MATH 3044, Numerical Analysis II
Spring 2019

Final Exam

The exam is out of 100 points. Show all relevant work to receive full credit. Partial credit is awarded for progress towards the solution. Submit your written solutions and MATLAB code by **Thursday, April 25** at **5:00 PM** (either emailed as a pdf or slid under my office door as a hard copy).

You are encouraged to use class notes, the course text, and any code that **you** have previously written. All other resources are prohibited, including, but not limited to, the internet and other people (regardless of their enrollment status in MATH 3044), except during the regularly-scheduled course time on April 25 in which you may consult with others in the class.

1. [20 points] A 6-cm by 5-cm rectangular silver plate has heat being uniformly generated at each point at the rate $q = 1.5\ \mathrm{cal/cm}^3 \cdot \mathrm{sec}$. Let x represent the distance along the edge of the plate of length 6 cm and y be the distance along the edge of the plate of length 5 cm. Suppose the temperature u along the edge is kept at the temperatures

$$\begin{aligned} u(x,0) &= x(6-x),\ u(x,5) = 0,\ \ 0 \le x \le 6, \\ u(0,y) &= y(5-y),\ u(6,y) = 0,\ \ 0 \le y \le 5, \end{aligned}$$

where the origin lies at a corner of the plate with coordinates $(0,0)$ and the edges lie along the positive x- and y-axes. The steady-state temperature $u = u(x,y)$ satisfies Poisson's equation

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = -\frac{q}{K},\ \ 0 < x < 6,\ 0 < y < 5,$$

where K, the thermal conductivity, is $1.04\ \ \mathrm{cal/cm} \cdot \mathrm{deg} \cdot \mathrm{sec}$. Approximate the temperature $u(x,y)$ with $\Delta x = 0.4$ and $\Delta y = \frac{1}{3}$ (use a direct method to solve the linear system that results from the discretized equation).

2. [20 points] Approximate the solution of

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0,\ \ 0 < x < 1,\ t > 0, \\ &u(0,t) = u(1,t) = 0,\ \ t > 0, \\ &u(x,0) = \begin{cases} 1, & 0 \le x \le \frac{1}{2}, \\ -1, & \frac{1}{2} < x \le 1, \end{cases} \\ &\frac{\partial u}{\partial t}(x,0) = 0,\ \ 0 \le x \le 1 \end{aligned}$$

with $\Delta x = \Delta t = 0.01$ and $t_{\max} = 3$, and produce a surface plot of the result.

3. [20 points] Approximate the solution of

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1,\ t > 0, \\
&u(x,0) = 1, \quad 0 \le x \le 1, \\
&u(0,t) = e^{-5t},\ u(1,t) = \left|\cos 2t\right| e^{-0.5t}, \quad t > 0,
\end{aligned}
$$

using the Crank-Nicolson method with $\Delta x = 0.05, \Delta t = 0.00125$, and $t_{\mathrm{max}} = 6$, and produce a surface plot of the result.

4. [20 points] Approximate the solution of Poisson's equation

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -e^{2.5(x^2+y^2)}, \quad -1 < x < 1,\ -1 < y < 1, \\
&u(x,-1) = u(x,1) = 0, \quad -1 \le x \le 1, \\
&u(-1,y) = u(1,y) = 0, \quad -1 \le y \le 1,
\end{aligned}
$$

using the nine-point nearest neighbor method (recall that this also requires a five-point method for the approximation of the right-hand side function $f(x,y)$) with $\Delta x = \Delta y = 0.1$, and produce a surface plot of the result.

5. [20 points] Approximate the solution of

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad 0 < x < 10,\ t > 0, \\
&u(x,0) = \begin{cases} \sin 2\pi x, & 0 \le x \le 3, \\ 0, & \text{otherwise}, \end{cases} \\
&u(0,t) = 0, \quad t > 0,
\end{aligned}
$$

using the Lax-Wendroff method for $a = 1, \Delta x = 0.005, \Delta t = 0.005$, and $t_{\mathrm{max}} = 5$, and plot the result as a function of x for $t = 0$ and $t = t_{\mathrm{max}}$ on the same coordinate axes.