Unit 4: Homework

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1 Comparing Estimators

Say that $X_1, ..., X_n$ is an i.i.d. sample from an exponetial distribution, with common density,

$$f_X(x) = \lambda e^{-\lambda x^2}$$

This distribution has mean $\frac{1}{\lambda}$ and variance $\frac{1}{\lambda^2}$. You are considering the following estimators for the mean:

1. $\hat{\theta}_1 = \frac{X_1+...+X_n}{n}$
2. $\hat{\theta}_2 = \frac{X_1+X_2}{2}$
3. $\hat{\theta}_3 = \frac{X_1+...+X_n}{n+1}$

- Compute the Bias of each estimator.
  1. $E[\hat{\theta}_1] - E[X] = E[\frac{1}{n}\sum_1^n X_i] - \frac{1}{\lambda} = \frac{1}{n}\sum_1^n E[X_i] - \frac{1}{\lambda} = \frac{1}{n}\frac{n}{\lambda} - \frac{1}{\lambda} = 0$
  2. $E[\hat{\theta}_2] - E[X] = E[\frac{1}{2}\sum_1^2 X_i] - \frac{1}{\lambda} = \frac{1}{2}\sum_1^2 E[X_i] - \frac{1}{\lambda} = \frac{1}{2}\frac{2}{\lambda} - \frac{1}{\lambda} = 0$
  3. $E[\hat{\theta}_3] - E[X] = E[\frac{1}{n+1}\sum_1^n X_i] - \frac{1}{\lambda} = \frac{1}{n+1}\sum_1^n E[X_i] - \frac{1}{\lambda} = \frac{1}{n+1}\frac{n}{\lambda} - \frac{1}{\lambda} = (\frac{n}{n+1} - 1)\frac{1}{\lambda}$
- Compute the sampling variance of each estimator
  1. $V[\hat{\theta}_1] = V[\frac{1}{n}\sum_1^n X_i] = \frac{1}{n^2}\sum_1^n V[X_i] = \frac{1}{n^2}\frac{n}{\lambda^2} = \frac{1}{n\lambda^2}$
  2. $V[\hat{\theta}_2] = V[\frac{1}{2}\sum_1^2 X_i] = \frac{1}{2^2}\sum_1^2 V[X_i] = \frac{1}{4}\frac{2}{\lambda^2} = \frac{1}{2\lambda^2}$
  3. $V[\hat{\theta}_3] = V[\frac{1}{n+1}\sum_1^n X_i] = \frac{1}{(n+1)^2}\sum_1^n V[X_i] = \frac{1}{(n+1)^2}\frac{n}{\lambda^2} = \frac{n}{(n+1)^2}\frac{1}{\lambda^2}$
- Compute the MSE of each estimator.
  1. $MSE_1 = V[\hat{\theta}_1] + Bias[\hat{\theta}_1, \theta]^2 = V[\hat{\theta}_1] = \frac{1}{n\lambda^2}$
  2. $MSE_2 = V[\hat{\theta}_2] + Bias[\hat{\theta}_2, \theta]^2 = V[\hat{\theta}_2] = \frac{1}{2\lambda^2}$
  3. $MSE_3 = V[\hat{\theta}_3] + Bias[\hat{\theta}_3, \theta]^2 = \frac{n}{(n+1)^2}\frac{1}{\lambda^2} + (\frac{n}{n+1} - 1)^2\frac{1}{\lambda^2}$
- Explain, in your own words, why the estimator 3 has the highest bias and the lowest MSE.
  Estimator 3 has the highest bias but the lowest MSE because of the bias-variance tradeoff. In this case, increasing the bias of the estimator resulted in a significantly smaller variance. This resulted in a net decrease of the MSE, which is the sum of these two quantities.

2 What does this mean mean?

Given an iid sample, $[X_1, ..., X_n]$, the geometric mean is defined as,

$$G_n = (X_1 X_2 ... X_n)^{\frac{1}{n}}$$

Another way of writing this is

$$G_n = \prod_{i=1}^{n} X_i^{\frac{1}{n}}$$

In this exercise, let's show that the geometric mean is not a consistent estimator for $E[X]$. For an easy counter example, assume that X has a uniform distribution on [0,1].

- Think of a common function that changes sums into products. Rewrite G_n in this form:

$$G_n = f\left(\frac{Y_1 + Y_2 + ... + Y_n}{n}\right) = f\left(\frac{1}{n}\sum_{1}^{n} Y_i\right)$$

  Here, each Y_i should be a function of X_i.

  Define $Y_i = ln(X_i)$.

$$ln(X_1 X_2 ... X_n)^{\frac{1}{n}} = \frac{1}{n} ln(X_1 X_2 ... X_n)$$

$$= \frac{1}{n}\sum_{1}^{n} ln(X_i)$$

$$= \frac{1}{n}\sum_{1}^{n} Y_i$$

  therefore,

$$G_n(n) = e^{\frac{1}{n}\sum_{1}^{n} Y_i}$$

- Apply the WLLN to compute the probability limit $lim_{n->\infty}\frac{1}{n}\sum_1^n Y_i$.

  The following conditions are satisfied to use the WLLN.

  - $Y_1, ..., Y_n$ are i.i.d because $X_1, ..., X_n$ are i.i.d and $Y = ln(X)$.
  - Finite variance can reasonably be assumed

  Therefore,

$$lim_{n->\infty}\frac{1}{n}\sum_1^n Y_i = E[Y]$$

  The pdf of Y is unknown, but the pdf of X is known. $E[Y]$ can be calculated using the Law of the Unthinking Statistician.

$$E[Y] = \int_{-\infty}^{\infty} y f_y dy = \int_{-\infty}^{\infty} ln(x) f_x dx$$

  Applying the uniform distribution over the support of X [0,1].

$$\int_0^1 ln(x)dx = -1$$

- Apply the Continuous Mapping Theorem to find the probability limit $\text{plim}_{n\to\infty} G_n$. Is it the same as $E[X]$?

  The continuous mapping theorem states that if $X_n \to^p c$, then $f(X_n) \to^p f(c)$

  Therefore,

$$plim_{n->\infty}G_n = plim_{n->\infty}f(\bar{Y}) = plim_{n->\infty}e^{\bar{Y}} = e^{-1}$$

  This is not the same as $E[X]$

3 In the World of Competitive Coin Flipping

You have a fair coin and flip it 100 times.

- Apply the central limit theorem and the R command pnorm to estimate the probability of getting between 54 and 60 heads.
- Write a function that simulates 100 fair coin flips and returns the number of heads. Run this simulation a bunch of times (e.g. 10,000) and compute the fraction of results between 54 and 60 (inclusive of the end points)
- Do your answers for part a) and b) match perfectly? Explain why there might be some difference.

See Attached RMD File