MATH 307: Individual Homework 18

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Problem 1

Bases:

$$A = U\Sigma V^*$$
$$A^* = (U\Sigma V^*)^* = V\Sigma^* U^*$$
$$\text{range}(A) = \{Ax | x \in \mathbb{F}^n\} = \text{span}(u_1, u_2, \ldots, u_r)$$
$$\text{range}(A^*) = \{A^*y | y \in \mathbb{F}^m\} = \text{span}(v_1, v_2, \ldots, v_r)$$

Proof that row rank = column rank:

Any $Ax = \sum_{j=1}^{r} (\sigma_j v_j^* x) u_j \in \text{range}(A)$ according to its SVD.

and any $A^*y = \sum_{i=1}^{r} (\sigma_i^* u_i^* y) v_i \in \text{range}(A^*)$ according to its SVD.

Both sets of linearly independent vectors must have r members.
Thus, for any A, the column space and row space both have dimension r, so their ranks are equal.

Problem 2

a)

The column (and therefore row) rank of A is five, because it has five nonzero singular values.
The set $\{v_1, v_2, \ldots, v_r\}$ is an orthonormal basis of range(A^*)

Proof:

$$A^* = (U\Sigma V^*)^* = V\Sigma^* U^*$$

b)

The nullity of $A^* = m - r = 1$ (by rank-nullity theorem).
Furthermore, a basis for null$(A^*) = \{u_{r+1}, u_{r+2}, \ldots u_n\}$

Proof:

$$\begin{aligned} A^* &= V\Sigma^* U^* \\ A^* U &= V\Sigma^* \\ A^* y &= \sum_{i=1}^{r} (\sigma_i u_i^* y) v_i \end{aligned}$$

And a vector y can only be in the null space if $A^* y = \sum_{i=1}^{r} (\sigma_i u_i^* y) v_i = 0$, which would be only those values with $\sigma_i = 0$, A.K.A: $\{u_{r+1}, u_{r+2}, \ldots u_n\}$.