# Week 3

John Mastroberti
Qual Study

July 9, 2021

**Solution 2018.2.** a. Let $\ell$ be the length of the rope on the right, $L$ be the length of rope on the left when $\ell = 0$, and $\theta$ be the angle that the right mass makes with the vertical. The Lagrangian is given by

$$\begin{aligned}
T &= \frac{1}{2}m\dot{\ell}^2 + \frac{1}{2}m\dot{\ell}^2 + \frac{1}{2}m\ell^2\dot{\theta}^2 \\
V &= mg(\ell - L) - mg\ell\cos\theta \\
\mathcal{L} &= m\dot{\ell}^2 + \frac{1}{2}m\ell^2\dot{\theta}^2 + mgL + mg\ell(1 - \cos\theta)
\end{aligned}$$

The equations of motion are then

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\ell} &= m\ell\dot{\theta}^2 + mg(1 - \cos\theta) \\
\frac{\partial\mathcal{L}}{\partial\dot{\ell}} &= 2m\dot{\ell} \\
2m\ddot{\ell} &= m\ell\dot{\theta}^2 + mg(1 - \cos\theta) \\
\frac{\partial\mathcal{L}}{\partial\theta} &= mg\ell(1 + \sin\theta) \\
\frac{\partial\mathcal{L}}{\partial\dot{\theta}} &= m\ell^2\dot{\theta} \\
m\ell^2\ddot{\theta} + 2m\ell\dot{\ell}\dot{\theta} &= mg\ell(1 + \sin\theta)
\end{aligned}$$

b. If the right side initially exhibits small oscillations of the form

$$\theta(t) = \epsilon\sin\omega t$$

where $\omega = \sqrt{g/\ell}$, then the average acceleration on the left side obeys

$$
\begin{aligned}
2\left\langle \ddot{\ell} \right\rangle &= \ell \left\langle (\epsilon\omega \cos\omega t)^2 \right\rangle + g - g \left\langle \cos(\epsilon \sin\omega t) \right\rangle \\
&= \ell\epsilon^2\omega^2(1/2) + g - g \\
\left\langle \ddot{\ell} \right\rangle &= \frac{1}{4}\epsilon^2 g
\end{aligned}
$$

Since the left mass is moving upward when $\ell$ is increasing, the acceleration is in the upward direction.

**Solution 2018.15.** a. Since the three-momenta are identical and sum to zero, the momenta must lie in a plane and be 120° apart from each other. Therefore, $\theta_1$ and $\theta_2$ are $2\pi/3$, and $\phi_1$ and $\phi_2$ can be anything as long as $\phi_1 - \phi_2 = \pi$.

b.

**Solution 2017.1.** a. The stable circular orbit occurs when the potential is minimized.

$$
\begin{aligned}
V(r) &= -\frac{mM}{r} + \frac{L^2}{2mr^2} - \frac{ML^2}{mr^3} \\
\frac{\partial V}{\partial r} &= \frac{mM}{r^2} - \frac{L^2}{mr^3} + \frac{3ML^2}{mr^4} = 0 \\
0 &= m^2Mr^2 - L^2 r + 3ML^2 \\
r &= \frac{1}{2m^2M}\left(L^2 \pm \sqrt{L^4 - 12m^2M^2L^2}\right) \\
&= \frac{L^2}{2m^2M}\left(1 \pm \sqrt{1 - 12\frac{m^2M^2}{L^2}}\right) \\
\frac{\partial^2 V}{\partial r^2} &= -\frac{2mM}{r^3} + \frac{3L^2}{mr^4} - \frac{12ML^2}{mr^5} \\
&= -\frac{1}{mr^5}\left(2m^2Mr^2 - 3L^2 r + 12ML^2\right)
\end{aligned}
$$

By sketching the potential, we can check that the larger root of $\frac{\partial V}{\partial r}$ corresponds to the only local minimum of $V(r)$. Therefore, the stable circular orbit has

$$
r_0 = \frac{L^2}{2m^2M}\left(1 + \sqrt{1 - 12\frac{m^2M^2}{L^2}}\right)
$$

b. The potential can be expanded around the stable equilibrium as

$$\begin{aligned}
V(r) &\approx V(r_0) + \frac{1}{2}V''(r_0)(r-r_0)^2 \\
V''(r_0) &= -\frac{1}{mr_0^5}\left(2m^2Mr_0^2 - 3L^2r_0 + 12ML^2\right) \\
&= -\frac{1}{mr_0^5}\left(2(L^2r_0 - 3ML^2) - 3L^2r_0 + 12ML^2\right) \\
&= -\frac{1}{mr_0^5}\left(-L^2r_0 + 6ML^2\right) \\
&= \frac{L^2}{mr_0^5}\left(r_0 - 6M\right)
\end{aligned}$$

Note that $r_0 > 6M$, so $V''(r_0) > 0$ as it should be for a stable equilibrium. By comparison with the SHO potential, the angular frequency of small oscillations is

$$\omega = \sqrt{\frac{V''(r_0)}{m}} = \frac{L}{mr_0^2}\sqrt{1 - \frac{6M}{r_0}}$$

**Solution 2017.3.**

**Solution 2016.2.**

**Solution 2016.15.**

**Solution 2015.1.** a. Take the origin to be the point marked fixed. The coordinates $(x_1, y_1)$ and $(x_2, y_2)$ of the masses $m_1$ and the mass $m_2$ are

$$\begin{aligned}
x_1 &= \pm a\sin\theta \\
x_2 &= 0 \\
y_1 &= -a\cos\theta \\
y_2 &= -2a\cos\theta
\end{aligned}$$

so the Lagrangian is

$$T = \frac{1}{2}m_1(2\dot{x}_1^2 + 2\dot{y}_1^2) + m_1x_1^2\Omega^2 + \frac{1}{2}m_2\dot{y}_2^2$$

$$
\begin{aligned}
&= m_1((a\dot\theta\cos\theta)^2 + (a\dot\theta\sin\theta)^2) + m_1 a^2\sin^2\theta\Omega^2 + 2m_2a^2\dot\theta^2\sin^2\theta \\
&= m_1a^2\dot\theta^2 + m_1a^2\sin^2\theta\Omega^2 + 2m_2a^2\dot\theta^2\sin^2\theta \\
V &= -m_1ga\cos\theta - 2m_2ga\cos\theta \\
\mathcal{L} &= m_1a^2\dot\theta^2 + m_1a^2\sin^2\theta\Omega^2 + 2m_2a^2\dot\theta^2\sin^2\theta + m_1ga\cos\theta + 2m_2ga\cos\theta
\end{aligned}
$$

b. The equation of motion is

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\theta} &= m_1a^2\Omega^2\sin 2\theta + 2m_2a^2\dot\theta^2\sin 2\theta - (m_1ga + 2m_2ga)\sin\theta \\
\frac{\partial\mathcal{L}}{\partial\dot\theta} &= 2m_1a^2\dot\theta + 4m_2a^2\dot\theta\sin^2\theta \\
0 &= m_1a^2\Omega^2\sin 2\theta + 2m_2a^2\dot\theta^2\sin 2\theta - (m_1ga + 2m_2ga)\sin\theta \\
&\quad - 2m_1a^2\ddot\theta - 4m_2a^2\ddot\theta\sin^2\theta - 4m_2a^2\dot\theta^2\sin 2\theta
\end{aligned}
$$

c. The system is in dynamical equilibrium when $\ddot\theta = 0$ given $\dot\theta = 0$:

$$
\begin{aligned}
0 &= m_1a^2\Omega^2\sin 2\theta - (m_1ga + 2m_2ga)\sin\theta \\
&= \sin\theta(2m_2a^2\Omega^2\cos\theta - m_1ga - 2m_2ga) \\
\theta &= 0,\ \cos^{-1}\frac{m_1ga + 2m_2ga}{2m_2a^2\Omega^2}
\end{aligned}
$$

d.

**Solution 2015.3.**

**Solution 2014.2.** a. While the wheel is slipping, the force and torque from friction are

$$
\begin{aligned}
F_{fr} &= Mg\mu \\
\Gamma_{fr} &= Mg\mu a
\end{aligned}
$$

Therefore, if $x$ is the horizontal position of the wheel and $\theta$ is it's rotational coordinate, the equations of motion while slipping are

$$
M\ddot{x} = Mg\mu
$$

$$I\ddot{\theta} = -Mg\mu a$$

which are easily solved:

$$\begin{aligned} x(t) &= x(0) + \dot{x}(0)t + \frac{1}{2}g\mu t^2 \\ \theta(t) &= \theta(0) + \dot{\theta}(0)t - \frac{1}{2}\frac{Mg\mu a}{I}t^2 \end{aligned}$$

Given the initial conditions, we see that

$$\begin{aligned} x(t) &= \frac{1}{2}g\mu t^2 \\ \theta(t) &= \omega_0 t - \frac{1}{2}\frac{Mg\mu a}{I}t^2 \end{aligned}$$

b. Slipping stops when $x$ and $\theta$ reach the condition for rolling without slipping:

$$\begin{aligned} \dot{x} &= a\dot{\theta} \\ g\mu t &= a\omega_0 - \frac{Mg\mu a^2}{I}t \\ \tau &= \frac{a\omega_0}{g\mu + Mg\mu a^2/I} \end{aligned}$$

c. The center of mass speed for $t > \tau$ is just

$$\dot{x}(\tau) = \frac{a\omega_0}{1 + Ma^2/I}$$

**Solution 2014.15.**

**Solution 2013.1.** a. The equations of motion for $\boldsymbol{\omega}$ are

$$\begin{aligned} I_{11}\dot{\omega}_1 &= \omega_2\omega_3(I_{22} - I_{33}) \\ I_{22}\dot{\omega}_2 &= \omega_3\omega_1(I_{33} - I_{11}) \\ I_{33}\dot{\omega}_3 &= \omega_1\omega_2(I_{11} - I_{22}) \end{aligned}$$

which simplify to

$$\dot{\omega}_1 = \omega_2\omega_3\frac{I_{22} - I_{33}}{I_{22}}$$

$$
\begin{aligned}
\dot{\omega}_2 &= \omega_1\omega_3\frac{I_{33}-I_{22}}{I_{22}} \\
\dot{\omega}_3 &= 0
\end{aligned}
$$

Since $\omega_3$ is constant, $\omega_3 = \omega_r$ always. Defining $\Omega = \omega_r\frac{I_{22}-I_{33}}{I_{22}}$, we have

$$
\begin{aligned}
\dot{\omega}_1 &= \Omega\omega_2 \\
\dot{\omega}_2 &= -\Omega\omega_1
\end{aligned}
$$

The solutions to these differential equations are obviously $\sin\Omega t$ and $\cos\Omega t$. Therefore, $\omega_p = \Omega$.

b.

**Solution 2013.3.**