

# **Introduction to Symbolic Logic**

**J.-M. Kuczynski**

[1.0 The concept of an inference](#)

[1.1 Inductive inference vs. deductive inference](#)

[1.2 Deduction in relation to entailment](#)

[1.3 Induction in relation to confirmation](#)

[1.4 Validity vs. soundness](#)

[1.5 Two kinds of induction](#)

[1.6 A fact about induction by enumeration](#)

[1.7 Two kinds of entailment](#)

[2.0 Some notational conventions relevant to formal logic](#)

[Biconditionals and “ \$\leftrightarrow\$ ”](#)

[Parentheses as indicators of scope](#)

[Negation](#)

[Negation in relation to the use of parentheses](#)

[The modal operators—the box and the pound sign](#)

[The non-exclusive meaning of “or”](#)

[The Sheffer Stroke](#)

[Compatibility and incompatibility](#)

[2.1 Some general principles relevant to formal logic](#)

[Modality](#)

[2.3 Principles and theorems of formal logic](#)

### [3.0 Some meta-logical principles: Model theoretic entailment](#)

#### [3.1 Formal entailment](#)

#### [3.2 Formal entailment a relation between expressions](#)

#### [3.3 The one genuinely important kind of entailment: The informal, non-model-theoretic kind](#)

#### [3.5 The ampliative character of deduction](#)

#### [3.6 The most important kind of entailment: the non-formal, non-model theoretic kind](#)

#### [3.7 Entailment not a limiting case of confirmation](#)

#### [3.8 Entailment both an intersentential and an interpropositional relation](#)

##### [3.8.1 Four kinds of logical equivalence](#)

#### [3.9 The irrelevance of formal logic to thought](#)

### [4.0 Models](#)

#### [5.0 Proof that all knowledge is inferential](#)

#### [5.1 Proof that some truths are non-empirical](#)

### [6.0 Set-theoretic characterizations of truth, meaning, and entailment](#)

### [7.0 Outline of the contents of the sequel to this volume](#)

#### [7.1 Function](#)

#### [7.2 Recursive definition](#)

#### [7.3 Cardinals, rationals, and reals](#)

#### [7.4 Formal languages](#)

#### [7.5 Incompleteness](#)

#### [7.6 Axiom-sets in relation to incompleteness](#)

#### [7.7 Compactness vs. continuity](#)

## [7.8 Set-measure](#)

## [7.9 Nil vs. infinitesimal](#)

## [7.10 Dimensionality](#)

### 1.0 The concept of an inference

To make an inference is to form a new belief on the basis of an old one. Here is an example. My friend Larry has been evicted from his apartment. I let him stay with me until he finds a new place to live. I notice that, shortly after Larry moves in, all of my money and valuables start disappearing. I also notice that purchases that I didn't make are appearing on my credit card bills. I know that nobody other than Larry had access to my money, valuables, and credit cards. So I infer that Larry has stolen my money and valuables and illicitly used my credit cards.

Inferential knowledge is indirect knowledge. Non-inferential knowledge is direct knowledge. There is some reason to believe that *all* knowledge is inferential. (This issue is discussed in Section 5.) In any case, it's clear that some knowledge is *relatively* direct. My knowledge that I am now typing on a keyboard is *more* direct than my knowledge that heat is molecular motion.

### 1.1 Inductive inference vs. deductive inference

A **premise** is a belief that one either has or is willing to accept for argument's sake. An inference always begins with certain premises and ends with a conclusion that one accepts on the basis of those premises. An inference is legitimate if the premises warrants acceptance of the conclusion.

There are two very different reasons why acceptance of P might warrant acceptance of Q. On the one hand, it might be impossible, for reasons of a strictly logical nature, that Q should be false if P is true. (P: Smith owns 127 cars. Q: Smith owns more than 100 cars.) On the other hand, it might be unlikely, but not impossible, that Q should be false if P is true. (P: Smith owns 127 cars. Q: Smith is very wealthy.)

P *entails* Q if it is logically impossible that Q should be false if P is true.

P *confirms* Q if P increases the likelihood that Q is true.

Given only that P confirms Q, it doesn't follow that acceptance of P warrants acceptance of Q. Smith is more likely to find a cure for cancer if he is intelligent than if he is unintelligent, and

*P: Smith is intelligent*

therefore confirms

*Q: Smith will find a cure for cancer.*

But acceptance of P obviously doesn't warrant acceptance of Q, since Q is more likely than not to be false even if P is true.

By the same token, P *does* warrant acceptance of

*R: Smith has not had more than 10 massive strokes,*

given that that, if P is true, there is only a small chance that R is false.

## 1.2 Deduction in relation to entailment

If P entails Q, then Q can be *deduced* from P. If a given person knows that P entails Q and on that basis infers Q

*P: x is a square.*

*Q: x has more than two sides.*

It is impossible for Q to be false if P is true. Therefore, P entails Q.

Given only that P entails Q, it doesn't follow that, if one infers Q from P, one is thereby making a legitimate inference.

If Smith knows that P is true and Smith correctly believes that P entails Q, Smith's inference does not yield knowledge of Q *unless* Smith's reason for believing that P entails Q is the right one. If Smith believes that squares have 90 sides and *for that reason* infers Q from P, Smith has not made a legitimate inference, it being a coincidence that he arrived at the right conclusion.

## 1.3 Induction in relation to confirmation

P confirms Q if the probability of Q, given P, is higher than the probability of Q, given not-P. Equivalently, P confirms Q if P, if true, raises the probability of Q without giving Q a probability of 100%.

It follows that a *sufficient* condition for P's confirming Q is that Q be capable of being legitimately inductively inferred from P:

*P: Smith drives a Rolls Royce, wears extremely expensive clothes, has an excellent credit rating, has an excellent employment history, all of his jobs being extremely lucrative ones; and, finally, Smith has no criminal record.*

*Q: Smith is wealthy.*

P, if true, makes it sufficiently improbable that Q is false that we may infer Q from P. Such an inference is not *deductive*, given that Smith might be an extraordinarily talented but penniless con-artist.

Even though Q's being capable of being legitimately inductively inferred from P is *sufficient* for P's confirming Q, it is not *necessary* for it.

*P: Smith is the most intellectually gifted person in Russia.*

*Q: Smith will find a cure for cancer.*

P raises the probability of Q, whereas not-P lowers the probability of Q. Thus, P confirms Q, even though P by itself does not warrant acceptance of Q.

True statements can *confirm* false ones, but they cannot *entail* them. *Smith's fingerprints are on the murder weapon*, though true, confirms the falsehood that *Smith is the murderer*. But *Smith's fingerprints are on the murder weapon* does not entail *Smith is the murderer*.

If P entails Q, Q is a *logical consequence* of P.

Thus, if P is true, no falsehood is a logical consequence of P.

At the same time, if Q is true, it is a logical consequence of some falsehood P:

*P: Smith's fingerprints are on every weapon in existence* entails the truth that

*Q: Smith's fingerprints are on the murder weapon.*

Indeed, given any true proposition Q, there are *infinitely* many falsehoods

P such that Q is a logical consequence of P.

$P_1$ : Either Smith's fingerprints are on every weapon in existence or  $1+2=1$ .

$P_2$ : Either Smith's fingerprints are on every weapon in existence or  $1+2=1$ .

$P_3$ : Either Smith's fingerprints are on every weapon in existence or  $1+3=1$ .

.

.

.

$P_n$ : Either Smith's fingerprints are on every weapon in existence or  $1+n=1$

.

.

.

Q is a logical consequence of  $P_i$ , for any  $i \geq 1$ .

## 1.4 Validity vs. soundness

A *valid* argument is a deductive argument whose premises, if true, *in fact* give a probability of 100% to the conclusion. In other words, a valid argument is an argument whose premises are supposed to give 100% probability to the conclusion and whose premises *in fact* give 100% probability to the conclusion.

A valid argument can have false premises. For an argument to be valid it is necessary only that if the premises *were* true, then there *would be* a 100% chance that the conclusion is true. It may be that Smith doesn't drive a Rolls Royce or any kind of car for that matter. But it's still the case that if Smith did drive a Rolls Royce, then, given that all Rolls Royces are cars, there would be no chance that Smith didn't drive a car.

A *sound* argument is a valid argument with true premises. (For example: JMK is a human; all humans are mammals; therefore, JMK is a mammal.)

A valid argument cannot possibly have true premises and a false conclusion, but it can have false premises and a false conclusion or false premises and a true conclusion.

## 1.5 Two kinds of induction

There are two kinds of inductive inference: ***induction by enumeration*** and ***inference to the best explanation***.

If, on the basis of the fact that you know of many x's that are y's and of no x's that aren't y's, you infer that *all* x's are y's or that the *next* x you encounter is a y, you've performed a case of induction by enumeration.

Here is an example. You've seen a million swans. They were all white. (Moreover, you *knew* in each case that what you were seeing was a white swan—you didn't think it was, for example, a beige duck.) You've never seen, or otherwise come to know of, any non-white swan. On this basis, you infer that *all* swans are white and, therefore, that, if you ever encounter another swan, it will be white.

If, in order to account for something of which you have knowledge, you posit the existence of something of which you don't have knowledge; you have made an *inference to the best explanation*.

All theories are cases of inference to the best explanation. All the theories put forth by Einstein, Darwin, Freud are instances of this mode of inference. Inference to the best explanation is a very powerful form of inference. (In fact, we will see in Section 1.7. that it is the *only* form of non-deductive inference. So-called "induction by enumeration," to the extent that it isn't a spurious method of inference, collapses into inference to the best explanation.)

The conclusions of inferences to the best explanation are always *causal* statements. You posit the existence of X, which you don't know of, in order to explain Y, which you do know of, because, in your judgment, X, if it existed, would *cause* Y to occur.

Example #3 is a case of inference to the best explanation. I infer that Larry stole my credit cards (etc.) because the relevant data is easy to explain on the assumption that he did, the reason being that his doing so would have *generated* that data, and hard to explain on the assumption that he didn't, the reason being that it's unclear what else could have generated that data.

To take another example: if you infer that a mouse has been eating the cheese in your cupboard, it's because a mouse's doing so would have generated the relevant data (e.g., the tiny little footprints inside the cupboard), and it's not clear what else could have generated that data.

In some cases, the conclusions of inferences to the best explanation aren't

*themselves* causal statements. Our knowledge that water consists of H<sub>2</sub>O molecules isn't itself a causal statement. Cause-effect relations hold between distinct entities (e.g., thunder and lightning). Water and H<sub>2</sub>O aren't distinct.

That said, it is believed that water consists of H<sub>2</sub>O molecules only because its being so composed would cause various things to happen that *do* in fact happen. So even though the conclusion of an inference to the best explanation need not *itself* be a causal statement, all inferences to the best explanation are causal inferences.

## 1.6 A fact about induction by enumeration

Induction by enumeration isn't nearly as powerful a form of inference as inference to the best explanation. Given only that every single one of the 5,000 first editions you've ever come across smelled like pipe tobacco, you cannot reasonably infer that all first editions smell like pipe tobacco. What if all of those first editions were the property of your pipe-smoking friend Larry? In that case, you'd have good reason to suspect that their smelling like pipe tobacco had to do, not with their being first editions, but with their being things that had been around Larry.

By the same token, so far as you are entitled to infer that all first editions smell like pipe tobacco, it's only to the extent that you have reason to believe that a thing's smelling like pipe tobacco is *rooted* in its being a first edition. But that means that, if the data entitles you to infer that all first editions smell like pipe tobacco, it's only because it also entitles you to accept some inference to the best explanation (one to the effect that a thing's being a first edition is, for some reason or other, responsible for its smelling like pipe tobacco).

In general, induction by enumeration is parasitic on inference to the best explanation. In Chapter 12 of *Analytic Philosophy*, it is shown how a failure to see this underlies the spurious belief that there is no non-deductive, inferential knowledge.

## 1.7 Two kinds of entailment



Just as there are two kinds of inductive inference, so there are two kinds of deductive inference. In other words, there are two kinds of entailment: formal entailment and informal entailment.

An entailment is “informal” if the syntactic structures of the sentences involved are not what make the entailment go through. Consider the following entailment:

*# If Smith is tall and Jones snores, then Smith is tall.*

This sentence has the form:

*If P and Q, then P.*

If a sentence has that form, it constitutes a valid inference. So # is made true by its form—that is, by its syntax.

But not all valid entailments hold in virtue of syntax. In fact, most do not. Consider the sentence:

*^ If Smith weighs 1000 pounds, and Jones weighs 80 pounds, then Smith weighs more than Jones.*

Obviously ^ is valid. But its syntax isn’t what makes it true, since that sentence has the same syntax as:

*^^ If Smith weighs 80 pounds, and Jones weighs 1000 pounds, then Smith weighs more than Jones,*

which is obviously invalid.

## 2.0 Some notational conventions relevant to formal logic

Statements of the form “if P, then Q” are known as “conditionals.” The expression “ $\rightarrow$ ” stands for “if...then.” So “ $P \rightarrow Q$ ” means “if P, then Q.” Thus,

*(1) Smith has four cars  $\rightarrow$  Smith has more than one car*

means that

*(2) If Smith has four cars, then Smith has more than one car.*

Thus, “ $P \rightarrow Q$ ” can be taken to mean that  $P$  implies  $Q$ . Thus “ $\rightarrow$ ” expresses implication.

As we will use the expression “ $\rightarrow$ ,” the expression “ $P \rightarrow Q$ ” means that  $P$  *entails*  $Q$ , i.e., that it is *impossible for  $P$  to be true if  $Q$  is false*. In other words, “ $\rightarrow$ ” expresses what is sometimes referred to as **strict entailment**. ( $P$  strictly entails  $Q$  if, supposing that  $P$  is true, there is no way that  $Q$  can be false. In this work, “entailment” and “strict entailment” are synonymous expressions.)

This isn’t how “ $\rightarrow$ ” is typically used. It is typically used to express a much weaker notion known as **material implication**. This term is defined below.

### Biconditionals and “ $\leftrightarrow$ ”

Statements of the form “if  $P$  then  $Q$ , and if  $Q$  then  $P$ ” are known as “biconditionals.” When stated in non-artificial notation, “if  $P$  then  $Q$ , and if  $Q$  then  $P$ ” is typically compressed into “ $P$  iff  $Q$ ” (or “ $P$  just in case  $Q$ ” or “ $P$  exactly if  $Q$ ”). We’ll compress it into: “ $P \leftrightarrow Q$ .” So “Smith is a father  $\leftrightarrow$  Smith is a male parent” says the same thing as “Smith is a father iff Smith is a male parent.”

### Parentheses as indicators of scope

Parentheses are used to indicate how sentences are to be grouped together. Consider the sentence:

(3) “*snow is white and grass is green  $\rightarrow$  grass is green.*”

(3) is ambiguous between:

(4) *it’s the case that snow is white; and it’s also the case that, if grass is green, then grass is green*

and

(5) *supposing that it’s the case that snow is white and that grass is green,*

*it follows that grass is green.*

(4) and (5) have different meanings. (5) doesn't say that snow is white. Nor does it say that grass is green. It says that *if it's the case that both grass is green and snow is white, then it's also the case that grass is green.* (4), on the other hand, says (among other things) that snow is white. So (4) is false in universes where snow is green. But (5) is true in such universes. (In fact, (5) is true in every possible universe.) Since there are circumstances where (4) is false but (5) is true, they don't have the same meanings.

Modern symbolic logic gives us a way of avoiding the laborious *ad hoc* methods just used to disambiguate (3). To express (4), we say:

*(3.4) snow is white and (grass is green  $\rightarrow$  grass is green).*

And to express (5), we say:

*(3.5) (snow is white and grass is green)  $\rightarrow$  grass is green.*

In general, any two co-parenthetical sentences are more tightly bonded to each other than either is to anything that isn't co-parenthetical with those sentences.

## Negation

$\sim P$  is the negation of P. Thus,

*“ $\sim$ (snow is not white)”*

is the negation of

*“snow is white.”*

This means that

*“ $\sim$ (snow is not white)”*

is meaning-equivalent with

*“it is not the case that snow is white.”*

Be it noted that

*“~(snow is not white)”*

does not say that snow is black. Nor does it say that snow is green. It says only that snow isn't white; everything else is left open.

In general, the negation of a sentence isn't the same thing as the “opposite” of that sentence. In modern logic, the word “opposite” has no meaning. The use of the “~” makes it easy to negate statements that would otherwise be hard to negate without prejudging important issues or using cumbersome constructions. Consider the sentence:

*(%) If John likes to listen to Mozart, then Larry likes to listen to Brahms or the sky is blue.*

In natural language one way to negate (%) is to put the expression “it is not the case that” in front of it. The result of this operation is:

*(%<sub>1</sub>) “it is not the case that if John likes to listen to Mozart, then Larry likes to listen to Brahms or the sky is blue.”*

But there's a problem. (%<sub>1</sub>) is ambiguous, as it could mean either:

*(%<sub>2</sub>) either the sky is blue or it is not the case Larry likes to listen to Brahms if John likes to listen to Mozart*

or

*(%<sub>3</sub>) it isn't the case that the sky is blue; nor is it the case that Larry likes to listen to Brahms if John likes to listen to Mozart.*

It takes work to come up with each of these two sentences. Each of the original two sentences had to be reparsed; and in each case this reparsing involved the use of methods that, being *ad hoc*, could be arrived at only

through a creative insight and logical inference. Such Herculean measures shouldn't have been necessary, given that the objective was only to perform a purely grammatical operation.

This problem doesn't arise in our special symbolic notation. % can be unambiguously disambiguated in a mechanical, thought-free manner. One need only put a '~' before % and then enclose the part one wants negated in parentheses. Thus, the symbolic analogue of:

$(\%_{2S}) \sim (\text{Larry likes to listen to Brahms if John likes to listen to Mozart}) \text{ or the sky is blue};$

and the symbolic analogue of  $(\%_3)$  is:

$(\%_{3S}) \sim (\text{either the sky is blue or (Larry likes to listen to Brahms if John likes to listen to Mozart)})$ .

In  $(\%_{3S})$ , the **scope** of the negation-sign is % in its entirety. (Were it not for the parentheses immediately flanking its second disjunct,  $(\%_{3S})$  would be ambiguous between a conditional whose consequent was a disjunction and a disjunction one of whose disjuncts was a conditional.) In  $(\%_{2S})$ , the scope of the negation-sign is confined to the sentence "if John likes to listen to Mozart, then Larry likes to listen to Brahms."

Thus, in our symbolic notation, parentheses are used is to make it clear how much scope is to be given to a given **operator**. An "operator" is an expression that, given one or more sentences, yields a new sentence. Examples of operators are: "it is not the case that," "or," "and," "because," "it is possible that," and "Fred believes that."

"Fred believes that" is an operator because, when given the sentence "snow is white," it yields a new sentence, namely: "Fred believes that snow is white." "And" is an operator because, when given the sentences "snow is white" and "grass is green," it yields the sentence "snow is white and grass is green." "Because" is an operator for the same reason *mutatis mutandis*. (Since "snow is white because grass is green" has a very different meaning from "grass is green because snow is white," "because" operates on *ordered* pairs of sentences.)

When an operator O occurs in some sentence S, where S has one or more

sentences as proper parts, the scope of that occurrence is identical with the set of sentences thereby combined into a single sentence.

Consider the sentence:

(i) *“Jim is tired because he went jogging and his wife Sally is a very demanding person.”*

This is ambiguous. It could mean either:

(ii) *Jim’s wife Sally is a very demanding person; moreover, Jim is tired because he went jogging*

or

(iii) *There are two reasons why Jim is tired: first he went jogging; second, Jim’s wife Sally is a very demanding person.*

Thus, in English (or any other natural language), use of the expression “because” may yield a sentence that is ambiguous and, in addition, can be disambiguated only by successfully doing a certain amount of syntax-chopping and, therefore, logic-chopping.

But in our symbolic notation, one can disambiguate (i) without having to do any logic-chopping. One need only put parentheses around the two sentences that one wants joined by the “because.” Thus, the symbolic analogue of (ii) is:

(ii<sub>s</sub>) *(Jim is tired because he went jogging) and Sally is a very demanding person.*

And the symbolic analogue of (iii) is:

(iii<sub>s</sub>) *Jim is tired because (he went jogging and Sally is a very demanding person).*

## Negation in relation to the use of parentheses

Parentheses make it clear what is being negated. For example,

*(A)  $\sim$ snow is white or grass is green*

is ambiguous. It could mean either

*(B) Either it's the case that grass is green or it's the case that snow isn't white*

or

*(C) it isn't the case that (snow is white or grass is green).*

(C) is equivalent with:

*(D) snow isn't white and grass isn't green.*

(B) isn't equivalent with (D), since (B), unlike (D), is true in a world where grass is green.

*(A<sub>1</sub>)  $\sim$ (snow is white or grass is green)*

says the same thing as (C), and

*(A<sub>2</sub>)  $\sim$ (snow is white) or grass is green*

says the same thing as (B).

Thus, parentheses make it clear how much scope a given occurrence of “ $\sim$ ” has—that is, they make it clear how much it is negating.

## The modal operators—the box and the pound sign

“ $\Box$ P” means “necessarily P” and “ $\Diamond$ P” means “possibly P.”

Typically, the possibility-operator is a diamond, not a pound sign, but this publication software doesn't support use of the diamond.

*Note concerning the modal operators:* The box and the pound sign are to be given minimal scope. So “ $\Box P \rightarrow Q$ ” is *not* to mean that it's necessarily the case that P implies Q. Rather, it is to mean that, if it's necessarily true that P is the case, then it follows that Q is the case. “ $\Box(P \rightarrow Q)$ ,” on the other hand, *does* say that it's necessarily the case that P implies Q.

## The non-exclusive meaning of “or”

We’ll use “or” in the *non-exclusive* sense. So, as we’ll be using it “either Smith is having dinner or Smith is in London” is consistent with Smith’s being in London while having dinner. In order for “P or Q” to be true, all that is necessary is *one* of those two statements be true. So there are three circumstances under which “P or Q” is true:

- (i) *P is true and Q is false.*
- (ii) *Q is true and P is false.*
- (iii) *P is true and Q is true.*

In contemporary logic, both formal and informal, “or” is always used in the non-exclusive sense.

## “Unless”

(a) *We work unless it is raining*

is equivalent with

(b) *If we aren’t working, it’s raining.*

In general,

(A) *P unless Q*

means

(B)  $\sim P \rightarrow Q$

## The Sheffer Stroke

“/” is a logical operator known as the “Sheffer Stroke.” The Sheffer Stroke is defined thus:

$P/Q \equiv_{\text{DF}} \sim P \text{ and } \sim Q$



Hence the following theorems:

$$\sim P \leftrightarrow P/P$$

$$P \leftrightarrow (P/P)/(P/P)$$

$$(P \text{ and } Q) \leftrightarrow (P/P)/(Q/Q)$$

$$(P \text{ or } Q) \leftrightarrow (P/Q)/(P/Q)$$

## Compatibility and incompatibility

Statements are compatible when neither entails the negation of the other.

Statements are incompatible when the one entails the negation of the other. P entails Q just in case P is incompatible with  $\sim Q$ .

### 2.1 Some general principles relevant to formal logic

**Deduction is truth-preserving:** A true statement cannot entail a false one. In other words, a false statement cannot be validly inferred from a true one.

**Deduction is transitive** (a corollary of the fact that it's truth-preserving): R is a transitive relation if, supposing that x bears R to y and y bears R to z, then x bears R to z. The relation of being less tall than is transitive. So is the relation of logical-consequence. Therefore, deduction is transitive. A statement entails anything entailed by any one of its consequences. For example:

*If Smith is an elephant, then Smith is a mammal.*

*If Smith is a mammal, then Smith has hair.*

*Therefore, if Smith is an elephant, then Smith has hair.*

**Induction isn't truth-preserving:** A true statement may **confirm** a false one. For practically any value of "x", the statement

*P: x is a heavy smoker*

certainly confirms

*Q: x won't live to be a hundred.*

But for some values of x, P is true and Q is false, as there have been heavy smokers who lived to be a 100.

*Commentary:* This principle is a simple of corollary of the fact that multiplying n by a proper fraction yields a number  $m < n$ . The same is true of the next principle.

***Induction isn't transitive:*** A corollary of the non-truth-preserving character of induction is that induction is not transitive. In other words, given only that P confirms Q and that Q confirms R, it doesn't follow that P confirms R:

*P: Smith is a morbidly obese man with a heart condition and terminal cancer who is due to be executed in five hours.*

*Q: Smith is a morbidly obese man with a heart condition and terminal cancer who is going to die soon.*

*R: Smith will soon die of some ailment related to his obesity, his heart-condition, or his cancer.*

P confirms Q; Q confirms R. But P doesn't confirm R. P makes it a veritable certainty that Smith will be *executed* in five hours and, therefore, that he *won't* die for reasons having to do with his heart or weight or cancer-situation. So P disconfirms R, even though it confirms Q, which, unlike P, does confirm R.

### **Contraposition, conversion, inversion**

These are operations that are performed on conditional statements (statements of the form "if P, then Q").

The ***contrapositive*** of  $(P \rightarrow Q)$  is  $(\sim Q \rightarrow \sim P)$ .

Thus, the contrapositive of

*"If Bob is a dog, then Bob is a mammal"*

is

*“If Bob is not a mammal, then Bob is not a dog.”*

Statements are equivalent with their contrapositives. In other words,  $(P \rightarrow Q) \leftrightarrow (\sim Q \rightarrow \sim P)$ . Bob’s being a dog entails his being a mammal iff his being a non-mammal entails his being a non-dog.

The **converse** of  $(P \rightarrow Q)$  is  $(Q \rightarrow P)$ . Thus, the converse of

*“If Bob is a human, then Bob is a mammal”*

is

*“If Bob is a mammal, then Bob is human.”*

As this example shows, statements are not generally equivalent with their converses. It’s only when both **antecedent** and **consequent** are equivalent that a given conditional is equivalent with its converse. (In  $(P \rightarrow Q)$ , P is the antecedent and Q is the consequent.) Thus, the converse of

*“If Bob is a father then Bob is a male parent is true”*

is true, since antecedent and consequence are equivalent.

The **inverse** of  $(P \rightarrow Q)$  is  $(\text{not-}P \rightarrow \text{not-}Q)$ . Thus, the inverse of

*“If Bob is a human, then Bob is a mammal”*

is

*“if Bob is not a human, then Bob is not a mammal.”*

As this example shows, statements are not generally equivalent with their inverses. It’s only when both antecedent and consequent are equivalent that conditional is equivalent with its inverse. This is because the converse of a proposition is equivalent with its inverse. In other words,

$(Q \rightarrow P) \leftrightarrow (\text{not-}P \rightarrow \text{not-}Q)$ .

The reason why they’re equivalent is that the second is the contrapositive of the first. (As we know, contraposing a conditional produces an equivalent conditional.)

A sentence-level operation is **truth-preserving** if, when given a true sentence as input, it cannot possibly yield a false sentence as output. Contraposition is truth-preserving. Conversion and inversion are not.

*Exercise:* Prove that, given any conditional sentence  $S$ , the contrapositive of  $S$  is equivalent with  $S$ . Further, prove that it is not the case that, given any conditional sentence  $S$ , the converse of  $S$  is equivalent with  $S$ . Finally, prove that, given any conditional sentence  $S$ , the inverse of  $S$  is equivalent with the converse of  $S$ .

### **Material vs. strict vs. formal entailment**

Logicians distinguish between “formal” and “material” implication, and they distinguish each from “strict” implication. “Strict implication” is synonymous with “entailment”, and “strict entailment” is therefore a pleonasm.

$P$  formally entails  $Q$  if the sentence  $\langle \text{if } P, \text{ then } Q \rangle$  is an instance of an open-sentence that is true for all values of its free variables. Thus, a formal truth is a truth that is an instance of an open-sentence that is true for all values of its free variables. “ $2 = 2$ ” is a ‘formal’ truth, because it is an instance of the open-sentence, namely:

“ $x = x$ ”

which is true for all values of its variables.

Formal implication is a *kind* of strict implication. Any case of formal implication is a case of strict implication, but not *vice versa*. The reasons for this are given in *Analytic Philosophy*, Chapters 1, 6, and 18, and also in Section 4.5 of the present work.

“ $P$  materially implies  $Q$ ” is true if any one of the following three conditions is met:

- (i)  $P$  is false and  $Q$  is false;
- (ii)  $P$  is false and  $Q$  is true;
- (iii)  $P$  is true and  $Q$  is true.

It is only if

- (iv)  $P$  is true and  $Q$  is false

that P does *not* materially imply Q.

If P is any false statement P, and Q is any statement at all, “P is true *and* Q is false” is false. For this reason, any false statement materially implies all statements.

Material implication isn’t a form of implication at all. So, while it’s a fact that, for any statement Q, any given falsity “materially implies” Q, that fact is an innocuous consequence of a linguistic convention that, although it concerns the word “implication,” has nothing to do with implication at all.

Material and strict entailment may hold between propositions or between expressions. But formal entailment holds only among expressions. It is typically held to hold among sentence-types. My strongly held view is that it holds only among sentence-tokens. (See Chapter 4 of *Analytic Philosophy* for a defense of this claim.) This is the view of Peter Strawson, Jon Barwise, and John Perry. But for the sake of brevity we will use the term ‘sentence’ to stand for sentence-type and sentence-tokens.

## Modality

“Modal” means “having to do with necessity or possibility.” Possibility and necessity are properties of statements, not objects. Any statement of the form  $\Box P$  (‘it’s necessarily the case that P’) or  $\# P$  (‘it is possible that P’) is a modal statement.

“Necessary” and “possible” are interdefinable. “Necessary” can be defined as “not possibly not,” and “possibly” can be defined as “not necessarily not.” Thus,

$$\Box P \text{ iff } \sim \# \sim Q.$$

Statements that affirm *bearing relations*, whether deductive or inductive, are modal. Thus, causal statements are modal statements, since they have the form  $\Box(\text{if } P, \text{ then } Q)$ .

Causal necessity isn’t logical necessity, of course. Given that

(i) I threw gasoline on the fire,

it is *causally* necessary that

(ii) The fire flared up.

But (ii) isn't a logical consequence of (i). In other words, it can't be known through meaning-analysis or syntax-chopping that (ii) cannot be false if (i) is true.

## 2.3 Principles and theorems of formal logic

1. For any  $x$  and any  $y$ ,  $((x = y) \rightarrow (\text{for any property } F, (x \text{ has } F \leftrightarrow y \text{ has } F)))$ .  
(Leibniz's Law.)

*Given anything  $x$  and anything  $y$ , if  $x$  is the very same thing as  $y$ , then  $x$  has a given property just in case  $y$  has that property.*

*If  $x$  is the very same thing as  $y$ , then anything true of the one is true of the other.*

*Anything true of one thing is true of another whenever those things are one and the same.*

*A necessary condition for identity is sameness of properties.*

*Commentary:* In Chapter 16 of *Analytic Philosophy*, it is argued that, although this principle is clearly true for "synchronic" identity-claims, it is clearly false for "diachronic" identity claims.

2. For any  $x$  and any  $y$ ,  $(\text{given any property } F, ((x \text{ has } F \leftrightarrow y \text{ has } F) \rightarrow (x = y)))$   
(The identity of indiscernibles.)

*Given anything  $x$  and anything  $y$ , if it the case that  $x$  has a given property just in case  $y$  has that property, then it follows that  $x$  is the very same thing as  $x$ .*

*If  $x$  has no properties that  $y$  lacks and  $y$  has no properties that  $x$  lacks, then  $x$  is the very same thing as  $y$ .*

*A sufficient condition for identity is sameness of properties.*

*Commentary:* Though it has been questioned, this principle is clearly correct.

Given any object  $x$ ,  $x$  has the property of being identical with  $x$ . So, given any object  $y$ , if  $y$  has every property that  $x$  has, then  $y$  has the property of being identical with  $x$  and is therefore itself identical with  $x$ .

But, while it's true that anything having that property is identical with  $x$ , it's also trivial. And my suspicion is that those who claim to be asking whether the identity of indiscernibles is correct are really asking a distinct but related question, namely: given an object  $y$  that has every property had by  $x$ , setting aside those properties that are *trivially* had by anything identical with  $x$  (e.g., the property of being identical with  $x$ ), is  $y$  identical with  $x$ ?

*That* question, unlike the question of whether the identity of indiscernibles is correct, isn't trivial. That said, the answer to it does seem, quite clearly, to be "yes." If, at any given moment,  $x$  is in the same place as  $y$ , has the same mass, is subjected to the same forces, and so on, then surely  $x$  is identical with  $y$ .

*Additional commentary:* The identity of indiscernibles is often said to be the "converse" of Leibniz's Law.

This isn't correct. For  $P$  to be the converse of  $Q$ , it is necessary that  $P$  be a conditional statement and that  $Q$  also be a conditional statement. Leibniz's Law isn't expressed by a conditional statement, and neither is the identity of indiscernibles.

Each is expressed by a *universal generalization*. In each case, that universal generalization is to the effect that each member of an infinitely large class of conditional statements is correct. But neither is itself expressed by a conditional statement, and neither is itself a conditional proposition.

Leibniz's Law is to the effect that:

*(LL) for any  $x$  and any  $y$ , if  $x$  is identical with  $y$ , it's impossible for  $x$  to lack any property had by  $y$  or vice versa.*

LL is not itself a conditional statement. For the same reason *mutatis mutandis*, the identity of indiscernibles isn't given by a conditional statement. Therefore, neither principle is the converse of the other.

3.  $P$  or  $\sim P$ . (*The law of excluded middle.*)

*Any given statement is either true or false.*

*What isn't true or false is not a statement.*

*S is a statement only if S is either true or false.*

*S's being a statement is sufficient for S's being either true or false.*

*S's being either true or false is necessary for S's being a statement.*

*Commentary:* Whether or not 3 is correct depends on what the word “statement” is taken to mean. If it's taken to mean the same thing as “sentence,” then 3 is false. The sentence (in other words, the sentence-type) “I am tired” is neither true nor false. (Certain *utterances* of it are true and certain utterances of it are false.)

If the word “statement” is taken to mean the same thing as “sentence-utterance” (or, more generally, “**sentence-token**”), then, once again, 3 is false. If, while pointing to an empty space, I say “that elephant is in terrible pain,” what I've said is neither true nor false. There is no such elephant. Therefore, there is no elephant  $x$  such that I am saying that  $x$  is in terrible pain. If I am saying anything at all, there is some elephant  $x$  such that what I am saying is true iff  $x$  is in terrible pain. Since there is no such elephant, I'm not saying anything; my utterance is neither true nor false.

If the word “statement” is taken to mean the same thing as “proposition,” then 3 is correct. What we are seeing is that the so-called truths of symbolic logic—the “laws of logic”—aren't true at all. They're statement-*forms*, not statements, and therefore aren't true or false. And it isn't easy to find interpretations of those statement-forms that validate them. In other words, it isn't easy to figure out what sorts of constants must replace the variables occurring in these “laws” if true sentences are to result. This suggests that modern symbolic logic (*a.k.a.* mathematical logic, *a.k.a.* formal logic) isn't logic at all. It's actually a small and rather unimportant branch of a discipline known as “model theory.” In Section 3.4, we'll see why, given this fact, it follows that formal logic has little to do with the way in which any actual or even possible being reasons.

4.  $\sim(P \text{ and } \sim P)$ . (*The law of non-contradiction.*)

*No statement is both true and false.*



*The class of statements that are both true and false is empty.*

*What isn't true, when not a belief, is either false or a non-statement.*

*What isn't false, when not a belief, is either true or a non-statement.*

5.  $(P \text{ and } (P \rightarrow Q)) \rightarrow Q$ . (Modus Ponens.)

*Supposing that P, and supposing that P entails Q, it follows that Q.*

*Any consequence of a truth is a truth.*

*The consequent of a conditional statement with a true antecedent is a truth.*

*If the antecedent of a conditional is true, so is the consequent.*

*The class of truths is closed under the relation of logical consequence.*

Example (henceforth "SD"):

Premise: Smith is a dog.

Premise: If Smith is a dog, then Smith is a mammal.

Conclusion: Smith is a mammal.

Commentary: Consider the following argument (henceforth "SD#"):

(1) Smith is a dog.

(2) All dogs are mammals.

Conclusion: (3) Smith is a mammal.

SD# is *not* an instance of modus ponens. To be an instance of modus ponens, an argument must have the form:

(1\*) P

(2\*) If P, then Q

Conclusion: (3\*) Q.

Let's refer to *this* argument form as "AF\*."

If we replace the occurrence of "P" in AF\* with "Smith is a dog" and we replace the occurrence of "Q" with "Smith is a mammal," the result is SD,

not SD#. There are no sentences such that, if the sentence-level variables in AF are replaced with those sentences, the result is SD#. Therefore, SD# is not an instance of modus ponens.

SD# is a valid argument. But it isn't an instance of modus ponens. It's an instance of the following argument form:

(1#) A has property P.

(2#) For any x, if x has P, then x has Q.

Conclusion: (3#) A has Q.

Let's refer to this argument form as "AF#."

SD# makes a statement *about* SD. SD# says that SD is an instance of modus ponens. If S is a statement that says of some argument A that A is an *instance* of modus ponens, S is not *itself* an instance of modus ponens. Thus, SD# is not itself an instance of modus ponens.

6.  $(P \rightarrow Q) \text{ and } \sim Q \rightarrow \sim P$ . (Modus tollens)

*A given statement is false if any one of its logical consequences is false.*

Premise: If Smith is a dog, then Smith is a mammal.

Premise: Smith isn't a mammal.

Conclusion: Smith is not a dog.

7.  $(P \text{ and } \sim Q) \rightarrow \sim(P \rightarrow Q)$ .

*Deduction is truth-preserving.*

*No true statement entails a false one.*

*Truth implies truth.*

*Truth follows from truth.*

*Explanation:* If P is *actually* true and Q is *actually* false, there is obviously a *possible* circumstance where P is true and Q is false. (What is actual must be possible. The impossible isn't actual.) An immediate consequence is that P doesn't entail Q.

8.  $((P \rightarrow Q) \text{ and } (Q \rightarrow R)) \rightarrow (P \rightarrow R)$  (If  $P$  entails  $Q$  and  $Q$  entails  $R$ , then  $P$  entails  $R$ .)

*Entailment is transitive.*

*A statement entails anything that any given one of its consequences entail.*

*The property of being true is hereditary with respect to the consequence-relation.*

*If  $K$  is the smallest class containing a given truth  $P$  and the logical consequence any given truth that it contains, then each of  $K$ 's members is a truth.*

*Commentary: (8) is a corollary of Principle (7).*

*Also, (8) is equivalent with*

$(8^*) (P \rightarrow Q) \rightarrow ((Q \rightarrow R) \rightarrow (P \rightarrow R)).$

*If  $P$  entails  $Q$ , then  $P$  entails  $R$  provided that  $Q$  entails  $R$ .*

*Each of (8) and  $(8^*)$  is equivalent with each of the following:*

*$Q$  is a consequence of  $P$  only if every consequence of  $Q$  is a consequence of  $P$ .*

*$Q$  is not a consequence of  $P$  unless every consequence of  $Q$  is a consequence of  $P$ .*

*Consequences of consequences are consequences.*

9.  $P \rightarrow \sim\sim P$ . (The law of double-negation.)

*Statements entail the negations of their negations.*

*A necessary condition for the truth of a given statement is that the negation of its negation be true.*

*A statement isn't true unless the negation of its negation is true.*

Premise: Smith is a professor.

Conclusion: It's not the case that Smith is not a professor.

10.  $P \rightarrow (P \text{ or } Q)$ .

*No false disjunction contains a true disjunct.*

*A necessary condition for the truth a disjunction is that one of the disjuncts be true.*

*A sufficient condition for the falsity of a disjunction is that each of its disjuncts be false.*

11.  $\sim(P \text{ or } Q) \leftrightarrow (\sim P \text{ and } \sim Q)$ .

*A disjunction is false iff all of its disjuncts are false.*

*A disjunction is false exactly if each of its disjuncts is false.*

*A disjunction is false unless at least one of its disjuncts is true.*

12.  $\sim P \rightarrow \sim(P \text{ and } Q)$ .

*No true conjunction contains a false conjunct.*

*A conjunction is false unless each conjunct is true.*

13.  $\sim(P \text{ and } Q) \leftrightarrow (\sim P \text{ or } \sim Q)$ .

*A conjunction is false just in case at least one of its conjuncts is false.*

14.  $((P \text{ or } Q) \text{ and } \sim P) \rightarrow Q$  (DeMorgan's Law)

*Given a true disjunction, the one disjuncts follows from the negation of the other.*

Premise: Either Jones is a plant or Smith is in the house.

Premise: Jones isn't a plant.

Conclusion: Therefore Smith is in the house.

15.  $\sim((\sim P \text{ and } Q) \rightarrow \sim(P \rightarrow Q))$

*A false statement may entail a true one.*

*The property of being false is not hereditary with respect to the consequence-relation.*

*Explanation:* "Paris is in France and Spain is north of Sweden" entails "Paris in France." Any given true statement is entailed by each of the infinitely many false conjunctions of which it is a conjunct. For that reason, and others, a false statement may entail a true one. (But a true one cannot entail a false one.)

16.  $\sim((P \rightarrow Q) \rightarrow (Q \rightarrow P))$

*Propositions aren't always consequences of their own consequences.*

*Conversion isn't truth-preserving.*

"Paris is in France and Spain is north of Sweden" entails "Paris in France," but not *vice versa*.

17.  $P \rightarrow P$

*Statements self-entail.*

*Explanation:* P entails P iff it isn't possible for (P and  $\sim P$ ) to be true. It isn't possible for (P and  $\sim P$ ) to be true. (The law of non-contradiction forbids it.) Therefore, P entails P.

18.  $\sim Q \rightarrow ((P \rightarrow Q) \rightarrow \sim P)$

*Nothing but falsity entails falsity.*

19.  $(Q \rightarrow (P \text{ and } \sim P)) \rightarrow \sim Q$ .

*Nothing true entails a contradiction.*

*Explanation:* No true statement entails a false one. (Principle 18.)  $(P \text{ and } \sim P)$  is false. (Principle 4.) So for any proposition P, no true statement entails  $(P \text{ and } \sim P)$ .

20.  $(P \rightarrow \sim P) \rightarrow \sim P$

*A statement is false if it entails its own negation.*

*No true statement is incompatible with itself.*

*Only false statements are incompatible with themselves.*

*Explanation:* Any given statement entails itself. (Principle 17.) So if a statement P entails its own negation, it follows that P entails  $(P \text{ and } \sim P)$ .  $(P \text{ and } \sim P)$  is false. (Principle 4.) Therefore, a statement is false if it entails its own negation.

21.  $(P \text{ and } \sim P) \rightarrow Q$

*Nothing doesn't follow from a contradiction.*

*Explanation:* There is no possible circumstance where  $(P \text{ and } \sim P)$  is true. Therefore, for any proposition Q, there is no possible circumstance where Q is false *and* where  $(P \text{ and } \sim P)$  is true.

### **Some principles of modal logic**

1M.  $P \rightarrow \#P$ .

*Whatever is actual is possible.*

2M.  $\Box P \rightarrow P$ .

*Whatever is necessary is actual.*

3M.  $\Box P \rightarrow \#P$ .

*Whatever is necessary is possible.*

Given that squares must have four sides, they do have four sides.

*Commentary:* This shows that there is a difference between the possible and the contingently true. The contingently true doesn't have to be true; but the possible is sometimes necessarily, and therefore non-contingently, true.

4M.  $(P \rightarrow Q) \leftrightarrow \sim \#(P \text{ and } \sim Q)$ .

*P entails Q iff P can't be true if Q is false.*

*Explanation:* This is a definition, not a principle. To say that P entails Q is to say that it's impossible for P to be true if Q is false.

5M.  $\Box(P \text{ and } Q) \leftrightarrow (\Box P \text{ and } \Box Q)$ .

*A conjunction is necessarily true iff each conjunct is necessarily true.*

*Explanation:* If P isn't necessarily true, there is some possible circumstance where it's false and where, therefore, (P and Q) is false, no matter what Q is. If (P and Q) is necessarily true, there is no circumstance where (P and Q) is false or, therefore, where either P is false or Q is false.

6M.  $\Box \sim(P \text{ or } Q) \leftrightarrow (\Box \sim P \text{ and } \Box \sim Q)$ .

*A disjunction is necessarily false iff each disjunct is necessarily false.*

*Explanation:* Each of P and Q must be false in every possible world in order for  $\sim(P \text{ or } Q)$  to be false in every possible world; and there's no way that  $\sim(P \text{ or } Q)$  can be true in any possible world if each of P and Q is false in every

possible world.

7M.  $(\Box P \text{ and } \# \sim Q) \rightarrow \sim(P \rightarrow Q)$ .

*Nothing that can be false is a consequence of anything that cannot be false.*

*Explanation:* If P is necessarily true (true in all circumstances), whereas Q is possibly false (false in some circumstances), there are circumstances where P is true and Q is false (i.e., P doesn't entail Q).

8M.  $\sim((\# P \text{ and } \# Q) \rightarrow \# (P \text{ and } Q))$ .

*Conjoining two possible statements may yield a necessarily false statement.*

*Explanation:* "On June 12, 2009, JMK did not at any time leave the state of Virginia" is possible, and so is "On June 12, 2009, JMK did at some point leave the state of Virginia." But they're not compossible (i.e., they cannot both be true).

9M.  $(\# P \text{ and } \Box \sim Q) \rightarrow \sim(P \rightarrow Q)$

*Nothing that can be true entails anything that cannot be true.*

*Explanation:* If P is true in some possible circumstance, whereas Q is true in none, there is a possible circumstance where P is true and Q is false, which means that P doesn't entail Q.

10M.  $\Box P \rightarrow (Q \rightarrow P)$

*Given any necessarily true statement, there is no statement that doesn't entail it.*

*Explanation:* If P is necessarily true, there are no possible circumstances where P is false. *A fortiori*, there are no possible circumstances where, for any proposition Q, P is false and Q is true.

11M.  $\sim \Box P \rightarrow (P \rightarrow Q)$

*Given any statement at all, be it true or false, there is no necessarily false*



*statement that doesn't entail it.*

*Explanation:* If P is necessarily false, there is no possible circumstance where P is true. *A fortiori*, there is no possible circumstance where, for any proposition Q, P is true *and* Q is false.

12M.  $\Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$ .

*A statement is necessary if it's entailed by a necessary conditional with a necessary antecedent.*

*Exercise:* 13M can be seen as a generalization of Modus ponens. Explain why this is so. *Exercise:* Is " $(\Box P \rightarrow \Box Q) \rightarrow \Box(P \rightarrow Q)$ " a correct principle? Explain why or why not.

13M.  $\Box P \rightarrow \Box \Box P$ . ("S5.")

*Whatever is necessary is necessarily necessary.*

*Explanation:* Point #1: If  $\sim \Box \Box P$ , then there is some possible circumstance where  $\sim \Box P$ . By contraposition, if there is *no* possible circumstance where  $\sim \Box P$ , then  $\Box \Box P$  is true. Point #2: If  $\sim \Box P$ , then there is some possible circumstance where  $\sim P$ . By contraposition, if there is *no* possible circumstance where  $\sim P$ , then  $\Box P$  is true. Point #3: Given Points #1 and #2, it follows that, if  $\Box P$ , then  $\Box \Box P$ . Conclusion: S5 is correct.

14M.  $\# P \rightarrow \#\# P$ .

*What is possible is possibly possible.*

### 3.0 Some meta-logical principles: Model theoretic entailment

P model-theoretically entails Q just in case there is no coherently conceivable scenario where P is true and Q is false.

Thus,  $1 + 1 = 2$  model-theoretically entails *triangles have three sides*, even though the two propositions do not, apart from that, have anything to do

with each other.

This last point is important. There isn't anything about shapes, let alone triangles specifically, in  $1 + 1 = 2$ ; and there isn't anything about addition in *triangles have three sides*. So even though, technically, each entails the other, there is no significant sense in which the one can be *inferred* from the other—no sense in which a knowledge of the one yields a knowledge of the other.

Of course, somebody who knows that  $1 + 1 = 2$  will probably also know that triangles have three sides, but he won't know the one on the basis of the other. By contrast, somebody who knows that  $x$  is a case of true belief could potentially know that on the basis of the fact that  $x$  is knowledge.

*Commentary:* Model-theoretic entailment is an over-valued notion that doesn't correspond at all well to the conception of entailment embodied in our inferential practices. Nonetheless, because, in the early '60's, some branches of philosophy were (wrongly) thought to be successfully assimilated to a branch of mathematics known as "model theory"—the purpose of this branch being to study model-theoretic entailment—philosophers now tend to treat model-theoretic entailment as *the* fundamental kind of entailment.

Model-theoretic entailment isn't really a kind of entailment. It is a *theory* of entailment. Like other scientists, logicians model data. Where logic is concerned, the data in question are intuitions on our part to the effect that one statement is a consequence of another.

$(K_1)$  "*x is knowledge*"

entails

$(K_2)$  "*x is a true belief.*"

Logicians want to make it clear what it is that we know in knowing this; they want to make it clear what it is for the one statement to entail the second.

Here is the proposal that they came up with:

*(MT) for P to entail Q is for there to be no possible circumstance in which the first is true and the second is false.*

Why did logicians choose MT when trying to model our intuitions concerning entailment? Because MT is *extensionally* correct. In other words,

for any statements  $P$  and  $Q$ ,  $P$  entails  $Q$  iff there is no possible circumstance where  $P$  is true and  $Q$  is false.

But even though there are no counter-examples to it, MT is false. It is obviously in virtue of facts about the structures of  $K_1$  and  $K_2$  that the former entails the latter. The fact that there are no models in which  $K_1$  is true and  $K_2$  is false is a mere *consequence* of that structural fact.

There is a related point. MT defines entailment in terms of “possible.” This makes it viciously circular.  $P$  and  $Q$  are compossible (in other words, *both  $P$  and  $Q$  is possible*) iff neither entails the negation of the other. We understand the concept of possibility *only* to the extent that we can understand it in terms of entailment (and other bearing-relations). So unless we’re willing to take the radically implausible view that “possible” is a **primitive expression**, MT defines “entailment” in terms of itself. (A primitive expression is one that doesn’t consist of other expressions. Examples of such expressions are “red” or “sweet.”)

To make MT non-circular, we must rid it of the word “possible.” Thus modified, MT is:

*(MT\*) for  $P$  to “entail”  $Q$  is for there to be no circumstance in which the first is true and the second is false.*

In MT\*, ‘circumstance’ must be mean *actual* circumstance. If it means anything else, MT\* will be circular, like MT.

MT\* is false. There is no circumstance in which the moon is made of cheese. And there is no circumstance in which I have ten cars. So there is no circumstance in which “the moon is made of cheese” is true and in which “JMK has ten cars” is false. So, according to MT\*, the former entails the latter. But the former doesn’t entail the latter. (The former “materially implies” the latter. But that’s irrelevant, since material implication isn’t entailment.)

So model-theoretic entailment isn’t entailment at all. It’s what a wrong *theory* of entailment says that it is.

Some of the principles stated in Section 2.3, not all of them, go through only if “entails” (i.e. “ $\rightarrow$ ”) is taken to refer to model-theoretic entailment.

### 3.1 Formal entailment

In some cases, one statement's following from another can be determined entirely on the basis of the structures of those sentences. For any statements P and Q,  $\langle P \text{ and } Q \rangle$  entails  $\langle P \rangle$ . For any statement P,  $\langle \text{not not } P \rangle$  entails  $\langle P \rangle$ . In other words, "if not-not-P, then P" is true.

If it can be determined that Q follows from P entirely on the basis of the structural properties of those sentences, P **formally entails** Q.

For a while, it was thought that all entailments could be formalized—that is, that whenever one statement entails another, it is entirely because of the syntactic or structural properties of those sentences. That viewpoint is simply false. Why it is false is explained in Chapter 1 of *Analytic Philosophy*, Section 6.0.

### 3.2 Formal entailment a relation between expressions

Only expressions have syntax. So, since P formally entails Q only if P and Q have certain *syntactic* properties, the relation of formal entailment holds between *sentences* (or sentence-tokens), whereas the relations of model-theoretic entailment holds between propositions, the same being true of "informal, non-model-theoretic entailment," which I will now define.

### 3.3 The one genuinely important kind of entailment: The informal, non-model-theoretic kind

Let  $P_1$  be the statement *John knows that  $1 + 1 = 2$* , and let  $Q_1$  be the statement *John believes that  $1 + 1 = 2$* .

$P_1$  model-theoretically entails  $Q_1$ . In other words, there is no coherently conceivable scenario where  $P_1$  is true and  $Q_1$  is false.

But the entailment relation that holds between  $P_1$  and  $Q_1$  is not *just* an instance of model-theoretic entailment; it is also an instance of another, more significant kind of entailment. This is made clear by the fact that there is a marked difference between, on the one hand, the relation that holds between

$P_1$  and  $Q_1$  and, on the other hand, the relation that holds between:

$$P_2: 1 + 1 = 2$$

and

$Q_2$ : *triangles have three sides.*

The difference is clear. Given the information contained in  $P_1$ , it can be deduced that  $Q_1$  is true.  $Q_1$  is, in some important sense, “implicit in” or “contained in”  $P_1$ . But  $Q_2$  isn’t implicit in  $P_2$ . There isn’t anything concerning triangles in  $P_2$ —not implicitly or explicitly.

When we make inferences, they are *not* of the strictly model-theoretic kind. Rather, they are of the kind where the statement that is inferred is, in some way or other, contained in the statement from which the inference is made. So the inference from:

$$P_1: \text{John knows that } 1 + 1 = 2$$

to

$$Q_1: \text{John believes that } 1 + 1 = 2$$

is an instance of the conception of entailment that is embodied in our actual inferential practices. But that inference isn’t formal; it isn’t like the inference from *snow is white and grass is green* to *snow is white*: the formal properties of  $P_1$  and  $Q_1$  leave it open whether  $Q_1$  follows from  $P_1$ . This is because those sentences are formally identical with, respectively,

$$P_3: \text{John is sure that } 1 + 1 = 2$$

and

$$Q_3: \text{John doubts that } 1 + 1 = 2.$$

$P_1$  is syntactically just like  $P_3$ , and  $Q_1$  is syntactically just like  $Q_3$ . But there is

no legitimate inference from  $P_3$  to  $Q_3$ ; and since those sentences are syntactically just like  $P_1$  and  $Q_1$ , respectively, it follows there is no legitimate, purely syntactic inference from  $P_1$  to  $Q_1$ .

Although, as previously discussed,  $P_1$  model-theoretically entails  $Q_1$ ,  $P_1$  also entails  $Q_1$  in some other, much more robust sense. In Kuczynski (2007), I put forth a theory as to the nature of this other, more important sort of entailment.

### 3.5 The ampliative character of deduction

It is often said that, if  $P$  entails  $Q$ , there is nothing in  $Q$  that is not also in  $P$ . So deduction (i.e., the identifying of entailments) is, according to this, “non-ampliative”—the consequent doesn’t “amplify” the antecedent (i.e., it doesn’t say anything not said by the antecedent).

This position is obviously false where model-theoretic entailment is concerned. (*Triangles are shapes* model-theoretically entails *Socrates is identical with Socrates*, but there is nothing in the content of the first that has anything to do with Socrates.) And it’s obviously false where formal entailment is concerned. (“Smith is tall” formally entails “either Smith is tall or grass is green,” but there is nothing about grass in “Smith is tall.”)

### 3.6 The most important kind of entailment: the non-formal, non-model theoretic kind

The statement

$$(P) \ 1+1=2$$

model-theoretically entails

(Q) *the interior angles of a Euclidean triangle add up to  $180^\circ$ .*

And  $Q$  model-theoretically entails  $P$ .

This is all very well. But it’s patently obvious that you can’t *infer* the truth

of P from that of Q, or *vice versa*. In general, model-theoretical entailment is inferentially useless. A *precondition* for knowing that P entails Q is that you know, first, that P cannot be false and, second that Q cannot be false. But if you know this, there's no need to infer the one from the other. What this shows is that knowledge of model-theoretic entailments are parasitic on some other, much more fundamental sort of entailment. And what *this* shows is that knowledge of model-theoretic entailments have no significant role in the thought-processes of any possible cogitator.

Unlike model theoretic entailment, formal entailment isn't inferentially useless. But our ability to make formal inferences is parasitic on our ability to make informal inferences. Actually, so-called "formal" inferences *are* informal ones.

Formal entailment is a relation that holds between expressions (more specifically, sentences or sentence-tokens). This means that we have to know what sentences mean to make inferences that are licensed by formal entailments. To infer

$(S_1)$  "snow is white"

from

$(S_2)$  "snow is white and grass is green,"

I have to know what those two sentences mean, at least up to a point. I have to know that "and" doesn't mean what is in fact meant by "or." I also have to know that "snow is white" and "grass is green" are complete sentences—that they aren't drivel, like "bunga berka der."

So in order to infer  $S_1$  from  $S_2$ , I have to know the truth of some proposition of the form:

$(S_3)$  *given that  $S_2$  means such and such and that  $S_1$  means thus and such, it follows that  $S_1$  is true, given that  $S_2$  is true.*

But  $S_3$  is not formally true. First of all, it's a proposition, not a sentence. But even the corresponding sentence (the result of putting the verbiage in question in quotation marks) isn't formally true. It's an analytic, logical, *informal* truth.

Second reason: Formal truths, and thus formally true entailments, are *instances* of universal generalizations that, although analytically true, are not themselves formal truths. The sentence:

$(S_4)$  “if snow is white and grass is green, then snow is white”

is formally true. But why? Because

$(S_5)$  for all sentences  $P$  and  $Q$ ,  $\langle (P \text{ and } (P \rightarrow Q)) \rightarrow P \rangle$ ,

though true, isn't *formally* true. It *instances* (e.g., “if snow is white and grass is green, then snow is white”), but not it itself. This is because it has the same form as:

$(S_6)$  For no sentences  $P$  and  $Q$ ,  $\langle (P \text{ and } (P \rightarrow Q)) \rightarrow P \rangle$ ,

which is false.

Some logicians say that, because  $S_5$  has an “all” in the place where  $S_6$  has a “no,” they *don't* have the same form. But that's just a way of saying that they don't have the same form because, if they did, then  $S_5$  wouldn't be a formal truth anymore. It's a fact that, in any sense of the expression “same form” that isn't circularly constructed to validate preexisting theories,  $S_5$  and  $S_6$  have the same form. They certainly have the same syntactic form, at least as linguists use the word “syntax.” The rules of sentence construction involved in the one case precisely parallel those used in the other. It's a simple fact, then, that  $S_5$  and  $S_6$  do not, in any significant sense, differ in form and that  $S_5$ , though true, isn't formally so.

Given that formal truth is parasitic on informal truth, the reason being that instances of formal truths are instances of informal ones, entailment cannot ultimately, or even proximately, be understood in formal terms and, in addition, only a fraction of the inferences that we make are formal. (This is setting aside the fact noted a moment ago that so-called formal reasoning is really informal reasoning in disguise.)

### 3.7 Entailment not a limiting case of confirmation



For P to entail Q is *not* for P to confirm Q to the highest possible degree.

Entailment is a logical notion. Confirmation is an explanatory notion.

For P to confirm Q to be a high degree is for (*P and not-Q*) to be counter-explanatory. In other words, it is for (*P and not-Q*) to be false *unless* deeply rooted presumptions about how the world works are uprooted and replaced.

For P to entail Q is for (*P and not-Q*) to be counter-conceptual. In other words, it is for the structures of concepts, unassisted by presumptions on our part as to how the world works, to prohibit the falsity of Q, given the truth of P.

Therefore, P's confirming Q to the highest possible degree is different from P's entailing Q.

### 3.8 Entailment both an intersentential and an interpropositional relation

Sometimes the word “entailment” refers to a relationship that holds between propositions (sentence-*meanings*), and sometimes it refers to a relation that holds between expressions of some kind. Some logicians hold that those expressions are sentence-*types*. Others, myself included, hold that those relations may hold between sentence-*types* *and also* between sentence-*tokens*. (Peter Strawson (1950) was the first to distinguish token-entailment from type-entailment. In so doing, he performed a major service for philosophy.)

With rare exceptions (Strawson being one of them), authors who discuss “entailment” don't make it clear whether they are talking about a relationship that holds between propositions, sentences, or neither.

Because this is an introductory text, we are leaving it open whether entailment refers to

- (i) *an intersentential relation,*
- (ii) *an interpropositional relation,*
- (iii) *both,*
- (iv) *some third notion, or*
- (v) *all of the above*

The answer is (v). We will now outline the reasons for this. (These reasons are discussed in detail in *Intermediate Set-theory and Logic*.)

### 3.8.1 Four kinds of logical equivalence

There are four kinds of logical equivalence:

- (i) *The kind that holds between propositions,*
- (ii) *The kind that holds between sentence-tokens, and*
- (iii) *The kind that holds between sentence-types.*
- (iv) *The kind that holds between predicates.*

The following two propositions are equivalent:

- (1) *Jim has 2 cars.*
- (1\*) *If  $n$  is the number of Jim's cars,  $n+1=3$ .*

(1) cannot be true unless (1\*) is true, and (1\*) cannot be true unless (1) is true. In general, two propositions  $P$  and  $Q$  are equivalent iff the one cannot be true unless the other is true.

To say of two sentence-tokens,  $S$  and  $S^*$ , that they are 'equivalent' is to say that, if  $P$  and  $P^*$  are the propositions *meant* by  $S$  and  $S^*$ , respectively,  $P$  is equivalent with  $P^*$ . Thus, if  $S$  is an utterance of "Jim has 2 cars" and  $S^*$  is an utterance, in the very same context, of "if  $n$  is the number of Jim's cars,  $n+1=3$ ", then, given that the propositions borne by  $S$  and  $S^*$  are (1) and (1\*), respectively, it follows that what is meant by  $S$  is equivalent with what is meant by  $S^*$ , in the sense that what is meant by  $S$  is true just in case what is meant by  $S^*$  is true.

Sentence-types are neither true nor false. The sentence-type "I am tired" is neither true nor false. It is particular tokens of that type that are true or false. The meaning of that type is a rule to the effect that if somebody  $x$  utters that sentence in a given context  $C$ , then that utterance is true exactly if, in  $C$ ,  $x$  is tired. Thus, the meaning of the sentence-type "I am tired" is a function from contexts of utterance to propositions. The same is true of any given sentence-type of natural language. Any given such sentence contains a tense-marker and therefore has for its meaning a function from times of utterance to propositions. In some cases, e.g. "2+2 equals 4", the function in question is a

constant function: any two tokens of that sentence-type encode the same proposition. But a function, even a constant function, from contexts of utterance to propositions is neither true nor false. Therefore, supposing *arguendo* that two sentence-types, T and T\*, can be ‘equivalent’, it is not in the same sense as S and S\*. Nor, obviously, is it in the same sense as T and T\*.

That said, there is a sense in which two sentence-types can be equivalent. Let T be the sentence-type “for any n, if triangles have n-many sides, then  $n^2=9$ ”, and let T\* be the sentence-type “for any n, no triangle has n-many sides unless  $n+1=4$ .” There is a sense in which T and T\* are ‘equivalent.’ They are equivalent in the sense that, the semantic rules of English semantic being what they are, no token of T can bear a proposition that is non-equivalent with any token of T\*. So, if  $p_1...p_n$  are the semantic rules of English, then to say of T and T\* that they are equivalent is to say: *A consequence of  $p_1...p_n$  is that no proposition thereby assigned T is non-equivalent with any proposition thereby assigned to T\*.*

The fourth kind of equivalence is predicate-equivalence. Consider the following two predicates:

- (a) “x is tall,”
- (b) “Either x is tall or x is a square circle.”

Since “x” is a variable, neither (a) nor (b) is either true or false. But (a) and (b) are equivalent, in the sense that the one is true for the same values of “x” as the other.

Technically, a distinction is to be made between predicate-type equivalence and predicate-token equivalence. And a distinction is to be made between each of those two kinds of equivalence and propositional-schema equivalence, a propositional-schema being what is *meant* by a predicate (or, strictly speaking, predicate-token).

### 3.9 The irrelevance of formal logic to thought

For reasons that were discussed in Chapters 1, 7, and 18 of *Analytic Philosophy*, any assistance that knowledge of formal logic can provide one in

the way of acquiring knowledge is parasitic on one's knowledge of informal analytic truths. This has nothing to do with the idiosyncrasies of human psychology. It's an epistemological consequence of a strictly logical point. Given any open-sentence (or open-proposition) all of whose instances are true, for example:

(1)  $\langle (P \text{ and } (P \rightarrow Q)) \rightarrow Q \rangle$ ,

the *fact* that it's true for all its instances is a consequence of the fact that some informal truth holds—for example:

(2) *given any two propositions  $P$  and  $Q$ ,  $\langle (P \text{ and } (P \rightarrow Q)) \rightarrow Q \rangle$ .*

It must be stressed that (2) is an informal truth. (2) has the same form as the false statement that:

(2#) *given no two propositions  $P$  and  $Q$ ,  $\langle (P \text{ and } (P \rightarrow Q)) \rightarrow Q \rangle$ .*

Nothing that has the same form as a false statement is formally true. Therefore, (2), though true, isn't formally so. (2#) is a statement *about* a class of formal truths; but, like most statements *about* formal truths, it is not itself a formal truth.

Bearing this in mind, consider the following statement, which, unlike (2), is formally true:

(3) *“if snow is white, and (if snow is white then snow is not pink), then snow is not pink).”*

There are two different reasons why one might accept (3). One is that one knows it to be an instance of (1) and one accepts it for that reason. But in that case, one is accepting (3) on the grounds that it's a consequence of (2). Since (2) is informal, one's acceptance of (3), under those circumstances embodies knowledge of an informal truth.

The same thing holds if one's acceptance of (3) is based on one's understanding of the concepts meant by the expressions composing it. Suppose that your reason for accepting (3) is that, given your understanding of the concept expressed by “if...then...,” you know that it would be absurd to deny (3). In that case, your acceptance of (3) is based on your knowledge that, *the structure of that concept being what it is*, any proposition of that

form must be true. Thus, your acceptance of (3) is based on your acceptance of (2). Given that (2) is an informal truth, your acceptance of (3) under those circumstances constitutes informal knowledge. There is no circumstance, therefore, under which your knowledge of (3) is any sense formal. The concept of “formal” (or “mechanical”) thought is therefore an incoherent one.

## 4.0 Models

An open-sentence is an expression that contains a free variable, and is therefore neither true nor false, but is otherwise just like a sentence. So  $\langle x \text{ is an even number} \rangle$  is an open-sentence, since it contains a variable (“ $x$ ”) where there should be a “2” or a “4” or some such. It often happens that, given a set of one or more open-sentences, we wish to find some way of replacing the variables with constants that yield true sentences. For example, let  $S_1$  be the set containing the following two open-sentences (I’ll henceforth omit the use of quasi-quotation marks):

- (1)  $x$  is an even number that is greater than zero.
- (2)  $x$  is less than 20.

If we assign the number two or the number four to  $x$ , the result is that both sentences are true. So each of these assignments *validates*  $S_1$ .

In general, given a set of open-sentences, an assignment of constants to the free variables validates that set iff two conditions are met. First, that assignment turns *each* open-sentence in that set into an actual sentence. Second, every sentence is true.

An assignment of constants to a set of open-sentences is an *interpretation* of that set. An interpretation of such a set *validates* it iff, under that interpretation, every sentence in that set comes out true.

Let  $S_2$  be the set containing only the following open sentences:

- (1)  $x$  is an odd number.
- (2)  $x^2$  is greater than 80.
- (3)  $\text{root-}x$  is irrational.

If we assign the number nine to  $x$ , (1) and (2) come out true; but (3) does not. But if we assign the number 13 to  $x$ , (1)–(3) all come out true. (It is assumed that the universe of discourse is the set of numbers. This assumption will continue to be made until further notice.)

An assignment of constants to such a set of open-sentences is an **interpretation** of that set. Not all such sets are validated by all interpretations of them, as we just saw.

False interpretations must be distinguished from partial interpretations. A partial interpretation is one that is incomplete but is correct as far as it goes. Let  $S_3$  be the set containing only the following open-sentences:

- (1)  $x$  is a *phi*.
- (2)  $x$  is larger than  $y$ .
- (3)  $y$  is a *phi*.
- (4)  $y$  is a *psi*.

If we assign the numbers 10 and 8 to  $x$  and  $y$  respectively, and the property of being even to “*phi*,” we’ve partially interpreted  $S_3$ , since, under that interpretation, (1)–(3) come out true. But since we haven’t yet assigned a meaning to “*psi*,” we haven’t completely interpreted it. We *have* completely interpreted that set of open sentences if we make the further stipulation that for a thing to be a *psi* is for it to have the property of being less than 9. In fact, supposing that we make that additional stipulation, we have provided a complete interpretation of them that validates them. If we had instead stipulated that for a thing to be a *psi* is for it to be less than 5, we would have provided a complete interpretation of those open sentences that failed to validate them.

Oftentimes, the interpretations that are of interest are very abstract ones. Let  $S_4$  be the set containing just the following open-sentences:

- (1)  $x$  is a *phi*.
- (2)  $x$  bears  $R$  to  $y$ .
- (3)  $y$  is a *psi*.
- (3)  $R$  is “*reflexive*” (any given thing bears  $R$  to itself).
- (4)  $y$  bears  $R$  to  $z$ .
- (5)  $z$  is a *phi*.
- (6)  $R$  is “*transitive*” (if  $a$  bears it to  $b$ , and  $b$  bears it to  $c$ , then  $a$  bears it to

c).

Here's an interpretation of  $S_4$ .  $x$ ,  $y$ , and  $z$  are, respectively, the numbers two, three, and four.  $\phi$  and  $\psi$ , respectively, are the properties of being even and odd.  $R$  is the less relation of being less than or equal to.  $y$  is the number.

Question: Does the following set ( $S_5$ ) have an interpretation that validates it? If so, what is it?

- (1)  $x$  is a  $\phi$ .
- (2)  $x$  bears  $R$  to  $y$ .
- (3) Anything that bears  $R$  to anything bears  $R$  to itself.
- (4)  $y$  bears  $R$  to  $z$ .
- (5) Nothing bears  $R$  to anything that has  $\phi$ .

Some sets of open-sentences are true under *all* interpretations of them. An example is the set containing: " $x$  is identical with  $x$ " and "if, for some  $\phi$ ,  $x$  has  $\phi$  and  $y$  does not have  $\phi$ , then  $x$  is not identical with  $y$ ." Others (e.g., any set containing " $x$  is not identical with  $x$ ") are true under none.

Any interpretation of a set of the first kind is a formal truth. Thus "JMK is identical with JMK, and anything identical with a professor is identical to nothing that isn't a professor" is a formal truth.

## 5.0 Proof that all knowledge is inferential

The objects of knowledge are *truths* (e.g.  $2+2=4$ ), not *things* (stable-patterns of mass-energy displacements, e.g. rocks, trees, stars). One is *aware* of things; one does not in the relevant sense *know* them. (In Spanish, German, and French, there is one word for truth-knowledge (*saber*, *wissen*, *savoir*) and a different one for thing-knowledge (*conocer*, *kennen*, *connaître*). It is truths, and truths alone, that one knows.

We obviously acquire a great deal of knowledge on the basis of sensory observation. But because it is states of affairs (these being compendia of things: rocks, trees, and the like), not truths, that are the objects of sensory observation, the contents of our observations must be converted into propositions if those observations are to avail us of knowledge, as opposed to mere awareness.

The conversion of non-propositional (observational, objectual) content into propositional content involves knowledge of relations  $R_1 \dots R_n$  between non-propositional data-sets, on the one hand, and propositions, on the other, whereby, given such and such data-sets,  $R_1 \dots R_n$  warrant the acceptance of thus and such propositions.  $R_1 \dots R_n$  cannot possibly be given by empirical propositions (propositions known on the basis of sense-perception), since it is only if we first know  $R_1 \dots R_n$  that observation can avail us of any knowledge.

Therefore,

- (i)  $R_1 \dots R_n$  are analytic truths, and
- (ii) Our knowledge of  $R_1 \dots R_n$  is a priori, meaning that said knowledge is constitutive of our ability to acquire knowledge.

Thus, Kant's contention that we have *a priori* knowledge is correct, and Leibniz's contention that there are analytic truths is also correct.

## 5.1 Proof that some truths are non-empirical

A truth is non-empirical if it cannot be known on the basis of sensory observation. A proposition is "analytic" if its own structure guarantees its truth. All non-empirical truths are analytic and *vice versa*.

*Proof that not all truths are empirical and that not all knowledge is strictly observation-based:* The proposition that

- (1) All truth is empirical truth

is true exactly if

- (2) The class  $K$  of non-empirical truths is empty.

(2) cannot be known to be true on empirical grounds, since it concerns the scope of what *cannot* be known on such grounds. So (1) is not true unless it is a non-empirical truth. And if (1) it is non-empirical truth, then (1) is false. Therefore, it is false.

It follows that there are analytic truths (e.g. the negation of 1), and also



that we have knowledge of analytic truths (e.g. our knowledge of the negation of 1).

## 6.0 Set-theoretic characterizations of truth, meaning, and entailment

Truths are true propositions. A proposition is a class of properties. More precisely, a given proposition  $P$  is the smallest class of properties  $p_1 \dots p_n$  such that

- (i)  $P$  is true exactly if  $p_1 \dots p_n$  are jointly instantiated and
- (ii) It must be assumed that  $p_1 \dots p_n$  are  $P$ 's constituents, lest  $P$ 's inferential properties be inexplicable.

Let  $P$  be the proposition *Sam is smart*.  $P$  is true exactly if the following properties are instantiated:

$p_1$ : The property of being identical with Sam.

$p_2$ : The property of being smart.

$p_3$ : The property of being a thing  $x$  such that  $x = \text{Sam}$  and  $x$  is smart.

Of course,  $P$  is true exactly  $p_3$  by itself is instantiated. (This is consistent with our analysis, since, if  $p_3$  instantiated, so are  $p_1$  and  $p_2$ . ) But  $p_1$  and  $p_2$  must be reckoned as being among the constituents of  $P$ . This is because  $P$ 's composition must parallel that of the sentence "Sam is tall." For  $P$  is *by definition* the meaning of (that which is expressed by) the sentence "Sam is tall" (henceforth " $S$ "); and  $S$ 's constituents (setting aside the tense-marker, which is not important in this context) are

(1) "Sam"

(2) "is tall" (or, more accurately, " $x$  is tall"---the " $x$ ", though typically orthographically suppressed, always being semantically present), and

(3) "Sam is tall."

If it is assumed that P's composition parallels S's, then P must be assumed to be a set whose members  $p_1$ ,  $p_2$ , and  $p_3$ . And unless P's composition is assumed to parallel S's, it is impossible to explain P's inferential properties. For example, when it is said, as it sometimes is, that the proposition (not the sentence)

*(P<sub>2</sub>) something is smart*

is a *formal* consequence of P, this can be taken to mean that:

*(P<sub>3</sub>) Being a member of P,  $p_2$  cannot be instantiated unless P is true.*

In general, cases of formal entailment among propositions can be identified with cases of set-membership. (Formal entailment among sentences is an entirely different matter.) Given two propositions  $p$  and  $q$ ,  $p$  formally entails  $q$  if  $q$  is a member or the property-class with which  $p$  is identical.

$p$  *informally* entails  $q$  if

*(a)  $q$  is not a member of  $p$ , but*

*(b) There is some property class  $Q$  such that  $q$  is a member of  $Q$  and, moreover, the members of  $Q$  are jointly instantiated just in case the members of  $p$  are jointly instantiated.*

For example,

*(P<sub>4</sub>) Sam is both sentient (aware, but not necessarily truth-aware) and sapient (truth-aware, as opposed to object-aware)*

is an informal consequence of P.  $x$  is *sentient* is not a constituent of P. But  $x$  is *sentient* is a constituent of many a proposition  $Q$  that is equivalent with P, e.g.

*(P<sub>5</sub>) Sam is sentient, sapient, and, relative to other sapient creatures, exceptionally adept at processing information.*

Not only is  $P_5$  a consequence of  $P$ ;  $P$  is a consequence of  $P_5$ . They are equivalent, since each is an informal consequence of the other.

In general, *if  $P$  then  $Q$*  is necessarily true if the members of  $P$  cannot be jointly instantiated unless the members of  $Q$  are jointly instantiated, and it is *formally* true if the members of  $Q$  are subset of the members of  $P$ .

The sequel to the present volume contains an analysis of molecular propositions (propositions that are either quantified generalizations or that contain other propositions as proper parts), which, though a natural extension of the just-stated analysis, is not an obvious such extension. The sequel also contains rigorous and general analyses of *function*, *recursion*, *number*, *formal language*, *continuity*, and *dimensionality*. We will now put forth rough and ready analyses of these concepts, it being the purpose of the sequel to this text to justify as well as refine these analyses.

## 7.0 Outline of the contents of the sequel to this volume

### 7.1 Function

A function is a rule that assigns no more than one output to any given input. A function therefore generates, and may therefore be identified with, a class  $K$  of ordered pairs such that if  $(x,y)$  and  $(x,z)$  both belong to  $K$ , then  $y=z$ .

A function of more than one variable can be represented as a function one variable, where that variable is an ordered  $n$ -tuple. Thus, the addition function can be regarded as a function of two variables (e.g. one that assigns 7 to 3 when the value of the other independent variable is 4) or as a function of one variable (one that assigns 7 to  $(3,4)$ ).

### 7.2 Recursive definition

A recursive definition is one to the effect that some class  $K$  is the *posterity of  $\alpha$  with respect to  $R$* , for some  $\alpha$  and some *ordering relation*  $R$ .

$R$  is an ordering relation if  $R$  with respect to  $K$  if, for any members  $x$ ,  $y$ , and  $z$  of  $K$ ,  $\neg xRx$  ( $x$  doesn't bear  $R$  to itself:  *$R$  is non-reflexive*), asymmetric

$(xRy \rightarrow \neg yRx)$  ( $y$  doesn't bear  $R$  to  $x$  if  $x$  bears  $R$  to  $x$ :  $R$  is *asymmetric*) and  $(xRy \rightarrow (yRz \rightarrow xRz))$  ( $x$  bears to whenever  $y$  bears  $R$  to  $z$  and  $x$  bears  $R$  to  $y$ :  $R$  is *transitive*).

$K$  is the posterity of  $\alpha$  with respect to  $R$  if  $K$  is the smallest class such that

- (i)  $\alpha \in K$ , and
- (ii)  $\delta \in K$  whenever  $\beta R \delta$ ,

where  $\beta \in K$ .

### 7.3 Cardinals, rationals, and reals

A cardinal number is the size of a class. The meaning of

(a) *Jim has 3 houses*

is

(b)  *$K$  has 3 members,*

where  $K$  is the class of Jim's houses.

The class  $\mathbb{N}$  of cardinal numbers is defined inductively:  $x$  belongs to  $\mathbb{N}$  iff either

- (A)  $x=0$  or
- (B)  $x=n+1$ ,

where  $n \in \mathbb{N}$ .

Given a class  $K$ ,  $K$  has 0 members if, for all  $x$ ,  $x \notin K$ ; and  $K$  has  $n+1$  members if, for some  $y$  such that  $y \in K$ , any class  $K^*$  has  $n$  members if  $y \notin K^*$  but  $K^*$  otherwise has the same members as  $K$ .

$\mathbb{Q}$  can be recursively defined, where  $\mathbb{Q}$  is the class of rationals, since the members of  $\mathbb{Q}$  are the members of the posterity of  $k_2$  with respect to  $\rho$ , where  $k_n$  is the class of all rationals (with denominator  $n>0$ ) having index  $n$ , meaning that  $k_n = \{p/q: p+q=n\}$  (e.g.  $k_5 = \{1/4, 4/1, 2/3, 3/2\}$ , and where  $\rho$  is the

relation borne by  $k_m$  to  $k_{m+1}$ , for arbitrary  $m$ .

Equivalently,  $\mathbb{Q}$  is the smallest class whose members occurring on the following series:

$1/1, 1/2, 2/1, 1/3, 3/1, 2/2, 1/4, 4/1, 2/3, 3/2, \dots$

The reals cannot be recursively defined since, given any list  $L$  of real numbers (represented as repeating decimals),  $D(L) \notin L$ , where  $D(L)$  is the number that results when, if  $x$  is the figure  $n$  the  $n$ th decimal place of the  $n$ th member of  $L$ ,  $x$  is replaced with  $x+1$  when  $x < 9$  and  $x$  is replaced with 0 when  $x=9$ .

$\mathbb{R}$  is the class that contains every rational, along with every number that is the *limit* of an initial segment of  $(\mathbb{Q}, <)$ , where  $(\mathbb{Q}, <)$  is the series rationals arranged from lesser to greater. An initial segment of  $(\mathbb{Q}, <)$  is one that doesn't contain a given rational unless it contains every smaller rational.

## 7.4 Formal languages

A *formal language* is a recursively defined expression-class.

If  $K$  is a class of truths,  $K$  can be formalized if there is some formal language  $L$  such that, for some  $t$ ,  $t \in K$  whenever  $s \in L$ , where  $t$  is the meaning of  $s$ .

Therefore, a class  $K$  of truths can be formalized exactly if  $K$  can be recursively defined.

## 7.5 Incompleteness

Any given recursive definition generates a discrete series (a series each of whose members has an immediate successor). If  $S$  and  $S^*$  are any two discrete series,  $S$  and  $S^*$  are *isomorphic*, meaning that any truth concerning the relation of any  $S$ -member  $x$  to any  $S$ -member  $y$  is equivalent to some truth about some  $S^*$ -member  $x^*$  to some  $S^*$ -member  $y^*$ .

Any given discrete series is isomorphic with  $(\mathbb{N}, <)$ , this being the series:  $0, 1, 2, 3, \dots$

Therefore, any given recursive definition, and therefore any uninterpreted calculus (any formal language), is modeled by a structure that is

structure-identical with  $\mathbb{N}$ .

This means that  $(\mathbb{N}, <)$  is a model of any formal language  $L$  of which  $(\mathbb{R}, <)$  is also a model.

This means that there is no way to formalize arithmetic, since  $(\mathbb{R}, <)$  is described by a subset of the class of arithmetical truths.

This means that any consistent model of arithmetic is *incomplete*.

## 7.6 Axiom-sets in relation to incompleteness

An axiom-set is a finite class of sentence-schema. A sentence-schema (plural: *schemata*) is an expression that contains a free variable, and is therefore neither true nor false, but is otherwise just like a sentence. *x is tall* is a sentence-schema. Other examples of sentence-schemata are: *x has  $\diamond\diamond$* , *x bears R to y*, and *P or not-P*.

An axiom is a member of an axiom-set.

An axiom is neither true nor false, since an axiom is a sentence-schema and sentence-schemata are neither true nor false.

Given a class  $K$  of sentence-schemata,  $K^*$  is a *model* of  $K$  if, for any member  $s^*$  of  $K^*$ , there is some member  $s$  of  $K$  such that  $s^*$  is the result of coupling  $s$  with a specific meaning.

A class  $K^*$  of truths can be formalized if, and only if,  $K^*$  is the model of a class  $K$  sentence-schemata all of whose models are isomorphic with each other. If, for any class  $K$  of sentence-schemata modelled by a class  $K^*$  of truths, there are non-isomorphic models, then  $K^*$  cannot be formalized. If, for some such class  $K$ , any models of  $K$  are isomorphic, then  $K^*$  can be formalized.

If  $K_{\mathbb{R}}$  models a recursively defined class  $K$  of sentence-schemata, where  $K_{\mathbb{R}}$  contains the arithmetic of real numbers,  $K_{\mathbb{N}}$  also models  $K$ , where  $K_{\mathbb{N}}$  contains the arithmetic of cardinals but not of reals. There is no one-one correspondence between  $K_{\mathbb{R}}$ , and  $K_{\mathbb{N}}$ . Therefore,  $K_{\mathbb{N}}$  is not isomorphic with  $K_{\mathbb{R}}$ . Therefore,  $K_{\mathbb{R}}$  cannot be formalized.

## 7.7 Compactness vs. continuity

A series  $S$  is compact if it is not discrete.  $(\mathbb{Q}, <)$  is compact.  $S$  is *continuous* if it contains each of its own limiting points.  $(\mathbb{R}, <)$  is continuous.  $(\mathbb{Q}, <)$  is not continuous.

Continuous series are always compact, but compact series are not always continuous.

## 7.8 Set-measure

The *measure* of a class  $K$  is *minimal* if  $K$  has  $n$ -many members, for some  $n \leq \aleph_0$ , where  $\aleph_0$ -many is the number of members of any class that can be recursively defined.

The measure of a class is *nil* if  $K$  has no members.

The measure of a non-empty class is non-minimal if  $K$  has  $\aleph_1$ -many members, where  $\aleph_1$  is the number of members of  $\mathbb{R}$ .

The values of direct measurements are always given by rational numbers. But given two direct measurements whose values are  $m_1$  and  $m_2$ , it may follow that  $m_3$  is irrational. For example, the length of  $m_3 = \sqrt{2}$  units when  $m_3$  the length of the hypotenuse of a right triangle each of whose other sides have been shown through direct measurement to have a length of one unit.

If  $M_1$  and  $M_2$  are the numbers corresponding any two distinct lengths (or velocities or masses or volumes...), there are infinitely many irrational numbers  $M_3$  in between  $M_1$  and  $M_2$ . Therefore, if  $\phi$  is any degree property (e.g. length, mass, volume), the degree to which  $x$  has  $\phi$  cannot change at all without assuming infinitely many irrational values. It follows that the number of any class  $K$  that does not contain irrationals is too small to be the measure of any magnitude that is had to any degree other than an infinitesimally small one, this being why a class whose cardinality is  $n$ , where  $0 \leq n \leq \aleph_0$ , is minimal (but not nil).

Equivalently, measurement is neither identical nor identifiable with *enumeration*. To enumerate a class is to assign each of its members an integer, on the condition no two members are assigned the same integer. Since  $[\mathbb{Q}] = [\mathbb{N}]$ , where  $[k]$  is  $k$ 's cardinality,  $\mathbb{Q}$  is as measurement-useless

as  $\aleph$ . And since  $[\mathbb{R}]$  is by definition the number of the smallest non-measurement-useless class, it follows that the values of direct and indirect measurement are always real numbers.

## 7.9 Nil vs. infinitesimal

Real numbers are *degrees*. Cardinals are class-sizes. Rationals are relations between class sizes. ( $m/n$  is the relation that  $x$  bears to  $y$  when  $x \times n = y \times m$ .)

Given an event  $e$ , when it is said that  $p(e)=0$ , where  $p(e)$  is the probability of  $e$ , this is ambiguous, as it could mean either

- (A)  $e$ 's probability is *nil* or
- (B)  $e$ 's probability is *minimal but not nil*.

Given some quarter  $C$  and some surface  $S$  larger than  $C$  on which  $C$  is being dropped, there are  $\aleph_1$ -many regions on  $S$  that  $C$  can occupy. Therefore, the probability that  $C$  will occupy this or that specific region is less than any positive amount, since  $1/\aleph_1$  is less than any positive rational. But there is some region  $R$  such that  $C$  *does* occupy.

$p(e)$  is *minimal* but not *nil*, where  $e$  is  $C$ 's occupying  $R$ , for some specific region  $R$ . Equivalently,  $0_C \neq 0_R$ , where  $0_C$  is cardinal 0 and  $0_R$  is real 0.

A magnitude is  $0_R$  if that magnitude's *degree* is 0. If  $D$  is a given magnitude's degree,  $D$  is 0 if  $D \leq \aleph_0$ . Probabilities are given by degrees, like mass and length. If probabilities were given by enumerable quantities, it would indeed be a paradox that there should occur events of probability 0.

## 7.10 Dimensionality

$S$  is a 1-dimensional series if  $S$  is a continuous series whose members are non-series.  $S$  is an  $n+1$ -dimensional series if  $S$  is a series whose members are  $n$ -dimensional series.

*Fin*