MATH 307: Individual Homework 15

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Problem 1

First Inequality:

$$||A|| = \sup_{x \neq 0} \frac{||Ax||}{||x||}$$

Because $||A||$ is the minimum upper bound of the possible values of the fraction, $\frac{||Ax||}{||x||}$, we can say:

$$\frac{||Ax||}{||x||} \leq ||A|| \implies ||Ax|| \leq ||A||||x||$$

Second Inequality:

$$||AB|| = \sup_{||x||=1} ||(AB)x||$$

Then, using the first inequality, we can determine that:

$$||Ax|| \leq ||A||||x||,$$
$$||Bx|| \leq ||B||||x||$$

as well as:

$$\sup_{||x||=1} ||(AB)x|| \leq \sup_{||x||=1} ||A||||Bx||$$

then, from there:

$$\begin{aligned}
\sup_{||x||=1} ||(AB)x|| \leq \sup_{||x||=1} ||A||||Bx|| \leq \sup_{||x||=1} ||A||||B||||x|| &\implies \\
\sup_{||x||=1} ||(AB)x|| \leq \sup_{||x||=1} ||A||||B||||x|| &\implies \\
||AB|| \leq \sup_{||x||=1} ||A||||B||||x|| &\implies \\
||AB|| \leq ||A||||B||
\end{aligned}$$

Problem 2

$$A = \begin{pmatrix} 2 & 4 & -11 & 4 & -9 \\ 4 & -1 & 6 & 2 & -1 \\ 1 & 5 & 6 & -7 & -8 \\ -20 & 20 & 2 & -2 & 0 \end{pmatrix}$$

Infinity Norm:

$$||A||_\infty = \max_i \sum_{j=1}^{n} |a_{ij}| = |-20| + |20| + |2| + |-2| + |0| = 44$$

1 Norm:

$$||A||_1 = \sup_{||x||_1=1} ||Ax||_1 = \max_j \sum_{i=1}^{m} |a_{ij}| = |4| + |-1| + |5| + |20| = 30$$

Frobenius Norm:

$$||A||_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2} = (4 + 16 + 121 + 16 + 81 + 16 + 1 + 36 + 4 + 1 + 1 + 25 + 36 + 49 + 64 + 400 + 400 + 4 + 4 + 0)^{\frac{1}{2}} = \sqrt{1279}$$

Problem 3

Before this proof, it is necessary to understand that, because A has dimensions $m \times n$, P has dimensions $m \times m$, and Q has dimensions $n \times n$, the operations PA and QA will both succeed as the respective inner dimensions are equal, and furthermore, they will both yield a matrix of dimensions $m \times n$, the same as A.

First Equality:

$$\begin{aligned} ||PA||_2 &= \max_{x \neq 0} \frac{||PAx||_2}{||x||_2} \\ &= \max_{x \neq 0} \frac{||P||_2 ||Ax||_2}{||x||_2} \\ &= ||P||_2 \max_{x \neq 0} \frac{||Ax||_2}{||x||_2} \\ &= \max_{x \neq 0} \frac{||Px||_2}{||x||_2} \max_{x \neq 0} \frac{||Ax||_2}{||x||_2} \end{aligned}$$

And for any orthogonal matrix, P, $||Px||_2 = ||x||_2 \implies \max_{x \neq 0} \frac{|Px||_2}{|x||_2} = 1$

Therefore,

$$
\begin{aligned}
&= \max_{x\neq 0} \frac{||Px||_2}{||x||_2} \max_{x\neq 0} \frac{||Ax||_2}{||x||_2} \\
&= 1 \max_{x\neq 0} \frac{||Ax||_2}{||x||_2} \\
&= \max_{x\neq 0} \frac{||Ax||_2}{||x||_2} \\
||PA||_2 &= ||A||_2
\end{aligned}
$$

Second Equality:

$$
\begin{aligned}
||AQ||_2 &= \max_{x\neq 0} \frac{||AQx||_2}{||x||_2} \\
&= \max_{x\neq 0} \frac{||A||_2||Qx||_2}{||x||_2} \\
&= ||A||_2 \max_{x\neq 0} \frac{||Qx||_2}{||x||_2} \\
&= \max_{x\neq 0} \frac{||Ax||_2}{||x||_2} \max_{x\neq 0} \frac{||Qx||_2}{||x||_2}
\end{aligned}
$$

And for any orthogonal matrix, Q, $||Qx||_2 = ||x||_2 \implies \max_{x\neq 0} \frac{|Qx||_2}{|x||_2} = 1$

Therefore,

$$
\begin{aligned}
&= \max_{x\neq 0} \frac{||Ax||_2}{||x||_2} \max_{x\neq 0} \frac{||Qx||_2}{||x||_2} \\
&= \max_{x\neq 0} \frac{||Ax||_2}{||x||_2} 1 \\
&= \max_{x\neq 0} \frac{||Ax||_2}{||x||_2} \\
||AQ||_2 &= ||A||_2
\end{aligned}
$$

Problem 4

First Assignment:

The assignment $\max(A) = \max\{a_{ij}\}$ does not define a norm, because it fails to satisfy the requisite of nonnegativity.

Proof (by counterexample):

Say $A = \begin{pmatrix} -4 & -3 \\ -1 & -2 \end{pmatrix}$.
$\max(A) = \max\{-4, -3, -1, -2\} = -1$, which is negative and therefore invalidates nonnegativity.

Second Assignment:

The assignment $\max(A) = \max\{|a_{ij}|\}$ defines a matrix norm, because it satisfies all three requisites of a matrix norm.

Proof:

1. Nonnegativity

Since the only possible results are the magnitudes of elements of the matrix, which can only be nonnegative, the result of the max function must also be nonnegative. Furthermore, if $|a_{ij}| = 0$, then $a_{ij} = 0$.

2. Scaling

$\max(\alpha A) \implies \forall a_{ij} \in A$, the ij-th entry in $\alpha A = \alpha a_{ij}$.
Therefore,

$$\begin{aligned}
\max(\alpha A) &= \max\{|\alpha a_{ij}|\} \\
&= \max\{|\alpha||a_{ij}|\} \\
&= |\alpha| \max\{|a_{ij}|\} \\
\max(\alpha A) &= |\alpha| \max(A)
\end{aligned}$$

3. Triangle Inequality

Assume the opposite is true:

$$\begin{aligned}
\max(A + B) &> \max(A) + \max(B) \\
\max\{|a_{ij} + b_{ij}|\} &> \max\{|a_{ij}|\} + \max\{|b_{ij}|\}
\end{aligned}$$

Then there are two cases:
1. The a_{ij} in $\max(A)$ and the b_{ij} in $\max(B)$ have the same sign. If this is the case, then those are also the two elements contained within $\max(A + B)$,

in which case $\max(A+B) = \max(A) + \max(B)$, which *contradicts* the original assumption.
2. The a_{ij} in $\max(A)$ and the b_{ij} in $\max(B)$ have the opposite sign. Then, either those remain as the a_{ij} and b_{ij} in $\max(A+B)$, in which case $\max(A+B) < \max(A) + \max(B)$ (*contradiction*), or two different elements are chosen. However, since their individual magnitudes are not greatest, the absolute value of their sum must be less than the sum of the two greatest absolute values (*contradiction*).
The assumption leads to a contradiction in all cases, therefore we must conclude the opposite:

$$\max(A+B) \leq \max(A) + \max(B)$$