MATH 307: Individual Homework 1

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1. a) ($\mathbb{N}$, +, 0) is a monoid, because it satisfies all three monoid conditions
   1. Closure: any addition of two natural numbers must be a natural number
   2. Associativity: order does not matter in the addition of natural numbers
   3. Identity: there exists the identity element $0 \in \mathbb{N}$.
      $a \in \mathbb{N}$, $0 + a = a + 0 = a$

   b) ($\mathbb{N}$, +) is a not a group. It is a monoid, yet it fails to satisfy the fourth additional condition for groups:
   4. Inverse: Because the natural numbers only contain positive integers, there are no two elements besides (0,0) that can be added to equal 0, the identity element.

   c) ($\mathbb{N}$, +, 0) is not an Abelian group, because it is not a group.

2. ($\mathbb{Z}$, +) is an Abelian group, because it satisfies all five requirements of Abelian groups:
   1. Closure: any addition of two integers must also be an integer.
   2. Associativity: order of addition does not matter when adding integers
   3. Identity: there exists the identity element $0 \in \mathbb{Z}$.
      $a \in \mathbb{Z}$, $0 + a = a + 0 = a$
   4. Inverse: For all $a \in \mathbb{Z}$, there exists $b \in \mathbb{Z}$ to satisfy the equations a + b = 0 and b + a = 0.
   5. Commutativity: For all $a,b \in \mathbb{Z}$, a + b = b + a.

   ($\mathbb{Z}$, $\cdot$) is a monoid, because it satisfies the three monoid conditions:
   1. Closure: any multiplication of two integers must also be an integer.
   2. Associativity: No matter the order in which a set of three integers is multiplied, the product will always be the same.
   3. Identity: there exists the identity element $1 \in \mathbb{Z}$.
      $a \in \mathbb{Z}$, $1 \cdot a = a \cdot 1 = a$

   (cont. on next page)

However, $(\mathbb{Z}, \cdot)$ is not an Abelian group, because it is not a group. It does not satisfy the inverse condition:
Proof by counterexample:

Claim: For all $a \in \mathbb{Z}$, there exists $b \in \mathbb{Z}$, for which $a \cdot b = id = 1$.

Take $3 \in \mathbb{Z}$.

$3 \cdot b = 1$

b could only be $1/3 \notin \mathbb{Z}$.

Therefore not all integers have an inverse in this monoid.