Week 2

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Qual Study

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Solution 2018.11. The ground state wave function should be symmetric about $x = 0$. In the regions of interest ($|x| > a$ and $0 < |x| < a$, Schrödinger's equation reads

$$\begin{cases} -\dfrac{\hbar^2}{2m}\psi'' = E\psi & |x| > a \\ -\dfrac{\hbar^2}{2m}\psi'' + V_1\psi = E\psi & 0 < |x| < a \end{cases}$$

Defining k_1 and k_2 by

$$k_1 = \sqrt{-\frac{2mE}{\hbar^2}}$$
$$k_1 = \sqrt{-\frac{2m(E - V_1)}{\hbar^2}}$$

we see that the solutions in the two regions are

$$\psi(x) = \begin{cases} Ae^{-k_1|x|} & |x| > a \\ Be^{-k_2|x|} + Ce^{k_2|x|} & 0 < |x| < a \end{cases}$$

Applying matching conditions at $|x| = a$ gives us

$$\begin{aligned} Ae^{-k_1 a} &= Be^{-k_2 a} + Ce^{k_2 a} \\ -k_1 Ae^{-k_1 a} &= -k_2 Be^{-k_2 a} + k_2 Ce^{k_2 a} \\ 0 &= \frac{k_2}{k_1}e^{-k_2 a}B - \frac{k_2}{k_1}e^{k_2 a}C \\ B &= e^{2k_2 a}C \end{aligned}$$

Due to the delta function at $x = 0$ in the potential, ψ' is not continuous at $x = 0$. In the region $0 < |x| < a$, we have

$$\psi'(x) = \left(-Bk_2 e^{-k_2|x|} + Ck_2 e^{k_2|x|}\right)\frac{|x|}{x}$$

$$\Delta\psi' \equiv \lim_{\epsilon\to 0}(\psi'(\epsilon) - \psi'(-\epsilon)) = 2k_2(C - B)$$

We also see from integrating the Schrödinger equation over the region $(-\epsilon, \epsilon)$ that

$$-\frac{\hbar^2}{2m}\Delta\psi' + V_0\psi(0) = 0$$

Thus,

$$\begin{aligned} 2k_2(C - B) &= \frac{2m}{\hbar^2}V_0\psi(0) \\ &= \frac{2m}{\hbar^2}V_0(B + C) \\ 2k_2(1 - e^{2k_2 a}) &= \frac{2m}{\hbar^2}V_0(e^{2k_2 a} + 1) \end{aligned}$$

This equation can be solved for k_2, which, using the definition of k_2, yields the ground state energy E.

Solution 2018.13. The wave function in each region is

$$\begin{aligned} \psi(x) &= \begin{cases} Ae^{-ikx} & x < 0 \\ B\cos(k'x) + C\sin(k'x) & 0 < x < a \\ De^{ikx} + Ee^{-ikx} & x > a \end{cases} \\ k &= \frac{\sqrt{2mE}}{\hbar} \\ k' &= \frac{\sqrt{2m(E - V_0)}}{\hbar} \end{aligned}$$

Ee^{-ikx} corresponds to the incident wave, so the transmission coefficient is $|A|^2 / |E|^2$. Applying matching conditions at $x = 0$ and $x = a$, we find that

$$\begin{aligned} A &= B + 0 \\ -ikA &= k'C \\ B\cos(k'a) + C\sin(k'a) &= De^{ika} + Ee^{-ika} \\ -k'B\sin(k'a) + k'C\cos(k'a) &= ikDe^{ika} - ikEe^{-ika} \end{aligned}$$

Clearly, $B = A$ and $C = -i(k/k')A$. Then

$$\begin{aligned} De^{ika} - Ee^{-ika} &= \frac{1}{ik}\left[-k'B\sin(k'a) + k'C\cos(k'a)\right] \\ 2Ee^{-ika} &= B\cos(k'a) + C\sin(k'a) - \frac{1}{ik}\left[-k'B\sin(k'a) + k'C\cos(k'a)\right] \\ &= A\cos(k'a) - i\frac{k}{k'}A\sin(k'a) - \frac{1}{ik}\left[-k'A\sin(k'a) - i\frac{k}{k'}k'A\cos(k'a)\right] \end{aligned}$$

$$
\begin{aligned}
&= A\cos(k'a) - i\frac{k}{k'}A\sin(k'a) - i\frac{k'}{k}A\sin(k'a) + A\cos(k'a) \\
\frac{E}{A} &= e^{ika}\left[\cos(k'a) - i\frac{k}{k'}\sin(k'a)\right] \\
\frac{|E|^2}{|A|^2} &= \cos^2(k'a) + \left(\frac{k}{k'}\right)^2 \sin^2(k'a)
\end{aligned}
$$

Therefore, the transmission coefficient is 100% whenever

$$
\begin{aligned}
1 &= \cos^2(k'a) + \left(\frac{k}{k'}\right)^2 \sin^2(k'a) \\
\sin^2(k'a) &= \left(\frac{k}{k'}\right)^2 \sin^2(k'a) \\
k &= k'
\end{aligned}
$$

Probably an algebra mistake somewhere.

Solution 2017.12. a. It should be even under parity

b. I don't wanna

Solution 2017.14. Since

$$
|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{1}{\sqrt{2}}|u_2\rangle
$$

we have

$$
|\psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-iE_1t/\hbar}|u_1\rangle + \frac{1}{\sqrt{2}}e^{-iE_2t/\hbar}|u_2\rangle
$$

Finally, we write A in the energy eigenbasis,

$$
\begin{aligned}
\hat{A} &= x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + y\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\hat{A}|\phi_1\rangle &= x|\phi_1\rangle + y|\phi_1\rangle = a_1|\phi_1\rangle \\
\hat{A}|\phi_2\rangle &= x|\phi_2\rangle - y|\phi_2\rangle = a_2|\phi_2\rangle \\
x &= \frac{a_1 + a_2}{2} \\
y &= \frac{a_1 - a_2}{2} \\
\hat{A} &= \frac{1}{2}\begin{pmatrix} a_1 + a_2 & a_1 - a_2 \\ a_1 - a_2 & a_1 + a_2 \end{pmatrix}
\end{aligned}
$$

so that we have

$$
\langle A(t)\rangle = \left\langle \psi(t) \middle| \hat{A} \middle| \psi(t) \right\rangle
$$

$$
\begin{aligned}
&= \frac{1}{4}\begin{pmatrix} e^{iE_1t/\hbar} & e^{iE_2t/\hbar} \end{pmatrix}\begin{pmatrix} a_1+a_2 & a_1-a_2 \\ a_1-a_2 & a_1+a_2 \end{pmatrix}\begin{pmatrix} e^{-iE_1t/\hbar} \\ e^{-iE_2t/\hbar} \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} e^{iE_1t/\hbar} & e^{iE_2t/\hbar} \end{pmatrix}\begin{pmatrix} (a_1+a_2)e^{-iE_1t/\hbar}+(a_1-a_2)e^{-iE_2t/\hbar} \\ (a_1-a_2)e^{-iE_1t/\hbar}+(a_1+a_2)e^{-iE_2t/\hbar} \end{pmatrix} \\
&= \frac{1}{4}\left[(a_1+a_2)+(a_1-a_2)e^{i(E_1-E_2)t/\hbar}+(a_1-a_2)e^{-i(E_1-E_2)t/\hbar}+(a_1+a_2)\right] \\
&= \frac{1}{2}\left[(a_1+a_2)+(a_1-a_2)\cos\frac{(E_1-E_2)t}{\hbar}\right] \\
&= a_1\cos\frac{(E_1-E_2)t}{2\hbar}+a_2\sin\frac{(E_1-E_2)t}{2\hbar}
\end{aligned}
$$

Solution 2016.11. a. Schrödinger's equation reads

$$-\frac{\hbar^2}{2m}\psi''-\frac{e^2}{4x}\psi=E\psi$$

Taking the hint's suggestion, we let $\psi(x)=u(x)e^{-kx}$ for $x>0$ ($\psi(x)=0$ for $x<0$), so that

$$
\begin{aligned}
\psi'(x) &= u'(x)e^{-kx}-ku(x)e^{-kx} \\
\psi''(x) &= u''(x)e^{-kx}-2ku'(x)e^{-kx}+k^2u(x)e^{-kx} \\
Eu &= -\frac{\hbar^2}{2m}(u''-2ku'+k^2u)-\frac{e^2}{4x}u
\end{aligned}
$$

Solution 2016.13. a. First, we write the energy eigenstates in terms of the flavor eigenstates:

$$
\begin{aligned}
\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix} \\
\begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix} &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix}
\end{aligned}
$$

Then, since

$$H=E_1\,|\nu_1\rangle\,\langle\nu_1|+E_2\,|\nu_2\rangle\,\langle\nu_2|$$

we can write

$$
\begin{aligned}
H &= E_1\left(\cos\theta\,|\nu_e\rangle-\sin\theta\,|\nu_\mu\rangle\right)\left(\cos\theta\,\langle\nu_e|-\sin\theta\,\langle\nu_\mu|\right) \\
&\quad+E_2\left(\sin\theta\,|\nu_e\rangle+\cos\theta\,|\nu_\mu\rangle\right)\left(\sin\theta\,\langle\nu_e|+\cos\theta\,\langle\nu_\mu|\right) \\
&= (E_1+E_2)\,|\nu_e\rangle\,\langle\nu_e|+(E_2+E_1)\,|\nu_\mu\rangle\,\langle\nu_\mu| \\
&\quad+(E_2-E_1)\sin\theta\cos\theta\,|\nu_\mu\rangle\,\langle\nu_e|+(E_2-E_1)\sin\theta\cos\theta\,|\nu_e\rangle\,\langle\nu_\mu|
\end{aligned}
$$

or, as a matrix in the flavor basis $\{|\nu_e\rangle\,,|\nu_\mu\rangle\}$,

$$H=\begin{pmatrix} E_1+E_2 & (E_2-E_1)\sin\theta\cos\theta \\ (E_2-E_1)\sin\theta\cos\theta & E_1+E_2 \end{pmatrix}$$

Defining $a = E_1 + E_2$ and $b = (E_2 - E_1)\sin\theta\cos\theta$ for convenience, we then have that the time evolution operator in the flavor basis is

$$\begin{aligned}
U(t, t_0 = 0) &= e^{-iHt/\hbar} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{it}{\hbar}\right)^n \begin{pmatrix} a & b \\ b & a \end{pmatrix}^n \\
\begin{pmatrix} a & b \\ b & a \end{pmatrix}^2 &= \begin{pmatrix} a^2 + b^2 & 2ab \\ 2ab & a^2 + b^2 \end{pmatrix} \\
2ab &= (E_2^2 - E_1^2)\sin 2\theta \\
a^2 + b^2 &= (E_1 + E_2)^2 + \frac{1}{4}(E_2 - E_1)^2 \sin^2 2\theta
\end{aligned}$$

Solution 2015.12. The ground state of H_0 is

$$\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$$

In free space, Schrödinger's equation is

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t)$$

Taking the Fourier transform of this equation gives us

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(x,t)e^{-ikx}\,dx &= -\frac{\hbar^2}{2m}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{\partial^2}{\partial x^2}\Psi(x,t)\right)e^{-ikx}\,dx \\
i\hbar\frac{\partial}{\partial t}\Phi(k,t) &= \frac{\hbar^2 k^2}{2m}\Phi(k,t) \\
\Phi(k,t) &= \Phi(k,0)e^{-i\frac{\hbar k^2}{2m}t} \\
\Phi(k,0) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}e^{-ikx}\,dx
\end{aligned}$$

To finish, you would do the integral for $\Phi(k,0)$, then take the inverse Fourier transform of $\Phi(k,t)$ to get $\Psi(x,t)$. $|\Psi(x,t)|^2\,dx$ would then be the desired probability.

Solution 2015.14.

Solution 2014.11. a. In this case,

$$\begin{aligned}
f^{(1)}(\mathbf{k}', \mathbf{k}) &= -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d^3x'\, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}'}V(\mathbf{x}') \\
&= -\frac{1}{4\pi}\frac{2m}{\hbar^2}\left[A - \frac{\hbar^2}{4}B\right] \\
\frac{d\sigma}{d\Omega} &= \left[\frac{1}{4\pi}\frac{2m}{\hbar^2}\left(A - \frac{\hbar^2}{4}B\right)\right]^2
\end{aligned}$$

b. In this case,

$$
\begin{aligned}
f^{(1)}(\mathbf{k}', \mathbf{k}) &= -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d^3x'\, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}'}V(\mathbf{x}') \\
&= -\frac{1}{4\pi}\frac{2m}{\hbar^2}\left[A + \frac{\hbar^2}{4}B\right] \\
\frac{d\sigma}{d\Omega} &= \left[\frac{1}{4\pi}\frac{2m}{\hbar^2}\left(A + \frac{\hbar^2}{4}B\right)\right]^2
\end{aligned}
$$

c.

Solution 2014.13. a. Using the ordering $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, the cNOT gate is written as

$$
U_{cNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
$$

b. The state after this operation is

$$
\begin{aligned}
|\psi_f\rangle &= U_{cNOT}U_H|\psi_i\rangle \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)
\end{aligned}
$$

c. You are equally likely to measure 0 or 1 for the first bit. However, after making a measurement of the first bit, the second bit must be in the same state as the first bit.

Solution 2013.12.

Solution 2013.14.

- $N\epsilon_1$

- $2\epsilon_1 + 2\epsilon_2 + \cdots + 2\epsilon_{N/2}$

- (a) $|\phi_1\rangle \otimes |\phi_1\rangle \otimes |\phi_1\rangle$

  (b) Two fold degeneracy

  $$\frac{1}{\sqrt{6}}\left(|1\uparrow;1\downarrow;2\uparrow\rangle + |1\downarrow;2\uparrow;1\uparrow\rangle + |2\uparrow;1\uparrow;1\downarrow\rangle - |1\downarrow;1\uparrow;2\uparrow\rangle - |2\uparrow;1\downarrow;1\uparrow\rangle - |1\uparrow;2\uparrow;1\downarrow\rangle\right)$$