MATH 3044, Numerical Analysis II
Spring 2019

Midterm Exam

The exam is out of 100 points. Show all relevant work to receive full credit. Partial credit is awarded for progress towards the solution. Submit your written solutions and MATLAB code by **Thursday, February 28** at **5:00 PM** (either emailed as a pdf or slid under my office door as a hard copy).

You are encouraged to use class notes, the course text, and any code that **you** have previously written. All other resources are prohibited, including, but not limited to, the internet and other people (regardless of their enrollment status in MATH 3044), except during the regularly-scheduled course time on February 28 in which you may consult with others in the class.

1. [15 points] (a) Use Taylor's method of order four to approximate the solution of the initial-value problem

$$\begin{aligned} y'(t) &= e^{t-y},\ 0 \le t \le 1 \\ y(0) &= 1 \end{aligned}$$

with $N = 10$ subintervals.

(b) Repeat part (a) for $N = 20, 40, 80, 160$, and discuss the behavior of the maximum of the global error on the domain as N increases [the exact solution is $y(t) = \ln(e^t + e - 1)$].

2. [20 points] Comment on the consistency, order, zero stability, and convergence of the multistep method

$$y_{i+1} = \alpha y_i + (1-\alpha)y_{i-1} + 2hf_i + \frac{h\alpha}{2}(f_{i-1} - 3f_i),\ i = 1, \dots, N-1$$

where $\alpha \in \mathbb{R}$.

3. [20 points] Implement a fourth-order AB, AM predictor-corrector method with tolerance 10^{-6} and step size $h \in [0.02, 0.5]$ to approximate the solution of the initial-value problem

$$\begin{aligned} y'(t) &= (t + 2t^3)y^3 - ty,\ 0 \le t \le 2 \\ y(0) &= \frac{1}{3}, \end{aligned}$$

and compare the results to the exact solution

$$y(t) = \left(3 + 2t^2 + 6e^{t^2}\right)^{-1/2}.$$

Use $h = 0.5$ for the initial step size; when the step size is updated, use RK4 to produce necessary starting values.

4. [20 points] Determine the local truncation error for the (one-step) theta method

$$y_{i+1} = y_i + h\Big(\theta f(t_i, y_i) + (1-\theta) f(t_{i+1}, y_{i+1})\Big),\ i = 0, 1, \dots, N-1$$

where $\theta \in [0, 1]$.

5. [25 points] Backward Differentiation Formulas (BDFs) are a family of implicit multistep methods of the form

$$y_{i+1} = \sum_{j=0}^{p} a_{j+1} y_{i-j} + h b_0 f_{i+1} \, (b_0 \neq 0), \; i = p, p+1, \ldots, N-1$$

generated by approximating $y'(t_{i+1})$ using $P'_{p+1}(t_{i+1})$, where $P_{p+1}(t)$ is the polynomial of degree $p+1$ ($p \geq 0$) that interpolates y at the $p+2$ nodes $t_{i-p}, t_{i-p+1}, \ldots, t_i, t_{i+1}$.

(a) Construct the $p = 1$ BDF method

$$y_{i+1} = \frac{4}{3} y_i - \frac{1}{3} y_{i-1} + \frac{2}{3} h f_{i+1}, \; i = 1, 2, \ldots, N-1.$$

Hint: Use the following steps:

- Find $P_2(t)$, the quadratic interpolating polynomial that interpolates (t_{i-1}, y_{i-1}), (t_i, y_i), (t_{i+1}, y_{i+1}). You may assume that t_{i-1}, t_i, t_{i+1} are equally spaced with step size h.
- Compute $P'_2(t_{i+1})$.
- Assume $P'_2(t_{i+1}) \approx y'(t_{i+1}) = f(t_{i+1}, y_{i+1}) = f_{i+1}$ and solve for y_{i+1}.

(b) Discuss the consistency, order, zero stability, and convergence of this method.