MATH 307: Individual Homework 11

John Mays

03/15/21, Dr. Guo

Problem 1

The function $f : P^3 \to \mathbb{R}$, where $f(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3) = \alpha_3$, is a linear mapping, because it satisfies both conditions of a linear mapping.

Proof:

Let $\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$ and $\beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3$ be vectors in P^3, and let $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2, \beta_3$ and λ be scalars in $\mathbb{R}$.
1. Scalar Multiplication:

$$\begin{aligned}
&f(\lambda(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3)) \\
&= f(\lambda\alpha_0 + \lambda\alpha_1 x + \lambda\alpha_2 x^2 + \lambda\alpha_3 x^3)) \\
&= \lambda\alpha_3 \\
&= \lambda f(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3)
\end{aligned}$$

2. Vector Addition:

$$\begin{aligned}
&f((\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3) + (\beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3)) \\
&= f((\alpha_0 + \beta_0) + (\alpha_1 + \beta_1)x + (\alpha_2 + \beta_2)x^2 + (\alpha_3 + \beta_3)x^3) \\
&= \alpha_3 + \beta_3 \\
&= f(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3) + f(\beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3)
\end{aligned}$$

Problem 2

A: Symmetric, Hermitian, Diagonal: (Upper-Triangular, Lower-Triangular)
B: Lower-Triangular
C: Upper-Triangular
D: Symmetric
E: Hermitian
F: None

Problem 3

$$(\alpha A + \beta B)^T = \alpha A^T + \beta B^T$$

LHS:

The matrix $\alpha A + \beta B$ has a ji-th entry $= \alpha a_{ji} + \beta b_{ji}$.
By the mechanics of a transpose, the ij-th entry of the $(\alpha A + \beta B)^T$ will equal the ji-th entry of $(\alpha A + \beta B)$.
In other words, the ij-th entry of the LHS $= \alpha a_{ji} + \beta b_{ji}$.

RHS:

The matrix A has a ji-th entry $= a_{ji}$
The matrix B has a ji-th entry $= b_{ji}$
By the definition of a transpose, the matrix A^T has a ij-th entry $= a_{ji}$
By the definition of a transpose, the matrix B^T has a ij-th entry $= b_{ji}$
Then by the scalar multiplication and matrix addition properties, we can say that the ij-th entry of $\alpha A^T + \beta B^T$, the RHS $= \alpha a_{ji} + \beta b_{ji}$.

Conclusion:

For all legal indices i, j, the ij-th entry of $\alpha A^T + \beta B^T$ = the ij-th entry of $(\alpha A + \beta B)^T$.
Therefore $\alpha A^T + \beta B^T = (\alpha A + \beta B)^T$.

Problem 4

For the sake of notation, let's call B the conjugate transpose of A, s.t. $B = A^*$
Assume the matrix A is Hermitian.
$\implies A = B$
$\implies a_{ij} \in A = b_{ij} \in B$
If $a_{ij} \in A$ is on the diagonal, then $i = j$. Therefore $b_{ji} \in B = b_{ij} \in B$
This implies that the conjugate transpose of any $a_{ij} \in A$ on the diagonal of A is mapped to the same position, (i, j).
Therefore $a_{ij} = b_{ij} = \overline{a_{ij}}$. This can only be true if $\mathrm{Im}(a_{ij}) = 0$.
Therefore the diagonal entries of a Hermitian matrix must be real-valued.