Introduction to Symbolic Logic
J.-M. Kuczynski PhD

Week 1: Basic Concepts, Notation, and Logical Operators

Introduction to Logic
Logic is the study of valid reasoning. In this course, we'll focus on symbolic logic, which uses special symbols and precise rules to analyze and evaluate arguments. Unlike informal logic, symbolic logic gives us tools to determine validity with mathematical precision.

Basic Concepts
Statement (Proposition): A sentence that is either true or false, but not both
Argument: A sequence of statements where one (the conclusion) is claimed to follow from the others (the premises)
Validity: An argument is valid if it's impossible for all premises to be true while the conclusion is false
Soundness: A valid argument with true premises

Basic Logical Symbols
1. Negation (¬): "not"
   Example: If p = "It is raining"
   Then ¬p = "It is not raining"

2. Conjunction (∧): "and"
   Example: If p = "It is cold" and q = "It is windy"
   Then p ∧ q = "It is cold and windy"

3. Disjunction (∨): "or"
   Example: If p = "I will study" and q = "I will fail"
   Then p ∨ q = "I will study or I will fail"

4. Conditional (→): "if...then"
   Example: If p = "It rains" and q = "The ground is wet"
   Then p → q = "If it rains, then the ground is wet"

Material vs. Strict Implication
1. Material Implication (→)
   Truth-functional: defined purely by truth values
   False only when antecedent is true and consequent is false
   Example: "If it rains, then the ground is wet"

2. Strict Implication (⊃)
   Modal: involves necessary connection
   Example: "If x is a bachelor, then x is unmarried"

Represents logical necessity rather than mere material implication

Translation Practice

From English to Symbolic Notation:

1. "It's not both sunny and warm"
   Let p = "It's sunny", q = "It's warm"
   Translation: $\neg(p \wedge q)$

2. "Either I'll go to the party or I'll stay home and study"
   Let p = "I'll go to the party", q = "I'll stay home", r = "I'll study"
   Translation: $p \vee (q \wedge r)$

From Symbolic Notation to English:

1. $p \rightarrow (q \vee r)$
   Let p = "It rains", q = "I take an umbrella", r = "I stay inside"
   Translation: "If it rains, then I'll either take an umbrella or stay inside"

Practice Exercises

Translate these statements into symbolic notation:
1. "If I study hard and get enough sleep, then I'll pass the exam"
2. "It's not true that I'll both go to the movies and save money"
3. "Either I'll finish my homework or I won't watch TV"

[Solutions will be discussed next class]

Key Takeaways

1. Symbolic logic provides precise tools for analyzing arguments
2. Basic operators ($\neg$, $\wedge$, $\vee$, $\rightarrow$) can combine to express complex statements
3. Translation between natural language and logical notation requires careful attention to meaning
4. Material implication differs from strict implication in important ways

Next week, we'll explore truth tables and begin formal proofs using these concepts.

Symbolic Logic - Homework 1

Basic Concepts and Notation

Total Points: 50

Part 1: Translation (20 points)

Translate the following English statements into symbolic logic notation. Let:
p = "It rains"
q = "The streets are wet"
r = "People use umbrellas"

s = "Traffic slows down"

1. "If it rains, then the streets are wet and people use umbrellas." (5 points)

2. "Either traffic slows down or it's not raining." (5 points)

3. "It's not true that when it rains, traffic slows down." (5 points)

4. "If the streets are wet and people use umbrellas, then it must be raining." (5 points)

Part 2: Symbolization (20 points)
Create appropriate symbols and translate these statements into logical notation:

5. "If John studies and gets tutoring, then he will pass the class. John studies and gets tutoring. Therefore, John will pass the class." (10 points)

6. "Either Mary goes to the gym or she goes for a run, but not both. If she goes for a run, she will be tired." (10 points)

Part 3: Analysis (10 points)
7. Explain the difference between material implication and strict implication using an example of each. (10 points)

Grading Rubric:
Part 1: 5 points per correct translation
Part 2: 10 points per correct symbolization and translation
Part 3:
  5 points for correct explanation of the difference
  5 points for appropriate examples

Symbolic Logic - Homework 1
Basic Concepts and Notation (WITH ANSWERS)
Total Points: 50

Part 1: Translation (20 points)
Translate the following English statements into symbolic logic notation. Let:
p = "It rains"
q = "The streets are wet"
r = "People use umbrellas"
s = "Traffic slows down"

1. "If it rains, then the streets are wet and people use umbrellas." (5 points)
   Answer: $p \rightarrow (q \wedge r)$

2. "Either traffic slows down or it's not raining." (5 points)
   Answer: $s \lor \neg p$

3. "It's not true that when it rains, traffic slows down." (5 points)
   Answer: $\neg(p \rightarrow s)$

4. "If the streets are wet and people use umbrellas, then it must be raining." (5 points)
   Answer: $(q \land r) \rightarrow p$

Part 2: Symbolization (20 points)
Create appropriate symbols and translate these statements into logical notation:

5. "If John studies and gets tutoring, then he will pass the class. John studies and gets tutoring. Therefore, John will pass the class." (10 points)
   Answer:
   Let s = "John studies"
   Let t = "John gets tutoring"
   Let p = "John will pass the class"
   Translation: $((s \land t) \rightarrow p) \land (s \land t) \rightarrow p$

6. "Either Mary goes to the gym or she goes for a run, but not both. If she goes for a run, she will be tired." (10 points)
   Answer:
   Let g = "Mary goes to the gym"
   Let r = "Mary goes for a run"
   Let t = "Mary will be tired"
   Translation: $((g \lor r) \land \neg(g \land r)) \land (r \rightarrow t)$

Part 3: Analysis (10 points)
7. Explain the difference between material implication and strict implication using an example of each. (10 points)

Answer:
Material implication ($\rightarrow$) is truth-functional and is false only when the antecedent is true and the consequent is false. Example: "If it rains, then the ground is wet" ($p \rightarrow q$). This is a material implication because it's based on observed correlation.

Strict implication ($\supset$) involves logical necessity. Example: "If x is a square, then x has four equal sides." This is a strict implication because it's based on the definition of a square - it's impossible for the antecedent to be true and the consequent false by logical necessity.

Grading Rubric:
Part 1: 5 points per correct translation
Part 2: 10 points per correct symbolization and translation

Part 3:
5 points for correct explanation of the difference
5 points for appropriate examples

Propositional Calculus and Truth Tables
Week 2: Truth Tables and Elementary Proofs

The Propositional Calculus
Propositional calculus is a formal system for representing and analyzing logical relationships between propositions. It provides rules for:
1. Constructing well-formed formulas
2. Evaluating truth values
3. Making valid inferences

Truth Tables
Truth tables systematically show all possible truth value combinations for propositional statements.

Basic Truth Tables

Negation (¬)

p	¬p
T	F
F	T

Conjunction (∧)

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction (∨)

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

Conditional (→)

p	q	p → q

T	T	T
T	F	F
F	T	T
F	F	T

Elementary Proofs

Proofs in propositional logic use rules of inference to derive conclusions from premises.

Key Rules of Inference:

1. Modus Ponens (MP)
   From $p \to q$ and p, infer q
   Example:
   If it rains, the ground is wet ($p \to q$)
   It is raining (p)
   Therefore, the ground is wet (q)

2. Modus Tollens (MT)
   From $p \to q$ and $\neg q$, infer $\neg p$
   Example:
   If it rains, the ground is wet ($p \to q$)
   The ground is not wet ($\neg q$)
   Therefore, it is not raining ($\neg p$)

3. Disjunctive Syllogism (DS)
   From $p \lor q$ and $\neg p$, infer q
   Example:
   Either I'll study or I'll fail ($p \lor q$)
   I'm not studying ($\neg p$)
   Therefore, I'll fail (q)

4. Double Negation (DN)
   From $\neg\neg p$, infer p (and vice versa)
   Example:
   It is not the case that it is not raining ($\neg\neg p$)
   Therefore, it is raining (p)

De Morgan's Laws

These important laws show the relationship between negation and conjunction/disjunction:

1. $\neg(p \land q) \equiv (\neg p \lor \neg q)$
2. $\neg(p \lor q) \equiv (\neg p \land \neg q)$

Example:
"It's not true that both the sun is shining and it's warm"
≡ "Either the sun is not shining or it's not warm"

Practice Problems

Let's verify De Morgan's first law using a truth table:

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Notice how the columns for $\neg(p \wedge q)$ and $(\neg p \vee \neg q)$ are identical, proving their logical equivalence.

Key Takeaways

1. Truth tables provide a mechanical method for determining validity
2. Rules of inference allow us to construct valid proofs
3. De Morgan's laws show how negation interacts with conjunction and disjunction
4. Complex statements can be analyzed by breaking them down into components

Next week, we'll explore Boolean algebra and its relationship to propositional logic.

Symbolic Logic - Homework 2

Truth Tables and Elementary Proofs

Total Points: 50

Part 1: Truth Tables (20 points)

1. Construct complete truth tables for the following expressions: (5 points each)

a) $(p \rightarrow q) \wedge \neg p$

b) $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$

Part 2: Proofs (20 points)

2. Construct proofs for the following arguments using rules of inference. Show each step. (10 points each)

a) Prove: q from premises $p \rightarrow q$, $r \rightarrow \neg p$, r

b) Prove: $\neg p$ from premises $p \rightarrow q$, $p \rightarrow r$, $\neg q \vee \neg r$

Part 3: Applications (10 points)

3. Express these arguments in symbolic form and determine their validity using either a truth table or proof: (5 points each)

a) "If it's sunny, I'll go to the beach. If I go to the beach, I'll get a sunburn. It's sunny. Therefore, I'll get a sunburn."

b) "Either the butler or the maid is guilty. The butler has an alibi. Therefore, the maid is guilty."

Symbolic Logic - Homework 2

Truth Tables and Elementary Proofs (WITH ANSWERS)

Total Points: 50

Part 1: Truth Tables (20 points)

1. Construct complete truth tables for the following expressions: (5 points each)

a) $(p \rightarrow q) \land \neg p$

p	q	$p \rightarrow q$	$\neg p$	$(p \rightarrow q) \land \neg p$
T	T	T	F	F
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

b) $\neg(p \lor q) \leftrightarrow (\neg p \land \neg q)$

p	q	$p \lor q$	$\neg(p \lor q)$	$\neg p$	$\neg q$	$\neg p \land \neg q$	$\neg(p \lor q) \leftrightarrow (\neg p \land \neg q)$
T	T	T	F	F	F	F	T
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	T
F	F	F	T	T	T	T	T

Part 2: Proofs (20 points)

2. Construct proofs for the following arguments using rules of inference. Show each step. (10 points each)

a) Prove: q from premises $p \rightarrow q$, $r \rightarrow \neg p$, r

Answer:

1. $p \rightarrow q$ (premise)
2. $r \rightarrow \neg p$ (premise)

3. r (premise)
4. $\neg p$ (from 2,3 by MP)
5. $p \to q$ (from 1)
6. q (from 4,5 by MT)

b) Prove: $\neg p$ from premises $p \to q$, $p \to r$, $\neg q \lor \neg r$

Answer:
1. $p \to q$ (premise)
2. $p \to r$ (premise)
3. $\neg q \lor \neg r$ (premise)
4. Assume p (for reductio)
5. q (from 1,4 by MP)
6. r (from 2,4 by MP)
7. $q \land r$ (from 5,6 by ∧I)
8. Contradiction with 3
9. Therefore, $\neg p$

Part 3: Applications (10 points)

3. Express these arguments in symbolic form and determine their validity using either a truth table or proof: (5 points each)

a) "If it's sunny, I'll go to the beach. If I go to the beach, I'll get a sunburn. It's sunny. Therefore, I'll get a sunburn."

Answer:
Let p = "It's sunny"
Let q = "I'll go to the beach"
Let r = "I'll get a sunburn"

Symbolic form: $(p \to q) \land (q \to r) \land p \therefore r$

Proof:
1. $p \to q$ (premise)
2. $q \to r$ (premise)
3. p (premise)
4. q (from 1,3 by MP)
5. r (from 2,4 by MP)

Valid argument

b) "Either the butler or the maid is guilty. The butler has an alibi. Therefore, the maid is guilty."

Answer:
Let b = "The butler is guilty"
Let m = "The maid is guilty"

Symbolic form: (b ∨ m) ∧ ¬b ∴ m

This is a valid argument by disjunctive syllogism:
1. b ∨ m (premise)
2. ¬b (premise)
3. m (from 1,2 by DS)

Symbolic Logic - Homework 2
Truth Tables and Elementary Proofs (WITH ANSWERS)
Total Points: 50

Part 1: Truth Tables (20 points)
1. Construct complete truth tables for the following expressions: (5 points each)

a) (p → q) ∧ ¬p

p	q	p → q	¬p	(p → q) ∧ ¬p
T	T	T	F	F
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

b) ¬(p ∨ q) ↔ (¬p ∧ ¬q)

p	q	p ∨ q	¬(p ∨ q)	¬p	¬q	¬p ∧ ¬q	¬(p ∨ q) ↔ (¬p ∧ ¬q)
T	T	T	F	F	F	F	T
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	T
F	F	F	T	T	T	T	T

Part 2: Proofs (20 points)
2. Construct proofs for the following arguments using rules of inference. Show each step. (10 points each)

a) Prove: q from premises p → q, r → ¬p, r

Answer:
1. p → q (premise)

2. $r \to \neg p$ (premise)
3. r (premise)
4. $\neg p$ (from 2,3 by MP)
5. $p \to q$ (from 1)
6. q (from 4,5 by MT)

b) Prove: $\neg p$ from premises $p \to q$, $p \to r$, $\neg q \lor \neg r$

Answer:
1. $p \to q$ (premise)
2. $p \to r$ (premise)
3. $\neg q \lor \neg r$ (premise)
4. Assume p (for reductio)
5. q (from 1,4 by MP)
6. r (from 2,4 by MP)
7. $q \land r$ (from 5,6 by ∧I)
8. Contradiction with 3
9. Therefore, $\neg p$

Part 3: Applications (10 points)

3. Express these arguments in symbolic form and determine their validity using either a truth table or proof: (5 points each)

a) "If it's sunny, I'll go to the beach. If I go to the beach, I'll get a sunburn. It's sunny. Therefore, I'll get a sunburn."

Answer:
Let p = "It's sunny"
Let q = "I'll go to the beach"
Let r = "I'll get a sunburn"

Symbolic form: $(p \to q) \land (q \to r) \land p \therefore r$

Proof:
1. $p \to q$ (premise)
2. $q \to r$ (premise)
3. p (premise)
4. q (from 1,3 by MP)
5. r (from 2,4 by MP)

Valid argument

b) "Either the butler or the maid is guilty. The butler has an alibi. Therefore, the maid is guilty."

Answer:
Let b = "The butler is guilty"
Let m = "The maid is guilty"

Symbolic form: $(b \vee m) \wedge \neg b \therefore m$

This is a valid argument by disjunctive syllogism:
1. $b \vee m$ (premise)
2. $\neg b$ (premise)
3. m (from 1,2 by DS)

Boolean Algebra
Week 3: Boolean Operations and Laws

Introduction to Boolean Algebra
Boolean algebra, developed by George Boole, is a mathematical structure dealing with operations on logical values. It forms the foundation of digital circuit design and computer science.

Basic Operations
1. NOT (Complement)
   Symbol: ¬ or '
   $0' = 1$
   $1' = 0$

2. AND (Conjunction)
   Symbol: ∧ or ·
   $1 \wedge 1 = 1$
   $1 \wedge 0 = 0$
   $0 \wedge 1 = 0$
   $0 \wedge 0 = 0$

3. OR (Disjunction)
   Symbol: ∨ or +
   $1 \vee 1 = 1$
   $1 \vee 0 = 1$
   $0 \vee 1 = 1$
   $0 \vee 0 = 0$

Fundamental Laws of Boolean Algebra

1. Identity Laws
   $x \vee 0 = x$

$x \land 1 = x$

2. Null Laws
$x \lor 1 = 1$
$x \land 0 = 0$

3. Idempotent Laws
$x \lor x = x$
$x \land x = x$

4. Complement Laws
$x \lor \neg x = 1$
$x \land \neg x = 0$

5. Commutative Laws
$x \lor y = y \lor x$
$x \land y = y \land x$

6. Associative Laws
$(x \lor y) \lor z = x \lor (y \lor z)$
$(x \land y) \land z = x \land (y \land z)$

7. Distributive Laws
$x \lor (y \land z) = (x \lor y) \land (x \lor z)$
$x \land (y \lor z) = (x \land y) \lor (x \land z)$

8. De Morgan's Laws
$\neg(x \lor y) = \neg x \land \neg y$
$\neg(x \land y) = \neg x \lor \neg y$

Boolean Functions and Truth Tables
Any boolean function can be represented as a truth table:

Example: $f(x,y) = x \land (\neg y)$

x	y	¬y	x ∧ (¬y)
0	0	1	0
0	1	0	0
1	0	1	1
1	1	0	0

Boolean Expressions and Simplification
Boolean expressions can be simplified using the laws above.

Example:
Simplify: $(x \land y) \lor (x \land \neg y) \lor (\neg x \land y)$

Step 1: Factor out common terms
$(x \land y) \lor (x \land \neg y) = x \land (y \lor \neg y) = x$ (by complement law)
So our expression is: $x \lor (\neg x \land y)$

Step 2: Use distributive law
$x \lor (\neg x \land y) = (x \lor \neg x) \land (x \lor y) = 1 \land (x \lor y) = (x \lor y)$

Applications in Digital Logic
Boolean algebra is fundamental to digital circuit design:

1. AND Gate
```
   A
   ---|
Output = A∧B
   ---|
   B
```

2. OR Gate
```
   A
   ---|
Output = A∨B
   ---|
   B
```

3. NOT Gate
```
Input ---|>o--- Output
```

Key Takeaways
1. Boolean algebra provides a mathematical framework for logical operations
2. The fundamental laws allow for systematic simplification of complex expressions
3. Truth tables can represent any boolean function
4. Applications extend to digital circuit design and computer science

Next week, we'll begin exploring quantification and how it extends our logical toolkit beyond simple true/false values.

Symbolic Logic - Homework 3
Boolean Algebra
Total Points: 50

Part 1: Boolean Laws and Simplification (20 points)
1. Simplify the following Boolean expressions. Show your work and cite the laws used at each step. (5 points each)

a) $(x \land y) \lor (x \land \neg y) \lor (x \land y)$

b) $\neg(\neg x \lor y) \land (x \lor \neg y)$

c) $(x \lor y) \land (x \lor \neg y) \land (\neg x \lor y)$

d) $\neg(x \land y) \lor (x \land \neg y)$

Part 2: Truth Tables and Boolean Functions (20 points)
2. Complete the truth tables for these Boolean functions and determine if they are equivalent: (10 points each)

a) $f(x,y) = (x \to y) \land \neg y$
$g(x,y) = x \land \neg y$

b) $f(x,y,z) = x \land (y \lor z)$
$g(x,y,z) = (x \land y) \lor (x \land z)$

Part 3: Applications (10 points)
3. Design a Boolean circuit using AND, OR, and NOT gates to implement the following function:
$f(x,y) = (x \land \neg y) \lor (\neg x \land y)$

Draw the circuit diagram and explain what the function does.
Symbolic Logic - Homework 3
Boolean Algebra
Total Points: 50

Part 1: Boolean Laws and Simplification (20 points)
1. Simplify the following Boolean expressions. Show your work and cite the laws used at each step. (5 points each)

a) $(x \land y) \lor (x \land \neg y) \lor (x \land y)$

Answer:
(x ∧ y) ∨ (x ∧ ¬y) ∨ (x ∧ y)
= (x ∧ y) ∨ (x ∧ ¬y) [Idempotent Law]
= x ∧ (y ∨ ¬y) [Distributive Law]
= x ∧ 1 [Complement Law]
= x [Identity Law]

b) ¬(¬x ∨ y) ∧ (x ∨ ¬y)

Answer:
¬(¬x ∨ y) ∧ (x ∨ ¬y)
= (x ∧ ¬y) ∧ (x ∨ ¬y) [De Morgan's Law]
= x ∧ ¬y [Absorption Law]

c) (x ∨ y) ∧ (x ∨ ¬y) ∧ (¬x ∨ y)

Answer:
(x ∨ y) ∧ (x ∨ ¬y) ∧ (¬x ∨ y)
= x ∨ (y ∧ ¬y) [Distributive Law]
= x ∨ 0 [Complement Law]
= x [Identity Law]

d) ¬(x ∧ y) ∨ (x ∧ ¬y)

Answer:
¬(x ∧ y) ∨ (x ∧ ¬y)
= (¬x ∨ ¬y) ∨ (x ∧ ¬y) [De Morgan's Law]
= ¬y ∨ (x ∧ ¬y) [Absorption Law]
= ¬y [Absorption Law]

Part 2: Truth Tables and Boolean Functions (20 points)
2. Complete the truth tables for these Boolean functions and determine if they are equivalent: (10 points each)

a) f(x,y) = (x → y) ∧ ¬y
g(x,y) = x ∧ ¬y

Answer:

x	y	x → y	¬y	(x → y) ∧ ¬y	x ∧ ¬y
0	0	1	1	1	0
0	1	1	0	0	0
1	0	0	1	0	1

1	1	1	0	0	0

The functions are not equivalent.

b) $f(x,y,z) = x \wedge (y \vee z)$
$g(x,y,z) = (x \wedge y) \vee (x \wedge z)$

Answer:

x	y	z	$y \vee z$	$x \wedge (y \vee z)$	$x \wedge y$	$x \wedge z$	$(x \wedge y) \vee (x \wedge z)$
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

The functions are equivalent (by distributive law).

Part 3: Applications (10 points)

3. Design a Boolean circuit using AND, OR, and NOT gates to implement the following function:
$f(x,y) = (x \wedge \neg y) \vee (\neg x \wedge y)$

Answer:
```
   x -------|AND|----
       |         |
       |NOT|      |OR|---- Output
       |         |
   y ---|AND|----
```

This is known as the XOR (exclusive or) function.

The function outputs 1 when exactly one input is 1, and 0 otherwise.

Symbolic Logic - First Midterm Examination
Weeks 1-3: Basic Logic, Truth Tables, and Boolean Algebra
Total Points: 250
Time Allowed: 120 minutes

Part I: Fundamentals of Logic (50 points)

1. (25 points) Define and give an example of each of the following:
   a) Material implication
   b) Strict implication
   c) Valid argument
   d) Sound argument
   e) Well-formed formula

2. (25 points) Translate the following into symbolic notation AND its negation:
"If either John or Mary goes to the store, then Bill will cook dinner, but only if Sarah helps."

Part II: Truth Tables and Validity (100 points)

3. (50 points) Use a truth table to determine if these arguments are valid:
   a) $p \to q$, $q \to r \therefore p \to r$
   b) $p \lor q$, $\neg p \land \neg q \therefore r$

4. (50 points) Complete a truth table for:
$((p \to q) \land \neg r) \leftrightarrow (p \land (q \lor \neg r))$

Part III: Boolean Algebra (100 points)

5. (50 points) Simplify using Boolean algebra laws (show all steps):
   $((x \land y) \lor (x \land \neg y)) \land (x \lor y)$

6. (50 points) Design a Boolean circuit for:
$f(x,y,z) = (x \land y) \lor (\neg x \land z)$

Draw the circuit diagram and explain how it implements the given function.

Symbolic Logic - First Midterm Examination
Weeks 1-3: Basic Logic, Truth Tables, and Boolean Algebra
Total Points: 250
Time Allowed: 120 minutes

Part I: Fundamentals of Logic (50 points)

1. (25 points) Define and give an example of each of the following:
   a) Material implication
   b) Strict implication
   c) Valid argument
   d) Sound argument
   e) Well-formed formula

Answer:
a) Material implication: A truth-functional conditional that is false only when the antecedent is true and consequent false.
Example: "If it rains, then the ground is wet." ($p \to q$)

b) Strict implication: A conditional expressing logical necessity.
Example: "If x is a triangle, then x has three sides." (Necessarily true by definition)

c) Valid argument: An argument where it's impossible for all premises to be true and conclusion false.
Example:
All men are mortal.
Socrates is a man.
Therefore, Socrates is mortal.

d) Sound argument: A valid argument with true premises.
Example: The above argument about Socrates is also sound because its premises are true.

e) Well-formed formula: An expression that follows the formal rules of logical syntax.
Example: $(p \land q) \to r$ is well-formed; $p \land \to q$ is not.

2. (25 points) Translate the following into symbolic notation AND its negation:
"If either John or Mary goes to the store, then Bill will cook dinner, but only if Sarah helps."

Answer:
Let: j = "John goes to the store"
m = "Mary goes to the store"
b = "Bill will cook dinner"
s = "Sarah helps"

Translation: $((j \lor m) \to b) \land (b \to s)$
Negation: $\neg(((j \lor m) \to b) \land (b \to s))$
$= \neg((j \lor m) \to b) \lor \neg(b \to s)$
$= ((j \lor m) \land \neg b) \lor (b \land \neg s)$

Part II: Truth Tables and Validity (100 points)

3. (50 points) Use a truth table to determine if these arguments are valid:
a) $p \to q,\ q \to r \therefore p \to r$
b) $p \lor q,\ \neg p \land \neg q \therefore r$

Answer:

a)

p	q	r	$p \to q$	$q \to r$	$p \to r$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

Valid: In all rows where premises ($p \to q$, $q \to r$) are true, conclusion ($p \to r$) is true.

b)

p	q	r	$p \lor q$	$\neg p$	$\neg q$	$\neg p \land \neg q$
T	T	T	T	F	F	F
T	T	F	T	F	F	F
T	F	T	T	F	T	F
T	F	F	T	F	T	F
F	T	T	T	T	F	F
F	T	F	T	T	F	F
F	F	T	F	T	T	T
F	F	F	F	T	T	T

Invalid: Premises can be true ($p \lor q$ = F, $\neg p \land \neg q$ = T) while conclusion (r) is false.

4. (50 points) Complete a truth table for:
$((p \to q) \land \neg r) \leftrightarrow (p \land (q \lor \neg r))$

Answer:

p	q	r	$p \to q$	$\neg r$	$(p \to q) \land \neg r$	$q \lor \neg r$	$p \land (q \lor \neg r)$	$((p \to q) \land \neg r) \leftrightarrow (p \land (q \lor \neg r))$
T	T	T	T	F	F	T	T	T
T	T	F	T	T	T	T	T	T
T	F	T	F	F	F	F	F	T
T	F	F	F	T	F	T	T	F
F	T	T	T	F	F	T	F	T
F	T	F	T	T	T	T	F	F
F	F	T	T	F	F	F	F	T
F	F	F	T	T	T	T	F	F

Part III: Boolean Algebra (100 points)

5. (50 points) Simplify using Boolean algebra laws (show all steps):
$((x \land y) \lor (x \land \neg y)) \land (x \lor y)$

Answer:
$((x \land y) \lor (x \land \neg y)) \land (x \lor y)$
$= [x \land (y \lor \neg y)] \land (x \lor y)$ [Distributive Law]
$= (x \land 1) \land (x \lor y)$ [Complement Law]
$= x \land (x \lor y)$ [Identity Law]
$= x$ [Absorption Law]

6. (50 points) Design a Boolean circuit for:
$f(x,y,z) = (x \land y) \lor (\neg x \land z)$

Answer:
```
  x -------|AND|----
      |          |
  y ----         |OR|---- Output
      |          |
  x ---|NOT|------|AND|----
      |
  z --------------
```

Explanation:
- Top path computes $(x \land y)$
- Bottom path computes $(\neg x \land z)$
- Final OR gate combines the results

Introduction to Quantification

Week 4: Quantifier Logic and Translation

Basic Concepts of Quantification

1. Universal Quantifier (∀)
   Means "for all" or "for every"
   Example: ∀x(Px) means "For all x, P is true of x"
   Common natural language expressions:
   "Every"
   "All"
   "Each"
   "Any"
   "No matter which"

2. Existential Quantifier (∃)
   Means "there exists" or "there is at least one"
   Example: $\exists x(Px)$ means "There exists an x such that P is true of x"
   Common natural language expressions:
     "Some"
     "There exists"
     "There is"
     "At least one"
     "For some"

Domain of Discourse
The set of all objects that variables can range over
Must be specified or understood from context
Examples:
  All real numbers
  All humans
  All integers
  All students in this class

Predicates
Functions that return true or false for given arguments
Examples:
  $P(x)$: "x is prime"
  $L(x,y)$: "x loves y"
  $G(x)$: "x is greater than 5"

Basic Translation Examples

1. "All dogs are mammals"
   Domain: All animals
   Let $D(x)$: x is a dog
   Let $M(x)$: x is a mammal
   Translation: $\forall x(D(x) \rightarrow M(x))$

2. "Some students study logic"
   Domain: All students
   Let $S(x)$: x studies logic
   Translation: $\exists x(S(x))$

3. "No one likes taxes"
   Domain: All people
   Let $L(x)$: x likes taxes
   Translation: $\neg\exists x(L(x))$
   Equivalent to: $\forall x(\neg L(x))$

Multiple Quantifiers

1. Order Matters
   $\forall x \exists y$ vs $\exists y \forall x$ can mean different things
   Example:
   "Everyone loves someone" vs "Someone is loved by everyone"
   $\forall x \exists y(L(x,y))$ vs $\exists y \forall x(L(x,y))$

2. Examples with Multiple Quantifiers:

   a) "Everyone has a mother"
      Domain: All people
      $M(x,y)$: x is the mother of y
      Translation: $\forall x \exists y(M(y,x))$

   b) "Some people love everyone"
      Domain: All people
      $L(x,y)$: x loves y
      Translation: $\exists x \forall y(L(x,y))$

Advanced Translation Examples

1. "Between any two real numbers there is another real number"
   Domain: Real numbers
   Translation: $\forall x \forall y \exists z((x < y) \rightarrow ((x < z) \land (z < y)))$

2. "For every positive number there is a smaller positive number"
   Domain: Real numbers
   Translation: $\forall x((x > 0) \rightarrow \exists y((y > 0) \land (y < x)))$

Negating Quantified Statements

1. Rules:
   $\neg \forall x(Px) \equiv \exists x(\neg Px)$
   $\neg \exists x(Px) \equiv \forall x(\neg Px)$

2. Examples:
   "Not everyone is happy"
   $\neg \forall x(H(x)) \equiv \exists x(\neg H(x))$
   "There is no free lunch"
   $\neg \exists x(F(x)) \equiv \forall x(\neg F(x))$

Practice Problems

1. Translate:
"Every student who studies passes"
Domain: All students
$S(x)$: x studies
$P(x)$: x passes
Answer: $\forall x(S(x) \rightarrow P(x))$

2. Negate:
"All mathematicians are logical"
Original: $\forall x(M(x) \rightarrow L(x))$
Negation: $\exists x(M(x) \land \neg L(x))$

Key Takeaways
1. Quantifiers allow us to make statements about collections of objects
2. Order of quantifiers matters
3. Translation requires careful attention to logical structure
4. Negation of quantified statements follows specific rules

Next week, we'll explore more advanced quantification concepts and their applications to mathematical statements, including the famous epsilon-delta definition of limits.

Symbolic Logic - Homework 4
Introduction to Quantification
Total Points: 50

Part 1: Basic Translation (20 points)
Translate each statement into symbolic logic using quantifiers. Clearly state your predicate symbols and domain. (5 points each)

1. "Every philosopher is wise"

2. "Some students failed but some students passed"

3. "No one loves everyone"

4. "Between any two distinct points, there is another point"

Part 2: Negations (15 points)
Negate each statement, then simplify using quantifier negation rules. Show your work. (5 points each)

1. $\forall x(P(x) \rightarrow Q(x))$

2. $\exists x \forall y(R(x,y))$

3. $\forall x \exists y(P(x) \rightarrow Q(y))$

Part 3: Analysis and Applications (15 points)

Consider these two statements:
1. "Everyone loves someone"
2. "Someone is loved by everyone"

a) (5 points) Translate both statements into symbolic logic.

b) (5 points) Are these statements logically equivalent? Explain why or why not.

c) (5 points) Write the negation of both statements.

Note: For full credit, explain your reasoning and show all steps in your work.

Symbolic Logic - Homework 4
Introduction to Quantification
Total Points: 50

Part 1: Basic Translation (20 points)
Translate each statement into symbolic logic using quantifiers. Clearly state your predicate symbols and domain. (5 points each)

1. "Every philosopher is wise"

Answer:
Domain: All people
P(x): x is a philosopher
W(x): x is wise
Translation: $\forall x(P(x) \rightarrow W(x))$

2. "Some students failed but some students passed"

Answer:
Domain: All students
F(x): x failed
P(x): x passed
Translation: $\exists x(F(x)) \land \exists x(P(x))$

3. "No one loves everyone"

Answer:
Domain: All people
$L(x,y)$: x loves y
Translation: $\neg\exists x\forall y(L(x,y))$
Alternative: $\forall x\exists y(\neg L(x,y))$

4. "Between any two distinct points, there is another point"

Answer:
Domain: All points
Translation: $\forall x\forall y\exists z(x \neq y \to ((x < z) \land (z < y)))$

Part 2: Negations (15 points)
Negate each statement, then simplify using quantifier negation rules. Show your work. (5 points each)

1. $\forall x(P(x) \to Q(x))$

Answer:
$\neg\forall x(P(x) \to Q(x))$
$\exists x\neg(P(x) \to Q(x))$
$\exists x(P(x) \land \neg Q(x))$

2. $\exists x\forall y(R(x,y))$

Answer:
$\neg\exists x\forall y(R(x,y))$
$\forall x\neg\forall y(R(x,y))$
$\forall x\exists y(\neg R(x,y))$

3. $\forall x\exists y(P(x) \to Q(y))$

Answer:
$\neg\forall x\exists y(P(x) \to Q(y))$
$\exists x\neg\exists y(P(x) \to Q(y))$
$\exists x\forall y\neg(P(x) \to Q(y))$
$\exists x\forall y(P(x) \land \neg Q(y))$

Part 3: Analysis and Applications (15 points)

Consider these two statements:
1. "Everyone loves someone"
2. "Someone is loved by everyone"

a) (5 points) Translate both statements into symbolic logic.

Answer:
L(x,y): x loves y
Statement 1: $\forall x \exists y(L(x,y))$
Statement 2: $\exists y \forall x(L(x,y))$

b) (5 points) Are these statements logically equivalent? Explain why or why not.

Answer:
No, they are not equivalent. Statement 1 allows each person to love a different person (everyone picks someone to love), while Statement 2 requires the existence of a single person who is loved by everyone (one person is universally loved).

To see this isn't equivalent, consider a world with just Alice and Bob where Alice loves Alice and Bob loves Bob. This satisfies Statement 1 but not Statement 2.

c) (5 points) Write the negation of both statements.

Answer:
Negation of 1: $\exists x \forall y(\neg L(x,y))$
"There is someone who loves no one"

Negation of 2: $\forall y \exists x(\neg L(x,y))$
"For every person, there is someone who doesn't love them"

Advanced Quantification and Mathematical Applications
Week 5: Complex Translation and Mathematical Logic

Advanced Translation Patterns

1. Uniqueness Quantifier ($\exists!$)
   Means "there exists exactly one"
   Example: $\exists! x(Px)$
   Equivalent to: $\exists x(Px \land \forall y(Py \rightarrow y = x))$
   Example: "There is exactly one even prime number"
   $\exists x(Prime(x) \land Even(x) \land \forall y((Prime(y) \land Even(y)) \rightarrow y = x))$

2. At Least/At Most/Exactly n
   "At least n": $\exists x_1 \ldots \exists x_n(P(x_1) \land \ldots \land P(x_n) \land x_1 \neq x_2 \land \ldots \land x_{n-1} \neq x_n)$
   "At most n": $\forall x_1 \ldots \forall x_{n+1}(P(x_1) \land \ldots \land P(x_{n+1}) \rightarrow (x_1 = x_2 \lor \ldots \lor x_n = x_{n+1}))$
   "Exactly n": Conjunction of "at least n" and "at most n"

Complex Mathematical Statements

1. Continuity
   A function f is continuous at a point c if:
   $\forall\varepsilon>0\ \exists\delta>0\ \forall x(|x-c|<\delta \rightarrow |f(x)-f(c)|<\varepsilon)$

2. Limit Definition
   The limit of f(x) as x approaches c is L if:
   $\forall\varepsilon>0\ \exists\delta>0\ \forall x(0<|x-c|<\delta \rightarrow |f(x)-L|<\varepsilon)$

The Epsilon-Delta Definition in Detail

1. Components
   ε (epsilon): measures closeness to limit value
   δ (delta): measures closeness to input value
   Relationship: For any ε, we can find δ

2. Breaking Down the Definition
   $\forall\varepsilon>0$: "For any positive error margin"
   $\exists\delta>0$: "There exists some positive distance"
   $\forall x$: "For all input values"
   $0<|x-c|<\delta$: "Within δ of c (but not equal to c)"
   $|f(x)-L|<\varepsilon$: "f(x) is within ε of L"

3. Example: $\lim(x\rightarrow 2)\ x^2 = 4$
   $\forall\varepsilon>0\ \exists\delta>0\ \forall x(0<|x-2|<\delta \rightarrow |x^2-4|<\varepsilon)$
   To prove: For given ε, find δ where:
   $|x-2|<\delta$ implies $|x^2-4|<\varepsilon$
   $|x^2-4| = |(x+2)(x-2)| = |x+2||x-2|$
   When $|x-2|<1$, $|x+2|<5$
   Choose $\delta = \min(1, \varepsilon/5)$

Advanced Translation Examples

1. Infinite Sequence Properties
   Bounded: $\exists M\forall n(|a_n|\leq M)$
   Convergent: $\exists L\forall\varepsilon>0\exists N\forall n(n>N \rightarrow |a_n-L|<\varepsilon)$
   Cauchy: $\forall\varepsilon>0\exists N\forall m\forall n(m,n>N \rightarrow |a_m-a_n|<\varepsilon)$

2. Function Properties
   One-to-one: $\forall x\forall y(f(x)=f(y) \rightarrow x=y)$
   Onto: $\forall y\exists x(f(x)=y)$
   Monotonic increasing: $\forall x\forall y(x<y \rightarrow f(x)<f(y))$

Practical Application Examples

1. Database Queries
   "Find all students who passed all courses"
   $\exists x(\text{Student}(x) \land \forall y(\text{Course}(y) \rightarrow \text{Passed}(x,y)))$

2. Software Specifications
   "Every request eventually gets a response"
   $\forall x(\text{Request}(x) \rightarrow \exists y(\text{Response}(y,x)))$

Common Translation Patterns

1. Only if
   "P only if Q" becomes $P \rightarrow Q$
   Example: "A number is even only if it's divisible by 2"
   $\forall x(\text{Even}(x) \rightarrow \text{Divisible}(x,2))$

2. Unless
   "P unless Q" becomes $P \lor Q$
   Example: "I'll study unless I'm tired"
   $\text{Study}(x) \lor \text{Tired}(x)$

3. Neither/Nor
   "Neither P nor Q" becomes $\neg P \land \neg Q$
   Example: "Neither prime nor composite"
   $\neg\text{Prime}(x) \land \neg\text{Composite}(x)$

Key Takeaways
1. Advanced quantification allows precise expression of complex ideas
2. Mathematical definitions often involve multiple nested quantifiers
3. Understanding quantifier order is crucial for correct interpretation
4. Epsilon-delta proofs follow a standard pattern
5. Translation requires attention to subtle linguistic differences

Next week, we'll explore models and their use in proving invalidity and consistency.

Symbolic Logic - Homework 5
Advanced Quantification and Mathematical Analysis
Total Points: 50

Part 1: Advanced Translation (20 points)

1. (10 points) Translate each statement into symbolic logic. State your predicates and domain.

a) "There is exactly one largest prime number less than 10."

b) "For any two different real numbers, their average lies between them."

2. (10 points) Negate and simplify the following statements:

a) $\forall x \exists y \forall z (P(x,y) \rightarrow Q(y,z))$

b) $\exists! x \forall y (P(x) \land Q(x,y))$

Part 2: Epsilon-Delta Analysis (20 points)

3. (10 points) Prove that $\lim(x \rightarrow 3)\ x^2 = 9$ using the epsilon-delta definition.

4. (10 points) Write the formal epsilon-delta definition for continuity of $f(x) = x^2$ at $x = 2$.

Part 3: Complex Mathematical Statements (10 points)

5. (5 points) Express the statement "A function f is uniformly continuous on an interval [a,b]" in symbolic logic.

6. (5 points) Translate the definition of a Cauchy sequence into English and explain why it doesn't use a limit L in its definition.

Note: For full credit:
- Show all work in epsilon-delta proofs
- Clearly state any predicates or domains used in translations
- Provide step-by-step reasoning for complex negations

Symbolic Logic - Homework 5
Advanced Quantification and Mathematical Analysis
Total Points: 50

Part 1: Advanced Translation (20 points)

1. (10 points) Translate each statement into symbolic logic. State your predicates and domain.

a) "There is exactly one largest prime number less than 10."

Answer:
Domain: Natural numbers
P(x): x is prime
L(x,y): x is less than y

$\exists x(P(x) \land L(x,10) \land \forall y((P(y) \land L(y,10)) \rightarrow (x \geq y)) \land$
$\forall z((P(z) \land L(z,10) \land (x \geq z)) \rightarrow x = z))$

b) "For any two different real numbers, their average lies between them."

Answer:
Domain: Real numbers
Translation: $\forall x \forall y(x \neq y \rightarrow \exists z((z = (x+y)/2) \land ((x < z \land z < y) \lor (y < z \land z < x))))$

2. (10 points) Negate and simplify the following statements:

a) $\forall x \exists y \forall z(P(x,y) \rightarrow Q(y,z))$

Answer:
$\neg \forall x \exists y \forall z(P(x,y) \rightarrow Q(y,z))$
$\equiv \exists x \forall y \exists z \neg(P(x,y) \rightarrow Q(y,z))$
$\equiv \exists x \forall y \exists z(P(x,y) \land \neg Q(y,z))$

b) $\exists! x \forall y(P(x) \land Q(x,y))$

Answer:
$\neg \exists! x \forall y(P(x) \land Q(x,y))$
$\equiv \neg \exists x(P(x) \land \forall y(Q(x,y)) \land \forall z((P(z) \land \forall y(Q(z,y))) \rightarrow z = x))$
$\equiv \forall x \neg(P(x) \land \forall y(Q(x,y))) \lor \exists z \exists w((P(z) \land \forall y(Q(z,y))) \land (P(w) \land \forall y(Q(w,y))) \land z \neq w)$

Part 2: Epsilon-Delta Analysis (20 points)

3. (10 points) Prove that $\lim(x\rightarrow 3)\ x^2 = 9$ using the epsilon-delta definition.

Answer:
We need to prove: $\forall \varepsilon>0\ \exists \delta>0\ \forall x(0<|x-3|<\delta \rightarrow |x^2-9|<\varepsilon)$

Proof:
$|x^2-9| = |x+3||x-3|$
When $|x-3| < 1$, we have $2 < x < 4$, so $|x+3| < 7$

Let $\delta = \min(1, \varepsilon/7)$
Then $|x-3| < \delta$ implies $|x^2-9| = |x+3||x-3| < 7\delta \leq \varepsilon$

4. (10 points) Write the formal epsilon-delta definition for continuity of $f(x) = x^2$ at $x = 2$.

Answer:
$\forall \varepsilon>0\ \exists \delta>0\ \forall x(|x-2|<\delta \rightarrow |x^2-4|<\varepsilon)$

To find δ given ε:
$|x^2-4| = |x+2||x-2|$
When $|x-2| < 1$, $|x+2| < 5$
Choose $\delta = \min(1, \varepsilon/5)$

Part 3: Complex Mathematical Statements (10 points)

5. (5 points) Express the statement "A function f is uniformly continuous on an interval [a,b]" in symbolic logic.

Answer:
$\forall\varepsilon>0\ \exists\delta>0\ \forall x\forall y((x\in[a,b] \land y\in[a,b] \land |x-y|<\delta) \to |f(x)-f(y)|<\varepsilon)$

6. (5 points) Translate the definition of a Cauchy sequence into English and explain why it doesn't use a limit L in its definition.

Answer:
Symbolic form: $\forall\varepsilon>0\ \exists N\forall m\forall n(m,n>N \to |a_m-a_n|<\varepsilon)$

English: "For any positive error margin ε, there exists some point N in the sequence after which any two terms in the sequence differ by less than ε."

Explanation: The Cauchy criterion doesn't reference a limit L because it defines convergence based on the behavior of the sequence terms relative to each other, not to some external value. This makes it useful in spaces where limits might not exist or be easily definable.

Grading Rubric:
- Part 1: 10 points each for correct translations and negations
- Part 2: 5 points for correct formula, 5 points for correct proof/explanation
- Part 3: 5 points each for correct translation and explanation

Models and Proof Theory
Week 6: Using Models to Prove Invalidity and Consistency

Introduction to Models

1. Definition of a Model
   A mathematical structure that interprets:
   - Domain of discourse
   - Predicates
   - Functions
   - Constants

Assigns truth values to formulas

2. Components of a Model M
   Universe/Domain (|M|)
   Interpretation function (I)
   Variable assignment (v)

Using Models to Prove Invalidity

1. Strategy
   To prove an argument invalid:
     Find a model where premises are true
     But conclusion is false
   One counterexample suffices

2. Example 1: Proving Invalidity
   Argument: $\forall x(P(x) \rightarrow Q(x)), \exists xP(x) \therefore \forall xQ(x)$
   Countermodel:
     Domain = {a, b}
     P = {a}
     Q = {a}
   Analysis:
     $\forall x(P(x) \rightarrow Q(x))$ is true (vacuously for b)
     $\exists xP(x)$ is true (because of a)
     $\forall xQ(x)$ is false (because $b \notin Q$)

3. Example 2: More Complex Invalidation
   Argument: $\forall x\exists yR(x,y) \therefore \exists y\forall xR(x,y)$
   Countermodel:
     Domain = {1, 2}
     R = {(1,1), (2,2)}
   Analysis:
     $\forall x\exists yR(x,y)$ is true (each x relates to itself)
     $\exists y\forall xR(x,y)$ is false (no y relates to all x)

Consistency Proofs

1. Definition
   A set of sentences is consistent if:
     There exists a model satisfying all sentences
     No contradiction can be derived

2. Methods to Prove Consistency
   Construct a model satisfying all sentences

Show inability to derive contradiction
Use semantic tableaux

3. Example: Proving Consistency
   Sentences:
   $\forall x \exists y R(x,y)$
   $\forall x \neg R(x,x)$
   $\forall x \forall y \forall z((R(x,y) \land R(y,z)) \to R(x,z))$
   Model:
   Domain = {1, 2, 3}
   R = {(1,2), (2,3), (1,3)}
   Verification:
   Each x has a y it relates to
   No element relates to itself
   Transitivity holds

Model Theory Techniques

1. Finite Models
   Useful for disproving universally quantified statements
   Easy to verify all cases
   Example:
   To disprove "All groups are abelian"
   Use finite non-abelian group as model

2. Infinite Models
   Required for some mathematical structures
   Often use familiar structures ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$)
   Example:
   Natural numbers with usual < relation
   Models "No largest element exists"

3. Model Construction Strategies
   Start with smallest possible domain
   Add elements only as needed
   Use diagrams or tables for relations
   Check all required properties

Advanced Applications

1. Independence Proofs
   Show axiom not derivable from others
   Need two models:
   One satisfying all axioms

One satisfying all but the target axiom

2. Categoricity
   All models are isomorphic
   Example:
     Complete ordered fields
     All isomorphic to real numbers

Practical Examples

1. Graph Theory
   Model: Vertices and edges
   Properties:
     Connectivity
     Coloring
     Paths

2. Number Theory
   Model: Numbers with operations
   Properties:
     Divisibility
     Primality
     Ordering

Common Mistakes to Avoid

1. Domain Selection
   Don't make domain unnecessarily large
   Ensure domain includes all needed elements
   Check for implicit assumptions

2. Relation Definition
   Must specify complete relation
   Check all pairs/tuples
   Verify properties (transitivity, etc.)

Key Takeaways
1. Models provide concrete examples or counterexamples
2. One countermodel sufficient for invalidity
3. Consistency requires showing possible satisfaction
4. Start with simple models when possible
5. Check all required properties systematically

Next week, we'll explore recursivity and definable classes.

Symbolic Logic - Homework 6
Models and Proofs
Total Points: 50

Part 1: Invalidity Proofs (20 points)

1. (10 points) Provide a countermodel to prove each argument is invalid:

a) $\forall x(P(x) \rightarrow Q(x)), \forall x(Q(x) \rightarrow R(x)) \therefore \forall x(P(x) \rightarrow P(x))$

b) $\forall x \exists y L(x,y) \therefore \exists y \forall x L(x,y)$
($L(x,y)$ means "x loves y")

2. (10 points) Show that these statements are not equivalent by providing a countermodel:
"Everyone has a friend" and "Someone is friends with everyone"

Part 2: Consistency Proofs (20 points)

3. (10 points) Prove the following set of sentences is consistent by providing a model:
$\forall x \exists y(x < y)$
$\forall x \forall y \forall z((x < y \land y < z) \rightarrow x < z)$
$\forall x \neg(x < x)$

4. (10 points) Show the following set of sentences is consistent:
$\exists x P(x)$
$\exists x \neg P(x)$
$\forall x \forall y(R(x,y) \rightarrow (P(x) \leftrightarrow P(y)))$

Part 3: Model Analysis (10 points)

5. (5 points) Given the model M with domain {1,2,3} and relation R = {(1,2), (2,3), (1,3)}, determine whether each statement is true or false:

a) $\forall x \exists y R(x,y)$
b) $\forall x \forall y \forall z((R(x,y) \land R(y,z)) \rightarrow R(x,z))$
c) $\exists x R(x,x)$
d) $\forall x \exists y(R(x,y) \land \neg R(y,x))$

6. (5 points) For the following properties, give the smallest possible domain size needed for a model where the property holds:

a) $\exists x \exists y(x \neq y \land \forall z(z = x \lor z = y))$
b) $\exists x \exists y \exists z(R(x,y) \land R(y,z) \land \neg R(x,z))$

Note: For full credit:
- Clearly specify domains and interpretations in your models
- Verify that your models satisfy all required conditions
- Explain why your counterexamples work

Symbolic Logic - Homework 6
Models and Proofs
Total Points: 50

Part 1: Invalidity Proofs (20 points)

1. (10 points) Provide a countermodel to prove each argument is invalid:

a) $\forall x(P(x) \rightarrow Q(x))$, $\forall x(Q(x) \rightarrow R(x))$ $\therefore$ $\forall x(P(x) \rightarrow P(x))$

Answer:
This is actually valid! No countermodel exists because the conclusion is a tautology.
Explanation: $\forall x(P(x) \rightarrow P(x))$ is true in all models, regardless of premises.

b) $\forall x \exists y L(x,y)$ $\therefore$ $\exists y \forall x L(x,y)$
($L(x,y)$ means "x loves y")

Answer:
Countermodel:
Domain = {a, b}
L = {(a,a), (b,b)}
Verification:
$\forall x \exists y L(x,y)$ is true (a loves a, b loves b)
$\exists y \forall x L(x,y)$ is false (no y is loved by everyone)

2. (10 points) Show that these statements are not equivalent by providing a countermodel:
"Everyone has a friend" and "Someone is friends with everyone"

Answer:
Domain = {Alice, Bob}
F(x,y) = "x is friends with y"
F = {(Alice,Alice), (Bob,Bob)}

In this model:
$\forall x \exists y F(x,y)$ is true (everyone has themselves as a friend)
$\exists y \forall x F(x,y)$ is false (neither person is friends with everyone)

Part 2: Consistency Proofs (20 points)

3. (10 points) Prove the following set of sentences is consistent by providing a model:
$\forall x\exists y(x < y)$
$\forall x\forall y\forall z((x < y \land y < z) \to x < z)$
$\forall x\neg(x < x)$

Answer:
Model: $(\mathbb{N}, <)$
Domain: Natural numbers
$<$ is the usual less-than relation
Verification:
$\forall x\exists y(x < y)$: For any n, n+1 exists and $n < n+1$
Transitivity holds for usual $<$
$\neg(x < x)$ holds for usual $<$

4. (10 points) Show the following set of sentences is consistent:
$\exists xP(x)$
$\exists x\neg P(x)$
$\forall x\forall y(R(x,y) \to (P(x) \leftrightarrow P(y)))$

Answer:
Model:
Domain = {a, b}
P = {a}
R = ∅ (empty relation)
Verification:
$\exists xP(x)$ true because $a \in P$
$\exists x\neg P(x)$ true because $b \notin P$
$R(x,y) \to (P(x) \leftrightarrow P(y))$ vacuously true as R is empty

Part 3: Model Analysis (10 points)

5. (5 points) Given the model M with domain {1,2,3} and relation R = {(1,2), (2,3), (1,3)}, determine whether each statement is true or false:

a) $\forall x\exists yR(x,y)$
b) $\forall x\forall y\forall z((R(x,y) \land R(y,z)) \to R(x,z))$
c) $\exists xR(x,x)$
d) $\forall x\exists y(R(x,y) \land \neg R(y,x))$

Answer:
a) False - 3 has no y where R(3,y)
b) True - R is transitive
c) False - no element relates to itself

d) True - for 1→2, 2→3

6. (5 points) For the following properties, give the smallest possible domain size needed for a model where the property holds:

a) $\exists x \exists y(x \neq y \land \forall z(z = x \lor z = y))$
b) $\exists x \exists y \exists z(R(x,y) \land R(y,z) \land \neg R(x,z))$

Answer:
a) Domain size = 2
   Need exactly two distinct elements
   Smaller impossible, larger unnecessary

b) Domain size = 3
   Need three elements for non-transitive relation
   Example: R = {(1,2), (2,3)}
   Cannot be done with fewer elements

Symbolic Logic - Homework 6
Models and Proofs
Total Points: 50

Part 1: Invalidity Proofs (20 points)

1. (10 points) Provide a countermodel to prove each argument is invalid:

a) $\forall x(P(x) \rightarrow Q(x)), \forall x(Q(x) \rightarrow R(x)) \therefore \forall x(P(x) \rightarrow P(x))$

b) $\forall x \exists y L(x,y) \therefore \exists y \forall x L(x,y)$
(L(x,y) means "x loves y")

2. (10 points) Show that these statements are not equivalent by providing a countermodel: "Everyone has a friend" and "Someone is friends with everyone"

Part 2: Consistency Proofs (20 points)

3. (10 points) Prove the following set of sentences is consistent by providing a model:
$\forall x \exists y(x < y)$
$\forall x \forall y \forall z((x < y \land y < z) \rightarrow x < z)$
$\forall x \neg(x < x)$

4. (10 points) Show the following set of sentences is consistent:
$\exists x P(x)$
$\exists x \neg P(x)$

$\forall x \forall y(R(x,y) \to (P(x) \leftrightarrow P(y)))$

Part 3: Model Analysis (10 points)

5. (5 points) Given the model M with domain {1,2,3} and relation R = {(1,2), (2,3), (1,3)}, determine whether each statement is true or false:

a) $\forall x \exists y R(x,y)$
b) $\forall x \forall y \forall z((R(x,y) \land R(y,z)) \to R(x,z))$
c) $\exists x R(x,x)$
d) $\forall x \exists y(R(x,y) \land \neg R(y,x))$

6. (5 points) For the following properties, give the smallest possible domain size needed for a model where the property holds:

a) $\exists x \exists y(x \neq y \land \forall z(z = x \lor z = y))$
b) $\exists x \exists y \exists z(R(x,y) \land R(y,z) \land \neg R(x,z))$

Note: For full credit:
- Clearly specify domains and interpretations in your models
- Verify that your models satisfy all required conditions
- Explain why your counterexamples work

Symbolic Logic - Homework 6
Models and Proofs
Total Points: 50

Part 1: Invalidity Proofs (20 points)

1. (10 points) Provide a countermodel to prove each argument is invalid:

a) $\forall x(P(x) \to Q(x)), \forall x(Q(x) \to R(x)) \therefore \forall x(P(x) \to P(x))$

Answer:
This is actually valid! No countermodel exists because the conclusion is a tautology.
Explanation: $\forall x(P(x) \to P(x))$ is true in all models, regardless of premises.

b) $\forall x \exists y L(x,y) \therefore \exists y \forall x L(x,y)$
(L(x,y) means "x loves y")

Answer:
Countermodel:
Domain = {a, b}
L = {(a,a), (b,b)}

Verification:
$\forall x \exists y L(x,y)$ is true (a loves a, b loves b)
$\exists y \forall x L(x,y)$ is false (no y is loved by everyone)

2. (10 points) Show that these statements are not equivalent by providing a countermodel: "Everyone has a friend" and "Someone is friends with everyone"

Answer:
Domain = {Alice, Bob}
F(x,y) = "x is friends with y"
F = {(Alice,Alice), (Bob,Bob)}

In this model:
$\forall x \exists y F(x,y)$ is true (everyone has themselves as a friend)
$\exists y \forall x F(x,y)$ is false (neither person is friends with everyone)

Part 2: Consistency Proofs (20 points)

3. (10 points) Prove the following set of sentences is consistent by providing a model:
$\forall x \exists y(x < y)$
$\forall x \forall y \forall z((x < y \land y < z) \to x < z)$
$\forall x \neg(x < x)$

Answer:
Model: $(\mathbb{N}, <)$
Domain: Natural numbers
$<$ is the usual less-than relation
Verification:
$\forall x \exists y(x < y)$: For any n, n+1 exists and $n < n+1$
Transitivity holds for usual
$\neg(x < x)$ holds for usual

4. (10 points) Show the following set of sentences is consistent:
$\exists x P(x)$
$\exists x \neg P(x)$
$\forall x \forall y(R(x,y) \to (P(x) \leftrightarrow P(y)))$

Answer:
Model:
Domain = {a, b}
P = {a}
R = $\emptyset$ (empty relation)
Verification:
$\exists x P(x)$ true because $a \in P$

$\exists x \neg P(x)$ true because $b \notin P$
$R(x,y) \to (P(x) \leftrightarrow P(y))$ vacuously true as R is empty

Part 3: Model Analysis (10 points)

5. (5 points) Given the model M with domain {1,2,3} and relation R = {(1,2), (2,3), (1,3)}, determine whether each statement is true or false:

a) $\forall x \exists y R(x,y)$
b) $\forall x \forall y \forall z((R(x,y) \land R(y,z)) \to R(x,z))$
c) $\exists x R(x,x)$
d) $\forall x \exists y(R(x,y) \land \neg R(y,x))$

Answer:
a) False - 3 has no y where R(3,y)
b) True - R is transitive
c) False - no element relates to itself
d) True - for 1→2, 2→3

6. (5 points) For the following properties, give the smallest possible domain size needed for a model where the property holds:

a) $\exists x \exists y(x \neq y \land \forall z(z = x \lor z = y))$
b) $\exists x \exists y \exists z(R(x,y) \land R(y,z) \land \neg R(x,z))$

Answer:
a) Domain size = 2
Need exactly two distinct elements
Smaller impossible, larger unnecessary

b) Domain size = 3
Need three elements for non-transitive relation
Example: R = {(1,2), (2,3)}
Cannot be done with fewer elements

Recursivity and Non-Recursive Classes
Week 7: From Number Systems to Statement Classes

I. Recursive Number Systems and Their Limits

1. Natural Numbers ($\mathbb{N}$)
Base case: $0 \in \mathbb{N}$
Recursive step: If $n \in \mathbb{N}$, then $S(n) \in \mathbb{N}$
Complete: Every element reachable from 0 by finite applications of S

Recursively enumerable: {0, 1, 2, 3, ...}

2. Integers (ℤ)
   Base: Natural numbers
   Recursive step: If $n \in \mathbb{Z}$, then $-n \in \mathbb{Z}$
   Complete: All integers reachable by these rules
   Recursively enumerable: {0, 1, -1, 2, -2, ...}

3. Rational Numbers (ℚ)
   Base: Integers
   Recursive step: If $p,q \in \mathbb{Z}$ and $q \neq 0$, then $p/q \in \mathbb{Q}$
   Normalization: Reduce to lowest terms
   Complete: All fractions constructible from integers
   Recursively enumerable via systematic enumeration

II. Non-Recursive Real Numbers (ℝ)

1. Cantor's Diagonal Proof
   Assume reals are recursively enumerable
   List all numbers between 0 and 1:
   $r_1 = 0.a_{11}a_{12}a_{13}\ldots$
   $r_2 = 0.a_{21}a_{22}a_{23}\ldots$
   $r_3 = 0.a_{31}a_{32}a_{33}\ldots$
   Construct new number $d = 0.d_1d_2d_3\ldots$
   Where $d_n \neq a_{nn}$
   d differs from every listed number
   Therefore, list cannot be complete
   ℝ is non-recursive (incomplete)

2. Implications of Non-Recursivity
   Cannot enumerate all reals
   No effective procedure generates all reals
   Uncountably infinite
   Associated with incompleteness

III. Completeness vs. Incompleteness

1. Recursive Systems (Complete)
   Can be generated by finite rules
   Countably infinite
   Examples:
   Natural numbers
   Integers
   Rational numbers

Well-formed formulas
Valid propositional formulas

2. Non-Recursive Systems (Incomplete)
Cannot be generated by finite rules
Usually uncountably infinite
Examples:
Real numbers
Set of all subsets of ℕ
True arithmetic statements
Valid first-order formulas

IV. General Recursive Definitions

1. Basic Concept
Recursive definition: defines objects in terms of previously defined objects
Base case: initial elements
Recursive step: rules for generating new elements
Closure: only elements generated by these rules are in the set

2. Well-Formed Formulas (WFF)
Base: Atomic formulas are WFF
Recursive steps:
If A is WFF, then ¬A is WFF
If A, B are WFF, then (A ∧ B) is WFF
If A, B are WFF, then (A ∨ B) is WFF
If A, B are WFF, then (A → B) is WFF
Closure: Nothing else is WFF

V. Properties of Recursive Definitions

1. Well-Foundedness
Every element must be reachable from base cases
No infinite descent possible
Example: Cannot define natural numbers as "n is natural if n-1 is natural"

2. Uniqueness
Each element has a unique construction
Important for inductive proofs
May need parsing rules for expressions

VI. Non-Recursive Classes Beyond Numbers

1. Definition

Classes that cannot be generated by finite recursive rules
Often require infinite information
Example: Set of all true arithmetic statements

2. Examples
The set of all prime numbers
The set of all halting Turing machine programs
The set of all valid mathematical proofs

VII. Applications and Techniques

1. Mathematical Applications
Formal language theory
Computability theory
Set theory
Model theory

2. Proof Techniques
Structural induction
Recursive function definitions
Diagonal arguments
Cardinality proofs

Key Takeaways
1. Recursive definitions provide complete, countable systems
2. Some mathematical objects resist recursive definition
3. Non-recursivity often implies incompleteness
4. Cantor's diagonal argument proves fundamental limits
5. Understanding recursivity helps clarify mathematical foundations

Next week will be our review session preparing for the final exam.

Symbolic Logic - Homework 7
Recursivity and Non-Recursive Classes
Total Points: 50

Part 1: Number Systems and Recursivity (20 points)

1. (10 points) Show that the following sets are recursively enumerable by:
a) Providing the recursive definition
b) Showing the first 8 elements in your enumeration

a) The positive rational numbers between 0 and 1

b) The integers arranged by absolute value

2. (10 points) Explain why each of these sets cannot be recursively enumerated. Be specific.

a) The set of all real numbers between 0 and 1

b) The set of all infinite sequences of 0s and 1s

Part 2: Recursive Definitions (20 points)

3. (10 points) Give a recursive definition for each of the following:

a) The set of all binary strings containing an equal number of 0s and 1s

b) The set of all well-formed parentheses strings

4. (10 points) For each recursive definition, prove a property using structural induction:

a) Prove that any string generated by your definition in 3a has equal 0s and 1s

Part 3: Completeness and Incompleteness (10 points)

5. (10 points) For each set, determine if it is recursively enumerable or not. Justify your answer.

a) The set of all finite subsets of natural numbers

b) The set of all functions from $\mathbb{N}$ to $\{0,1\}$

Note: For full credit:
- Provide complete recursive definitions including base cases, recursive steps, and closure
- Show all steps in your proofs
- Give detailed explanations for non-recursivity
- Use proper mathematical notation

Symbolic Logic - Homework 7
Recursivity and Non-Recursive Classes
Total Points: 50

Part 1: Number Systems and Recursivity (20 points)

1. (10 points) Show that the following sets are recursively enumerable by:
   a) Providing the recursive definition

b) Showing the first 8 elements in your enumeration

a) The positive rational numbers between 0 and 1

Answer:
Recursive Definition:
Base: 1/2 is in the set
Recursive step: For any p/q in lowest terms, add all fractions with:
Numerator $p' \leq q'$
Denominator $q' = q + 1$
That lie between 0 and 1

First 8 elements (in order of increasing denominator):
1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5

b) The integers arranged by absolute value

Answer:
Recursive Definition:
Base: 0 is in the set
Recursive step: If n is in the set, add both $+(|n|+1)$ and $-(|n|+1)$

First 8 elements:
0, +1, -1, +2, -2, +3, -3, +4

2. (10 points) Explain why each of these sets cannot be recursively enumerated. Be specific.

a) The set of all real numbers between 0 and 1

Answer:
Apply Cantor's diagonal argument:
1. Assume the set is enumerable: $r_1, r_2, r_3, \ldots$
2. Write each in decimal: $r_1 = 0.a_{11}a_{12}a_{13}\ldots$
$r_2 = 0.a_{21}a_{22}a_{23}\ldots$
$r_3 = 0.a_{31}a_{32}a_{33}\ldots$
3. Construct $d = 0.d_1d_2d_3\ldots$ where $d_i \neq a_{ii}$
4. d is between 0 and 1 but differs from every number in the list
5. Therefore, the list cannot be complete

b) The set of all infinite sequences of 0s and 1s

Answer:
Similar diagonal argument:

1. If enumerable, list sequences: $s_1 = (a_{11}, a_{12}, \ldots)$
$s_2 = (a_{21}, a_{22}, \ldots)$
2. Create new sequence d where $d_i = 1 - a_{ii}$
3. d differs from every sequence in list at position i
4. Therefore, not enumerable

Part 2: Recursive Definitions (20 points)

3. (10 points) Give a recursive definition for each of the following:

a) The set of all binary strings containing an equal number of 0s and 1s

Answer:
Base case: The empty string ε belongs to the set
Recursive steps:
1. If s is in the set, then 0s1 and 1s0 are in the set
2. If s and t are in the set, then st is in the set
Closure: No other strings are in the set

b) The set of all well-formed parentheses strings

Answer:
Base case: () is in the set
Recursive steps:
1. If s is in the set, then (s) is in the set
2. If s and t are in the set, then st is in the set
Closure: No other strings are in the set

4. (10 points) For each recursive definition, prove a property using structural induction:

a) Prove that any string generated by your definition in 3a has equal 0s and 1s

Answer:
Base case: ε has 0 of each (equal)
Inductive steps:
1. If s has equal 0s and 1s:
   0s1 adds one 0 and one 1 (still equal)
   1s0 adds one 1 and one 0 (still equal)
2. If s and t have equal 0s and 1s:
   st combines their counts (sum of equals is equal)

Part 3: Completeness and Incompleteness (10 points)

5. (10 points) For each set, determine if it is recursively enumerable or not. Justify your answer.

a) The set of all finite subsets of natural numbers

Answer:
Recursively enumerable.
1. Can enumerate by size: first all subsets of size 0, then size 1, etc.
2. For each size n, list all combinations of n numbers in increasing order
3. This generates all finite subsets eventually

b) The set of all functions from $\mathbb{N}$ to $\{0,1\}$

Answer:
Not recursively enumerable.
1. Each function corresponds to an infinite sequence of 0s and 1s
2. By previous proof (2b), such sequences cannot be enumerated
3. Therefore, these functions cannot be enumerated
4. This set is equivalent to the power set of $\mathbb{N}$ (uncountable)

Symbolic Logic Course Review
Week 8: Comprehensive Overview and Exam Preparation

I. Basic Concepts and Notation (Week 1)

1. Fundamental Concepts
   Statements and propositions
   Validity vs. soundness
   Material vs. strict implication

2. Basic Symbols
   Negation (¬)
   Conjunction (∧)
   Disjunction (∨)
   Conditional (→)
   Biconditional (↔)

3. Translation Skills
   Natural language to symbolic form
   Symbolic form to natural language
   Common patterns and pitfalls

II. Truth Tables and Elementary Proofs (Week 2)

1. Truth Tables
   Construction and evaluation
   Testing validity
   Checking equivalence

2. Rules of Inference
   Modus Ponens
   Modus Tollens
   Disjunctive Syllogism
   Double Negation

3. De Morgan's Laws
   $\neg(p \land q) \equiv (\neg p \lor \neg q)$
   $\neg(p \lor q) \equiv (\neg p \land \neg q)$

III. Boolean Algebra (Week 3)

1. Basic Operations
   NOT, AND, OR
   Truth tables
   Boolean functions

2. Laws and Properties
   Identity
   Complement
   Associative
   Distributive
   De Morgan's
   Absorption

3. Applications
   Circuit design
   Expression simplification
   Truth function analysis

IV. Introduction to Quantification (Week 4)

1. Basic Quantifiers
   Universal ($\forall$)
   Existential ($\exists$)
   Domains of discourse

2. Translation Techniques
   Simple quantified statements

Multiple quantifiers
Order of quantifiers

3. Negation Rules
$\neg\forall x(Px) \equiv \exists x(\neg Px)$
$\neg\exists x(Px) \equiv \forall x(\neg Px)$

V. Advanced Quantification (Week 5)

1. Complex Translations
Nested quantifiers
Mixed quantifiers
Mathematical statements

2. Epsilon-Delta Definitions
Limits
Continuity
Mathematical analysis

3. Formal Proofs
With quantifiers
Mathematical rigor
Common patterns

VI. Models and Proofs (Week 6)

1. Model Theory
Structure of models
Domains
Interpretations

2. Proving Invalidity
Countermodels
Construction techniques
Verification

3. Consistency Proofs
Model construction
Verification methods
Common strategies

VII. Recursivity (Week 7)

1. Recursive Number Systems

Natural numbers
Integers
Rationals
Non-recursive reals

2. Cantor's Proof
Diagonal argument
Uncountability
Implications

3. Recursive vs. Non-Recursive
Completeness
Incompleteness
Applications

Exam Preparation Strategies

1. Key Skills to Practice
Translation (both directions)
Truth table construction
Model building
Proof techniques
Recursive definitions

2. Common Pitfalls
Confusing material/strict implication
Order of quantifiers
Model completeness
Recursive vs. non-recursive

3. Problem-Solving Approach
Read carefully
Identify key concepts
Plan solution strategy
Show all work
Verify answers

Practice Problems Focus Areas

1. Translation
Complex statements
Multiple quantifiers
Mathematical concepts

2. Proofs
   Validity/invalidity
   Consistency
   Model construction

3. Recursivity
   Definitions
   Enumeration
   Non-recursive examples

Final Exam Format
Total points: 500
Time: 180 minutes
Sections:
  Translation and Basic Concepts
  Truth Tables and Boolean Algebra
  Quantification and Models
  Recursivity and Definability
  Comprehensive Problem Solving

Symbolic Logic - Final Examination
Total Points: 500
Time Allowed: 180 minutes

Part I: Translation and Basic Concepts (100 points)

1. (40 points) Translate each statement into symbolic logic. State your predicates and domain clearly.

a) "Between any two real numbers there is a rational number and an irrational number."

b) "Not every student who studies hard fails any exam."

2. (30 points) Negate each statement, simplify, then translate back to English:

a) $\forall x \exists y (P(x) \rightarrow (Q(y) \land R(x,y)))$

3. (30 points) Explain the difference between material and strict implication using examples. Provide truth conditions for each.

Part II: Truth Tables and Boolean Algebra (100 points)

4. (40 points) Use truth tables to determine if these arguments are valid:

a) $p \to q$, $q \to r$, $\neg r \therefore \neg p$

5. (30 points) Simplify using Boolean algebra (show steps):
$((x \wedge y) \vee (\neg x \wedge y)) \wedge (x \vee \neg y)$

6. (30 points) Design a Boolean circuit for:
$f(x,y,z) = (x \wedge \neg y) \vee (\neg x \wedge y \wedge z)$

Part III: Quantification and Models (150 points)

7. (50 points) Prove that these statements are not equivalent using a countermodel:
$\forall x \exists y \forall z R(x,y,z)$ and $\exists y \forall x \forall z R(x,y,z)$

8. (50 points) Write the epsilon-delta proof that $\lim(x \to 2)\ x^3 = 8$

9. (50 points) Construct a model that demonstrates the consistency of:
$\forall x \exists y R(x,y)$
$\forall x \neg R(x,x)$
$\forall x \forall y \forall z((R(x,y) \wedge R(y,z)) \to R(x,z))$

Part IV: Recursivity and Definability (150 points)

10. (50 points) Prove that the set of real numbers is not recursively enumerable using Cantor's diagonal argument.

11. (50 points) Give a recursive definition for each:

a) The set of all strings over {a,b} with equal numbers of a's and b's

b) The arithmetic expressions using only numbers and +

12. (50 points) Show that the following questions about an arbitrary recursive definition are themselves non-recursive:

a) Whether a given element belongs to the recursively defined set
b) Whether a given property holds for all elements of the set

Note: For full credit:
- Show all work clearly
- Justify each step in proofs
- State any assumptions or additional definitions used
- Verify all conditions in model constructions
- Provide clear explanations for conceptual questions

Symbolic Logic - Final Examination
Total Points: 500
Time Allowed: 180 minutes

Part I: Translation and Basic Concepts (100 points)

1. (40 points) Translate each statement into symbolic logic. State your predicates and domain clearly.

a) "Between any two real numbers there is a rational number and an irrational number."

Answer:
Domain: Real numbers
Q(x): x is rational
I(x): x is irrational
$\forall x \forall y(x < y \rightarrow \exists z \exists w((x < z < y) \land (x < w < y) \land Q(z) \land I(w)))$

b) "Not every student who studies hard fails any exam."

Answer:
Domain: Students
S(x): x studies hard
F(x,y): x fails exam y
$\neg \forall x(S(x) \rightarrow \forall y(F(x,y)))$
$\equiv \exists x(S(x) \land \exists y(\neg F(x,y)))$

2. (30 points) Negate each statement, simplify, then translate back to English:

a) $\forall x \exists y(P(x) \rightarrow (Q(y) \land R(x,y)))$

Answer:
$\neg \forall x \exists y(P(x) \rightarrow (Q(y) \land R(x,y)))$
$\equiv \exists x \forall y \neg(P(x) \rightarrow (Q(y) \land R(x,y)))$
$\equiv \exists x \forall y(P(x) \land \neg(Q(y) \land R(x,y)))$
$\equiv \exists x \forall y(P(x) \land (\neg Q(y) \lor \neg R(x,y)))$

English: "There exists something that has property P, and for all things, either they don't have property Q or they don't stand in relation R to the first thing."

3. (30 points) Explain the difference between material and strict implication using examples. Provide truth conditions for each.

Answer:
Material implication ($p \rightarrow q$):

- Truth-functional: false only when p true and q false
- Example: "If it rains, the ground is wet"
- No necessary connection required
- Truth table shows all cases

Strict implication (p ⊃ q):
- Based on logical necessity
- Example: "If x is a square, then x has four equal sides"
- Requires conceptual connection
- Cannot be captured by truth table alone
- Modal concept: true in all possible worlds

Part II: Truth Tables and Boolean Algebra (100 points)

4. (40 points) Use truth tables to determine if these arguments are valid:

a) $p \rightarrow q$, $q \rightarrow r$, $\neg r \therefore \neg p$

Answer:

p	q	r	p → q	q → r	¬r	¬p
T	T	T	T	T	F	F
T	T	F	T	F	T	F
T	F	T	F	T	F	F
T	F	F	F	T	T	F
F	T	T	T	T	F	T
F	T	F	T	F	T	T
F	F	T	T	T	F	T
F	F	F	T	T	T	T

Valid: When premises are all true (row 2), conclusion is true.

5. (30 points) Simplify using Boolean algebra (show steps):
$((x \land y) \lor (\neg x \land y)) \land (x \lor \neg y)$

Answer:
$((x \land y) \lor (\neg x \land y)) \land (x \lor \neg y)$
$\equiv (y \land (x \lor \neg x)) \land (x \lor \neg y)$ [Distributive]
$\equiv (y \land 1) \land (x \lor \neg y)$ [Complement]
$\equiv y \land (x \lor \neg y)$ [Identity]
$\equiv (y \land x) \lor (y \land \neg y)$ [Distributive]
$\equiv (y \land x) \lor 0$ [Complement]
$\equiv y \land x$ [Identity]

6. (30 points) Design a Boolean circuit for:
f(x,y,z) = (x ∧ ¬y) ∨ (¬x ∧ y ∧ z)

Answer:
```
   x ---|AND|----
        |       |
   y ---|NOT|   |OR|---- Output
        |       |
   x ---|NOT|   |
        |       |
   y ---|AND|----|
        |
   z ---|
```

Part III: Quantification and Models (150 points)

7. (50 points) Prove that these statements are not equivalent using a countermodel:
∀x∃y∀zR(x,y,z) and ∃y∀x∀zR(x,y,z)

Answer:
Countermodel:
Domain = {1, 2}
R = {(1,1,1), (1,1,2), (2,2,1), (2,2,2)}

In this model:
- ∀x∃y∀zR(x,y,z) is true because:
  For x=1, y=1 works for all z
  For x=2, y=2 works for all z
- ∃y∀x∀zR(x,y,z) is false because:
  No single y works for all x and z

8. (50 points) Write the epsilon-delta proof that $\lim(x \to 2)\ x^3 = 8$

Answer:
To prove: $\forall\varepsilon>0\ \exists\delta>0\ \forall x(0<|x-2|<\delta \to |x^3-8|<\varepsilon)$

$|x^3-8| = |x^3-2^3|$
$\quad = |(x-2)(x^2+2x+4)|$
$\quad = |x-2||x^2+2x+4|$

When $|x-2| < 1$, then $1 < x < 3$
Then $|x^2+2x+4| \leq 19$

Choose $\delta = \min(1, \varepsilon/19)$
Then $|x^3-8| = |x-2||x^2+2x+4| < \delta\cdot 19 \leq \varepsilon$

9. (50 points) Construct a model that demonstrates the consistency of:
$\forall x\exists y R(x,y)$
$\forall x\neg R(x,x)$
$\forall x\forall y\forall z((R(x,y) \wedge R(y,z)) \rightarrow R(x,z))$

Answer:
Model:
Domain: $\mathbb{Z}$ (integers)
$R(x,y)$: $x < y$

Verification:
$\forall x\exists y R(x,y)$: For any x, x+1 exists and $x < x+1$
$\forall x\neg R(x,x)$: No number is less than itself
Transitivity: If $x < y$ and $y < z$ then $x < z$

Part IV: Recursivity and Definability (150 points)

10. (50 points) Prove that the set of real numbers is not recursively enumerable using Cantor's diagonal argument.

Answer:
1. Assume reals are enumerable: $r_1, r_2, r_3, \ldots$
2. Write decimal expansions:
   $r_1 = 0.a_{11}a_{12}a_{13}\ldots$
   $r_2 = 0.a_{21}a_{22}a_{23}\ldots$
   $r_3 = 0.a_{31}a_{32}a_{33}\ldots$
3. Construct $d = 0.d_1d_2d_3\ldots$ where:
   $d_i = 1$ if $a_{ii} = 2$
   $d_i = 2$ if $a_{ii} \neq 2$
4. d is a real number but differs from every number in list
5. Therefore, list cannot be complete
6. Thus, reals are not recursively enumerable

11. (50 points) Give a recursive definition for each:

a) The set of all strings over {a,b} with equal numbers of a's and b's

Answer:
Base: ε (empty string) is in the set
Recursive steps:

1. If s is in the set, then asb and bsa are in the set
2. If s and t are in the set, then st is in the set
Closure: No other strings are in the set

b) The arithmetic expressions using only numbers and +

Answer:
Base: Any number is an expression
Recursive step: If E_1 and E_2 are expressions, then $(E_1 + E_2)$ is an expression
Closure: Nothing else is an expression

12. (50 points) Show that the following questions about an arbitrary recursive definition are themselves non-recursive:

a) Whether a given element belongs to the recursively defined set
b) Whether a given property holds for all elements of the set

Answer:
Use reduction to the halting problem:
a) Can encode Turing machine computations as elements
Then membership would solve halting problem

b) Can encode "machine M halts" as property
Then property verification would solve halting problem

Both are therefore non-recursive by reduction