Unit 3: Homework

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1 Best Game in the Casino

You flip a fair coin 3 times, and get a different amount of money depending on how many heads you get.

- For 0 heads, you get \$0.
- For 1 head, you get \$2.
- For 2 heads, you get \$4.
- Your expected winnings from the game are \$6.

1. How much do you get paid if the coin comes up heads 3 times?

$$E(x) = \sum_i x_i f(x_i)$$

For a random coin flipped three times, the values of x and $f(x)$ for all i are

Num. of Heads	Prob.(x)	Payout (f(x))
0	$\frac{1}{8}$	0
1	$\frac{3}{8}$	2
2	$\frac{3}{8}$	4
3	$\frac{1}{8}$	?

Applying the definition of expected value ot solve

$$6 = \frac{1}{8}0 + \frac{3}{8}2 + \frac{3}{8}4 + \frac{1}{8}f(x_3)$$

Solving

$$f(x_3) = 30$$

2. Write down a complete expression for the cumulative probability function for your winnings from the game.

   The table from part one can be used to express the PMF of number of heads

$$f_X(x) = \begin{cases} \frac{1}{8} & x = 0 \\ \frac{3}{8} & x = 1 \\ \frac{3}{8} & x = 2 \\ \frac{1}{8} & x = 3 \end{cases}$$

   This can then be multiplied by payouts to determine the PDF of winnings

$$f_X(x) = \begin{cases} \frac{1}{8} & x = 0 \\ \frac{3}{8} & x = 2 \\ \frac{3}{8} & x = 4 \\ \frac{1}{8} & x = 30 \end{cases}$$

   Summing payouts, the CDF is then determined to be

$$f_X(x) = \begin{cases} \frac{1}{8} & -\infty < x < 2 \\ \frac{1}{2} & 2 \leq x < 4 \\ \frac{7}{8} & 4 \leq x < 30 \\ 1 & 30 \leq x < \infty \end{cases}$$

2 Reciprocal Dice

Let X be a random variable representing the outcome of rolling a 6-sided die. Before the die is rolled, you are given two options. Either:

- You get $1/E(X)$ in dollars right away; or,
- You wait until the die is rolled, then get $1/X$ in dollars.

1. Which option is better for you, in expectation?

   For the first option, the expectation is

   $$E(X) = \frac{1}{6}(1 + 2 + 3 + 4 + 5) = 3.5$$

   Which would result in an expected payout of \$0.28.

   For the second option, the expectation is

   $$E(X) = \frac{1}{6}(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}) = 0.40833$$

   Which would result in a payout of \$0.41.

   The second option is better in expectation.

2. Knowing who you are, and your tolerance for risk, which of the two options would you take, and why?

   I am willing to take on more risk in a game of chance when I am not wagering any of my own money. As a result, I would choose the second option knowing that the expected payout was higher, despite the increased risk of receiving a low payout.

3 The Warranty is Worth It

Suppose the life span of a particular (shoddy) server is a continuous random variable, T, with a uniform probability distribution between 0 and 1 year. The server comes with a contract that guarantees you money if the server lasts less than 1 year. In particular, if the server lasts t years, the manufacturer will pay you $g(t) = \$100(1-t)^{1/2}$. Let $X = g(T)$ be the random variable representing the payout from the contract.

1. Compute the expected payout from the contract, $E(X) = E(g(T))$.

$$E[g(x)] = \int_{-\infty}^{\infty} g(t)f(t)dt$$

where

$$g(t) = 100(1-t)^{1/2}$$

$$f(t) = \begin{cases} 0 & -\infty < x < 0 \\ 1 & 0 \leq x < 1 \\ 0 & 1 \leq t < \infty \end{cases}$$

therefore

$$E[g(t)] = \int_{-\infty}^{0} [100(1-t)^{\frac{1}{2}}](0)dt + \int_{0}^{1} [100(1-t)^{\frac{1}{2}}](1)dt + \int_{1}^{\infty} [100(1-t)^{\frac{1}{2}}](0)dt$$

$$= 100 \int_{0}^{1} (1-t)^{\frac{1}{2}} dt$$

$$= 100[-\frac{2}{3}(1-x)^{\frac{3}{2}}|_0^1$$

$$= \frac{200}{3}$$

4 Great Time to Watch Async

Suppose your waiting time in minutes for the Caltrain in the morning is uniformly distributed on [0, 5], whereas waiting time in the evening is uniformly distributed on [0, 10]. Each waiting time is independent of all other waiting times.

1. If you take the Caltrain each morning and each evening for 5 days in a row, what is your total expected waiting time?

   The pdf of morning wait time M and evening wait time E are expressed as

$$f_M(t) = \begin{cases} 0 & -\infty < t < 0 \\ .2 & 0 \leq t \leq 5 \\ 0 & 5 < t < \infty \end{cases}$$

$$f_E(t) = \begin{cases} 0 & -\infty < t < 0 \\ .1 & 0 \leq t \leq 10 \\ 0 & 10 < t < \infty \end{cases}$$

   The expected values of M and E, $E[M]$ and $E[E]$, are expressed as

$$E[M] = \int_{-\infty}^{0} (0)tdt + \int_{0}^{5} (0.2)tdt + \int_{5}^{\infty} (0)tdt$$

$$E[E] = \int_{-\infty}^{0} (0)tdt + \int_{0}^{10} (0.1)tdt + \int_{10}^{\infty} (0)tdt$$

   Integration steps have been omitted, as the expected value of a uniform distribution over a single, continuous support is the midpoint of the support

$$E[M] = 2.5$$
$$E[E] = 5$$

To find the expected total wait time for 5 days of morning and evening travel requires the use of two properties of the expected value operator.

- $E[X+Y] = E[X] + E[Y], \text{ if X and Y are independent}$
- $E[aX] = aE[X]$

Therefore, the expected wait time for a single day is

$$E[M] + E[E] = 2.5 + 5 = 7.5$$

And the expected total wait time for all five days is

$$5(E[M] + E[E]) = 37.5 \text{ minutes}$$

2. What is the variance of your total waiting time?

   The variance for a uniform distribution between a and b can be solved by

$$V[X] = \frac{1}{12}(b-a)^2$$

   Therefore

$$V[M] = \frac{1}{12}(5-0)^2 = \frac{25}{12}$$

$$V[E] = \frac{1}{12}(10-0)^2 = \frac{100}{12}$$

   Given that all events are independent, the variance of the total waiting time is expressed as

$$\sum_{i}^{5} V[M] + \sum_{i}^{5} V[E]$$

   Therefore, the variance for all five days is

$$5\frac{25}{12} + 5\frac{100}{12} = 52.08 \text{ minutes}$$

3. What is the expected value of the difference between the total evening waiting time and the total morning waiting time over all 5 days?

   Using properties of expected value stated earlier in the problem

$$E[5(E - M)] = 5(E[E] - E[M]) = 5(5 - 2.5) = 12.5 \text{ minutes}$$

4. What is the variance of the difference between the total evening waiting time and the total morning waiting time over all 5 days?

   Using properties of variance stated earlier in the problem

$$V[E - M] = \sum_{i}^{5}(V[E_i] + V[M_i]) = 5\frac{100}{12} + 5\frac{25}{12} = 52.08 \text{ minutes}$$

5 Maximizing Correlation

Show that if $Y = a_1 X + b_1$ where X and Y are random variables and $a \neq 0$, $corr(X, Y) = -1$ or $+1$.

Two properties of correlation are:

- $\rho[X, X] = 1$
- $\rho[aX + c, bY + d] = \rho[X, Y]$

Seeing as the second property applies to all a, b, c, and d, assume

$$a = a_1$$

$$c = b_1$$

$$b = 1$$

$$d = 0$$

Which yields

$$\rho[a_1 X + b_1, Y] = \rho[X, Y]$$

Applying the definition of Y in the problem

$$\rho[Y, Y] = \rho[X, Y] = 1$$

It follows that if $b = -1$ then this would yield a $corr(X, Y) = -1$