MATH 307: Individual Homework 10

John Mays

03/10/21, Dr. Guo

Problem 1

$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$P_{v_1}(v_4) = \frac{\langle v_4, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 = \frac{1+2}{1+1+4} v_1 = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.5 \\ 0 \\ 1 \end{bmatrix}$

$P_{v_4}(v_1) = \frac{\langle v_1, v_4 \rangle}{\langle v_4, v_4 \rangle} v_4 = \frac{1+2}{1+1+1} v_1 = v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Problem 2

$\{1, x, x^2\}$ is a linearly independent set, because no vector in the set can be expressed as a linear combination of the others.

Proof:

$\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$.
$\alpha_1(1) + \alpha_2(x) + \alpha_3(x^2) = 0 \implies \alpha_1 = \alpha_2 = \alpha_3 = 0$

Furthermore, the vectors are not mutually orthogonal.

Proof by counterexample:

$\langle 1, x \rangle = \int_0^1 1x = 0.5$
$\langle v_j, v_k \rangle \neq 0$ for $j \neq k$, therefore the vectors cannot be orthogonal.

Problem 3

$v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$P_{v_3} v_1 = \frac{\langle v_1, v_3 \rangle}{\langle v_3, v_3 \rangle} v_3 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix}$

$P_{v_3} v_2 = \frac{\langle v_2, v_3 \rangle}{\langle v_3, v_3 \rangle} v_3 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix}$

Due to the linearity of the inner product,

$P_{v_3}(2v_1 + v_2) = 2P_{v_3} v_1 + P_{v_3} v_2 = 2 \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1.5 \\ 0 \end{bmatrix}$

Problem 4

$$e_1 = \frac{1}{||1||} = 1$$

Finding a vector orthogonal to e_1:

$$\begin{aligned} v_2 &= x - P_{e_1} x \\ &= x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} 1 \\ &= x - \frac{\int_0^1 x dx}{\int_0^1 1 dx} 1 \\ &= x - \frac{1}{2} \end{aligned}$$

Finding the magnitude of v_2 in order to normalize it:

$$\begin{aligned} \langle e_2, e_2 \rangle &= \int_0^1 (x - \frac{1}{2})^2 dx \\ &= \int_0^1 (x^2 - x + \frac{1}{4}) dx \\ &=_0^1 [\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{4}x] \\ &= \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12} \\ \sqrt{\langle e_1, e_2 \rangle} &= \frac{1}{2\sqrt{3}} \end{aligned}$$

Therefore $e_2 = \frac{x - \frac{1}{2}}{\frac{1}{2\sqrt{3}}} = 2\sqrt{3}x - \sqrt{3}$

Therefore the set $\{1, 2\sqrt{3}x - \sqrt{3}\}$ defines an orthonormal basis for V.