Qualifying Exam Cram Sheet

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Contents

1 Electromagnetism

1.1 Basics

1.1.1 Maxwell's equations

$$\nabla \cdot \mathbf{D} = \rho \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

1.1.2 Lorentz Force Law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

1.1.3 Definition of D and H

In terms of **P** and **M**,

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$$

In linear media,

$$\mathbf{D} = \epsilon \mathbf{E}$$

$$\mathbf{H} = \frac{1}{\mu}\mathbf{B}$$

1.1.4 Potentials

$$\mathbf{E} = -\nabla \Phi \qquad \Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' + \frac{1}{4\pi\epsilon_0}\oint \frac{\sigma(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} da$$

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x'$$

1.1.5 Charge Density

$$\sigma = -\epsilon_0 \frac{\partial \Phi}{\partial \hat{n}}$$

1.2 Laplace's Equation and Solutions

In a source free region, both **E** and **H** can be written as the gradient of a scalar potential which satisfies Laplace's equation,

$$\nabla^2 \Phi = 0$$

1.2.1 2D Rectangular Coordinates

$$\Phi(x, y) = \sum_k (A_k \sinh(kx) + B_k \cosh(kx))(C_k \sin(ky) + D_k \cos(ky))$$

Coefficients and allowed values of k are determined by boundary conditions

1.2.2 2D Polar Coordinates

$$\Phi(r, \phi) = \sum_n (A_n r^n + B_n r^{-n})(C_n \cos n\phi + D_n \sin n\phi)$$

1.2.3 3D Spherical with Azimuthal Symmetry

$$\Phi(r, \theta) = \sum_l \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

1.2.4 3D Spherical without Azimuthal Symmetry

$$\Phi(r, \theta) = \sum_{l,m} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) Y_{lm}(\theta, \phi)$$

1.3 The Addition Theorem

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = 4\pi \sum_l \sum_m \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi)$$

1.4 The Multipole Expansion

1.4.1 Spherical Coordinates

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \sum_l \sum_m \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}}$$

$$q_{lm} = \int Y_{lm}^*(\theta', \phi') r'^l \rho(\mathbf{x}') \, d^3x'$$

1.4.2 Rectangular Dipole and Quadrupole Moments

$$\mathbf{p} = \int \mathbf{x}' \rho(\mathbf{x}')\, d^3x'$$
$$Q_{ij} = \int (3x_i'x_j' - r'^2\delta_{ij})\rho(\mathbf{x}')\, d^3x'$$
$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{x}}{r^3} + \frac{1}{2}\sum_{i,j} Q_{ij}\frac{x_i x_j}{r^5}\right]$$

1.5 Magnetism

1.5.1 Magnetic Field Due to Current Distribution

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{x}')\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} d^3x'$$

1.5.2 Magnetic Dipole Moment

$$\mathbf{m} = \frac{1}{2}\int \mathbf{x}' \times \mathbf{J}(\mathbf{x}')\, d^3x'$$
$$m = IA \qquad \text{(for a loop)}$$
$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{3\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{m}) - \mathbf{m}}{r^3}$$

1.5.3 Magnetostatic Boundary Value Problems

1. Source Free

$$\mathbf{H} = -\boldsymbol{\nabla}\Phi_M$$
$$\nabla^2\Phi_M = 0$$

2. Hard Ferromagnets ($\mathbf{M}$ given)

$$\rho_M = -\boldsymbol{\nabla}\cdot\mathbf{M}$$
$$\sigma_M = \hat{\mathbf{n}}\cdot\mathbf{M}$$
$$\Phi_M(\mathbf{x}) = \frac{1}{4\pi}\int \frac{\rho_M(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d^3x' + \frac{1}{4\pi}\oint \frac{\sigma_M(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, da$$

1.6 Boundary Value Problem Matching Conditions

$$
\begin{aligned}
(\mathbf{D}_2 - \mathbf{D}_1) \cdot \hat{\mathbf{n}} &= \sigma \\
(\mathbf{E}_2 - \mathbf{E}_1) \times \hat{\mathbf{n}} &= 0 \\
(\mathbf{B}_2 - \mathbf{B}_1) \cdot \hat{\mathbf{n}} &= 0 \\
(\mathbf{H}_2 - \mathbf{H}_1) \times \hat{\mathbf{n}} &= -\mathbf{K} \text{ (surface current density)}
\end{aligned}
$$

1.7 Field Energy and Momentum

1.7.1 Energy Density

$$u = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$$

1.7.2 Momentum Density

$$
\begin{aligned}
\mathbf{g} &= \frac{1}{c^2}\mathbf{S} \\
\mathbf{S} &= \mathbf{E} \times \mathbf{H}
\end{aligned}
$$

1.7.3 Energy Conservation

$$\frac{\partial u}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E}$$

1.8 Electromagnetic Waves

1.8.1 Equations

$$
\begin{aligned}
\mathbf{E} &= \mathbf{E}_0 e^{i(k\hat{\mathbf{n}}\cdot\mathbf{x} - \omega t)} \\
\mathbf{H} &= \hat{\mathbf{n}} \times \mathbf{E}/Z
\end{aligned}
$$

1.8.2 Reflection/Refraction

1. Index of Refraction

$$n = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}$$

2. Normal Incidence

   - Transverse components of **E** and **H** are continuous

- Reflection and transmission coefficients:

$$
\begin{aligned}
T &= \frac{2n}{n' + n} \\
R &= \pm\frac{n - n'}{n + n'}
\end{aligned}
$$

1.9 Radiation

1.9.1 Power Radiated

$$
\begin{aligned}
\frac{dP}{d\Omega} &= \frac{1}{2}\mathrm{Re}[r^2\hat{\mathbf{n}} \cdot \mathbf{E} \times \mathbf{H}^*] \\
&= \frac{c^2 Z_0}{32\pi^2} k^4 \left|(\hat{\mathbf{n}} \times \mathbf{p}) \times \hat{\mathbf{n}}\right|^2 \ \text{(for dipole radiation)}
\end{aligned}
$$

1.10 Relativity

$$
\begin{aligned}
\gamma &= \frac{1}{\sqrt{1 - \beta^2}} \\
\Gamma &= \begin{pmatrix} \gamma & \pm\gamma\beta & 0 & 0 \\ \pm\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
p_\mu p^\mu &= m^2 \\
E &= \gamma m c^2 \\
\mathbf{p} &= \gamma m \mathbf{u} \\
U_\mu &= (\gamma c, \gamma \mathbf{u}) \\
p_\mu &= m U_\mu
\end{aligned}
$$

2 Mechanics

2.1 Basics

$$\mathbf{F} = \dot{\mathbf{p}}$$
$$\mathbf{\Gamma} = \mathbf{r} \times \mathbf{F}$$
$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

2.2 Orbital Motion

$$V_{eff}(r) = V(r) + \frac{\ell^2}{2mr^2}$$

- Circular orbits $\rightarrow$ minimum of V_{eff}
- Minimum and maximum r values for a non-circular orbit $\rightarrow$ solutions of $V_{eff} = E$

$$\phi = \pm \frac{\ell}{\sqrt{2m}} \int dr\, r^{-2} [E - V_{eff}(r)]^{-1/2}$$

2.3 Non-inertial Coordinate Systems

$$\ddot{\mathbf{r}}_{body} = \frac{\mathbf{F}^{(e)}}{m} - \ddot{\mathbf{a}}_{inertial} - 2\boldsymbol{\omega} \times \dot{\mathbf{r}}_{body} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \hat{\mathbf{r}}) - \dot{\boldsymbol{\omega}} \times \mathbf{r}$$

2.4 Lagrangian Dynamics

$$\mathcal{L} = T - V$$
$$\frac{\partial \mathcal{L}}{\partial q} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

With constraints

$$f_j(q_1, \ldots, q_n, t) = c_j,\ j = 1, \ldots, k$$

the Euler-Lagrange equation becomes

$$\frac{d\mathcal{L}}{d\dot{q}_\sigma} - \frac{\partial \mathcal{L}}{\partial q_\sigma} = \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_\sigma},\ \sigma = 1, \ldots, n$$

where the constraint forces are given by the right hand side,

$$Q_\sigma = \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_\sigma}$$

2.5 Small Oscillations

First, expand the coordinates around their equilibrium values:

$$q_\sigma = q_\sigma^0 + \eta_\sigma$$
$$\dot{q}_\sigma = \dot{\eta}_\sigma$$

Then, working to quadratic order in η, write T and V as

$$T = \frac{1}{2}\sum_\sigma \sum_\lambda m_{\sigma\lambda}\dot{\eta}_\sigma\dot{\eta}_\lambda$$
$$V = \frac{1}{2}\sum_\sigma \sum_\lambda v_{\sigma\lambda}\eta_\sigma\eta_\lambda$$

This determines the $\underline{m}$ and $\underline{v}$ matrices. Finally, the normal mode frequencies are found via

$$0 = \det[\underline{v} - \omega^2\underline{m}]$$

and the normal mode eigenvectors $\boldsymbol{\rho}$ via

$$0 = (\underline{v} - \omega^2\underline{m})\boldsymbol{\rho}$$

2.6 Rigid Body Rotations

$$\mathbf{L} = \underline{I}\boldsymbol{\omega}$$
$$T = \frac{1}{2}\mathbf{L}\cdot\boldsymbol{\omega}$$
$$I_1\dot{\omega}_1 = \omega_2\omega_3(I_2 - I_3) + \Gamma_1^{(e)} \text{ (and cyclic permutations)}$$

3 Quantum Mechanics

3.1 Basics

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{x},t) &= H\Psi(\mathbf{x},t) \\
H\psi(\mathbf{x}) &= E\psi(\mathbf{x}) \\
\psi(\mathbf{x},t) &= \sum_n c_n\psi_n(\mathbf{x})e^{-iEt/\hbar} \\
H &= \frac{p^2}{2m} + V \\
p &= -i\hbar\frac{\partial}{\partial x} \\
[x,p] &= i\hbar \\
\sigma_A\sigma_B &\geq \left|\frac{1}{2i}\langle[A,B]\rangle\right|
\end{aligned}
$$

- Matching conditions: $\psi(x)$ and $\psi'(x)$ are continuous
  - $\psi'(x)$ is discontinous in the presence of a delta function potential
  - In that case, determine the discontinuity in $\psi'(x)$ by integrating the Schrodinger equation over a small interval surrounding the delta function

3.2 Common Potentials

3.2.1 Particle in a Box

$$
\begin{aligned}
V(x) &= \begin{cases} 0 & x \in [0,a] \\ \infty & \text{otherwise} \end{cases} \\
\psi_n(x) &= \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a} \\
E_n &= \frac{\hbar^2 n^2 \pi^2}{2ma^2}
\end{aligned}
$$

3.2.2 Simple Harmonic Oscillator

$$
\begin{aligned}
V(x) &= \frac{1}{2}m\omega^2 x^2 \\
a &= \frac{1}{\sqrt{2\hbar m\omega}}(m\omega x + ip)
\end{aligned}
$$

$$
\begin{aligned}
[a, a^\dagger] &= 1 \\
N &= a^\dagger a \\
H &= \hbar\omega \left(N + \frac{1}{2} \right) \\
E_n &= \hbar\omega \left(n + \frac{1}{2} \right) \\
\psi_0(x) &= \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \exp \left(-\frac{m\omega}{2\hbar} x^2 \right) \\
\psi_n(x) &= \frac{1}{\sqrt{n!}} (a^\dagger)^n \psi_0(x) \\
a^\dagger |n\rangle &= \sqrt{n+1}\, |n+1\rangle \\
a |n\rangle &= \sqrt{n}\, |n-1\rangle \\
x &\propto a + a^\dagger \\
p &\propto a - a^\dagger
\end{aligned}
$$

3.2.3 Hydrogen Atom

$$
\begin{aligned}
V(\mathbf{x}) &= -\frac{Ze^2}{r} \\
\psi_{nlm}(\mathbf{x}) &= R_{nl}(r) Y_{lm}(\theta, \phi) \\
E_n &= -\frac{Z^2 e^2}{2n^2 a_0} \\
E_1 &= -13.6\text{eV}
\end{aligned}
$$

3.3 Angular Momentum

3.3.1 Basics

$$
\begin{aligned}
[L_x, L_y] &= i\hbar L_z \\
L_z |l, m\rangle &= \hbar m |l, m\rangle \\
L^2 |l, m\rangle &= \hbar^2 l(l+1) |l, m\rangle \\
L_\pm &= L_x \pm i L_y \\
L_\pm |l, m\rangle &= \hbar \sqrt{(l \mp m)(l \pm m + 1)}\, |l, m \pm 1\rangle \\
\sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\sigma_y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}
\end{aligned}
$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Allowed values of m: $-l, -l+1, \ldots, +l$

3.3.2 Addition of Angular Momentum

$$\begin{aligned}
\mathbf{J} &= \mathbf{L}_1 + \mathbf{L}_2 \\
J^2 &= L_1^2 + L_2^2 + 2\mathbf{L}_1 \cdot \mathbf{L}_2 \\
\mathbf{L}_1 \cdot \mathbf{L}_2 &= \frac{1}{2}(J^2 - L_1^2 - L_2^2) \\
|j, j_z\rangle &= \sum C_{l_1,l_2,m_1,m_2} |l_1, m_1; l_2, m_2\rangle \\
j_z &= m_1 + m_2 \\
j &= l_1 + l_2, l_1 + l_2 - 1, \ldots, |l_1 - l_2| \\
|l_1 + l_2, l_1 + l_2\rangle &= |l_1, l_1; l_2, l_2\rangle
\end{aligned}$$

- $|l_1 + l_2, j_z\rangle$ is constructed by repeatedly applying J_- to $|j, j\rangle$.
- $|j, j_z\rangle$ is constructed by walking down the allowed values of j starting from $l_1 + l_2$
  - Construct $|j-1, j-1\rangle$ from $|j, j-1\rangle$ by finding the linear combination of allowed m_1 and m_2 that is orthogonal to $|j, j-1\rangle$
  - Construct $|j-1, j_z\rangle$ by applying the lowering operator

3.4 Approximation Methods

3.4.1 Perturbation Theory

$$\begin{aligned}
H &= H_0 + V \\
V_{mn} &= \left\langle m^{(0)} \middle| V \middle| n^{(0)} \right\rangle \\
E_n^{(1)} &= V_{nn} \\
\left| n^{(1)} \right\rangle &= \sum_{m \neq n} \frac{V_{mn}}{E_n^{(0)} - E_m^{(0)}} \left| m^{(0)} \right\rangle \\
E_n^{(2)} &= \sum_{m \neq n} \frac{|V_{nm}|^2}{E_n^{(0)} - E_m^{(0)}}
\end{aligned}$$

- In a degenerate subspace where all states have energy E_D under H_0, the first order correction to the energy is found via

$$0 = \det[V - (E - E_D)]$$

3.4.2 Variational Principle

$$E_0 \leq \frac{\langle \psi \,|\, H \,|\, \psi \rangle}{\langle \psi \,|\, \psi \rangle}$$

3.4.3 WKB

$$p(x) = \sqrt{2m(E - V(x)}$$

$$\psi(x) \sim \frac{C}{\sqrt{p(x)}} e^{\pm \frac{i}{\hbar} \int p(x)\, dx}$$

$$\int_{x_1}^{x_2} p(x')\, dx' = \begin{cases} \left(n + \frac{1}{2}\right) \hbar\pi & \text{no hard walls} \\ \left(n + \frac{3}{4}\right) \hbar\pi & \text{one hard wall} \\ (n+1)\, \hbar\pi & \text{two hard walls} \end{cases}$$

where x_1 and x_2 are the classical turning points (roots of $p(x)$)

3.5 Scattering (Born Approximation)

$$f^{(1)}(\mathbf{k}, \mathbf{k}') = -\frac{m}{2\pi\hbar^2} \int d^3x'\, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}'} V(\mathbf{x}')$$

$$q = \left|\mathbf{k} - \mathbf{k}'\right| = 2k \sin\frac{\theta}{2}$$

$$f^{(1)}(\theta) = -\frac{2m}{\hbar^2}\frac{1}{q}\int_0^\infty rV(r)\sin qr\, dr$$

$$\frac{d\sigma}{d\Omega} = \left|f(\mathbf{k}', \mathbf{k})\right|^2$$

$$\mathrm{Im} f(\mathbf{k}, \mathbf{k}) = \frac{k}{4\pi}\sigma_{tot}$$

3.6 Many Body Theory

- Boson wave functions are symmetric under any particle exchange
- Fermion wave functions are antisymmetric under any particle exchange

- For wave-functions with a spatial component and a spin component,
  - Boson: symmetric in both or antisymmetric in both
  - Fermion: symmetric in one and antisymmetric in the other

4 Statistical Physics

4.1 Basics

$$S = k_B \log \Omega$$
$$\left(\frac{\partial S}{\partial E}\right)_{N,V} = \frac{1}{T}$$
$$dE = -PdV + TdS + \mu dN$$
$$C_V = \left(\frac{\partial U}{\partial T}\right)_{V,N}$$

- Isothermal: $dT = 0$
- Adiabatic: $dS = 0$

4.2 Canonical Ensemble

$$Z = \sum_{states} e^{-\beta E}$$
$$\langle O \rangle = \frac{1}{Z} \sum_s O(s) e^{-\beta E}$$
$$\sum_s \to \int \frac{d^3x\, d^3p}{h^3}$$
$$U = -\frac{\partial}{\partial \beta} \log Z$$
$$A = U - TS = -k_B T \log Z$$
$$dA = -PdV - SdT$$

- Partition function for an ideal gas molecule:

$$Z_1 = \frac{V}{\lambda^3}$$
$$\lambda = \sqrt{\frac{h^2}{2\pi m k_B T}}$$

- In general, for a system of non-interacting particles, the total partition function obeys

$$Z = \frac{1}{N!} Z_1^N$$

where Z_1 is the partition function for one particle, and the $1/N!$ factor is necessary if the particles are indistinguishable

4.3 Grand Canonical Ensemble

$$
\begin{aligned}
Z_G &= \sum_{states} e^{-\alpha N - \beta E} \\
U &= -\frac{\partial}{\partial \beta} \log Z_G \\
\mathcal{N} &= -\frac{\partial}{\partial \alpha} \log Z_G \\
\alpha &= -\beta\mu \\
y &= e^{-\alpha} = e^{\mu/k_B T} \\
Z_G &= \sum_N y^N Z(N, \beta) \\
\Phi_G &= -k_B T \log Z_G = U - TS - \mu\mathcal{N} = -PV
\end{aligned}
$$

4.4 Quantum Statistical Mechanics

$$
\begin{aligned}
Z_G &= \prod_n [1 - \eta e^{-\beta(E_n - \mu)}]^{-\eta} \\
\langle N_n \rangle &= \frac{1}{y^{-1} e^{\beta E_n} - \eta} \\
D(E) &= \frac{g(E)}{V} = \frac{1}{V} \sum_{\mathbf{k}} (E - E_{\mathbf{k}}) \\
\varrho &= \int \frac{D(E)}{y^{-1} e^{\beta E} - 1} \, dE \\
\log Z_G &= -\eta V \int_0^\infty \log(1 - \eta y e^{-\beta E}) D(E) \, dE
\end{aligned}
$$

- $\eta = +1$ for bosons, -1 for fermions

4.4.1 Bose Systems

- $y = 1$ at Bose-Einstein Condensation, and in all cases for photons

4.4.2 Fermi Systems

$$
\varrho = \int_0^{E_F} D(E) \, dE
$$

- Any other Fermi parameter (Fermi temperature, Fermi momentum, etc) is determined from the relationship between energy and that parameter (E.g. $T_F = E_F/k_B$)