MATH 307: Individual Homework 12

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Problem 1

$A = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$

Assume B is a right inverse of $A \implies AB = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Then,
$\begin{pmatrix} 1 & i \end{pmatrix} b_1 = 1$
$\begin{pmatrix} 1 & i \end{pmatrix} b_2 = 0$
$\begin{pmatrix} 1 & -i \end{pmatrix} b_1 = 0$
$\begin{pmatrix} 1 & -i \end{pmatrix} b_2 = 1$

Therefore $b_1 = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2}i \end{pmatrix}$ and $b_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2}i \end{pmatrix}$. So $B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2}i & \frac{1}{2}i \end{pmatrix}$.

To confirm that B is a left inverse of A, BA should be equal to I.

$BA = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2}i & \frac{1}{2}i \end{pmatrix} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} = \begin{pmatrix} (\frac{1}{2})(1) + (\frac{1}{2})(1) & (\frac{1}{2})(i) + (\frac{1}{2})(-i) \\ (-\frac{1}{2})(1) + (\frac{1}{2})(1) & (-\frac{1}{2})(i) + (\frac{1}{2})(-i) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

Since B is a left and right inverse of A, $B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2}i & \frac{1}{2}i \end{pmatrix}$ is *the* inverse of A.

Problem 2

$A = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 2\sqrt{2} \end{pmatrix}$

For notation purposes, $\mathbf{x}$ is a two dimensional vector s.t. $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$.

a.

Writing out $A\mathbf{x} = \mathbf{b}$ yields: $\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 2\sqrt{2} \end{pmatrix}$

This yields two equations for each row of the $\mathbf{x}$ vector:

$\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y = 0$

$\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y = 2\sqrt{2}$

From the first equation, we can determine that $x = y$.

Plugging into the second, we can ascertain that $\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}x = 2\sqrt{2} \implies \sqrt{2}x = 2\sqrt{2} \implies x = 2 \implies y = 2$

Therefore $\mathbf{x} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

b.

Since $\mathbf{b}$ is the vector $\mathbf{x}$ after having been rotated positive $45°$, $\mathbf{x}$ can be thought of as $\mathbf{b}$ rotated negative $45°$.

To calculate $\mathbf{x}$, we can first calculate the "reversed" rotation matrix for $\theta = -45° = -\frac{\pi}{4}$ as $\begin{pmatrix} \cos -\frac{\pi}{4} & -\sin -\frac{\pi}{4} \\ \sin -\frac{\pi}{4} & \cos -\frac{\pi}{4} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$

Then calculate the "reversed" rotation matrix times $\mathbf{b}$ to yield $\mathbf{x}$:

$\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 2\sqrt{2} \end{pmatrix} = \begin{pmatrix} (\frac{\sqrt{2}}{2})(0) + (\frac{\sqrt{2}}{2})(2\sqrt{2}) \\ -\frac{\sqrt{2}}{2})(0) + (\frac{\sqrt{2}}{2})(2\sqrt{2}) \end{pmatrix} = \mathbf{x} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

c.

To prove that A is an orthogonal matrix, we must prove that $AA^T = I = A^T A$.

The first necessary calculation is to find A^T. $A^T = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$.

Then we calculate the LHS and RHS:

LHS:

$AA^T = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} (\frac{\sqrt{2}}{2})(\frac{\sqrt{2}}{2}) + (-\frac{\sqrt{2}}{2})(-\frac{\sqrt{2}}{2}) & (\frac{\sqrt{2}}{2})(\frac{\sqrt{2}}{2}) + (-\frac{\sqrt{2}}{2})(\frac{\sqrt{2}}{2}) \\ (\frac{\sqrt{2}}{2})(\frac{\sqrt{2}}{2}) + (\frac{\sqrt{2}}{2})(-\frac{\sqrt{2}}{2}) & (\frac{\sqrt{2}}{2})(\frac{\sqrt{2}}{2}) + (\frac{\sqrt{2}}{2})(\frac{\sqrt{2}}{2}) \end{pmatrix} =$
$\begin{pmatrix} \frac{1}{2} + \frac{1}{2} & \frac{1}{2} - \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2} & \frac{1}{2} + \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

RHS:

$A^T A = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} (\frac{\sqrt{2}}{2})(\frac{\sqrt{2}}{2}) + (\frac{\sqrt{2}}{2})(\frac{\sqrt{2}}{2}) & (\frac{\sqrt{2}}{2})(-\frac{\sqrt{2}}{2}) + (\frac{\sqrt{2}}{2})(\frac{\sqrt{2}}{2}) \\ (-\frac{\sqrt{2}}{2})(\frac{\sqrt{2}}{2}) + (\frac{\sqrt{2}}{2})(-\frac{\sqrt{2}}{2}) & (-\frac{\sqrt{2}}{2})(\frac{\sqrt{2}}{2}) + (\frac{\sqrt{2}}{2})(\frac{\sqrt{2}}{2}) \end{pmatrix} =$
$\begin{pmatrix} \frac{1}{2} + \frac{1}{2} & -\frac{1}{2} + \frac{1}{2} \\ -\frac{1}{2} + \frac{1}{2} & \frac{1}{2} + \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Therefore A is an orthogonal matrix.

And since it is orthogonal, $A^{-1} = A^T = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$.

$$\begin{aligned}
A\mathbf{x} &= \mathbf{b} \\
A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} \\
I\mathbf{x} &= A^{-1}\mathbf{b} \\
\mathbf{x} &= A^{-1}\mathbf{b} \\
\mathbf{x} &= \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 2\sqrt{2} \end{pmatrix} \\
\mathbf{x} &= \begin{pmatrix} (\frac{\sqrt{2}}{2})(0) + (\frac{\sqrt{2}}{2})(2\sqrt{2}) \\ -\frac{\sqrt{2}}{2})(0) + (\frac{\sqrt{2}}{2})(2\sqrt{2}) \end{pmatrix} \\
\mathbf{x} &= \begin{pmatrix} 2 \\ 2 \end{pmatrix}
\end{aligned}$$