MATH 307: Individual Homework 5

John Mays

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Problem 1

(a)

A corollary to the Exchange Theorem states that if a vector space admits a finite basis, then the number of the elements in any basis is the same. We know that one basis for P^4 is $\{1, x, x^2, x^3, x^4\}$, which has five elements. However, $\{1, x, -x^2, x^3\}$ only has four elements, therefore it is not a basis of P^4.

(b)

$\{1, 1+x, 1+x+x^2, x^2+x^3, x^3-x^4\}$ is linearly independent because each each vector of five has a successively higher highest polynomial degree.
$\{1, 1+x, 1+x+x^2, x^2+x^3, x^3-x^4\}$ is a spanning set. To prove this we can translate these vectors into a known spanning set:

$$\begin{aligned}
1 &= 1 \\
x &= (1+x) - 1 \\
x^2 &= (1+x+x^2) - (1+x) \\
x^3 &= (x^2+x^3) - x^2 \\
x^4 &= -(x^3-x^4) + x^3
\end{aligned}$$

Therefore $\{1, 1+x, 1+x, 1+x+x^2, x^2+x^3, x^3-x^4\}$ is a basis of P^4, because it is both a spanning set of P^4 and linearly independent.

(c)

$v = \{-x^4, x^3, -x^2, x, -1\}$ is linearly independent because each vector of five is mapped to exactly one of the five polynomial degrees in P^4: 0,1,2,3,4,5. If $\sum_{j=1}^{4} \alpha_j v_j = 0$, then $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = alpha_5 = 0$.
$v = \{-x^4, x^3, -x^2, x, -1\}$ is a spanning set. To prove this we can translate these vectors into a known spanning set:

$$
\begin{aligned}
1 &= -(-1) \\
x &= x \\
x^2 &= -(-x^2) \\
x^3 &= x^3 \\
x^4 &= -(-x^4)
\end{aligned}
$$

Therefore $v = \{-x^4, x^3, -x^2, x, -1\}$ is a basis of P^4, because it is both a spanning set of P^4 and linearly independent.

(d)

A corollary to the Exchange Theorem states that if a vector space admits a finite basis, then the number of the elements in any basis is the same. We know that one basis for P^4 is $\{1, x, x^2, x^3, x^4\}$, which has five elements. However, $\{5, x^4, x^3 - x^2, x^2 - x, x + 10, x^2 - 5\}$ has six elements, therefore it is not a basis of P^4.

Problem 2

5 vectors in $\mathbb{R}^4$ cannot be linearly independent. The exchange theorem states that if $\mathbb{R}^4$, which is spanned by $\{e_1, e_2, e_3, e_4\}$, has any subset of linearly independent vectors, they can contain at most 4 elements. Another way of putting this is that, if another vector were to be added to the spanning set, it would essentially have to be a linear combination of e_1, e_2, e_3, e_4, which violates linear independence. Therefore four is the maximum size of a linearly independent set in $\mathbb{R}^4$.

Problem 3

5 vectors in $\mathbb{R}^6$ can be linearly independent. A set with five vectors would be allowed by the exchange theorem. Here is an example of one:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

All vectors are linearly independent.

Problem 4

The dimension of $\mathbb{C}^{3\times 2}$ over $\mathbb{C}$ is 6. $e_{11}, e_{12}, e_{21}, e_{22}, e_{31}, e_{32}, e_{32} - e_{11}$ is not a basis of $\mathbb{C}^{3\times 2}$. A quick proof is to look at the last vector, $e_{32} - e_{11}$, which is clearly expressed as a linear combination of two other vectors, e_{11} and e_{32}. In a set of vectors that is linearly independent, it is not possible to express any vector as a linear combination of other vectors. Therefore this set is not linearly independent, and therefore this set is not a basis of $\mathbb{C}^{3\times 2}$.