Week 1

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Qual Study

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Solution 2018.1. Since we need the constraint forces for part (c), we may as well work the problem using Lagrange multipliers from the start. In each case, the unconstrained Lagrangian is

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

a. The constraint is $f(x, y) = y - ax^2 = 0$, so we have

$$\begin{aligned}
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} &= \lambda \frac{\partial f}{\partial x} \\
m\ddot{x} &= -2\lambda a x \\
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{\partial \mathcal{L}}{\partial y} &= \lambda \frac{\partial f}{\partial y} \\
m\ddot{y} + mg &= \lambda
\end{aligned}$$

Let $\lambda' = \lambda/m$ to clean things up. Our system of equations is then

$$\begin{aligned}
\ddot{x} &= -2\lambda' a x \\
\ddot{y} &= -g + \lambda' \\
y &= ax^2
\end{aligned}$$

The constraint tells us that

$$\begin{aligned}
\dot{y} &= 2ax\dot{x} \\
\ddot{y} &= 2a(\dot{x}^2 + x\ddot{x})
\end{aligned}$$

so we have

$$\begin{aligned}
2a(\dot{x}^2 + x\ddot{x}) &= g + \lambda' \\
&= -g - \frac{\ddot{x}}{2ax}
\end{aligned}$$

$$4a^2x(\dot{x}^2 + x\ddot{x}) = -2gax - \ddot{x}$$

Thus, the equations of motion for x and y are

$$\begin{aligned}
0 &= (1 + 4a^2x^2)\ddot{x} + 4a^2x\dot{x}^2 + 2gax \\
y &= ax^2
\end{aligned}$$

b. The constraint is $f(x, y) = x^2 + y^2 = r^2$, so we have

$$\begin{aligned}
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} &= \lambda \frac{\partial f}{\partial x} \\
m\ddot{x} &= 2\lambda x \\
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{\partial \mathcal{L}}{\partial y} &= \lambda \frac{\partial f}{\partial y} \\
m\ddot{y} + mg &= 2\lambda y
\end{aligned}$$

This time, we use $\lambda' = 2\lambda/m$ to clean things up:

$$\begin{aligned}
\ddot{x} &= \lambda' x \\
\ddot{y} + g &= \lambda' y
\end{aligned}$$

We then eliminate λ' to get

$$\frac{\ddot{x}}{x} = \frac{\ddot{y} + g}{y}$$

At this point we're technically done with part (b); the equations of motion are

$$\begin{aligned}
\frac{\ddot{x}}{x} &= \frac{\ddot{y} + g}{y} \\
x^2 + y^2 &= r^2
\end{aligned}$$

c.

Solution 2018.3. Note: this problem was covered in the P521 lecture on September 21, 2020.

a. We use

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{body} = \mathbf{F}_e - m\left(\frac{d^2\mathbf{a}}{dt^2}\right)_{inertial} - 2m\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{body} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - m\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}$$

To first order in $\boldsymbol{\omega}$, this is

$$m\ddot{\mathbf{r}} = \mathbf{F}_e - 2m\boldsymbol{\omega} \times \dot{\mathbf{r}}$$

In the given coordinates,

$$\begin{aligned}
\boldsymbol{\omega} &= \omega(\cos\theta\hat{\mathbf{e}}_3 - \sin\theta\hat{\mathbf{e}}_1) \\
\mathbf{F}_e &= -mg\hat{\mathbf{e}}_3
\end{aligned}$$

We can then split the equation of motion out into its components:

$$\begin{aligned}
\ddot{x} &= 2\dot{y}\omega\cos\theta \\
\ddot{y} &= -2\dot{x}\omega\cos\theta - 2\dot{z}\omega\sin\theta \\
\ddot{z} &= -g + 2\dot{y}\omega\sin\theta
\end{aligned}$$

Working order by order, we write

$$\mathbf{r}(t) = \mathbf{r}_0(t) + \omega\tau\mathbf{r}_1(t) + \mathcal{O}(\omega^2)$$

where $\mathbf{r}_0$ and $\mathbf{r}_1$ are independent of ω. The order ω^0 equations read

$$\begin{aligned}
\ddot{x}_0 &= 0 \\
\ddot{y}_0 &= 0 \\
\ddot{z}_0 &= -g \\
\mathbf{r}_0(t) &= \left(v_0 t - \frac{g}{2}t^2\right)\hat{\mathbf{e}}_3
\end{aligned}$$

The time taken to reach the ground is then

$$\tau = \frac{2v_0}{g}$$

The order ω^1 equations read

$$\begin{aligned}
\omega\tau\ddot{x}_1 &= 2\dot{y}_0\omega\cos\theta \\
\omega\tau\ddot{y}_1 &= -2\dot{x}_0\omega\cos\theta - 2\dot{z}_0\omega\sin\theta \\
\omega\tau\ddot{z}_1 &= 2\dot{y}_0\omega\sin\theta
\end{aligned}$$

Since $x_0(t) = y_0(t) = 0$, this simplifies to

$$\begin{aligned}
\omega\tau\ddot{x}_1 &= 0 \\
\omega\tau\ddot{y}_1 &= -2(v_0 - gt)\omega\sin\theta \\
\omega\tau\ddot{z}_1 &= 0 \\
y_1(t) &= y_1(0) + \dot{y}_1(0)t - \frac{1}{\tau}\left(v_0 t^2 - \frac{g}{3}t^2\right)\sin\theta
\end{aligned}$$

Since $y(0) = 0$ and $\dot{y}(0) = 0$, the same must be true of y_1, giving us

$$y_1(t) = -\frac{1}{\tau}\left(v_0 t^2 - \frac{g}{3}t^2\right)\sin\theta$$

Put all together, this gives us

$$\mathbf{r}(t) = \left(v_0 t - \frac{g}{2}t^2\right)\hat{\mathbf{e}}_3 - \left(v_0 t^2 - \frac{g}{3}t^2\right)\omega\sin\theta\hat{\mathbf{e}}_2 + \mathcal{O}(\omega^2)$$

Therefore, the Coriolis deflection when the particle hits the ground is

$$\begin{aligned}\mathbf{r}(\tau) &= -\left(v_0\tau^2 - \frac{g}{3}\tau^2\right)\omega\sin\theta\hat{\mathbf{e}}_2 \\ &= -\frac{4}{3}\frac{v_0^3}{g^2}\omega\sin\theta\hat{\mathbf{e}}_2\end{aligned}$$

b. If the particle is dropped from rest at height h instead, we have

$$\begin{aligned}\mathbf{r}_0(t) &= \left(h - \frac{g}{2}t^2\right)\hat{\mathbf{e}}_3 \\ \tau &= \sqrt{\frac{2h}{g}} \\ \omega\tau\ddot{y}_1 &= -2(-gt)\omega\sin\theta \\ y_1(t) &= \frac{gt^3}{3\tau}\omega\sin\theta \\ \mathbf{r}(t) &= \left(h - \frac{g}{2}t^2\right)\hat{\mathbf{e}}_3 + \frac{1}{3}gt^3\omega\sin\theta + \mathcal{O}(\omega^2)\end{aligned}$$

and the Coriolis deflection is

$$\mathbf{r}(\tau) = \frac{1}{3}g\left(\frac{2h}{g}\right)^{3/2}\omega\sin\theta\hat{\mathbf{e}}_2$$

Solution 2017.2. Let $F(t)$ be the force applied to the sphere by the block via friction at time t. Newton's laws tell us

$$\begin{aligned}m\ddot{x} &= F(t) \\ M\ddot{X} &= -F(t) + f(t)\end{aligned}$$

Additionally, by conservation of energy, the total kinetic energy of the system must be equal to the work done on the system. The total kinetic energy is given by

$$\begin{aligned}T &= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\omega^2 \\ &= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\left(\frac{2}{5}mR^2\right)\left(\frac{\dot{x}}{R}\right)^2 \\ &= \frac{1}{2}M\dot{X}^2 + \frac{7}{10}m\dot{x}^2\end{aligned}$$

while the work done on the system between time zero and time t is

$$W = \int \mathbf{F}\cdot d\mathbf{r} = \int_0^t f(t')\dot{X}(t')\,dt'$$

This gives us a system of equations for $x(t)$ and $X(t)$:

$$\begin{aligned} M\ddot{X} + m\ddot{x} &= f(t) \\ \frac{1}{2}M\dot{X}^2 + \frac{7}{10}m\dot{x}^2 &= \int_0^t f(t')\dot{X}(t')\,dt' \end{aligned}$$

From the first equation, and the fact that $\dot{x}(0) = \dot{X}(0) = 0$, we have

$$M\dot{X} + m\dot{x} = -\frac{a}{\omega}\cos\omega t$$

If we differentiate both sides of the second equation, we get

$$M\dot{X}\ddot{X} + \frac{7}{5}m\dot{x}\ddot{x} = f(t)\dot{X}$$

Substituting what we know about $\dot{X}$ and $\ddot{X}$, we have

$$\begin{aligned} \left(-m\dot{x} - \frac{a}{\omega}\cos\omega t\right)\frac{1}{M}(f(t) - m\ddot{x}) + \frac{7}{5}m\dot{x}\ddot{x} &= -f(t)\frac{1}{M}\left(m\dot{x} + \frac{a}{\omega}\cos\omega t\right) \\ \left(m\dot{x} + \frac{a}{\omega}\cos\omega t\right)\frac{m}{M}\ddot{x} + \frac{7}{5}m\dot{x}\ddot{x} &= 0 \\ \left(\frac{m^2}{M} + \frac{7}{5}m\right)\dot{x} &= -\frac{a}{\omega}\cos\omega t \\ \left(\frac{m^2}{M} + \frac{7}{5}m\right)x &= -\frac{a}{\omega^2}\sin\omega t \end{aligned}$$

Solution 2017.15. In the rest frame of the pion, we have

$$\begin{aligned} \begin{pmatrix} m_\pi \\ 0 \end{pmatrix} &= \begin{pmatrix} p'_\nu \\ p'_\nu \end{pmatrix} + \begin{pmatrix} E_\mu \\ -p'_\nu \end{pmatrix} \\ E_\mu^2 &= m_\mu^2 + (p'_\nu)^2 \\ &= (m_\pi - p_\nu)^2 \\ &= m_\pi^2 - 2m_\pi p'_\nu + (p'_\nu)^2 \\ p'_\nu &= \frac{m_\pi^2 - m_\mu^2}{2m_\pi} \end{aligned}$$

In the lab frame, the neutrinos will clearly have the largest possible energy if they are emitted in the same direction as the momentum of the pion beam. Therefore, in the lab frame, the four momentum of the neutrinos with maximum energy is

$$p_\nu^\alpha = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} p'_\nu \\ p'_\nu \\ 0 \\ 0 \end{pmatrix}$$

$$= \gamma(1+\beta)\begin{pmatrix} p'_\nu \\ p'_\nu \\ 0 \\ 0 \end{pmatrix}$$

Finally, we must determine what γ is for the pion beam. Since 10 GeV $\gg m_\pi$, we may write

$$10\text{GeV} \approx p_\pi = m_\pi \gamma\beta$$

This gives us $\gamma\beta \approx 100$, which implies that $\beta \approx 1$. Thus, the maximum energy that the neutrinos can have in the lab frame is

$$\begin{aligned} E_\nu &\approx (100+100)\frac{m_\pi^2 - m_\mu^2}{2m_\pi} \\ &\approx 6.0\text{GeV} \end{aligned}$$

Solution 2016.1. a. The force F is the force resulting from the potential

$$V = \frac{1}{2}kr^2 = \frac{k}{2}(\rho^2 + z^2)$$

Also, the kinetic energy of the particle can be written as

$$T = \frac{m}{2}\left(\dot{z}^2 + R^2\dot{\phi}^2\right)$$

Thus, the Lagrangian for this system is

$$\mathcal{L} = T - V = \frac{m}{2}\left(\dot{z}^2 + R^2\dot{\phi}^2\right) - \frac{k}{2}(R^2 + z^2)$$

b. Lagrange's equations read

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) &= \frac{\partial \mathcal{L}}{\partial \phi} \\ 0 &= \frac{d}{dt}\left(mR^2\dot{\phi}\right) \\ &= mR^2\ddot{\phi} \\ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}\right) &= \frac{\partial \mathcal{L}}{\partial z} \\ -kz &= \frac{d}{dt}(m\dot{z}) = m\ddot{z} \end{aligned}$$

c. Cleaning things up a bit, we have

$$\begin{aligned} 0 &= mR^2\ddot{\phi} \\ 0 &= m\ddot{z} + kz \end{aligned}$$

Since $\frac{d}{dt}(mR^2\dot{\phi}) = 0$, we identify $mR^2\dot{\phi}$ as a conserved quantity, and recognize that it is the angular momentum of the particle. Defining $\omega^2 = k/m$, the z equation gives us

$$z(t) = A\cos\omega t + B\sin\omega t$$

where A and B are determined by the initial conditions. Thus, the particle exhibits harmonic oscillation around $z = 0$, and it revolves around the cylinder at a constant angular velocity.

Solution 2016.3. a. Let the equilibrium separation between the light atoms and the heavy atom be L. The total kinetic energy is clearly

$$T = \frac{1}{2}\left(M\dot{X}^2 + m\dot{x}_1^2 + m\dot{x}_2^2\right)$$

while the total potential energy is

$$V = \frac{k}{2}\left((X - x_1 - L)^2 + (x_2 - X - L)^2\right)$$

where I have assumed that $x_1 < X < x_2$.

b. Let

$$\begin{aligned} x_1 &= x_1^0 + \eta_1 \\ x_2 &= x_2^0 + \eta_2 \\ X &= X^0 + \eta_3 \end{aligned}$$

where x_1^0, x_2^0, and X^0 are the equilibrium values of x_1, x_2, and X, and η_i are their fluctuations from equilibrium. We then re-write the kinetic and potential energy as

$$\begin{aligned} T &= \frac{M}{2}\dot{\eta}_3^2 + \frac{m}{2}(\dot{\eta}_1^2 + \dot{\eta}_2^2) \\ V &= \frac{k}{2}\left((\eta_3 - \eta_1)^2 + (\eta_2 - \eta_3)^2\right) \\ &= \frac{k}{2}\left(2\eta_3^2 + \eta_1^2 + \eta_2^2 - 2\eta_1\eta_3 - 2\eta_2\eta_3\right) \end{aligned}$$

Thus, the mass and potential matrices are

$$\underline{m} = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & M \end{pmatrix}$$

$$\underline{v} = \begin{pmatrix} k & 0 & -k \\ 0 & k & -k \\ -k & -k & 2k \end{pmatrix}$$

The normal mode frequencies are then found via

$$
\begin{aligned}
0 &= \det(\underline{v} - \omega^2 \underline{m}) \\
&= \det \begin{pmatrix} k-\omega^2 m & 0 & -k \\ 0 & k-\omega^2 m & -k \\ -k & -k & 2k-\omega^2 M \end{pmatrix} \\
&= (k-\omega^2 m)((k-\omega^2 m)(2k-\omega^2 M) - k^2) - k(k(k-\omega^2 m)) \\
&= (k-\omega^2 m)\left[(k-\omega^2 m)(2k-\omega^2 M) - k^2 - k^2\right] \\
&= (k-\omega^2 m)\left[\omega^4 mM - 2k\omega^2 m - k\omega^2 M\right] \\
&= \omega^2(k-\omega^2 m)(mM\omega^2 - 2km - kM) \\
\omega^2 &= 0, \frac{k}{m}, \frac{2km+kM}{mM}
\end{aligned}
$$

The frequency $\omega = 0$ corresponds to the molecule moving with constant velocity and not oscillating. The frequency $\omega^2 = k/m$ corresponds to the two bonds oscillating in phase (i.e. $x_2 - X = X - x_1$) because this case is exactly analogous to the particles of mass m being attached to a fixed point by a spring with spring constant k. Therefore, the last frequency must correspond to the atoms oscillating out of phase (i.e. $x_2 - X$ is maximized when $X - x_1$ is minimized).

Solution 2015.2. a. The energy of the particle at any time is

$$E = \frac{1}{2}mv^2 + \frac{\beta}{r^2}$$

This is a conserved quantity, and it is equal to the initial energy $E_0 = \frac{1}{2}mv_0^2$. Additionally, the angular momentum of the particle is

$$\ell = mrv\sin\theta$$

where θ is the angle between the radius vector and the velocity vector. This is also a conserved quanitity, and it is equal to the initial angular momentum mbv_0. Since $\mathbf{r}$ and $\mathbf{v}$ are perpendicular when the particle makes its closest approach, we have a system equations for R_{min}, the distance of closest approach, and v_{min}, the speed of the particle at closest approach:

$$
\begin{aligned}
E &= \frac{1}{2}mv_{min}^2 + \frac{\beta}{R_{min}^2} \\
R_{min}v_{min} &= bv_0 \\
v_{min} &= \frac{bv_0}{R_{min}} \\
\frac{1}{2}mv_0^2 &= \frac{1}{2}m\left(\frac{bv_0}{R_{min}}\right)^2 + \frac{\beta}{R_{min}^2} \\
&= \frac{1}{2}mv_0^2\frac{b^2}{R_{min}^2} + \frac{\beta}{R_{min}^2}
\end{aligned}
$$

$$
\begin{aligned}
E &= E\frac{b^2}{R_{min}^2} + \frac{\beta}{R_{min}^2} \\
1 &= \frac{b^2 + \beta/E}{R_{min}^2} \\
R_{min} &= \sqrt{b^2 + \frac{\beta}{E}}
\end{aligned}
$$

b. The speed of the particle when it makes its closest approach is then

$$
v_{min} = bv_0 \left(b^2 + \frac{\beta}{E}\right)^{-1/2}
$$

Solution 2015.15. a. This is not possible. It is clearly energetically forbidden in the rest frame of the electron.

b. This is also not possible. Since the photon's energy is frame dependent, there exists a reference frame where the energy of the photon is less than the rest mass of an electron, making the process energetically forbidden.

Solution 2014.1. a. Let the horizontal position of m_1 be x_1, and let the angle between the pendulum arm and the vertical be ϕ. Denoting the x and y coordinates of m_2 as x_2 and y_2, we have

$$
T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2)
$$

We can write x_2 and y_2 in terms of x_1 and ϕ:

$$
\begin{aligned}
x_2 &= x_1 + \ell \sin\phi \\
y_2 &= -\ell \cos\phi
\end{aligned}
$$

(taking the zero point for y to be at the height of m_1), so

$$
\begin{aligned}
\dot{x}_2 &= \dot{x}_1 + \ell\dot{\phi}\cos\phi \\
\dot{y}_2 &= \ell\dot{\phi}\sin\phi
\end{aligned}
$$

Additionally, the potential energy of the system is

$$
V = m_2 g y_2 = -m_2 g \ell \cos\phi
$$

Thus, the Lagrangian for this system is

$$
\mathcal{L} = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\left[\left(\dot{x}_1 + \ell\dot{\phi}\cos\phi\right)^2 + (\ell\dot{\phi}\sin\phi)^2\right] + m_2 g\ell\cos\phi
$$

b. The equations of motion are then

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}_1}\right) &= \frac{\partial\mathcal{L}}{\partial x_1}\\
0 &= \frac{d}{dt}\left[m_1\dot{x}_1 + m_2\left(\dot{x}_1 + \ell\dot{\phi}\cos\phi\right)\right]\\
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) &= \frac{\partial\mathcal{L}}{\partial\phi}\\
\frac{\partial\mathcal{L}}{\partial\phi} &= m_2\left[\left(\dot{x}_1 + \ell\dot{\phi}\cos\phi\right)(-\ell\dot{\phi}\sin\phi) + (\ell\dot{\phi})^2\sin\phi\cos\phi\right] - m_2 g\ell\sin\phi\\
&= -m_2\ell\dot{x}_1\dot{\phi}\sin\phi - m_2 g\ell\sin\phi\\
\frac{\partial\mathcal{L}}{\partial\dot{\phi}} &= m_2\left[\left(\dot{x}_1 + \ell\dot{\phi}\cos\phi\right)\ell\cos\phi + \dot{\phi}\ell^2\sin^2\phi\right]\\
&= m_2\left[\dot{x}_1\ell\cos\phi + \ell^2\dot{\phi}\right]\\
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) &= m_2\left[\ddot{x}_1\ell\cos\phi - \dot{x}_1\ell\dot{\phi}\sin\phi + \ell^2\ddot{\phi}\right]\\
-g\ell\sin\phi &= \ddot{x}_1\ell\cos\phi + \ell^2\ddot{\phi}
\end{aligned}$$

c. TODO: normal modes

Solution 2014.3. Assuming a central potential, we have

$$\phi = \phi_0 \pm \frac{\ell}{\sqrt{2m}}\int dr\, r^{-2}[E - V_{eff}(r)]^{-1/2}$$

where $V_{eff}(r) = V(r) + \frac{\ell^2}{2mr^2}$. Using $u = 1/r$, we have

$$\begin{aligned}
du &= -\frac{1}{r^2}dr\\
\phi &= \phi_0 \pm \frac{\ell}{\sqrt{2m}}\int du\left(E - V(r = 1/u) - \frac{\ell^2}{2m}u^2\right)^{-1/2}\\
&= \phi_0 \pm \int du\left(\frac{2mE}{\ell^2} - \frac{2m}{\ell^2}V(u) - u^2\right)^{-1/2}
\end{aligned}$$

If the trajectory is a log spiral of the form $r = ke^{\alpha\phi}$, then

$$\phi = \frac{1}{\alpha}\log\frac{r}{k}$$

Therefore, we seek V such that

$$\frac{1}{\alpha}\log\frac{r}{k} = -\frac{1}{\alpha}\log ku = \pm\int du\left(\frac{2mE}{\ell^2} - \frac{2m}{\ell^2}V(u) - u^2\right)^{-1/2}$$

Differentiating both sides with respect to u yields

$$\begin{aligned}
-\frac{1}{\alpha u} &= \pm\left(\frac{2mE}{\ell^2} - \frac{2m}{\ell^2}V(u) - u^2\right)^{-1/2} \\
\frac{1}{\alpha^2 u^2} &= \frac{2mE}{\ell^2} - \frac{2m}{\ell^2}V(u) - u^2 \\
\frac{2m}{\ell^2}V(u) &= \frac{2mE}{\ell^2} - \frac{1}{\alpha^2 u^2} - u^2 \\
V(r) &= E - \frac{\ell^2}{2m\alpha^2}r^2 - \frac{\ell^2}{2mr^2}
\end{aligned}$$

Therefore, the necessary force law is

$$\mathbf{F} = -\boldsymbol{\nabla} V = \frac{\ell^2}{m}\left(\frac{r}{\alpha^2} - \frac{1}{r^3}\right)\hat{\mathbf{r}}$$

Solution 2013.2. Let the angle each pendulum makes with the vertical be ϕ_i. Taking $y = 0$ at the tops of the pendulums, we have

$$\begin{aligned}
x_i &= C_i + L\sin\phi_i \\
y_i &= -L\cos\phi_i \\
\dot{x}_i &= L\dot{\phi}_i\cos\phi_i \\
\dot{y}_i &= L\dot{\phi}_i\sin\phi_i
\end{aligned}$$

This lets us write the kinetic and potential energy as

$$\begin{aligned}
T &= \frac{1}{2}\sum_{i=1}^{3} m\left[\left(L\dot{\phi}_i\cos\phi_i\right)^2 + \left(L\dot{\phi}_i\sin\phi_i\right)^2\right] \\
V &= -\sum_{i=1}^{3} mgL\cos\phi_i + \frac{1}{2}k\left[(\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} - L_{12})^2 + (\sqrt{(x_3-x_2)^2 + (y_3-y_2)^2} - L_{23})^2\right]
\end{aligned}$$

where L_{12} and L_{13} are the equilibrium lengths of the springs connecting pendulums 1 and 2 and pendulums 2 and 3.

Writing $\phi_i = \phi_i^0 + \eta_i$, where ϕ_i^0 are the equilibrium values of the ϕ_i's, we have to second order in the displacements from equilibrium

$$\begin{aligned}
T &= \frac{1}{2}\sum_{i=1}^{3} m\left[\left(L\dot{\eta}_i\cos(\phi_i^0 + \eta_i)\right)^2 + \left(L\dot{\eta}_i\sin(\phi_i^0 + \eta_i)\right)^2\right] \\
&\approx \frac{1}{2}\sum_{i=1}^{3} m\left[L^2\dot{\eta}_i^2(\cos^2\phi_i^0 + \sin^2\phi_i^0)\right] \\
&= \frac{1}{2}\sum_{i=1}^{3} mL^2\dot{\eta}_i^2
\end{aligned}$$

$$
\begin{aligned}
V = &-\sum_{i=1}^{3} mgL\cos(\phi_i^0 + \eta_i) \\
&+ \frac{1}{2}k\left[\left(\sqrt{(C_2 + L\sin(\phi_2^0 + \eta_2) - C_1 - L\sin(\phi_1^0 + \eta_1))^2 + (-L\cos(\phi_2^0 + \eta_2) + L\cos(\phi_1^0 + \eta_1))^2} - L_{12}\right)^2\right. \\
&\left. + \left(\sqrt{(C_3 + L\sin(\phi_3^0 + \eta_3) - C_2 - L\sin(\phi_2^0 + \eta_2))^2 + (-L\cos(\phi_3^0 + \eta_3) + L\cos(\phi_2^0 + \eta_2))^2} - L_{23}\right)^2\right]
\end{aligned}
$$

To proceed, note that the equilibrium values of ϕ_i cannot be determined from the information given in the problem. Therefore, I assume that the normal mode frequencies do not depend on them, allowing me to choose $\phi_i^0 = 0$ for convenience. To second order in the η_i's, we then have

$$
\begin{aligned}
V \approx &-mgL\sum_{i=1}^{3}\left(1 + \frac{1}{2}\eta_i^2\right) \\
&+ \frac{k}{2}\left[\left(\sqrt{(C_2 + L\eta_2 - C_1 - L\eta_1)^2 + (-L + L\eta_2^2/2 + L - L\eta_1^2/2)^2} - L_{12}\right)^2\right. \\
&\left. + \left(\sqrt{(C_3 + L\eta_3 - C_2 - L\eta_2)^2 + (-L + L\eta_3^2/2 + L - L\eta_2^2/2)^2} - L_{23}\right)^2\right]
\end{aligned}
$$

Note that using my assumption about ϕ_i^0, we have $L_{12} = C_2 - C_1$ and $L_{23} = C_3 - C_2$. Also, we only have to work to linear order inside the parenthesis to retain second order in the η_i's overall. Continuing, we have

$$
\begin{aligned}
V &\approx \text{const} + \frac{1}{2}mgL\sum_{i=1}^{3}\eta_i^2 \\
&\quad + \frac{k}{2}\left[\left(\sqrt{L_{12}^2 + 2L_{12}L(\eta_2 - \eta_1)} - L_{12}\right)^2\right. \\
&\quad \left. + \left(\sqrt{L_{23}^2 + 2L_{23}L(\eta_3 - \eta_2)} - L_{23}\right)^2\right] \\
&\approx \text{const} + \frac{1}{2}mgL\sum_{i=1}^{3}\eta_i^2 + \frac{k}{2}\left[(L_{12} + L(\eta_2 - \eta_1) - L_{12})^2 + (L_{23} + L(\eta_3 - \eta_2) - L_{23})^2\right] \\
&\approx \text{const} + \frac{1}{2}mgL\sum_{i=1}^{3}\eta_i^2 + \frac{1}{2}kL^2\left[\eta_2^2 - 2\eta_1\eta_2 + \eta_1^2 + \eta_3^2 - 2\eta_3\eta_2 + \eta_2^2\right]
\end{aligned}
$$

Thus,

$$
\underline{m} = \begin{pmatrix} mL^2 & 0 & 0 \\ 0 & mL^2 & 0 \\ 0 & 0 & mL^2 \end{pmatrix}
$$

$$\underline{v} = \begin{pmatrix} mgL + kL^2 & kL^2 & 0 \\ kL^2 & mgL + 2kL^2 & kL^2 \\ 0 & kL^2 & mgL + kL^2 \end{pmatrix}$$

and the normal mode frequencies are found with

$$\begin{aligned}
0 &= \det[\underline{v} - \omega^2 \underline{m}] \\
&= \det \begin{pmatrix} mgL + kL^2 - \omega^2 mL^2 & kL^2 & 0 \\ kL^2 & mgL + 2kL^2 - \omega^2 mL^2 & kL^2 \\ 0 & kL^2 & mgL + kL^2 - \omega^2 mL^2 \end{pmatrix} \\
&= -(mL^2\omega^2 - mgL)(mL^2\omega^2 - 3kL^2 - mgL)(mL^2\omega^2 - mgL - kL^2) \\
\omega^2 &= \frac{g}{L}, 3\frac{k}{m} + \frac{g}{L}, \frac{k}{m} + \frac{g}{L}
\end{aligned}$$

Solution 2013.15. Let β_1 refer to the velocity of the first particle, β_2 to the velocity of the first two particles after they merge, and β_3 to the velocity of the final particle. Considering the four momenta before and after the first collision, we have

$$\begin{aligned}
\begin{pmatrix} \gamma_1 m \\ \gamma_1 \beta_1 m \end{pmatrix} + \begin{pmatrix} m \\ 0 \end{pmatrix} &= \begin{pmatrix} \gamma_2 m_2 \\ \gamma_2 \beta_2 m_2 \end{pmatrix} \\
(\gamma_1 + 1)m &= \gamma_2 m_2 \\
\gamma_1 \beta_1 m &= \gamma_2 \beta_2 m_2
\end{aligned}$$

Considering the four momenta before and after the second collision, we have

$$\begin{aligned}
\begin{pmatrix} \gamma_2 m_2 \\ \gamma_2 \beta_2 m_2 \end{pmatrix} + \begin{pmatrix} m \\ 0 \end{pmatrix} &= \begin{pmatrix} \gamma_3 M \\ \gamma_3 \beta_3 M \end{pmatrix} \\
(\gamma_1 + 2)m &= \gamma_3 M \\
\gamma_1 \beta_1 m &= \gamma_3 \beta_3 M \\
(\gamma_1 + 2)m &= \gamma_3 \frac{\gamma_1 \beta_1 m}{\gamma_3 \beta_3} \\
\gamma_1 + 2 &= \gamma_1 \frac{\beta_1}{\beta_3} \\
\beta_3 &= \beta_1 \frac{\gamma_1}{\gamma_1 + 2} \\
\beta_3^2 &= \frac{\beta_1^2 \gamma_1^2}{(\gamma_1 + 2)^2} \\
&= \frac{\gamma_1^2 - 1}{(\gamma_1 + 2)^2} \\
\gamma_3 &= \left(1 - \beta_3^2\right)^{-1/2} \\
M &= m \frac{\gamma_1 + 2}{\gamma_3}
\end{aligned}$$

a. Using $\beta_1 = 0.8$, we have

$$\begin{aligned}
\gamma_1 &= \frac{5}{3} \\
\beta_3 &= \frac{4}{11} \\
\gamma_3 &= \frac{11}{\sqrt{105}} \\
\frac{M}{m} &= \frac{\gamma_1 + 2}{\gamma_3} = \frac{\sqrt{105}}{3} \approx 3.4156
\end{aligned}$$

b. V/c is just β_3, which is $4/11$.