MATH 307: Individual Homework 23

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Problem 1

$$A = \begin{pmatrix} -2 & 1 \\ -1 & 2 \end{pmatrix}, b = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$x_1 = \frac{\det(b|a_2)}{\det(A)} = \frac{(3)(2) - (1)(3)}{(-2)(2) - (1)(-1)} = \frac{3}{-3} = -1$$

$$x_2 = \frac{\det(a_1|b)}{\det(A)} = \frac{(-2)(3) - (3)(-1)}{(-2)(2) - (1)(-1)} = \frac{-3}{-3} = 1$$

Check:

$$a_1 x = -2x_1 + x_2 = -2(-1) + (1) = 3$$

$$a_2 x = -x_1 + 2x_2 = -(-1) + 2(1) = 3$$

Therefore $x = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Problem 2

W.T.S: $Ax = 0$ has nontrivial solutions $\implies$ 0 is an eigenvalue of A.

If there exist an x such that $Ax = 0$, then $\lambda x = 0$ is true. And since the corresponding x is nonzero, λ must be 0.

W.T.S: 0 is an eigenvalue of A. $\implies$ $Ax = 0$ has nontrivial solutions.

Assume $Ax = 0$ does not have nontrivial solutions. If this is true, then $\lambda x = 0$ also has no solutions (if λ could be zero, that would be a solution.
Therefore $Ax = 0$ does not have nontrivial solutions $\implies$ $\lambda \neq 0$.
Contrapositively, $\lambda = 0$ $\implies$ that $Ax = 0$ does have nontrivial solutions.

Therefore we can say that $Ax = 0$ has nontrivial solutions $\iff$ 0 is an eigenvalue of A.

W.T.S: The determinant of $A = 0$ $\implies$ 0 is an eigenvalue of A.
$\det(A) = \lambda_1 \lambda_2 \ldots \lambda_i \ldots \lambda_n$.
If $\det(A) = 0$, then one or more of the terms in the product must also be 0.
Therefore $\det(A) = 0$ $\implies$ 0 is an eigenvalue of A.

W.T.S: 0 is an eigenvalue of A $\implies$ that the determinant of $A = 0$

$\det(A) = \lambda_1 \lambda_2 \ldots \lambda_i \ldots \lambda_n$.
If any $\lambda_i = 0$, then $\det(A)$ must also be 0.
Therefore 0 is an eigenvalue of A $\implies$ $\det(A) = 0$.
Therefore we can now say The determinant of $A = 0$ $\iff$ 0 is an eigenvalue of A.
Combining with our first iff. claim, we can now say that: The determinant of $A = 0$ $\iff$ 0 is an eigenvalue of A $\iff$ $Ax = 0$ has nontrivial solutions.

Problem 3

From **Homework 22**, we have proved that full rank $\implies$ the matrix is invertible. Since A is of full rank, A is invertible.
Therefore if we take the normal equation:

$$A^*Ax = A^*b \to (A^*A)^{-1}(A^*A)x = (A^*A)^{-1}A^*b \to x = (A^*A)^{-1}A^*b$$

we can conclude that x is unique for any given A and b because it is solely determined by A and b.