MATH 307: Individual Homework 22

John Mays

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Problem 1

A is invertible $\implies$ All of its singular values are nonzero.

Assume A is not invertible $\implies$ $\det(A) = 0$, and that it also has all nonzero singular values.

$$\det(A) = \det(U\Sigma V^T) = \det(U)\det(\Sigma)\det(V^T) = 0$$

Both U and V are orthogonal matrices, therefore their determinants are nonzero.
Therefore, $\det(\Sigma) = 0$.
And since Σ is a diagonal matrix, its determinant is the product of its diagonal entries. Therefore $\det(0) \implies$ at least one $\sigma_i = 0 \to$ contradiction.

Therefore if A is invertible, all of its singular values must be nonzero.

All of A's singular values are nonzero $\implies$ it is invertible.

If A is invertible, then $A^{-1} = (U\Sigma V^T) = V\Sigma^{-1}U^{-1}$.
Note: Σ is a diagonal matrix with the singular values along the diagonal. In order to take the inverse of Σ, it is simply the reciprocal of all of the diagonal entries.

Assume one of the singular values is zero $\implies$ one of Σ's diagonal entries is zero. Taking the inverse would require taking the reciprocal of zero, therefore the singular value decomposition would not exist $\to$ contradiction: every matrix has a singular value decomposition.

Therefore all of A's singular values must be nonzero in order for it to be invertible.

Conclusion:

Therefore A is invertible $\iff$ All of its singular values are nonzero.

Problem 2

From properties of eigendecomposition, we know that if $A = Q\Lambda Q^{-1}$, then $-A = Q(-\Lambda)Q^{-1}$. This implies that $-A$ will have the same eigenvalues as A, just negated.

Characteristic Polynomial:

$\det(\lambda I - (-A)) = (\lambda - (-\lambda_1))(\lambda - (-\lambda_2)) \dots (\lambda - (-\lambda_n))$
Say $\lambda = 0$. Then:
$\det(A) = (\lambda_1)(\lambda_2) \dots (\lambda_n)$
Therefore the determinant of A is the product of its eigenvalues.

Problem 3

W.T.S. A is invertible $\implies \det(A) \neq 0$.

If A is invertible, $AA^{-1} = A^{-1}A = I$.
If A is invertible, all of its eigenvalues must be nonzero, because $A^{-1}v = \frac{1}{\lambda}v$.
$\det(AA^{-1}) = \det(A^{-1}A) = \det(A^{-1})\det(A) = \det(I) = 1$
Note that A^{-1} is also invertible by definition, therefore its eigenvalues must also be nonzero.
Continuing, it is possible to write the determinant of A in terms of $\det(A^{-1})$ as such:

$$\det(A) = \frac{1}{\det(A^{-1})} = \frac{1}{\lambda_1^{-1}\lambda_2^{-1}\dots\lambda_n^{-1}} \neq 0$$

Therefore if A is invertible, its determinant must be nonzero.

W.T.S. $\det(A) \neq 0 \implies A$ is invertible.

From **Problem 1**, if A is not invertible, $A = U\Sigma V^T$, will have zero-values singular values (which are just $\sigma_i = \sqrt{|\lambda_i|}$.
$\det(A) = 0 = \det(U\Sigma V^T) = \det(U)\det\Sigma\det(V^T) = ((nonzero))\det(\Sigma)((nonzero)) \implies$
$\det(\Sigma) = \sigma_1\sigma_2\dots\sigma_n \implies \lambda_1\lambda_2\dots\lambda_n = 0$
Therefore if A is not invertible, it must have a zero-valued eigenvalue.

Contrapositively, if $\det(A) \neq 0$, A is invertible.

Conclusion:

$\det(A) \neq 0 \iff A$ is invertible.