MATH 307: Individual Homework 14

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Problem 1

$A \in F^{m \times n}$, where $F = \mathbb{R}$ or $\mathbb{C}$
Range$(A) = \{Ax | x \in F^n\}$
Null$(A^*) = \{x \in F^m | A^*x = 0\}$

Proof:

Suppose there are two arbitrary vectors, $x \in \text{Range}(A)$ s.t. $x = Az$ and $y \in \text{Null}(A^*)$ s.t. $A^*y = 0$.

Assume x and y are orthogonal. $\implies \langle x, y \rangle = 0$

$$\begin{aligned} \langle x, y \rangle &= \\ y^*x &= \\ y^*Az &= \\ (y^*A)^*z &= \\ (A^*y)z &= \\ (0)z &= 0 \end{aligned}$$

Since $\langle x, y \rangle = 0$, any two arbitrary vectors, $x \in \text{Range}(A)$ and $y \in \text{Null}(A^*)$, are orthogonal.

Problem 2

$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$, $A = QR$.

Gram-Schmidt:

$$e_1 = \frac{a_1}{||a_1||} = \frac{1}{\sqrt{1^2+3^2+1^2}} \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{11}} \\ \frac{3}{\sqrt{11}} \\ \frac{1}{\sqrt{11}} \end{pmatrix}$$

$$s_2 = a_2 - \langle a_2, e_1 \rangle e_1$$

$$= \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - (\frac{2}{\sqrt{11}} - \frac{3}{\sqrt{11}} + \frac{1}{\sqrt{11}}) \begin{pmatrix} \frac{1}{\sqrt{11}} \\ \frac{3}{\sqrt{11}} \\ \frac{1}{\sqrt{11}} \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$e_2 = \frac{s_2}{||s_2||} = \frac{1}{\sqrt{2^2+(-1)^2+1^2}} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$s_3 = a_3 - \langle a_3, e_1 \rangle e_1 - \langle a_3, e_2 \rangle e_2$$

$$s_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - (\frac{1}{\sqrt{11}} + \frac{3}{\sqrt{11}} + \frac{2}{\sqrt{11}}) \begin{pmatrix} \frac{1}{\sqrt{11}} \\ \frac{3}{\sqrt{11}} \\ \frac{1}{\sqrt{11}} \end{pmatrix} - (\frac{2}{\sqrt{6}} - \frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}}) \begin{pmatrix} \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - (\frac{6}{\sqrt{11}}) \begin{pmatrix} \frac{1}{\sqrt{11}} \\ \frac{3}{\sqrt{11}} \\ \frac{1}{\sqrt{11}} \end{pmatrix} - (\frac{3}{\sqrt{6}}) \begin{pmatrix} \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} \frac{6}{11} \\ \frac{18}{11} \\ \frac{6}{11} \end{pmatrix} - \begin{pmatrix} \frac{6}{6} \\ -\frac{3}{6} \\ \frac{3}{6} \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{6}{11} - 1 \\ 1 - \frac{18}{11} + \frac{1}{2} \\ 2 - \frac{6}{11} - \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{6}{11} \\ -\frac{3}{22} \\ \frac{21}{22} \end{pmatrix}$$

$$e_3 = \frac{s_3}{||s_3||} = \frac{\sqrt{22}}{\sqrt{27}} \begin{pmatrix} -\frac{6}{11} \\ -\frac{3}{22} \\ \frac{21}{22} \end{pmatrix} = \begin{pmatrix} -\frac{2\sqrt{2}}{\sqrt{33}} \\ -\frac{1}{\sqrt{66}} \\ \frac{7}{\sqrt{66}} \end{pmatrix}$$

Now, $Q = \begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{11}} & \frac{2}{\sqrt{6}} & -\frac{2\sqrt{2}}{\sqrt{33}} \\ \frac{3}{\sqrt{11}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{66}} \\ \frac{1}{\sqrt{11}} & \frac{1}{\sqrt{6}} & \frac{7}{\sqrt{66}} \end{pmatrix}$

And $R = \begin{pmatrix} a_1 \cdot e_1 & a_2 \cdot e_1 & a_3 \cdot e_1 \\ 0 & a_2 \cdot e_2 & a_3 \cdot e_2 \\ 0 & 0 & a_3 \cdot e_3 \end{pmatrix} = \begin{pmatrix} \sqrt{11} & 0 & \frac{6}{\sqrt{11}} \\ 0 & \sqrt{6} & \frac{3}{\sqrt{6}} \\ 0 & 0 & 3\frac{3}{22} \end{pmatrix}$

Problem 3

Since it is a reduced QR-decomposition, Q will have dimension $m \times n$ and R will have dimension $n \times n$.
Q can be equal to a normalized version of A where the columns are normalized as $\frac{a_i}{||a_i||}$ in order to be unitary. And R can be equal to a modified version of I, where the columns are defined as $||a_i||i_i$ (which qualifies as an upper-triangular matrix).