

# Qualifying Exam Cram Sheet

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# 1 Electromagnetism

## 1.1 Basics

### 1.1.1 Maxwell's equations

$$\begin{aligned}\nabla \cdot \mathbf{D} &= \rho & \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} & \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\end{aligned}$$

### 1.1.2 Lorentz Force Law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

### 1.1.3 Definition of $\mathbf{D}$ and $\mathbf{H}$

In terms of  $\mathbf{P}$  and  $\mathbf{M}$ ,

$$\begin{aligned}\mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} &= \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}\end{aligned}$$

In linear media,

$$\begin{aligned}\mathbf{D} &= \epsilon \mathbf{E} \\ \mathbf{H} &= \frac{1}{\mu} \mathbf{B}\end{aligned}$$

### 1.1.4 Potentials

$$\begin{aligned}\mathbf{E} &= -\nabla \Phi & \Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' + \frac{1}{4\pi\epsilon_0} \oint \frac{\sigma(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} da \\ \mathbf{B} &= \nabla \times \mathbf{A} & \mathbf{A}(\mathbf{x}) &= \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x'\end{aligned}$$

### 1.1.5 Charge Density

$$\sigma = -\epsilon_0 \frac{\partial \Phi}{\partial \hat{n}}$$

## 1.2 Method of Images

- Charge  $q$  a distance  $a$  from a grounded conducting plane
  - Image charge  $-q$  a distance  $a$  from the plane
- Charge  $q$  a distance  $y$  from the center of a grounded conducting sphere of radius  $a$ 
  - Image charge  $q' = -(a/y)q$  a distance  $a^2/y$  from the center of the sphere

## 1.3 Laplace's Equation and Solutions

In a source free region, both  $\mathbf{E}$  and  $\mathbf{H}$  can be written as the gradient of a scalar potential which satisfies Laplace's equation,

$$\nabla^2 \Phi = 0$$

### 1.3.1 2D Rectangular Coordinates

$$\Phi(x, y) = \sum_k (A_k \sinh(kx) + B_k \cosh(kx))(C_k \sin(ky) + D_k \cos(ky))$$

Coefficients and allowed values of  $k$  are determined by boundary conditions

### 1.3.2 2D Polar Coordinates

$$\Phi(r, \phi) = \sum_n (A_n r^n + B_n r^{-n})(C_n \cos n\phi + D_n \sin n\phi)$$

### 1.3.3 3D Spherical with Azimuthal Symmetry

$$\Phi(r, \theta) = \sum_l \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$$

### 1.3.4 3D Spherical without Azimuthal Symmetry

$$\Phi(r, \theta) = \sum_{l,m} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) Y_{lm}(\theta, \phi)$$

## 1.4 The Addition Theorem

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = 4\pi \sum_l \sum_m \frac{1}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi)$$

## 1.5 The Multipole Expansion

### 1.5.1 Spherical Coordinates

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \sum_l \sum_m \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}}$$

$$q_{lm} = \int Y_{lm}^*(\theta', \phi') r'^l \rho(\mathbf{x}') d^3x'$$

### 1.5.2 Rectangular Dipole and Quadrupole Moments

$$\mathbf{p} = \int \mathbf{x}' \rho(\mathbf{x}') d^3x'$$

$$Q_{ij} = \int (3x'_i x'_j - r'^2 \delta_{ij}) \rho(\mathbf{x}') d^3x'$$

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \left[ \frac{q}{r} + \frac{\mathbf{p} \cdot \mathbf{x}}{r^3} + \frac{1}{2} \sum_{i,j} Q_{ij} \frac{x_i x_j}{r^5} \right]$$

## 1.6 Magnetism

### 1.6.1 Magnetic Field Due to Current Distribution

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{x}') \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} d^3x'$$

### 1.6.2 Magnetic Dipole Moment

$$\mathbf{m} = \frac{1}{2} \int \mathbf{x}' \times \mathbf{J}(\mathbf{x}') d^3x'$$

$$m = IA \quad (\text{for a loop})$$

$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{m}) - \mathbf{m}}{r^3}$$

### 1.6.3 Magnetostatic Boundary Value Problems

#### 1. Source Free

$$\mathbf{H} = -\nabla \Phi_M$$

$$\nabla^2 \Phi_M = 0$$

## 2. Hard Ferromagnets ( $\mathbf{M}$ given)

$$\rho_M = -\nabla \cdot \mathbf{M}$$

$$\sigma_M = \hat{\mathbf{n}} \cdot \mathbf{M}$$

$$\Phi_M(\mathbf{x}) = \frac{1}{4\pi} \int \frac{\rho_M(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' + \frac{1}{4\pi} \oint \frac{\sigma_M(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} da$$

### 1.7 Boundary Value Problem Matching Conditions

$$(\mathbf{D}_2 - \mathbf{D}_1) \cdot \hat{\mathbf{n}} = \sigma$$

$$(\mathbf{E}_2 - \mathbf{E}_1) \times \hat{\mathbf{n}} = 0$$

$$(\mathbf{B}_2 - \mathbf{B}_1) \cdot \hat{\mathbf{n}} = 0$$

$$(\mathbf{H}_2 - \mathbf{H}_1) \times \hat{\mathbf{n}} = -\mathbf{K} \text{ (surface current density)}$$

### 1.8 Field Energy and Momentum

#### 1.8.1 Energy Density

$$u = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$$

#### 1.8.2 Momentum Density

$$\mathbf{g} = \frac{1}{c^2} \mathbf{S}$$

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

#### 1.8.3 Energy Conservation

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E}$$

### 1.9 Electromagnetic Waves

#### 1.9.1 Equations

$$\mathbf{E} = \mathbf{E}_0 e^{i(k\hat{\mathbf{n}} \cdot \mathbf{x} - \omega t)}$$

$$\mathbf{H} = \hat{\mathbf{n}} \times \mathbf{E}/Z$$

### 1.9.2 Reflection/Refraction

#### 1. Index of Refraction

$$n = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}$$

#### 2. Normal Incidence

- Transverse components of  $\mathbf{E}$  and  $\mathbf{H}$  are continuous
- Reflection and transmission coefficients:

$$T = \frac{2n}{n' + n}$$
$$R = \pm \frac{n - n'}{n + n'}$$

### 1.10 Radiation

#### 1.10.1 Power Radiated

$$\frac{dP}{d\Omega} = \frac{1}{2} \text{Re}[r^2 \hat{\mathbf{n}} \cdot \mathbf{E} \times \mathbf{H}^*]$$
$$= \frac{c^2 Z_0}{32\pi^2} k^4 |(\hat{\mathbf{n}} \times \mathbf{p}) \times \hat{\mathbf{n}}|^2 \quad (\text{for dipole radiation})$$

### 1.11 Relativity

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}$$
$$\Gamma = \begin{pmatrix} \gamma & \pm\gamma\beta & 0 & 0 \\ \pm\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$p_\mu p^\mu = m^2$$

$$E = \gamma mc^2$$

$$\mathbf{p} = \gamma m \mathbf{u}$$

$$U_\mu = (\gamma c, \gamma \mathbf{u})$$

$$p_\mu = m U_\mu$$

## 2 Mechanics

### 2.1 Basics

$$\begin{aligned}\mathbf{F} &= \dot{\mathbf{p}} \\ \mathbf{\Gamma} &= \mathbf{r} \times \mathbf{F} \\ \mathbf{L} &= \mathbf{r} \times \mathbf{p}\end{aligned}$$

### 2.2 Orbital Motion

$$V_{eff}(r) = V(r) + \frac{\ell^2}{2mr^2}$$

- Circular orbits  $\rightarrow$  minimum of  $V_{eff}$
- Minimum and maximum  $r$  values for a non-circular orbit  $\rightarrow$  solutions of  $V_{eff} = E$

$$\phi = \pm \frac{\ell}{\sqrt{2m}} \int dr r^{-2} [E - V_{eff}(r)]^{-1/2}$$

### 2.3 Non-inertial Coordinate Systems

$$\ddot{\mathbf{r}}_{body} = \frac{\mathbf{F}^{(e)}}{m} - \ddot{\mathbf{a}}_{inertial} - 2\boldsymbol{\omega} \times \dot{\mathbf{r}}_{body} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \hat{\mathbf{r}}) - \dot{\boldsymbol{\omega}} \times \mathbf{r}$$

### 2.4 Lagrangian Dynamics

$$\begin{aligned}\mathcal{L} &= T - V \\ \frac{\partial \mathcal{L}}{\partial q} &= \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}}\end{aligned}$$

With constraints

$$f_j(q_1, \dots, q_n, t) = c_j, j = 1, \dots, k$$

the Euler-Lagrange equation becomes

$$\frac{d\mathcal{L}}{d\dot{q}_\sigma} - \frac{\partial \mathcal{L}}{\partial q_\sigma} = \sum_{j=1}^k \lambda_j \frac{\partial f_j}{\partial q_\sigma}, \sigma = 1, \dots, n$$

where the constraint forces are given by the right hand side,

$$Q_\sigma = \sum_{j=1}^k \lambda_j \frac{\partial f_j}{\partial q_\sigma}$$



## 2.5 Small Oscillations

First, expand the coordinates around their equilibrium values:

$$\begin{aligned} q_\sigma &= q_\sigma^0 + \eta_\sigma \\ \dot{q}_\sigma &= \dot{\eta}_\sigma \end{aligned}$$

Then, working to quadratic order in  $\eta$ , write  $T$  and  $V$  as

$$\begin{aligned} T &= \frac{1}{2} \sum_\sigma \sum_\lambda m_{\sigma\lambda} \dot{\eta}_\sigma \dot{\eta}_\lambda \\ V &= \frac{1}{2} \sum_\sigma \sum_\lambda v_{\sigma\lambda} \eta_\sigma \eta_\lambda \end{aligned}$$

This determines the  $\underline{m}$  and  $\underline{v}$  matrices. Finally, the normal mode frequencies are found via

$$0 = \det[\underline{v} - \omega^2 \underline{m}]$$

and the normal mode eigenvectors  $\boldsymbol{\rho}$  via

$$0 = (\underline{v} - \omega^2 \underline{m}) \boldsymbol{\rho}$$

## 2.6 Rigid Body Rotations

$$\mathbf{L} = \underline{I} \boldsymbol{\omega}$$

$$T = \frac{1}{2} \mathbf{L} \cdot \boldsymbol{\omega}$$

$$I_1 \dot{\omega}_1 = \omega_2 \omega_3 (I_2 - I_3) + \Gamma_1^{(e)} \text{ (and cyclic permutations)}$$

## 3 Quantum Mechanics

### 3.1 Basics

$$\begin{aligned}i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{x}, t) &= H\Psi(\mathbf{x}, t) \\ H\psi(\mathbf{x}) &= E\psi(\mathbf{x}) \\ \psi(\mathbf{x}, t) &= \sum_n c_n \psi_n(\mathbf{x}) e^{-iEt/\hbar} \\ H &= \frac{p^2}{2m} + V \\ p &= -i\hbar \frac{\partial}{\partial x} \\ [x, p] &= i\hbar \\ \sigma_A \sigma_B &\geq \left| \frac{1}{2i} \langle [A, B] \rangle \right|\end{aligned}$$

- Matching conditions:  $\psi(x)$  and  $\psi'(x)$  are continuous
  - $\psi'(x)$  is discontinuous in the presence of a delta function potential
  - In that case, determine the discontinuity in  $\psi'(x)$  by integrating the Schrodinger equation over a small interval surrounding the delta function

### 3.2 Common Potentials

#### 3.2.1 Particle in a Box

$$\begin{aligned}V(x) &= \begin{cases} 0 & x \in [0, a] \\ \infty & \text{otherwise} \end{cases} \\ \psi_n(x) &= \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \\ E_n &= \frac{\hbar^2 n^2 \pi^2}{2ma^2}\end{aligned}$$

#### 3.2.2 Simple Harmonic Oscillator

$$\begin{aligned}V(x) &= \frac{1}{2} m \omega^2 x^2 \\ a &= \frac{1}{\sqrt{2\hbar m \omega}} (m \omega x + i p)\end{aligned}$$

$$\begin{aligned}
[a, a^\dagger] &= 1 \\
N &= a^\dagger a \\
H &= \hbar\omega \left( N + \frac{1}{2} \right) \\
E_n &= \hbar\omega \left( n + \frac{1}{2} \right) \\
\psi_0(x) &= \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} \exp \left( -\frac{m\omega}{2\hbar} x^2 \right) \\
\psi_n(x) &= \frac{1}{\sqrt{n!}} (a^\dagger)^n \psi_0(x) \\
a^\dagger |n\rangle &= \sqrt{n+1} |n+1\rangle \\
a |n\rangle &= \sqrt{n} |n-1\rangle \\
x &\propto a + a^\dagger \\
p &\propto a - a^\dagger
\end{aligned}$$

### 3.2.3 Hydrogen Atom

$$\begin{aligned}
V(\mathbf{x}) &= -\frac{Ze^2}{r} \\
\psi_{nlm}(\mathbf{x}) &= R_{nl}(r) Y_{lm}(\theta, \phi) \\
E_n &= -\frac{Z^2 e^2}{2n^2 a_0} \\
E_1 &= -13.6 \text{eV}
\end{aligned}$$

## 3.3 Angular Momentum

### 3.3.1 Basics

$$\begin{aligned}
[L_x, L_y] &= i\hbar L_z \\
L_z |l, m\rangle &= \hbar m |l, m\rangle \\
L^2 |l, m\rangle &= \hbar^2 l(l+1) |l, m\rangle \\
L_\pm &= L_x \pm iL_y \\
L_\pm |l, m\rangle &= \hbar \sqrt{(l \mp m)(l \pm m + 1)} |l, m \pm 1\rangle \\
\sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\sigma_y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}
\end{aligned}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Allowed values of  $m$ :  $-l, -l+1, \dots, +l$

### 3.3.2 Addition of Angular Momentum

$$\begin{aligned} \mathbf{J} &= \mathbf{L}_1 + \mathbf{L}_2 \\ J^2 &= L_1^2 + L_2^2 + 2\mathbf{L}_1 \cdot \mathbf{L}_2 \\ \mathbf{L}_1 \cdot \mathbf{L}_2 &= \frac{1}{2}(J^2 - L_1^2 - L_2^2) \\ |j, j_z\rangle &= \sum C_{l_1, l_2, m_1, m_2} |l_1, m_1; l_2, m_2\rangle \\ j_z &= m_1 + m_2 \\ j &= l_1 + l_2, l_1 + l_2 - 1, \dots, |l_1 - l_2| \\ |l_1 + l_2, l_1 + l_2\rangle &= |l_1, l_1; l_2, l_2\rangle \end{aligned}$$

- $|l_1 + l_2, j_z\rangle$  is constructed by repeatedly applying  $J_-$  to  $|j, j\rangle$ .
- $|j, j_z\rangle$  is constructed by walking down the allowed values of  $j$  starting from  $l_1 + l_2$ 
  - Construct  $|j-1, j-1\rangle$  from  $|j, j-1\rangle$  by finding the linear combination of allowed  $m_1$  and  $m_2$  that is orthogonal to  $|j, j-1\rangle$
  - Construct  $|j-1, j_z\rangle$  by applying the lowering operator

## 3.4 Approximation Methods

### 3.4.1 Perturbation Theory

$$\begin{aligned} H &= H_0 + V \\ V_{mn} &= \langle m^{(0)} | V | n^{(0)} \rangle \\ E_n^{(1)} &= V_{nn} \\ |n^{(1)}\rangle &= \sum_{m \neq n} \frac{V_{mn}}{E_n^{(0)} - E_m^{(0)}} |m^{(0)}\rangle \\ E_n^{(2)} &= \sum_{m \neq n} \frac{|V_{nm}|^2}{E_n^{(0)} - E_m^{(0)}} \end{aligned}$$

- In a degenerate subspace where all states have energy  $E_D$  under  $H_0$ , the first order correction to the energy is found via

$$0 = \det[V - (E - E_D)]$$

### 3.4.2 Variational Principle

$$E_0 \leq \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle}$$

### 3.4.3 WKB

$$p(x) = \sqrt{2m(E - V(x))}$$

$$\psi(x) \sim \frac{C}{\sqrt{p(x)}} e^{\pm \frac{i}{\hbar} \int p(x) dx}$$

$$\int_{x_1}^{x_2} p(x') dx' = \begin{cases} \left(n + \frac{1}{2}\right) \hbar\pi & \text{no hard walls} \\ \left(n + \frac{3}{4}\right) \hbar\pi & \text{one hard wall} \\ (n + 1) \hbar\pi & \text{two hard walls} \end{cases}$$

where  $x_1$  and  $x_2$  are the classical turning points (roots of  $p(x)$ )

## 3.5 Scattering (Born Approximation)

$$f^{(1)}(\mathbf{k}, \mathbf{k}') = -\frac{m}{2\pi\hbar^2} \int d^3x' e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}'} V(\mathbf{x}')$$

$$q = |\mathbf{k} - \mathbf{k}'| = 2k \sin \frac{\theta}{2}$$

$$f^{(1)}(\theta) = -\frac{2m}{\hbar^2} \frac{1}{q} \int_0^\infty r V(r) \sin qr dr$$

$$\frac{d\sigma}{d\Omega} = |f(\mathbf{k}', \mathbf{k})|^2$$

$$\text{Im} f(\mathbf{k}, \mathbf{k}) = \frac{k}{4\pi} \sigma_{tot}$$

## 3.6 Many Body Theory

- Boson wave functions are symmetric under any particle exchange
- Fermion wave functions are antisymmetric under any particle exchange

- For wave-functions with a spatial component and a spin component,
  - Boson: symmetric in both or antisymmetric in both
  - Fermion: symmetric in one and antisymmetric in the other

## 4 Statistical Physics

### 4.1 Basics

$$\begin{aligned} S &= k_B \log \Omega \\ \left( \frac{\partial S}{\partial E} \right)_{N,V} &= \frac{1}{T} \\ dE &= -PdV + TdS + \mu dN \\ C_V &= \left( \frac{\partial U}{\partial T} \right)_{V,N} \end{aligned}$$

- Isothermal:  $dT = 0$
- Adiabatic:  $dS = 0$

### 4.2 Canonical Ensemble

$$\begin{aligned} Z &= \sum_{states} e^{-\beta E} \\ \langle O \rangle &= \frac{1}{Z} \sum_s O(s) e^{-\beta E} \\ \sum_s &\rightarrow \int \frac{d^3x d^3p}{h^3} \\ U &= -\frac{\partial}{\partial \beta} \log Z \\ A &= U - TS = -k_B T \log Z \\ dA &= -PdV - SdT \end{aligned}$$

- Partition function for an ideal gas molecule:

$$\begin{aligned} Z_1 &= \frac{V}{\lambda^3} \\ \lambda &= \sqrt{\frac{h^2}{2\pi m k_B T}} \end{aligned}$$

- In general, for a system of non-interacting particles, the total partition function obeys

$$Z = \frac{1}{N!} Z_1^N$$

where  $Z_1$  is the partition function for one particle, and the  $1/N!$  factor is necessary if the particles are indistinguishable

### 4.3 Grand Canonical Ensemble

$$\begin{aligned}
Z_G &= \sum_{states} e^{-\alpha N - \beta E} \\
U &= -\frac{\partial}{\partial \beta} \log Z_G \\
\mathcal{N} &= -\frac{\partial}{\partial \alpha} \log Z_G \\
\alpha &= -\beta \mu \\
y &= e^{-\alpha} = e^{\mu/k_B T} \\
Z_G &= \sum_N y^N Z(N, \beta) \\
\Phi_G &= -k_B T \log Z_G = U - TS - \mu \mathcal{N} = -PV
\end{aligned}$$

### 4.4 Quantum Statistical Mechanics

$$\begin{aligned}
Z_G &= \prod_n [1 - \eta e^{-\beta(E_n - \mu)}]^{-\eta} \\
\langle N_n \rangle &= \frac{1}{y^{-1} e^{\beta E_n} - \eta} \\
D(E) &= \frac{g(E)}{V} = \frac{1}{V} \sum_{\mathbf{k}} (E - E_{\mathbf{k}}) \\
\varrho &= \int \frac{D(E)}{y^{-1} e^{\beta E} - 1} dE \\
\log Z_G &= -\eta V \int_0^\infty \log(1 - \eta y e^{-\beta E}) D(E) dE
\end{aligned}$$

- $\eta = +1$  for bosons,  $-1$  for fermions

#### 4.4.1 Bose Systems

- $y = 1$  at Bose-Einstein Condensation, and in all cases for photons

#### 4.4.2 Fermi Systems

$$\varrho = \int_0^{E_F} D(E) dE$$

- Any other Fermi parameter (Fermi temperature, Fermi momentum, etc) is determined from the relationship between energy and that parameter (E.g.  $T_F = E_F/k_B$ )