MATH 307: Individual Homework 6

John Mays

02/24/21, Dr. Guo

Problem 1

$\langle \cdot, \cdot \rangle : \mathbb{C}^4 \times \mathbb{C}^4 \to \mathbb{C}$, $\langle z, w \rangle = z_1 w_1 + 2z_2 w_2 + 3z_3 w_3 + 4z_4 w_4$ does not define an inner product, because it fails to satisfy the condition of positivity.

Proof:

Suppose $v \in \mathbb{C}^4 = (i, 0, 0, 0)$. Then $\langle v, v \rangle = (i)(i) + 2(0)(0) + 3(0)(0) + 4(0)(0) = -1 \neq 0$
$\langle v, v \rangle < 0$, yet $v \neq 0$.
This assignment does not satisfy positivity on $\mathbb{C}^4$, and therefore it does not define an inner product.

Problem 2

$\langle \cdot, \cdot \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $\langle x, y \rangle = 10x_1 y_1 + x_2 y_2 + 3x_3 y_3$ is an inner product, because it satisfies all three conditions of an inner product.

Proof:

1. Conjugate Symmetry:

$\langle x, y \rangle = 10x_1 y_1 + x_2 y_2 + 3x_3 y_3 = 10y_1 x_1 + y_2 x_2 + 3y_3 x_3 = \overline{\langle y, x \rangle}$

2. Positivity:

$\langle x, x \rangle = 10x_1 x_1 + x_2 x_2 + 3x_3 y_3 = 10(x_1)^2 + (x_2)^2 + 3(x_3)^2$
$\forall x \in \mathbb{R} \backslash 0, x^2 \geq 0$ and $\forall \{x, x_1, \ldots, x_n | \in \mathbb{R}, x_n \geq 0\}, \sum_{i=1}^{n} x_n \geq 0$ therefore the expression $10(x_1)^2 + (x_2)^2 + 3(x_3)^2 \geq 0$ when $x \in \mathbb{R}^3 \neq 0$.

3. Sesquilinearity:

$\langle \alpha x + y, z \rangle = 10(\alpha x_1 + y_1)(z_1) + (\alpha x_2 + y_2)(z_2) + 3(\alpha x_3 + y_3)(z_3) = 10(\alpha x_1)(z_1) + (\alpha x_2)(z_2) + 3(\alpha x_3)(z_3) + 10(+y_1)(z_1) + (y_2)(z_2) + 3(y_3)(z_3) = \alpha[10(x_1)(z_1) + (x_2)(z_2) + 3(x_3)(z_3)] + 10(y_1)(z_1) + (y_2)(z_2) + 3(y_3)(z_3) = \alpha\langle x, z \rangle + \langle y, z \rangle$

Lengths and Orthogonality

$|(1,-1,2)| = |v| = \sqrt{\langle v,v\rangle} = \sqrt{10(1)^2+(-1)^2+3(2)^2 = 10+1+12} = \sqrt{23}$
$|(0,3,-2)| = |w| = \sqrt{\langle w,w\rangle} = \sqrt{10(0)^2+(3)^2+3(-2)^2 = 0+9+12} = \sqrt{21}$
$\cos\theta = \dfrac{\langle v,x\rangle}{\sqrt{\langle v,v\rangle}\sqrt{\langle w,w\rangle}} = \dfrac{10(1)(0)+(-1)(3)+3(2)(-2)}{21\times 23} = \dfrac{-15}{\sqrt{23}\sqrt{21}} \neq 0,$
therefore they are not orthogonal.

The lengths of $(1,-1,2)$ and $(0,3,-2)$ are $\sqrt{23}$ and $\sqrt{21}$ respectively. The cosine between them is $\frac{-15}{\sqrt{23}\sqrt{21}}$, and they are not orthogonal.

Problem 3

$\langle x,y\rangle = x_1y_1 - x_2y_2$ is not an inner product on $\mathbb{R}^2$, because it does not satisfy the condition of positivity:

Proof:

Suppose $x = \{0,1\}$. Then $\langle x,x\rangle = (0)(0)-(1)(1) = -1 \ngeq 0$

Problem 4

$\langle A,B\rangle = \sum_j\sum_k a_{jk}\overline{b_{jk}}$ defines an inner product on $\mathbb{C}^{m\times n}|A,B\in\mathbb{C}^{m\times n}, m,n\in\mathbb{Z}$ because it satisfies all three conditions of an inner product:

Proof:

Suppose

$$A = \begin{bmatrix} a_{11,R}+ia_{11,I} \\ a_{21,R}+ia_{21,I} \\ \vdots \\ a_{n1,R}+ia_{n1,I} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11,R}+ib_{11,I} \\ b_{21,R}+ib_{21,I} \\ \vdots \\ b_{n1,R}+ib_{n1,I} \end{bmatrix}$$

1. Conjugate Symmetry:

$\langle A,B\rangle = \sum_j\sum_k a_{jk}\overline{b_{jk}} = \sum_j\sum_k a_{j,R}b_{k,R} + a_{j,I}b_{k,R} + ia_{j,I}b_{k,R} + ia_{j,R}b_{k,I} = \overline{\langle B,A\rangle}$

2. Positivity:

$\langle A, A \rangle = \sum_j \sum_k a_{j,R}^2 + a_{j,I}^2 \geq 0$ if $A \neq 0$

3. Sesquilinearity: