MATH 307: Individual Homework 19

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Problem 1

Finding E with elementary row operations:

$EA^{4\times 4} = B^{4\times 4}$

Multiply row 2 by (-3):

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1^* \\ a_2^* \\ a_3^* \\ a_4^* \end{bmatrix} = \begin{bmatrix} a_1^* \\ -3a_2^* \\ a_3^* \\ a_4^* \end{bmatrix} = A^{(1)}$$

Interchange rows 1 and 4 of the new matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1^{(1)*} \\ a_2^{(1)*} \\ a_3^{(1)*} \\ a_4^{(1)*} \end{bmatrix} = \begin{bmatrix} a_4^{(1)*} \\ a_2^{(1)*} \\ a_3^{(1)*} \\ a_1^{(1)*} \end{bmatrix} = A^{(2)}$$

Add 2 times row 2 to row 3 of the new matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1^{(2)*} \\ a_2^{(2)*} \\ a_3^{(2)*} \\ a_4^{(2)*} \end{bmatrix} = \begin{bmatrix} a_1^{(2)*} \\ a_2^{(2)*} \\ 2a_2^{(2)*} + a_3^{(2)*} \\ a_4^{(2)*} \end{bmatrix} = B$$

Substituting and Combining:

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&2&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&-3&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}a_1^*\\a_2^*\\a_3^*\\a_4^*\end{bmatrix}=B \implies$$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&2&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&-3&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}=E$$

$$\begin{bmatrix}0&0&0&1\\0&1&0&0\\0&2&1&0\\1&0&0&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&-3&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}=E$$

$$\begin{bmatrix}0&0&0&1\\0&-3&0&0\\0&-6&1&0\\1&0&0&0\end{bmatrix}=E$$

Finding the inverse of E:

$$EE^{-1}=I$$

$$\begin{bmatrix}e_1^* & =\{0,0,0,1\}\\ e_2^* & =\{0,-3,0,0\}\\ e_3^* & =\{0,-6,1,0\}\\ e_4^* & =\{1,0,0,0\}\end{bmatrix}\begin{bmatrix}e_1^{-1} & e_2^{-1} & e_3^{-1} & e_4^{-1}\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

Just from inspecting the products that $= 1$ in the identity matrix, it can be determined that E^{-1} must be something like $\begin{bmatrix}0&e&e&1\\e&-\frac{1}{3}&e&e\\e&e&e&e\\1&e&e&0\end{bmatrix}$ where the e's are yet-to-be determined values.

$$E^{-1}E=I$$

$$\begin{bmatrix}0&e&e&1\\e&-\frac{1}{3}&e&e\\e&e&e&e\\1&e&e&0\end{bmatrix}\begin{bmatrix}0&0&0&1\\0&-3&0&0\\0&-6&1&0\\1&0&0&0\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

The only row of E^{-1} with no information about it is row 3. And if we inspect the entry of the identity matrix at $2,3$, we can determine that $\begin{bmatrix}e_{31}^{-1} & e_{31}^{-1} & e_{31}^{-1} & e_{31}^{-1}\end{bmatrix}\begin{bmatrix}0\\-3\\-6\\0\end{bmatrix}=0$, and a reasonable guess at the third row of the inverse would be $\{0,2,-1,0\}$

or $\{0, -2, 1, 0\}$. However, when checked against the first operation, $\{0, -2, 1, 0\}$ is the only option that can yield the identity matrix's 1 value at $3, 3$.

If we now guess $E^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$,
we will find that the equations $EE^{-1} = I$ and $E^{-1}E = I$ are true.

Therefore $E^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.