**MATH 307: Individual Homework 1**

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1. a) (ℕ, +, 0) is a monoid, because it satisfies all three monoid conditions  
    1. Closure: any addition of two natural numbers must be a natural number  
    2. Associativity: order does not matter in the addition of natural numbers  
    3. Identity: there exists the identity element 0 ∈ ℕ.   
    *a* ∈ ℕ, 0 + *a* = *a* + 0 = *a*  
     
   b) (ℕ, +) is a not a group. It is a monoid, yet it fails to satisfy the fourth additional condition for groups:  
    4. Inverse: Because the natural numbers only contain positive integers, there   
    are no two elements besides (0,0) that can be added to equal 0, the identity  
    element.  
     
   c) (ℕ, +, 0) is not an Abelian group, because it is not a group.
2. (ℤ, +) is an Abelian group, because it satisfies all five requirements of Abelian groups:  
    1. Closure: any addition of two integers must also be an integer.  
    2. Associativity: order of addition does not matter when adding integers  
    3. Identity: there exists the identity element 0 ∈ ℤ.   
    *a* ∈ ℤ, 0 + *a* = *a* + 0 = *a*  
    4. Inverse: For all *a* ∈ ℤ, there exists *b* ∈ ℤ to satisfy the equations a + b = 0   
    and b + a = 0.  
    5. Commutativity: For all *a,b* ∈ ℤ , a + b = b + a.  
     
   (ℤ, ⋅) is a monoid, because it satisfies the three monoid conditions:  
    1. Closure: any multiplication of two integers must also be an integer.  
    2. Associativity: No matter the order in which a set of three integers is   
    multiplied, the product will always be the same.  
    3. Identity: there exists the identity element 1 ∈ ℤ.   
    *a* ∈ ℤ, 1 ⋅ *a* = *a* ⋅ 1 = *a*   
   (cont. on next page)  
   However, (ℤ, ⋅) is not an Abelian group, because it is not a group. It does not satisfy the inverse condition:  
   Proof by counterexample:  
    Claim: For all *a* ∈ ℤ, there exists *b* ∈ ℤ , for which *a* ⋅ *b = id = 1*.

Take 3 ∈ ℤ.  
 3 ⋅ *b*= 1  
 *b* could only be 1/3 ∉ ℤ.  
 Therefore not all integers have an inverse in this monoid.