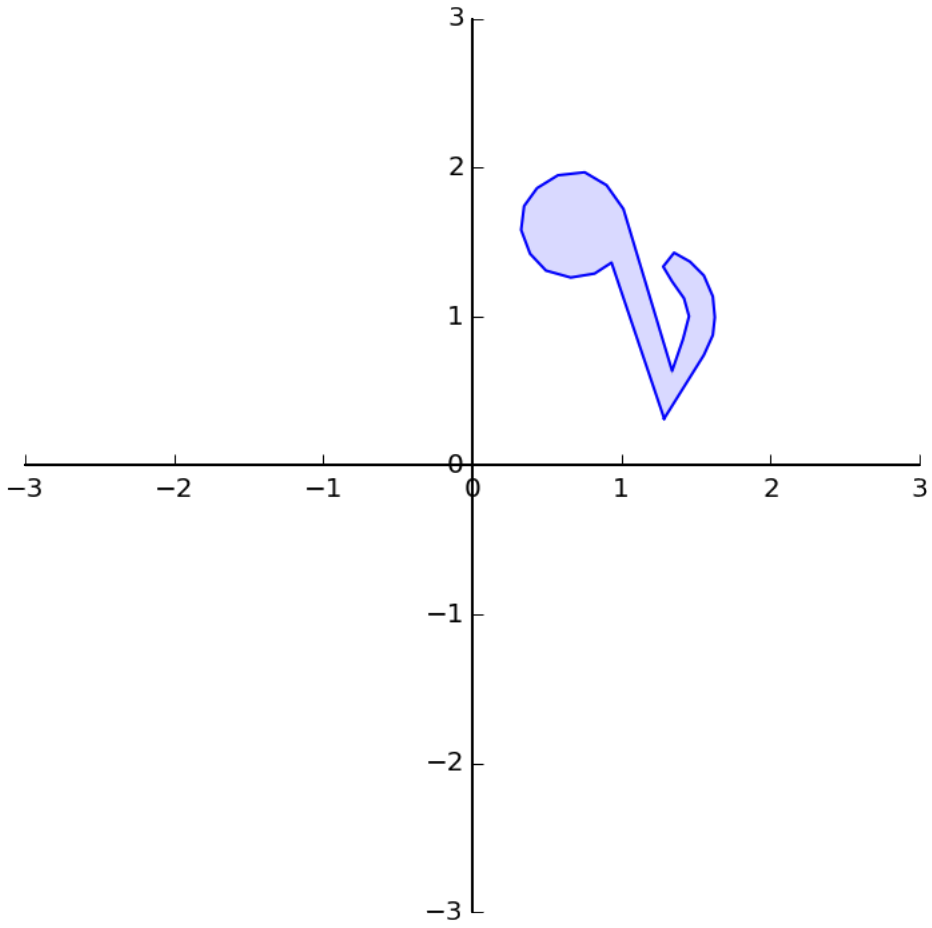

```
In [27]: A = np.array(
             [[1,0],
              [0,0]])
         ax = dm.plotSetup()
         # dm.plotSquare(square)
         dm.plotSquare(A.dot(square),'r')
         ax.arrow(1.0,1.0,0,-0.9,head_width=0.15, head_length=0.1, length_includes_head=True)
         ax.arrow(0.0,1.0,0,-0.9,head_width=0.15, head_length=0.1, length_includes_head=True)
         Latex(r'Projection onto the $x_1$ axis')
```

Out[27]:

Projection onto the x_1 axis



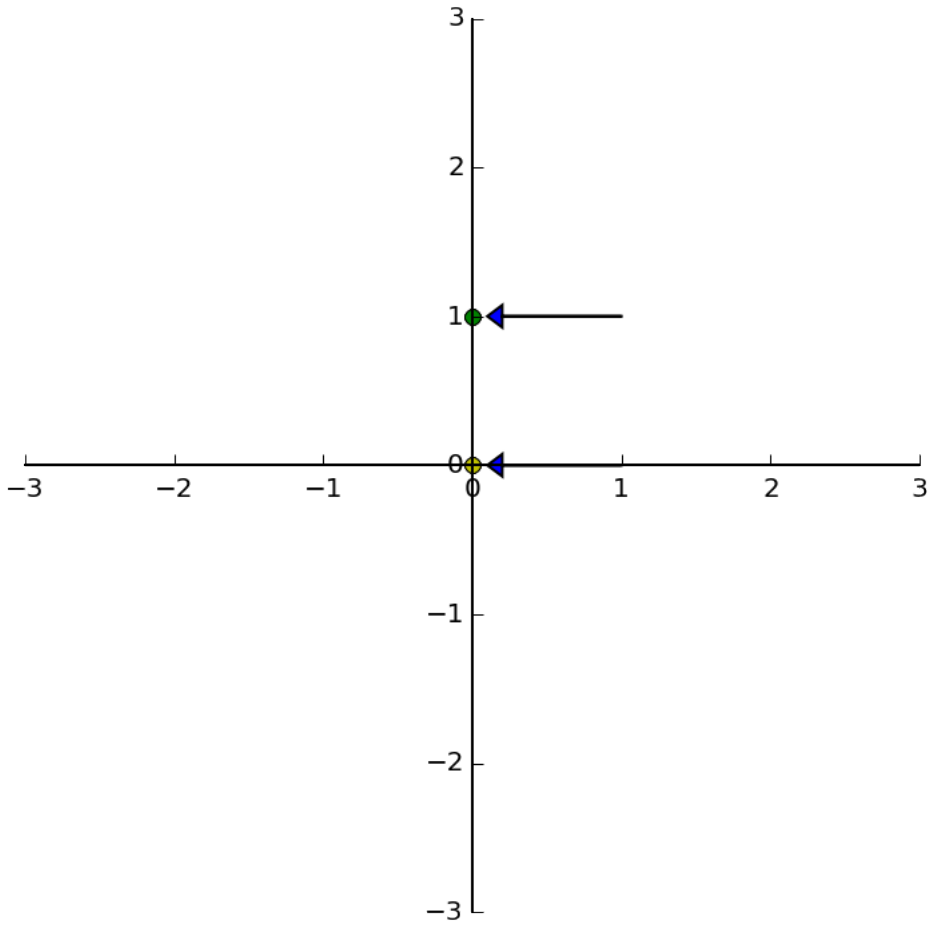

```
In [28]: dm.plotSetup()
         dm.plotShape(note)
         dm.plotShape(A.dot(note),'r')
```



In [29]:
```
A = np.array(
    [[0,0],
     [0,1]])
ax = dm.plotSetup()
# dm.plotSquare(square)
dm.plotSquare(A.dot(square))
ax.arrow(1.0,1.0,-0.9,0,head_width=0.15, head_length=0.1, length_includes_head=True)
ax.arrow(1.0,0.0,-0.9,0,head_width=0.15, head_length=0.1, length_includes_head=True)
Latex(r'Projection onto the $x_2$ axis')
```

Out[29]:

Projection onto the x_2 axis



1.3 Existence and Uniqueness

Notice that some of these transformations map multiple inputs to the same output, and some are incapable of generating certain outputs.

We need some terminology to understand these properties of linear transformations.

Definition. A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at least one* $\mathbf{x}$ in $\mathbb{R}^n$.

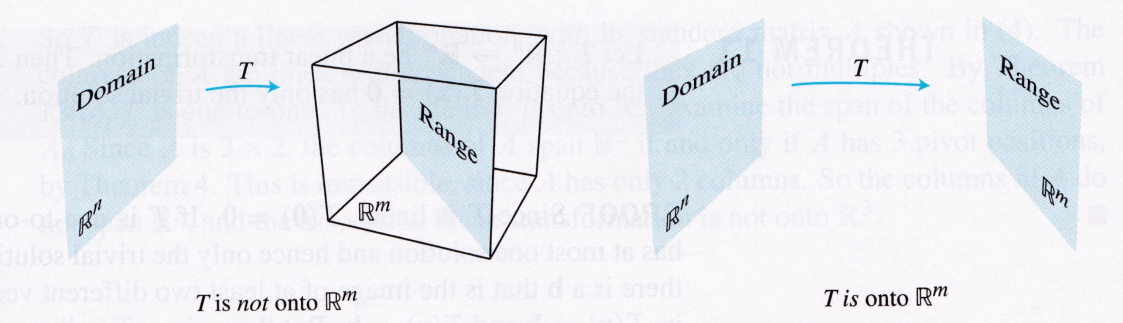
Informally, T is onto if every element of its codomain is in its range.

Another (important) way of thinking about this is that T is onto if there is a solution $\mathbf{x}$ of

$$T(\mathbf{x}) = \mathbf{b}$$

for all possible $\mathbf{b}$.

This is asking an **existence** question about a solution of the equation $T(\mathbf{x}) = \mathbf{b}$ for all $\mathbf{b}$.

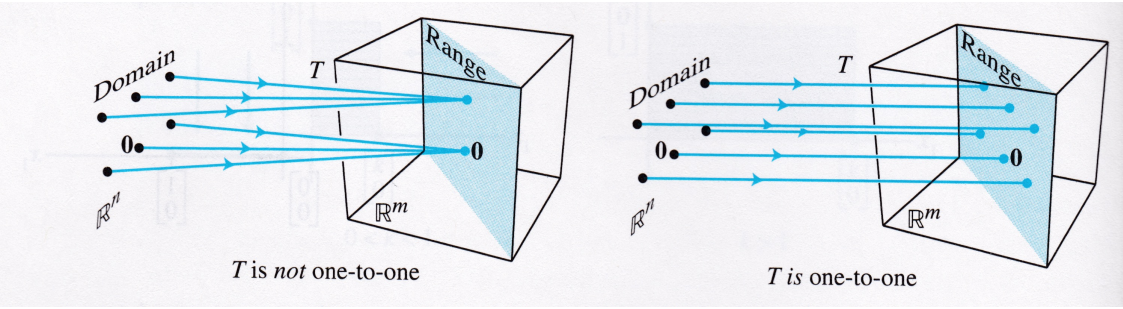


1.4 Question Time! Q8.X

Definition. A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at most one* $\mathbf{x}$ in $\mathbb{R}^n$.

If T is one-to-one, then for each $\mathbf{b}$, the equation $T(\mathbf{x}) = \mathbf{b}$ has either a unique solution, or none at all.

This is asking an **existence** question about a solution of the equation $T(\mathbf{x}) = \mathbf{b}$ for all $\mathbf{b}$.



Let's examine the relationship between these ideas and some previous definitions.

If $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$, is $T(\mathbf{x}) = A\mathbf{x}$ onto? one-to-one?

$T(\mathbf{x})$ is onto. $T(\mathbf{x})$ may or may not be one-to-one. If the system has multiple solutions for some $\mathbf{b}$, $T(\mathbf{x})$ is not one-to-one.

If $A\mathbf{x} = \mathbf{b}$ is consistent and has a unique solution for all $\mathbf{b}$, is $T(\mathbf{x}) = A\mathbf{x}$ onto? one-to-one?

Yes to both.

If $A\mathbf{x} = \mathbf{b}$ is not consistent for all $\mathbf{b}$, is $T(\mathbf{x}) = A\mathbf{x}$ onto? one-to-one?

$T(\mathbf{x})$ is **not** onto. $T(\mathbf{x})$ may or may not be one-to-one.

If $T(\mathbf{x}) = A\mathbf{x}$ is onto, is $A\mathbf{x} = \mathbf{b}$ consistent for all $\mathbf{b}$? is the solution unique for all $\mathbf{b}$?

If $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one, is $A\mathbf{x} = \mathbf{b}$ consistent for all $\mathbf{b}$? is the solution unique for all $\mathbf{b}$?