MATH 307: Individual Homework 2

John Mays

02/09/21, Dr. Guo

Problem 1

$(\mathbb{Q}(x), \times, +)$ is a ring, because it satisfies all the conditions of a ring:

For the sake of simplicity, let there be three polynomials, $a(x), b(x), c(x) \in \mathbb{Q}(x)$, which can be represented respectively as:
$a_0 + a_1x + \cdots + a_nx^n$
$b_0 + b_1x + \cdots + b_mx^m$
$c_0 + c_1x + \cdots + c_lx^l$

1. $(\mathbb{Q}(x), +)$ **is an Abelian group:**

1. Closure: $a(x) + b(x) = a_0 + b_0 + a_1x + b_1x + \cdots + a_nx^n + b_mx^m$, which is also a polynomial with rational coefficients.

2. Associativity: No matter the order in which the elements of polynomials are added, their sums are still equal. $a(x)+(b(x)+c(x)) = (a(x)+b(x))+c(x)$.

3. Identity: $\exists$ an identity element, $0 \in \mathbb{Q}(x)$ s.t. $\forall a(x) \in \mathbb{Q}(x), a(x) + 0 = 0 + a(x) = a(x)$.

4. Inverse: $\forall a(x) \in \mathbb{Q}(x), \exists b(x) \in \mathbb{Q}(x)$ s.t. $a(x) + b(x) = b(x) + a(x) = 0$, the identity element. The inverse of $a(x)$ would simply be $-a(x)$, which also belongs to $\mathbb{Q}(x)$.

5. Commutativity: $\forall a(x), b(x) \in \mathbb{Q}(x), a(x) + b(x) = b(x) + a(x)$. Order of addition is irrelevant.

2. $(\mathbb{Q}(x), \times)$ **is a monoid:**

1. Closure: $a(x) \times b(x) = a_0b_0 + a_0b_1x + a_1b_0x + a_1b_1x^2 \cdots + a_nb_mx^{mn}$, which is also a polynomial with rational coefficients.

2. Associativity: $\forall a(x), b(x), c(x) \in \mathbb{Q}(x), a(x) \times (b(x) \times c(x)) = (a(x) \times b(x)) \times c(x)$.

3. Identity: $\exists$ an identity element, $1 \in \mathbb{Q}(x)$ s.t. $\forall a(x) \in \mathbb{Q}(x), a(x) \times 1 = 1 \times a(x) = a(x)$.

3. The operation $\times$ is distributive over $+$ s.t:

$\forall a(x), b(x), c(x) \in \mathbb{Q}(x), a(x) \times (b(x) + c(x)) = a(x) \times b(x) + a(x) \times c(x) = a_0b_0 + a_0b_1x + a_1b_0x + a_1b_1x^2 + a_0c_0 + a_0c_1x + a_1c_0x + a_1c_1x^2 + \cdots + a_nb_mx^{mn} + a_nc_lx^{ln}$.

Problem 2

$(\mathbb{Z}, +, \times)$ is a not a field, because it fails the inverse property of fields. $(\mathbb{Z}, +, 0)$, is an Abelian group and $(\mathbb{Z}, \times, 1)$ is a commutative monoid; however, in order for this to be a field, the set of $\mathbb{Z}\backslash\{0\}$—the set of integers excluding the additive inverse—would have to have multiplicative inverses for all of its elements: $\forall a \in \mathbb{Z}\backslash\{0\}, \exists b \in \mathbb{Z}\backslash\{0\}$ s.t. $a \times b = b \times a = 1$, with 1 being the multiplicative identity w.r.t. $\mathbb{Z}$. As a proof by counterexample, take the integer 3. The inverse of 3 would have to be $1/3$ in order to satisfy the equation, but $1/3$ does not belong to the integers.