MATH 307: Individual Homework 21

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Problem 1

$L_1 = I + l_1 e_1^T$
$L_1^{-1} = I - l_1 e_1^T$
If this is true, then $L_1 L_1^{-1} = L_1^{-1} L_1 = I$

Proof:

First Part: $L_1 L_1^{-1}$

$$\begin{aligned} L_1 L_1^{-1} &= (I + l_1 e_1^T)(I - l_1 e_1^T) \\ &= II + l_1 e_1^T I - l_1 e_1^T I - l_1 e_1^T l_1 e_1^T \\ &= II - l_1 e_1^T l_1 e_1^T \end{aligned}$$

Multiplying $l_1 e_1^T$ by $l_1 e_1^T$ yields a 0 matrix, because every nonzero entry in the first row of $l_1 e_1^T$ is multiplied by a 0 in the calculation of the resultant matrix:

$$l_1 e_1^T l_1 e_1^T = \begin{bmatrix} 0 & \dots & 0 \\ -l_{12} & \dots & 0 \\ \vdots & \ddots & 0 \\ -l_{1n} & \dots & 0 \end{bmatrix} \begin{bmatrix} 0 & \dots & 0 \\ -l_{12} & \dots & 0 \\ \vdots & \ddots & 0 \\ -l_{1n} & \dots & 0 \end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}$$

Therefore,

$$\begin{aligned} &= II - l_1 e_1^T l_1 e_1^T \\ &= II - 0 \\ L_1 L_1^{-1} &= I \end{aligned}$$

Second Part: $L_1^{-1} L_1$

$$\begin{aligned} L_1^{-1} L_1 &= (I - l_1 e_1^T)(I + l_1 e_1^T) \\ &= II - l_1 e_1^T I + l_1 e_1^T I - l_1 e_1^T l_1 e_1^T \\ &= II - l_1 e_1^T l_1 e_1^T \end{aligned}$$

From the first part of the proof, we know that $II - l_1 e_1^T l_1 e_1^T = II - 0 = I$ Therefore,

$$L_1^{-1} L_1 = I$$

Conclusion:

$L_1 L_1^{-1} = L_1^{-1} L_1 = I$, therefore $L^{-1} = I - l_1 e_1^T$ is the inverse of $L_1 = I + l_1 e_1^T$.

Problem 2

$$A|b = \begin{pmatrix} -1 & 2 & 1 & 3 & 2 \\ -3 & 2 & 1 & 0 & -5 \\ -3 & 2 & 1 & 1 & 2 \end{pmatrix}$$

Subtract 3 times row 1 from row 2.
Subtract 3 times row 1 from row 3.

$$= \begin{pmatrix} -1 & 2 & 1 & 3 & 2 \\ 0 & -4 & -2 & -9 & -11 \\ 0 & -4 & -2 & -8 & -4 \end{pmatrix}$$

Subtract row 2 from row 3.

$$U|\widetilde{b} = \begin{pmatrix} -1 & 2 & 1 & 3 & 2 \\ 0 & -4 & -2 & -9 & -11 \\ 0 & 0 & 0 & 1 & 7 \end{pmatrix}$$

Finding a particular solution:

$x_4 = 7$. Say $x_3 = 0 \rightarrow -4x_2 - 2(0) - 9(7) = -11 \rightarrow x_2 = -13$. $-x_1 + 2(-13) + 1(0) + 3(7) = 2 \rightarrow x_1 = 7$.

Therefore $x_p = \begin{pmatrix} 7 \\ -13 \\ 0 \\ 7 \end{pmatrix}$

To find a vector in the null space of A, I will solve $\begin{pmatrix} -1 & 2 & 1 & 3 & 0 \\ 0 & -4 & -2 & -9 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

for $x_3 = 1 \rightarrow x = \begin{pmatrix} 0 \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$

Therefore a general solution is $\begin{pmatrix} 7 \\ -13 \\ 0 \\ 7 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$

Problem 3

a)

$$A|b = \begin{pmatrix} -1 & 2 & 1 & 0 & 2 & -1 \\ 2 & 0 & 0 & 3 & -1 & 0 \\ -1 & 6 & 3 & 3 & 5 & c \end{pmatrix}$$

Add 2 times row 1 to row 2
Subtract row 1 from row 3

$$= \begin{pmatrix} -1 & 2 & 1 & 0 & 2 & -1 \\ 0 & 4 & 2 & 3 & 3 & -2 \\ 0 & 4 & 2 & 3 & 3 & c+1 \end{pmatrix}$$

c must be $= -3$.

b)

Say $x_3 = x_4 = x_5 = 0$. $\rightarrow 4x_2 = -2 \rightarrow x_2 = -\frac{1}{2}$. $\rightarrow -1x_1 + 2(-\frac{1}{2}) = -1 \rightarrow x_1 = 0$

Therefore, $x_p = \begin{pmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}$