MATH 307: Individual Homework 15

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04/01/21, Dr. Guo

Problem 1

First Inequality:

$$||A|| = \sup_{x \neq 0} \frac{||Ax||}{||x||}$$

Because $||A||$ is the minimum upper bound of the possible values of the fraction, $\frac{||Ax||}{||x||}$, we can say:

$$\frac{||Ax||}{||x||} \leq ||A|| \implies ||Ax|| \leq ||A||||x||$$

Second Inequality:

$$||AB|| = \sup_{||x||=1} ||(AB)x||$$

Then, using the first inequality, we can determine that:

$$||Ax|| \leq ||A||||x||,$$
$$||Bx|| \leq ||B||||x||$$

as well as:

$$\sup_{||x||=1} ||(AB)x|| \leq \sup_{||x||=1} ||A||||Bx||$$

then, from there:

$$\begin{aligned}
\sup_{||x||=1} ||(AB)x|| \leq \sup_{||x||=1} ||A||||Bx|| \leq \sup_{||x||=1} ||A||||B||||x|| \implies \\
\sup_{||x||=1} ||(AB)x|| \leq \sup_{||x||=1} ||A||||B||||x|| \implies \\
||AB|| \leq \sup_{||x||=1} ||A||||B||||x|| \implies \\
||AB|| \leq ||A||||B||
\end{aligned}$$

Problem 2

$$A = \begin{pmatrix} 2 & 4 & -11 & 4 & -9 \\ 4 & -1 & 6 & 2 & -1 \\ 1 & 5 & 6 & -7 & -8 \\ -20 & 20 & 2 & -2 & 0 \end{pmatrix}$$

Infinity Norm:

$$||A||_\infty = \max_i \sum_{j=1}^{n} |a_{ij}| = |-20| + |20| + |2| + |-2| + |0| = 44$$

1 Norm:

$$||A||_1 = \sup_{||x||_1=1} ||Ax||_1 = \max_j \sum_{i=1}^{m} |a_{ij}| = |4| + |-1| + |5| + |20| = 30$$

Frobenius Norm:

$$||A||_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2} = (4 + 16 + 121 + 16 + 81 + 16 + 1 + 36 + 4 + 1 + 1 + 25 + 36 + 49 + 64 + 400 + 400 + 4 + 4 + 0)^{\frac{1}{2}} = \sqrt{1279}$$

Problem 3

Before this proof, it is necessary to understand that, because A has dimensions $m \times n$, P has dimensions $m \times m$, and Q has dimensions $n \times n$, the operations PA and QA will both succeed as the respective inner dimensions are equal, and furthermore, they will both yield a matrix of dimensions $m \times n$, the same as A.

First Equality:

$$\begin{aligned} ||PA||_2 &= \max_{x \neq 0} \frac{||PAx||_2}{||x||_2} \\ &= \max_{x \neq 0} \frac{||P||_2 ||Ax||_2}{||x||_2} \\ &= ||P||_2 \max_{x \neq 0} \frac{||Ax||_2}{||x||_2} \\ &= \max_{x \neq 0} \frac{||Px||_2}{||x||_2} \max_{x \neq 0} \frac{||Ax||_2}{||x||_2} \end{aligned}$$

And for any orthogonal matrix, P, $||Px||_2 = ||x||_2 \implies \max_{x \neq 0} \frac{|Px||_2}{|x||_2} = 1$

Therefore,

$$
\begin{aligned}
&= \max_{x\neq 0} \frac{||Px||_2}{||x||_2} \max_{x\neq 0} \frac{||Ax||_2}{||x||_2} \\
&= 1 \max_{x\neq 0} \frac{||Ax||_2}{||x||_2} \\
&= \max_{x\neq 0} \frac{||Ax||_2}{||x||_2} \\
||PA||_2 &= ||A||_2
\end{aligned}
$$

Second Equality:

$$
\begin{aligned}
||AQ||_2 &= \max_{x\neq 0} \frac{||AQx||_2}{||x||_2} \\
&= \max_{x\neq 0} \frac{||A||_2||Qx||_2}{||x||_2} \\
&= ||A||_2 \max_{x\neq 0} \frac{||Qx||_2}{||x||_2} \\
&= \max_{x\neq 0} \frac{||Ax||_2}{||x||_2} \max_{x\neq 0} \frac{||Qx||_2}{||x||_2}
\end{aligned}
$$

And for any orthogonal matrix, Q, $||Qx||_2 = ||x||_2 \implies \max_{x\neq 0} \frac{|Qx||_2}{|x||_2} = 1$

Therefore,

$$
\begin{aligned}
&= \max_{x\neq 0} \frac{||Ax||_2}{||x||_2} \max_{x\neq 0} \frac{||Qx||_2}{||x||_2} \\
&= \max_{x\neq 0} \frac{||Ax||_2}{||x||_2} 1 \\
&= \max_{x\neq 0} \frac{||Ax||_2}{||x||_2} \\
||AQ||_2 &= ||A||_2
\end{aligned}
$$

Problem 4

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