Rewritten Document

Section 1: Introduction

Chapter 1: Key Definitions and Principles of Set Theory and Mathematical Logic for the Real Number System

1.1 The Propositional Calculus: Foundations

In constructing the real number system, we rely heavily on logical principles and set-theoretic foundations. These structures provide the necessary framework for rigorously defining real numbers and establishing their properties.

The propositional calculus serves as our starting point, offering the basic logical tools required for mathematical reasoning. A proposition is a declarative statement that is either true or false, but not both. The truth value of a proposition forms the basis of logical operations, which in turn enable us to construct and analyze mathematical structures like the real number system. ## 1.2 Logical Equivalences and Their Connection to Real Numbers

1.2.1 Basic Logical Expressions in the Context of Real Numbers

1. "x is a real number such that $0 < x < 1$" can be written as $(0 < x) \wedge (x < 1)$.

Its negation is $\neg((0 < x) \wedge (x < 1)) \Leftrightarrow \neg(0 < x) \vee \neg(x < 1) \Leftrightarrow (x \leq 0) \vee (x \geq 1)$. 2. "x is not in the interval [a,b]" means $\neg(a \leq x \leq b) \Leftrightarrow \neg((a \leq x) \wedge (x \leq b)) \Leftrightarrow \neg(a \leq x) \vee \neg(x \leq b) \Leftrightarrow (x < a) \vee (x > b)$. 3.

Let P: "$x^2 > 4$" and Q: "$x > 2$ or $x < -2$". The statement is $P \rightarrow Q$. In the real number system, whenever $x^2 > 4$, we have $x > 2$ or $x < -2$.

This is because $x^2 > 4 \Leftrightarrow |x| > 2 \Leftrightarrow x > 2$ or $x < -2$. Since $P \rightarrow Q$ is true for all values of x, it's a tautology. 4. "All irrational numbers are transcendental" can be expressed as $\neg P \rightarrow \neg Q$, which is equivalent to $Q \rightarrow P$, where P is "x is rational" and Q is "x is algebraic." This statement is false because there exist algebraic irrational numbers, such as

2. ## 1.3 Logical Principles and Inference Rules

1.3.1 Fundamental Principles

- **Self-entailment**: $P \to P$
- **Double Negation**: $P \Leftrightarrow \neg\neg P$
- **Excluded Middle**: $P \vee \neg P$
- **Non-contradiction**: $\neg(P \wedge \neg P)$
- **Interchange**:
- $(P \wedge Q) \Leftrightarrow (Q \wedge P)$
- $(P \vee Q) \Leftrightarrow (Q \vee P)$

Connection to Real Numbers: The principle of excluded middle is essential in proving properties of real numbers.

For instance, in establishing that every real number is either rational or irrational, we rely on the fact that for any real number x, either x is rational or x is not rational. ### 1.3.2 Rules of Inference

- **Modus Ponens**: $(P \wedge (P \to Q)) \to Q$
- **Modus Tollens**: $((P \to Q) \wedge \neg Q) \to \neg P$
- **Disjunctive Syllogism**: $((P \vee Q) \wedge \neg P) \to Q$
- **Self-verification**: $(\neg P \to P) \to P$
- **Self-refutation**: $(P \to \neg P) \to \neg P$

These principles form the backbone of mathematical reasoning and proof techniques.

In the context of real numbers, they allow us to establish properties such as completeness, density, and order relations. ## 1.4 Application to the Real Number System

Before we delve into the real number system itself, it is crucial to understand how logical principles apply to real number properties.

Consider the following examples:

1.4.1 Density Property of Real Numbers

The density property states that between any two distinct real numbers, there exists another real number.

We can formalize this as:

$\forall x, y \in \mathbb{R}\, (x < y \to \exists z \in \mathbb{R}\, (x < z < y))$

To prove this, we might use the construction $z = \frac{x+y}{2}$ and apply logical principles to verify that $x < z < y$. ### 1.4.2 Completeness Property

The completeness property, which distinguishes real numbers from rational numbers, can be expressed using logical quantifiers:

$\forall S \subset \mathbb{R}\,(S \neq \emptyset \wedge S \text{ is bounded above} \to \exists \sup(S) \in \mathbb{R})$

This property requires careful application of logical principles, particularly when dealing with the existence of least upper bounds. ## 1.5 Exercises on Logic and Real Numbers

1.

Express the statement "The square of any real number is non-negative" using logical symbols and quantifiers. 2. Prove that if x and y are real numbers such that $x^2 = y^2$, then either $x = y$ or $x = -y$. 3.

Using the principle of excluded middle, prove that 2 is irrational. 4. Express the Archimedean property of real numbers using logical symbols and explain its significance. 5.

Prove that the set of rational numbers is dense in the real numbers using logical principles discussed in this chapter. By mastering these logical foundations, we establish the necessary framework for a rigorous development of the real number system, which we will explore in subsequent chapters.

Section 2: Section 2

Chapter 1: Key Definitions and Principles of Set-theory and Mathematical Logic for the Real Number System

1.1 The Propositional Calculus: Foundations

In constructing the real number system, we rely heavily on logical principles and set-theoretic foundations. The propositional calculus provides the fundamental logical framework upon which mathematical reasoning is built. ## 1.2 Logical Equivalences and Their Connection to Real Numbers

1.2.1 Examples of Logical Formulations in Real Analysis

1. "x is a real number such that 0 < x < 1" can be written as $(0 < x) \wedge (x < 1)$.

Its negation is $\neg((0 < x) \wedge (x < 1)) \Leftrightarrow \neg(0 < x) \vee \neg(x < 1) \Leftrightarrow (x \leq 0) \vee (x \geq 1)$. 2. "x is not in the interval [a,b]" means $\neg(a \leq x \leq b) \Leftrightarrow \neg((a \leq x) \wedge (x \leq b)) \Leftrightarrow \neg(a \leq x) \vee \neg(x \leq b) \Leftrightarrow (x < a) \vee (x > b)$. 3.

Let P: "$x^2 > 4$" and Q: "$x > 2$ or $x < -2$". The statement is $P \to Q$. In the real number system, whenever $x^2 > 4$, we have $x > 2$ or $x < -2$.

This is because $x^2 > 4 \Leftrightarrow |x| > 2 \Leftrightarrow x > 2$ or $x < -2$. Since $P \to Q$ is true for all values of x, it's a tautology. 4. "All irrational numbers are transcendental" can be expressed as $\neg P \to \neg Q$, which is equivalent to $Q \to P$. ## 1.3 Logical Principles and Inference Rules

1.3.1 Fundamental Principles

These principles form the backbone of mathematical reasoning and proof techniques:

- Self-entailment: $P \to P$
- Double Negation: $P \Leftrightarrow \neg\neg P$
- Excluded Middle: $P \vee \neg P$
- Non-contradiction: $\neg(P \wedge \neg P)$
- Interchange:
- $(P \vee Q) \Leftrightarrow (Q \vee P)$
- $(P \wedge Q) \Leftrightarrow (Q \wedge P)$

Connection to Real Numbers: The principle of excluded middle is essential in proving properties of real numbers, particularly in establishing completeness and continuity. ### 1.3.2 Rules of Inference

Before we delve into the real number system, it's crucial to understand these fundamental rules of inference:

- Modus Ponens: $(P \wedge (P \to Q)) \to Q$
- Modus Tollens: $((P \to Q) \wedge \neg Q) \to \neg P$
- Disjunctive Syllogism: $((P \vee Q) \wedge \neg P) \to Q$
- Self-verification: $(\neg P \to P) \to P$
- Self-refutation: $(P \to \neg P) \to \neg P$