MATH 307: Individual Homework 4

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02/16/21, Dr. Guo

Problem 1

The set $W = \{x \in \mathbb{R}^4 : |x_j| < 3, 1 \leq j \leq 4\}$ is not a vector space, because it does not satisfy the condition of closure for scalar multiplication:

Proof:

Take $x = \{-2, 2, 1, 2\} \in \mathbb{W}$ and $\alpha = 2 \in \mathbb{R}$. $\alpha x = \{-4, 4, 2, 4\} \notin \mathbb{W}$.
Therefore the operation of scalar multiplication does not produce an element in $\mathbb{W}$ for all of the elements in $\mathbb{W}$.
Therefore the operation of scalar multiplication is invalid for $\mathbb{W}$, and $\mathbb{W}$ cannot be a vector space over the field $\mathbb{R}$.

Problem 2

The set of all vectors in $\mathbb{R}^3$ whose components(x, y, z) satisfy the equation $x + 20y - 12z - 1 = 0$ (let's call this set $\mathbb{W}$) cannot be a vector space, because the set does not contain the additive identity 0 s.t. $\forall v \in \mathbb{W}, 0 + v = v + 0 = v$.

Proof:

A vector $v \in \mathbb{W}$ cannot be equal to the additive identity $(0, 0, 0)$ because it does not satisfy the conditional equation for the set $\mathbb{W}$:

$$(0) + 20(0) - 12(0) - 1 = -1 \neq 0$$

The set cannot contain the additive identity, therefore it cannot be a vector space.

Problem 3

The set $\{A \in \mathbb{R}^{n\times n}|tr(A) = 0\}$ is a subspace of the set $F^{n\times n}$ containing all of the $n \times n$ matrices composed of elements from $\mathbb{R}$, because it satisfies the two conditions required of a subspace.

Proof:

Condition 1. $\forall\{A, B \in \mathbb{R}^{n\times n}|tr(A) = tr(B) = 0\}, A+B \in \mathbb{R}^{n\times n}|tr(A+B) = 0$

Because $F^{n\times n}$ is a vector space over F, we know that $\mathbb{R}^{n\times n}$ has closure w.r.t. vector addition, we also know that $\{A + B \in \mathbb{R}^{n\times n}\}$
Only the satisfaction of $tr(A + B) = 0$ is left to prove:
If $tr(A) = 0$, the sum of all of the elements across the diagonal of the matrix is zero. The same applies to $tr(B)$. When A and B are added together, matrix addition dictates that the elements in the diagonal of the new matrix will just be the addition of the elements in that same row and column from the previous matrices. In other words,

$$tr(A + B) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = 0 + 0 = 0$$

Therefore, $\forall\{A, B \in \mathbb{R}^{n\times n}|tr(A) = tr(B) = 0\}, \{A+B \in \mathbb{R}^{n\times n}|tr(A+B) = 0\}$.
Closure holds w.r.t. vector addition.

Condition 2. $\forall A \in \mathbb{R}^{n\times n}, \alpha \in \mathbb{R}, \alpha A \in \mathbb{R}^{n\times n}$

Because $F^{n\times n}$ is a vector space over F, we know that $\mathbb{R}^{n\times n}$ has closure w.r.t scalar multiplication, we also know that $\{\alpha A \in \mathbb{R}^{n\times n}\}$
Only the satisfaction of $tr(\alpha A) = 0$ is left to prove:
The operation of scalar multiplication of a matrix dictates that any element in the new matrix is just the element from the same position in the old matrix multiplied by the scalar. Therefore, $tr(\alpha A)$ can be represented as: $\sum_{i=1}^{n} \alpha a_{ii}$, simplified as: $\alpha \sum_{i=1}^{n} a_{ii} = \alpha tr(A) = \alpha 0 = 0$.
Therefore $tr(\alpha A) = 0$ and closure holds w.r.t. scalar multiplication.