Week 3

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Qual Study

June 12, 2021

Solution 2018.6. The partition function for one link is

$$\begin{aligned}
Z_1 &= \sum_s e^{-\beta E_s} \\
&= \sum_{n=0}^{\infty} \left(e^{-\beta[(n+1/2)\hbar\omega_a - Fa]} + e^{-\beta[(n+1/2)\hbar\omega_b - Fb]}\right) \\
&= e^{-\beta\hbar\omega_a/2} e^{\beta F a} \sum_{n=0}^{\infty} e^{-\beta n\hbar\omega_a} + e^{-\beta\hbar\omega_b/2} e^{\beta F b} \sum_{n=0}^{\infty} e^{-\beta n\hbar\omega_b} \\
&= \frac{e^{-\beta\hbar\omega_a/2} e^{\beta F a}}{1 - e^{-\beta\hbar\omega_a}} + \frac{e^{-\beta\hbar\omega_b/2} e^{\beta F b}}{1 - e^{-\beta\hbar\omega_b}}
\end{aligned}$$

a. The average length for one link is given by

$$\begin{aligned}
\langle \ell \rangle &= \frac{1}{Z_1} \sum_s \ell_s e^{-\beta E_s} \\
&= \frac{1}{Z_1} \sum_{n=0}^{\infty} \left(a e^{-\beta[(n+1/2)\hbar\omega_a - Fa]} + b e^{-\beta[(n+1/2)\hbar\omega_b - Fb]}\right) \\
&= \frac{1}{Z_1} \left[a \frac{e^{-\beta\hbar\omega_a/2} e^{\beta F a}}{1 - e^{-\beta\hbar\omega_a}} + b \frac{e^{-\beta\hbar\omega_b/2} e^{\beta F b}}{1 - e^{-\beta\hbar\omega_b}}\right] \\
&= \left[a \frac{e^{-\beta\hbar\omega_a/2} e^{\beta F a}}{1 - e^{-\beta\hbar\omega_a}} + b \frac{e^{-\beta\hbar\omega_b/2} e^{\beta F b}}{1 - e^{-\beta\hbar\omega_b}}\right] \left[\frac{e^{-\beta\hbar\omega_a/2} e^{\beta F a}}{1 - e^{-\beta\hbar\omega_a}} + \frac{e^{-\beta\hbar\omega_b/2} e^{\beta F b}}{1 - e^{-\beta\hbar\omega_b}}\right]^{-1}
\end{aligned}$$

The links are non-interacting, so the average total length is just

$$L = N \left[a \frac{e^{-\beta\hbar\omega_a/2} e^{\beta F a}}{1 - e^{-\beta\hbar\omega_a}} + b \frac{e^{-\beta\hbar\omega_b/2} e^{\beta F b}}{1 - e^{-\beta\hbar\omega_b}}\right] \left[\frac{e^{-\beta\hbar\omega_a/2} e^{\beta F a}}{1 - e^{-\beta\hbar\omega_a}} + \frac{e^{-\beta\hbar\omega_b/2} e^{\beta F b}}{1 - e^{-\beta\hbar\omega_b}}\right]^{-1}$$

b. In the $T \to 0$ limit, the system should occupy its minimum energy state. Therefore, $\lim_{T\to 0} L = Na$ if $\frac{1}{2}\hbar\omega_a - Fa < \frac{1}{2}\hbar\omega_b - Fb$, otherwise the limit is Nb. In the $T \to \infty$ limit, $\beta \to 0$, and we have

$$\lim_{T\to\infty} L = N \lim_{T\to\infty} \left[a \frac{e^{-\beta\hbar\omega_a/2} e^{\beta F a}}{\beta\hbar\omega_a} + b \frac{e^{-\beta\hbar\omega_b/2} e^{\beta F b}}{\beta\hbar\omega_b}\right] \left[\frac{e^{-\beta\hbar\omega_a/2} e^{\beta F a}}{\beta\hbar\omega_a} + \frac{e^{-\beta\hbar\omega_b/2} e^{\beta F b}}{\beta\hbar\omega_b}\right]^{-1}$$

$$= N \lim_{T\to\infty} \left[a\frac{1}{\beta\hbar\omega_a} + b\frac{1}{\beta\hbar\omega_b} \right] \left[\frac{1}{\beta\hbar\omega_a} + \frac{1}{\beta\hbar\omega_b} \right]^{-1}$$
$$= N\frac{a/\hbar\omega_a + b/\hbar\omega_b}{1/\hbar\omega_a + 1/\hbar\omega_b}$$

Solution 2018.8. a. This is worked in box 6.2 of the P556 notes. I reproduce the solution here

$$\frac{N}{A} = \int \frac{1}{y^{-1}e^{\beta E} - \eta} \frac{d^2\mathbf{k}}{(2\pi)^2}$$
$$= \int_0^\infty \frac{1}{y^{-1}e^{\beta\hbar^2k^2/2m} - \eta}\frac{k\,dk}{2\pi} \qquad u = e^{-\beta\hbar^2k^2/2m},\ du = e^{-\beta\hbar^2k^2/2m}\frac{\beta\hbar^2}{2m}k\,dk$$
$$= \frac{2m}{\beta\hbar^2}\int_0^1 \frac{1}{y^{-1}/u - \eta}\frac{du}{u}$$
$$= \frac{m}{\pi\beta\hbar^2}\int_0^1 \frac{du}{y^{-1} - \eta u}$$
$$= -\frac{m}{\pi\beta\hbar^2\eta}\left[\log(y^{-1} - \eta u)\right]_0^1$$
$$= -\frac{m}{\pi\beta\hbar^2\eta}\left[\log(y^{-1} - \eta) - \log(y^{-1})\right]$$
$$= -\frac{m}{\pi\beta\hbar^2\eta}\left[\log(1 - \eta y)\right]$$
$$1 - \eta y = \exp\left(-\frac{\pi\hbar^2\eta}{mk_BT}\frac{N}{A}\right)$$
$$y = \eta - \eta\exp\left(-\frac{\pi\hbar^2\eta}{mk_BT}\frac{N}{A}\right)$$
$$\mu = k_BT\log\left[\eta - \eta\exp\left(-\frac{\pi\hbar^2\eta}{mk_BT}\frac{N}{A}\right)\right]$$

b.
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Solution 2017.7. a. The single particle partition function is

$$Z_1 = Z_{1,trans}Z_{1,rot}$$
$$Z_{1,trans} = \frac{V}{\lambda^3}$$
$$Z_{1,rot} = \frac{1}{h^2}\int \exp\left[-\beta\frac{1}{2I}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right)\right]\,d\phi\,d\theta\,dp_\phi\,dp_\theta$$
$$= \frac{1}{h^2}(2\pi)\int_0^\pi \sqrt{\frac{\pi}{1/2I}}\sqrt{\frac{\pi}{1/2I\sin^2\theta}}\,d\theta$$
$$= \frac{4I\pi^2}{h^2}\int_0^\pi \sin\theta\,d\theta = \frac{2I}{\hbar^2}$$

Then, the partition function for N particles is

$$Z(N, \beta) = \frac{1}{N!} Z_1^N$$

since the particles are indistinguishable, and the grand partition function is

$$\begin{aligned} Z_G &= \sum_{N=0}^{\infty} z^N Z(N, \beta) \\ &= \exp\left(z Z_1\right) \end{aligned}$$

b. The average number of particles is

$$\begin{aligned} \mathcal{N} &= -\frac{\partial}{\partial \alpha} \log Z_G \\ &= z \frac{\partial}{\partial z} \log Z_G \\ &= z \frac{\partial}{\partial z} \left(z Z_1\right) \\ &= z Z_1 \end{aligned}$$

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Solution 2017.16. ¡++¿

Solution 2016.6. ¡++¿

Solution 2016.8. ¡++¿

Solution 2015.7. ¡++¿

Solution 2015.16. ¡++¿

Solution 2014.6. ¡++¿

Solution 2014.8. ¡++¿

Solution 2013.7. ¡++¿

Solution 2013.16. ¡++¿