MATH 307: Individual Homework 16

John Mays

04/05/21, Dr. Guo

Problem 1

$A = \begin{pmatrix} -2 & -2 \\ -1 & -3 \end{pmatrix}$

Eigenvalues:

$$\det(A - \lambda I) = \det \begin{pmatrix} -2-\lambda & -2 \\ -1 & -3-\lambda \end{pmatrix} = 0$$
$$(-2-\lambda)(-3-\lambda) - (-2)(-1) = 6 + 5\lambda + \lambda^2 - 2 = 0$$
$$\lambda^2 + 5\lambda + 4 = 0$$
$$\lambda_1 = -1,$$
$$\lambda_2 = -4$$

Eigenvectors:

$$Av_1 = \lambda_1 v_1$$
$$\begin{pmatrix} -2 & -2 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda_1 \begin{pmatrix} x \\ y \end{pmatrix}$$
$$\begin{pmatrix} -2x - 2y \\ -1x - 3y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix} \implies x = -2y$$
$$v_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$Av_2 = \lambda_2 v_2$$
$$\begin{pmatrix} -2 & -2 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda_2 \begin{pmatrix} x \\ y \end{pmatrix}$$
$$\begin{pmatrix} -2x - 2y \\ -1x - 3y \end{pmatrix} = \begin{pmatrix} -4x \\ -4y \end{pmatrix} \implies x = y$$
$$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Problem 2

Finding Eigenpair for A^{-1}:

$$\begin{aligned}
Av &= \lambda v \\
A^{-1}Av &= \lambda A^{-1}v \\
v &= \lambda A^{-1}v \\
\frac{1}{\lambda}v &= A^{-1}v \\
\lambda^{-1}v &= A^{-1}v \implies \text{the eigenpair for } A^{-1} \text{ is } (\lambda^{-1}, v).
\end{aligned}$$

Finding Eigenpair for $(A^{-1})^3$:

(Proof by induction)

We know that $A^{-1}v = \lambda^{-1}v$ is true.
Or, put another way, $(A^{-1})^k v = (\lambda^{-1})^k v$ for $k = 1$.

Assume this statement, $(A^{-1})^{k-1}v = (\lambda^{-1})^{k-1}v$, is true for $k > 1$.

$$\begin{aligned}
(A^{-1})^k v &= (A^{-1})((A^{-1})^{k-1})v \\
&= (A^{-1})((\lambda^{-1})^{k-1})v \\
&= ((\lambda^{-1})^{k-1})(A^{-1})v \\
&= ((\lambda^{-1})^{k-1})(\lambda^{-1})v \\
(A^{-1})^k v &= (\lambda^{-1})^k v \text{ for all } k \geq 1.
\end{aligned}$$

In our case, $k = 3$, so a corresponding eigenvalue for $(A^{-1})^3$ is $(\lambda^{-1})^3 = \lambda^{-3}$.

Thus, a corresponding eigenpair for $(A^{-1})^3$ is (λ^{-3}, v).

Problem 3

We know that $Pv = \lambda v, v \neq 0$ and $P^2 = P$.

Proof (by induction):

We know that $P^k v = \lambda^k v$ for $k = 1$.

Assume this statement, $P^{k-1}v = P^{k-1}v$, is true for $k > 1$.

$$\begin{aligned}
P^k v &= PP^{k-1}v \\
&= P\lambda^{k-1}v \\
&= \lambda^{k-1}Pv \\
&= \lambda^{k-1}\lambda v \\
P^k v &= \lambda^k v \text{ for all } k \geq 1.
\end{aligned}$$

Now we know $P^2 v = \lambda^2 v$ (for $k = 2$).
$Pv = P^2 v = \lambda^2 v = \lambda v \implies \lambda^2 = \lambda$, which only has two solutions: $0, 1$.
Therefore the eigenvalues for a projector may only be 0 or 1.

Problem 4

$$\begin{aligned}
A^*Av &= \lambda v \\
v^*A^*Av &= v^*\lambda v \\
A^*v^*Av &= \lambda v^*v \\
(Av)^*(Av) &= \lambda v^*v \\
\langle Av, Av\rangle &= \lambda\langle v, v\rangle \\
(||Av||_2)^2 &= \lambda(||v||_2)^2 \\
\lambda &= \frac{(||Av||_2)^2}{(||v||_2)^2} = (||A||_2)^2 \geq 0, \text{ by nonnegativity of matrix norms.}
\end{aligned}$$