MATH 307: Individual Homework 17

John Mays

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Problem 1

Because U and V are orthogonal, $U^* = U^{-1}$ and $V^* = V^{-1}$. Therefore,

$$\begin{aligned}
A^{-1} &= (U\Sigma V^*)^{-1} \\
&= (V^*)^{-1}\Sigma^{-1}U^{-1} \\
&= (V^{-1})^{-1}\Sigma^{-1}U^{-1} \\
A^{-1} &= V\Sigma^{-1}U^*
\end{aligned}$$

Problem 2

Inner Product Case:

$$\begin{aligned}
A^*A &= (U\Sigma V^*)^*(U\Sigma V^*) \\
&= (V^*)^*\Sigma^*U^*U\Sigma V^* \\
&= V\Sigma^*U^*U\Sigma V^* \\
&= V\Sigma^*(U^*U)\Sigma V^* \\
&= V\Sigma^*I\Sigma V^* \\
A^*A &= V\Sigma^*\Sigma V^*
\end{aligned}$$

And the new singular matrix, $(\Sigma^*\Sigma)$, will look like this:

$$(\Sigma^*\Sigma) = \begin{bmatrix} \sigma_1 & & & \dots & 0 \\ & \ddots & & \dots & 0 \\ & & \sigma_n & \dots & 0 \end{bmatrix}^{n\times m} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}^{m\times n} = \begin{bmatrix} |\sigma_1|^2 & & \\ & \ddots & \\ & & |\sigma_n|^2 \end{bmatrix}^{n\times n}$$

The last diagonal entry is equal to $|\sigma_n|^2$, which, due to the fact that all $\sigma_i > 0$, must be a positive value. Therefore A^*A is invertible.

Outer Product Case:

$$\begin{aligned}
AA^* &= (U\Sigma V^*)(U\Sigma V^*)^* \\
&= U\Sigma V^*(V^*)^*\Sigma^*U^* \\
&= U\Sigma V^*V\Sigma^*U^* \\
&= U\Sigma(V^*V)\Sigma^*U^* \\
&= U\Sigma I\Sigma^*U^* \\
AA^* &= U\Sigma\Sigma^*U^*
\end{aligned}$$

And the new singular matrix, $(\Sigma\Sigma^*)$, will look like this:

$$(\Sigma\Sigma^*) = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}^{m\times n} \begin{bmatrix} \sigma_1 & & & \dots & 0 \\ & \ddots & & \dots & 0 \\ & & \sigma_n & \dots & 0 \end{bmatrix}^{n\times m} = \begin{bmatrix} \sigma_1 & & & \dots & 0 \\ & \ddots & & \dots & 0 \\ & & \sigma_n & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{m\times m}$$

The last diagonal entry is equal to 0, which is not a positive value, therefore AA^* is not invertible.