CHAPTER 5: LINEAR PROGRAMMING

LEARNING OBJECTIVES:

By the end of this section, the student will be able to

1. Define linear programming and optimization.
2. Identify the critical features of linear programming.
3. Use graphical method and simplex algorithm to show the optimum solution.
4. Elaborate the steps in applying linear programming.
5. Solve optimization problem using the concept of linear programming.
6. Perform sensitivity analysis on solutions to optimization problems.

5.1 LINEAR PROGRAMMING

It is a mathematical modelling technique useful for the allocation and decision making that involves scarce or limited resources, such as labor, material, machine, time, warehouse space, etc. to several competing activities such as product, service, job, new equipment, projects etc. based on given criteria or optimality. Some applications of Linear programming are product mix problem, diet problems, profit planning, make-or-buy problems, transportation problems, and agriculture problems.

The general procedure for solving linear programming problems is to graph the inequalities that represent the **constraints** and is expected to form a **closed region (feasible region**) bounded by the constraints. The intersection points of the boundary lines are the corner points of the feasible region. The solution set of this system of inequalities is set of points called the **feasible solution** and the objective function can be evaluated for different feasible solutions and the maximum and minimum values obtained.

The following are requirements when solving linear programming problem:

1. **Decision variables** are quantities to be determined, should be interrelated and non-negative.

2. **Objective function** are values that needs to be optimized, it is the basis for decision making.

3. **Constraints** represent how each decision variable would use the controlled amount of resources, either scarce or abundant. These are the limitations or restrictions on the decision variables.

4. **Data** gives the evaluation on the relationship between objective function and the constraints.

5. **Non-negative restriction** for all linear programs, which means the values for decision variables should be greater than or equal to 0.

For a problem to be a linear programming problem, the decision variables, objective function, and constraints must be linear functions. If all the three conditions are satisfied, it is called a **Linear Programming Problem.**

***Optimization –** A mathematical technique for finding a maximum or minimum value of a function of several variables subject to a set of constraints, as linear programming, or systems analysis.

5.2 STEPS IN THE GRAPHICAL METHOD OF LINEAR PROGRAMMING

A graphical method involves formulating a set of linear inequalities subject to the constraints. The following are suggested steps in the graphing linear programming model.

1. Identify the decision variables and write the objective function.

2. State the constraints and the non-negative restriction.

3. Graph the feasible region for the set of constraints in a rectangular coordinate system. The feasible region explains all values the problem can take, and it also gives the optimal solution.

Graphing can be done using slope-intercept form or by providing table of values of decision variables.

4. Determine the corner points, (Intersection of the plotted inequalities which shows the feasible region). This can be solved using any of the algebraic solutions such as Cramer's rule of determinants, Methods of elimination by addition or subtraction and Method of Substitution.

5. Calculate the optimum value of the objective function. This may be done by substituting the corner points obtained previously to the objective function and determine which corner points yield the maximum and minimum value.

Example 5.0: Use the graphical method to compute the maximum and minimum value of

P = 5x + 3y, given the following constraints,

$$x + 2y \leq 14$$

$$3x - y \geq 0$$

$$x - y \leq 2$$

Solutions:

Steps 1 - 2: Identifying the constraints with decision variables and the objective function.

Objective: Maximize P = 5x + 3y

Constraints: $x + 2y \leq 14$

$$3x - y \geq 0$$

$$x - y \leq 2$$

Non-negative restriction:

$$x \geq 0, \ \ y \geq 0$$

Step 3: Graph the given constraints in the rectangular coordinate system as follows:

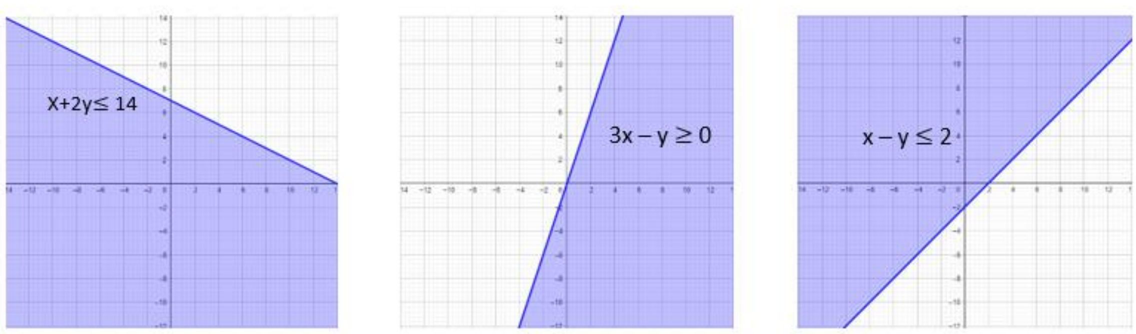

Figure 5.0

Take note that the solid lines represent the boundary of the given constraints (inequality) and the shaded portion of the graph are the feasible region.

When the graphs above are put together in a rectangular coordinate system as shown below, it shows the feasible region and its corner points.

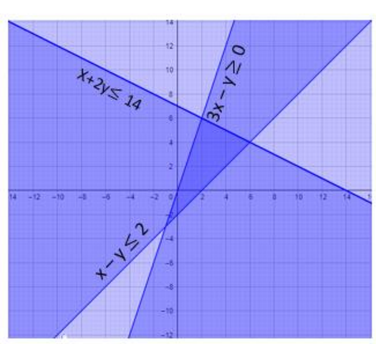

Figure 5.1

Step 4: The corner points can be solved algebraically. The corner points are assigned the names points A, B and C.

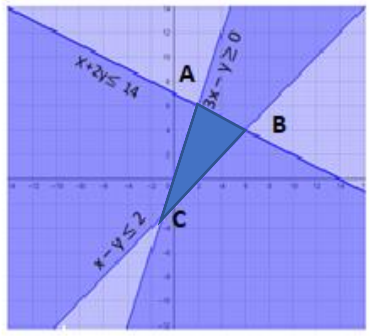

Figure 5.2

Using Cramer's Rule to solve for corner points A, B and C.

Corner point A is the intersection of lines x + 2y = 14 and 3x – y = 0,

$$x = \frac{\begin{vmatrix} 14 & 2 \\ 0 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}} = \frac{(14)(-1)-(0)(2)}{(1)(-1)-(3)(2)} = \frac{-14-0}{-1-6} = \frac{-14}{-7} = 2$$

$$y = \frac{\begin{vmatrix} 1 & 14 \\ 3 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}} = \frac{(1)(0)-(3)(14)}{(1)(-1)-(3)(2)} = \frac{-42}{-7} = 6$$ therefore **A (2,6)**

Corner point B is the intersection of lines x + 2y = 14 and x – y = 2,

$$x = \frac{\begin{vmatrix} 14 & 2 \\ 2 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}} = \frac{(14)(-1)-(2)(2)}{(1)(-1)-(1)(2)} = \frac{-14-4}{-1-2} = \frac{-18}{-3} = 6$$

$$y = \frac{\begin{vmatrix} 1 & 14 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}} = \frac{(1)(2)-(1)(14)}{(1)(-1)-(1)(2)} = \frac{2-14}{-1-2} = \frac{-12}{-3} = 4$$ therefore B **(6,4)**

Corner point C is the intersection of lines 3x – y = 0 and x – y = 2

$$x = \frac{\begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix}} = \frac{(2)(-1)-(0)(-1)}{(1)(-1)-(3)(-1)} = \frac{-2-0}{-1+3} = \frac{-2}{2} = -1$$

$$y = \frac{\begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix}} = \frac{(1)(0)-(3)(2)}{(1)(-1)-(3)(-1)} = \frac{0-6}{-1+3} = -3$$ therefore **C(-1,-3)**

Step 5: Finding the optimum value of the objective functions, by substituting each corner points to the **objective function P = 5x + 3y,** and the result is summarized in the table below:

Table 5.0

Corner Point	Coordinates	Value of P P = 5x + 3y
A	(2,6)	5(2) + 3(6) = 28
B	(6,4)	5(6) + 3(4) =42
C	(-1, -3)	5(-1) + 3(-3) = -14

It can be seen from table 5.0 that the **maximum value of P is obtained at corner point B (6,4), which yields a value 42** and the minimum value of P is obtained at C (-1, -3) which yields a value of -14.

Example 5.2: Some of the products of a Furniture manufacturing company are tables and chairs. Each table is sold at ₱1,900 and each chair at ₱650. One of the carpenters can afford to spend up to 40 hours per week working and takes six hours to make a table and three hours to make a chair. Clientele demand requires that he makes at least three times as many chairs as tables. Tables take up four times as much as storage space as chairs and there is room for at most four tables each week. Determine the number of tables and chairs to make per production to maximize profit of Furniture manufacturing company.

Express this problem as a linear programming problem and solve it graphically.

Solution:

Steps 1-2: Determining variables, constraints, and objective and the non-negative restrictions.

Let x – be the number of tables produced per week.
y – be the number of chairs produced per week.

Constraints

Total worked time

$6x + 3y \leq 40$

Clientele demand

$y \geq 3x$

Storage space

$\frac{Y}{4} + x \leq 4$

$x \geq 0$ and $y \geq 0$

Objective:

Maximize: $1900x + 650y$

Step 3: Construct the graph of the constraints as shown in figure 5.3.

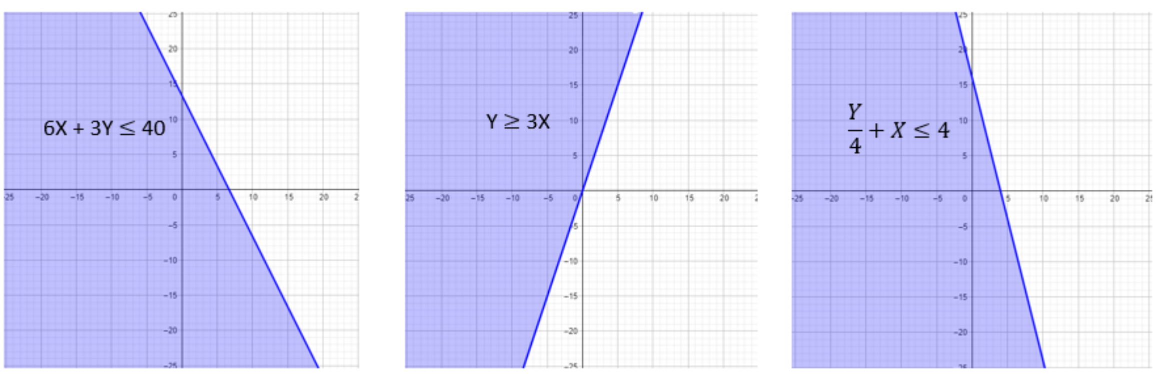

Figure 5.3

Figure 5.4 shows that the solution lies at the intersection of constraints.

Figure 5.4

Step 4: Solving simultaneously for the corners A, B and C of the constraints:

For corner A: The intersection of $6x + 3y = 40$ and $\frac{Y}{4} + X = 4$

Multiply equation with fraction by 4 then,

$$6x + 3y = 40 \quad \text{Equation 1}$$
$$4x + y = 16 \quad \text{Equation 2}$$

Multiplying equation 2 by 3, then subtract equation 1,

	12x	+	3y	=	48
-	6x	+	3y	=	40
	6x			**=**	**8**
	x			=	$\frac{8}{6} = \frac{4}{3}$

To find y, substitute value of x to equation 2,

$4(\frac{4}{3})+y = 16;\ y = \frac{32}{3}$

therefore, Corner **A** $(\frac{4}{3}, \frac{32}{3})$

For corner B: The intersection of y = 3x and $\frac{y}{4} + x = 4$

By Substituting y to $\frac{Y}{4} + X = 4$ $\quad \frac{3x}{4} + x = 4 \longrightarrow$ 3x + 4x = 16 $\quad x = \frac{16}{7}$

Substituting value of x to y, $\longrightarrow$ $y = 3(\frac{16}{7}) = \frac{48}{7}$

therefore, corner **B** $(\frac{16}{7}, \frac{48}{7})$

For corner C: The intersection of Y = 3X and 6x + 3y = 40

Substituting Y = 3X to 6x + 3y = 40 $\longrightarrow$ 6x + 3(3x) = 40

15x = 40 $\quad x = \frac{40}{15} = \frac{8}{3}$

Solving for y, $\quad y = 3(\frac{8}{3}) = 8$

therefore, corner **C** $(\frac{8}{3}, 8)$

Summarizing the coordinates of corner points in table 5.1, and substituting to P:

Table 5.1

Corner Point	Coordinates	Value of P P = 1900X + 650Y
A	$(\frac{4}{3}, \frac{32}{3})$	$1900(\frac{4}{3}) + 650(\frac{32}{3})$ = 9,466.67
B	$(\frac{16}{7}, \frac{48}{7})$	$1900(\frac{16}{7}) + 650(\frac{48}{7})$ = 8,800.00
C	$(\frac{8}{3}, 8)$	$1900(\frac{8}{3})$ + 650(8) = 10,266.67

Step 5: The optimum value are at C $(\frac{8}{3}, 8)$ which yields **maximum value** of ₱ **10,266.67** while at B$(\frac{16}{7}, \frac{48}{7})$ which yields a **minimum value** of **₱8,800.00**

Example 5.3: A Young Entrepreneur is choosing to invest his funds of P50,000.00 in either mutual funds or bank's time deposit or maybe both. The mutual fund pays 8% per year and time deposit pays 9%. A finance manager advised him to invest ₱ 30,000.00 in time deposit and at least ₱ 10,000.00 to mutual funds. How much should he allot to each investment in order to maximize returns on his investment?

Solution: To determine the amount that should be allotted to each investment and to maximize return on investment, the following are assumptions:

Let x – be the amount invested in the mutual fund.

Y – be the amount invested in time deposit.

Certainly, $x \geq 0$, and $y \geq 0$

Summarizing the given conditions in the table 5.2

Table 5.2

Return Requirements	Mutual Fund, x at 8%	Time Deposit, y at 9%	Amount, ₱
Total Investment	1	1	50,000.00
Mutual Fund	1		30,000.00
Time Deposit		1	10,000.00

So, the equations for total funds must be, x + y = 50,000

While the mutual fund has no more than 30,000 $x \leq 30{,}000$

And for time deposit $y \geq 10{,}000$

Linear programming model is to maximize P = 0.08x + 0.09y

Subject to the following: x + y = 50,000

$x \leq 30{,}000$

$y \geq 10{,}000$

$x \geq 0$ and $y \geq 0$

Graphing the feasible region for the constraints as shown in figure 5.5,

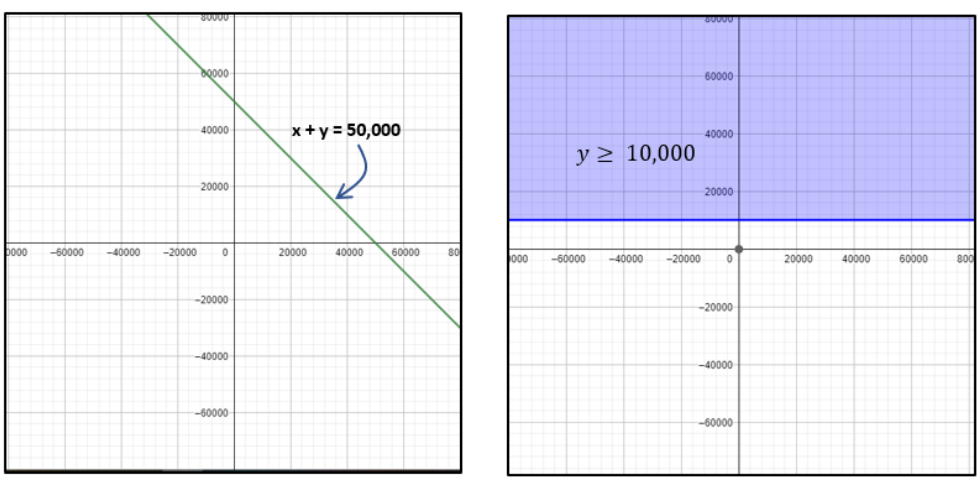

Figure 5.5

The graph on the right side of Figure 5.6 shows constraints when put together, take note that the corner points are labeled A, B, C and D.

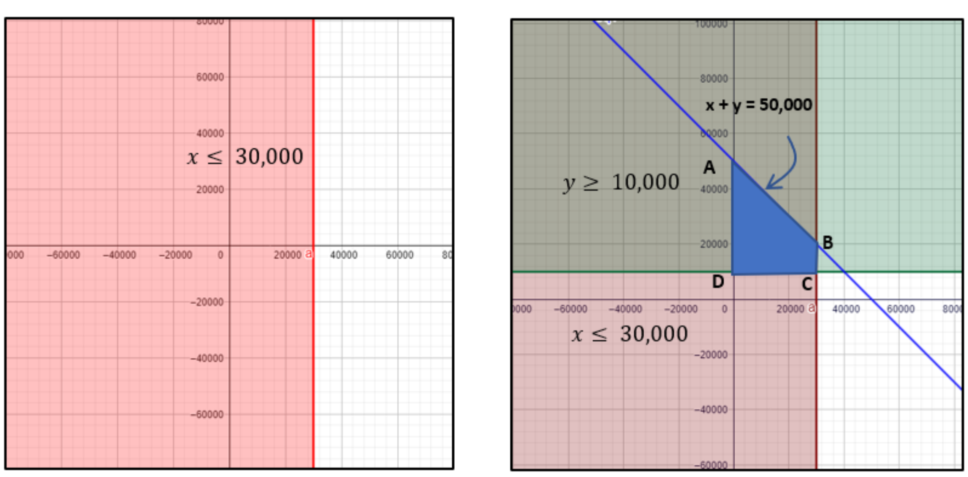

Figure 5.6

Solving for corner points A, B, C and D of the feasible region,

For A: It is obvious from the graph that **A (0, 50,000)**

For B: Substitute y = 10,000 to x + y = 50,000

X + 10,000 = 50,000

X = 40,000

Therefore **B (40,000, 10,000)**

For C: It is the point of intersection of the lines x = 30,000 and y = 10,000

Therefore **C (30,000, 10,000)**

For D: From the graph line y = 10,000 and intersect the y axis.

Therefore **D (0, 10,000)**

Table 5.3

Corner Point	Coordinates	Value of P P = 0.08x + 0.09y
A	(0, 50,000)	**0.08(0) + 0.09(50,000) = 4,500**
B	(40,000, 10,000)	0.08(40,000) + 0.09(10,000) = 4,100
C	(30,000, 10,000)	0.08(30,000) + 0.09(10,000) = 3,300
D	(0,10,000)	0.08(0) + 0.09(10,000) = 900

From the above computations, P is maximum at corner point A where the investment should only be made to time deposit.

Example 5.4: A Popular Ferry boat company offers cheap tickets for economy seats from Batangas port to Puerto Galera. Despite the cheap tickets the company still expect to gain a maximum profit, and to realize profit it must sell a minimum of 25 economy seats tickets and a minimum of 40 business seats tickets. The company makes a profit of ₱ 225 for each economy seats ticket and ₱200 for each business seats ticket. At most, the ferry boat has a capacity of 150 travelers. How many of each ticket should be sold in order to maximize company's profits?

Solution: Identifying unknown quantities

Let x – be the number of economy seats tickets.

Y – be the number of business class seats tickets.

Since the goal is to maximize profits, then the objective function, **P = 225x + 200y**

The company still expect the largest profit as possible, despite the following constraints,

- Sell at least 25 economy seats.
- Sell at least 40 business class tickets.
- No more than 150 tickets can be sold since it is the capacity of the boat.

Thus, the three mathematical constraints are,

$x \geq 25$; $y \geq 40$ $x + y \leq 150$

Next step is to graph the constraints, see figure 5.7.

And then determine the corners A, B and C.

Corner A: Substitute **x = 25** to x + y = 150

25 + y = 150 **y = 125**

Therefore A (25, 125)

Corner B: Is the intersection of x = 25 and y = 40

Therefore B (25,40)

Corner C: Substitute y = 40 to x + y =150

X + 40 = 150

x = 110

Therefore C (125, 25)

The graph of the constraints is shown in figure 5.7.

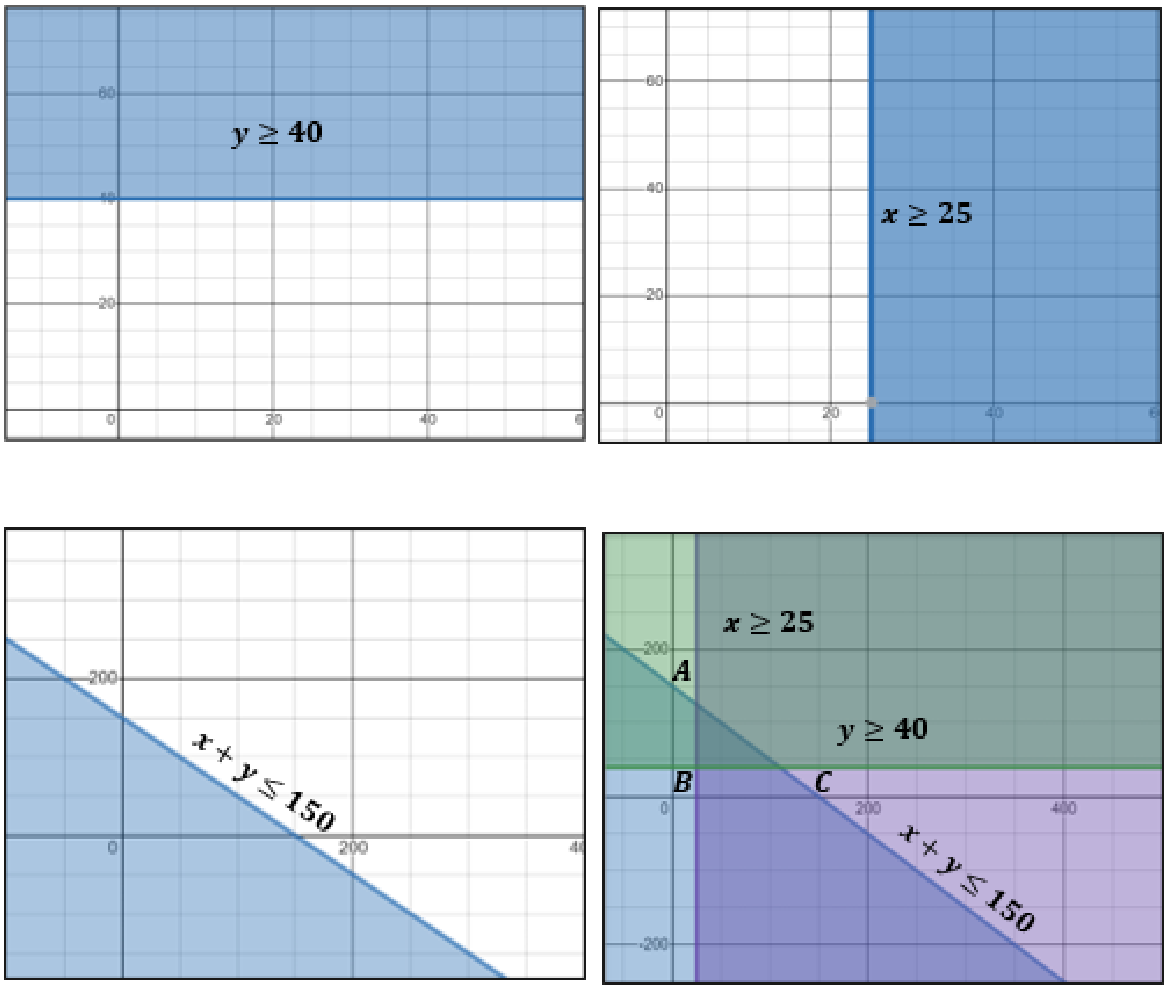

Figure 5.7

Table 5.4, summarizes values obtained for corner points A, B and C.

Table 5.4

Corner Point	Coordinates	Value of P P = 225 x + 200y
A	(25, 125)	225(425) + 200(125) = 120,625
B	(25, 40)	225(25) + 200(40) = 13,625
C	(110, 40)	225(125) + 200(25) = 33,125

Finally, this popular boat company should sell **25 economy seats tickets and 125 business class seats tickets to maximize profit.**

REVIEW EXERCISES 5.0

Name ________________________________ Score __________ Date _____________
Course, Year & Section ______________Student no. _________ Professor_____________

A. Calculate the maximum and minimum value of the following linear programs using graphical method.

1. Objective: $P = 3x + 4y$

Constraints: $x + 2y \leq 10$
$x - y \geq 0$
$x + y \leq 2$
$x \geq 0$
$y \geq 0$

2. Objective: $P = 8x + 2y$

Constraints: $x + y \geq 24$
$x \leq 10$
$y \leq 15$
$x \geq 0$
$y \geq 0$

3. Objective: $P = 30x + 40y$
Constraints: $4x + 2y \leq 32$
$x + 3y \leq 18$
$x \geq 0$
$y \geq 0$

4. Objective: $P = 12x + 8y$
Constraints: $5x + y \leq 32$
$x + 3y \leq 12$
$x \geq 0$
$y \geq 0$

5. Objective: $P = 5x + 3y$

Constraints: $2x + 6y \geq 36$
$x + y \geq 12$
$x + y \geq 12$
$x \geq 0$
$y \geq 0$

6. Objective: $P = 12x + 5y$

Constraints: $-x + y \leq 4$
$x - y \leq 4$
$x + y \geq 4$
$x + y \leq 12$
$x \geq 0$
$y \geq 0$

7. Objective: $P = x + 4y$

Constraints: $x + y \leq 6$
$2x + y \leq 8$
$x \geq 0$
$y \geq 0$

8. Objective: $P = 6x + 9y$

Constraints: $-x + y \leq 4$
$6x + 3y \geq 30$
$4x + 5y \geq 40$
$2x + 5y \geq 30$
$x \geq 0$
$y \geq 0$

9. Objective: $P = 5x + 14y$

Constraints: $2x + 3y = 24$
$2x + y \geq 12$
$4x + 18y \geq 36$
$4y \geq 0$
$x \geq 0$
$y \geq 0$

10. Objective: $P = 12x + 3y$

Constraints: $0.1x + 0.5y \leq 3.2$
$0.3x + 0.3y \geq 1.2$
$x \geq 0$
$y \geq 0$

B. Solve the following problems as indicated:

11. A teacher plans to invest ₱ 100,000in a mutual fund, time deposit savings and treasury bills. The teacher plans to invest a minimum of ₱ 30,000 in each of the investment to gain a highest profit?

12. A Pizza parlor is baking two flavors of Pizza, meat flavor and all veggies flavor. The meat flavor pizza contains pepperoni, bacon and beef with cheese and onions. The veggies flavor has pineapple, green pepper, mushroom, onions, capers, and cheese. The parlor sells at least 150 units a day of meat flavor and at least 120 units of veggies flavor. The meat flavor cost ₱ 200 per unit and veggie flavor cost ₱ 150 per unit. The meat flavor is paired with a special type of dough which cost ₱ 60 per unit and the dough for veggies flavor is ₱ 30 per unit. The parlor can spend no more than ₱ 12,000 per day on dough. Determine the daily sales volume that will realize a maximum return on both flavors of pizza.

13. A chocolate manufacturing company that produces two types of chocolate candies, M and N. Both candies need milk and chocolates as ingredients. To produce each unit of M and N chocolate candies requires the following:

M: 1 unit of milk and 3 units of chocolates

N: 1 unit of milk and 2 units of chocolates

The company's kitchen has a total of 5 units of milk and 12 units of chocolates. The company makes a profit of ₱ 60 per unit candy M sold and ₱ 50 unit candy N sold. How many units of candies M and N should be produced by the company in order to maximize profit?

14. To protect her skin from allergies and imperfections, a budding singer was advised by her dermatologist to take vitamins for her daily maintenance. Every day she must take 100 units of vitamin A, 500 units of vitamin C and 200 units of vitamin B complex. She has the option to chose between two brands of vitamins which contain all 3 vitamins per capsule. Brand Beauty contains 2 units of vitamin A, 250 units of vitamin C and 200 units of vitamin B complex while Brand Pretty contains 100 units of vitamin A, 500 units of vitamin C and 100 units of vitamin B complex. Brand Beauty vitamins cost ₱ 23.50 per capsule and Brand Pretty vitamins cost ₱ 27.00 per capsule. Determine the number of capsules she should purchase to minimize cost while maintaining good health.

15. A newly retired lawyer has recently acquired a 110 hectares piece of land in Bataan. He has decided to grow rice and corn on that land. Due to the quality of the sun and the region's excellent climate, the entire production of rice and corn can be sold. He wants to know how to plant each variety in the 110 hectares, given the costs, net profits and labor requirements according to the data shown below:

Variety	Cost (Price/Hectare)	Net Profit (Price/Hectare)	Man-days/Hectare
Rice	100	50	10
Corn	200	120	30

The lawyer has a budget of ₱ 1,000,000 and availability of 1,200 man- days during the planning stage. Find the optimal solution and the optimal value.

16. A stuffed toy manufacturer in Bulacan produces two types of stuffed toys such as stuffed dolls and stuffed animals. Stuffed dolls are sold at ₱ 100.00 at stuffed animals at ₱ 90.00. There are 2,000 resource units available every day from which the dolls require 20 units while animals require 12 units. Both stuffed dolls and animals require a production time 15 minutes with a total of 8 working a day. What quantity of stuffed dolls and stuffed animals would yield the maximum profit?

17. A diet chart shown below includes data on calories, proteins, carbohydrate, and fat content of some vegetables. Emily wants a diet with minimum cost.

Nutrients	Cabbage	Carrot	Cauliflower	Potato
Calories	11.1	35.1	14.3	57
Fat in (grams)	0.1	0.1	0.3	0
Carbohydrates (grams)	2.4	8.2	2.5	13
Protein (grams)	0.6	0.9	1.1	1
Cost in Peso per gram	0.06	0.07	0.2	0.08

Source: https://www.blogilates.com/6141902952/andhttp://www.magkano.com/market/vegetables_price.htm

The chart provides the nutrient as well as the per-unit cost of each food item. The diet has to be planned in such a way that it should contain at least 500 calories, 6 grams of protein, 10 grams of carbohydrates and 8 grams of fat.

18. A small bakery in Bulacan is baking our favorite breads like pandesal, monay and ensaymada. The bakery have an order of 100 pieces of pandesal for a breakfast meeting early morning the next day. The pandesal can be baked using the new and old oven or both. When bake in the old oven, would take 30 minutes of time and when bake in the new oven would take 25 minutes. The cost of using the old oven is ₱ 25 per minute and using new oven is ₱ 20. The old oven has only 10 hours to operate while the new oven can be operated as desired. Determine the number of hours to schedule the baking of pandesal in each oven that minimize the cost of operation?

19. Engr Isaac is investing her savings of ₱500,000 into Food cart franchises, the Milk Tea Craze and Fish balls and squid balls cart. The milk tea craze is offering a rate of returns of 12% while the fish balls and squid balls cart offers 11.5%. She plans to invest ₱ 280,000 on Milk Tea Craze and ₱ 150,000 on fish balls and squid balls carts. How much should Engr. Isaac invest to maximize profit? What is the maximum annual interest?

20. A bag factory started producing two products last month, product X and Y using two new machines A and B.

Machine	**The cost of producing each unit of X**	**The cost of producing each unit of Y**
A	50 minutes	24 minutes
B	30 minutes	33 minutes

Working plans for a week are:

- 40 hours of work for machine A
- 35 hours of work on machine B

The week starts with:

- A stock of 30 units of X and 90 units of Y
- A demand of 75 units of x and 95 units of Y

How to plan the production, to end the week with maximum stock?

THE FOLLOWING ARE TOPICS INCLUDED IN THE FULL COURSE OF LINEAR PROGRAMMING.

- ✓ **Linear Programming by Graphical Method**
- ✓ **Linear Programming by Simplex Method**
  - **Standard Maximization**
  - **Maximization with Mixed Constraints**
  - **Minimization**
- ✓ **Linear Programming by Transportation Method**
  - **Stepping stone Method**
  - **Modified Method**
- ✓ **Linear Programming by Assignment Method**
- ✓ **Break Even Analysis**
- ✓ **Network Analysis**
- ✓ **Inventory Models**
- ✓ **Forecasting**
- ✓ **Decision Theory**

5.3 THE SIMPLEX ALGORITHM

It is one of the most popular and powerful methods of linear programming. It is a repetitive procedure for obtaining the most feasible solution. It is carried out by performing elementary row operations on a matrix called **simplex tableau.** The tableau consists of augmented matrix corresponding to the constraint equations and the coefficients of objective functions.

A linear programming function is in its **standard form** if it seeks to maximize the objective function as follows:

$$P = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Subject to the following constraints,

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2$$

$$. \quad . \quad . \quad . \quad . \quad .$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m$$

Where $x_i \geq 0$ and $b_i \geq 0$

Adding slack variables, s_1, s_2…s_n to each of the constraints and equating to constant,

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + s_1 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + s_2 = b_2$$

$$. \quad . \quad . \quad . \quad . \quad .$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + s_m = b_m$$

The slack variables are the non-negative numbers that are added to the constraints to remove the inequalities from the equation.

A **basic solution** is a solution (x_1, x_2,…, x_n, s_1, s_2 ,…, s_m) for a linear programming problem in standard form in which at most *m* variables are nonzero.

A **basic variable** is a variable that corresponds to a column in tableau, in which column is cleared out and has only one **non-zero** element. The values of the basic variables are found by reading the solution from the matrix that results by deleting out the non-basic columns.

A **non-basic variable** is a variable that corresponds to a column that is not cleared out and has more than one non-zero element. **Its value is zero.**

A **Simplex tableau** consists of the augmented matrix corresponding to the constraint equations together with slack variables and the objective function.

Standard Maximum-Type Problem

A linear programming problem is a standard maximum type of problem if the following conditions are met:

1. The objective function is linear and to be maximized.
2. All variables are non-negative, (zero or positive).
3. All explicit constraints involve $\leq$.
4. All the constants in the constraints are positive, (right hand side of each constraint).

Steps in solving linear programming problem in standard form:

1. Add the slack variable to constraints and rewrite it in the **equation form**, the objective function is also rewritten the same way as the constraints.
2. Construct the Simplex Tableau.
3. Determination of Pivotal column, Pivotal row and the Pivot element.
4. Reduction of the value of pivot to 1 and reduction of all other entries in the pivotal column to zero using elementary row operations.
5. Iteration of steps 3 and 4 until the optimum solution is obtained. This is where all entries at the bottom row are zero or positive.
6. If you obtain a final tableau, then the linear programming problem has a maximum solution, which is given by the entry in the lower-right corner of the tableau.

Example 5.5: Use simplex algorithm to maximize P = 4x + 6y, subject to the following constraints.

$$-x + y \leq 11$$
$$X + y \geq 27$$
$$2x + 5y \leq 90$$
$$x \geq 0; \quad y \geq 0$$

Solution: The steps in solving linear programming problems is demonstrated using this example

Step 1: Add the slack variable to constraints and rewrite it in the equation form, the objective function is also rewritten as shown below,

The new linear program,

$$-x + y + S_1 = 11$$
$$x + y + S_2 = 27$$
$$2x + 5y + S_3 = 90$$
$$-4x - 6y + P = 0$$

Step 2: Construct the Simplex Tableau

Below is the initial simplex tableau, the number of rows should be equal to the equations in the new linear program + 1 and the number of columns is equal to the number of variables + 2. The entries to the table at the topmost row are the variables in the new linear program, the 2nd, third and 4th row are coefficients of the constraints arranged according to the way they are listed. The last row entries are coefficients of the variables of objective function and its constant value.

Basic	x	y	S_1	S_2	S_3	P	Q
S_1	-1	1	1	0	0	0	11
S_2	1	1	0	1	0	0	27
S_3	2	5	0	0	1	0	90
	-4	-6	0	0	0	1	0

On the other hand, the first column is called the **basic column**, this is the column where the decision is based, while the last column is called the **quantity (Q) column** where the constants are listed and where the values of the decision variables are obtained. And the columns in between are called **coefficient column**.

In the table that follows, S_1, S_2, S_3 and P are basic variables since they correspond to a column which is cleared out and has only one non-zero element. So, the initial values of basic variables are $S_1 = 1$, $S_2 = 0$, $S_3 = 2$ and P = 0.

S_1	S_2	S_3	P	Q
1	0	0	0	11
0	1	0	0	27
0	0	1	0	90
0	0	0	1	0

Variables x and y are non-basic variables, so their values are x = 0 and y = 0.

Step 3: Determination of Pivotal column, Pivotal row and the Pivot element.

The **pivotal column (entering column**) is the *most negative entry* in the objective row, (bottom) with entry under the column of variable y is -6, **column y is the pivotal column**. In case of tied entries, any one of them can be used to determine the pivotal column. However, if all entries in the entering column are 0 or negative, there is no maximum solution.

Pivotal column

Basic Variables	x	y	S_1	S_2	S_3	P	Q
S_1	-1	1	1	0	0	0	11
S_2	1	1	0	1	0	0	27
S_3	2	5	0	0	1	0	90
	-4	-6	0	0	0	1	0

Pivotal row

The **pivotal row (departing row)** can be determined by the taking the smallest positive ratio of the entries in the quantity column to its corresponding entries in the pivotal column.

$$11 \div 1 = \mathbf{11} \qquad 27 \div 1 = 27 \qquad 90 \div 5 = 18$$

In the above table, the smallest positive ratio is 11 which is in 1st row, this indicates that **1st row is the pivotal row.** If no non-negative ratios can be found, the problem does not have a solution. If one of the ratios is 0, that qualifies as a non-negative value.

Encircle the pivot column and pivot row, the entry at the intersection of the pivotal row and pivotal column is **1 becomes the pivot element or simply pivot.**

Step 4: Reduction of the value of pivot to 1 and reduction of all other entries in the pivotal column to zero using elementary row operations. This process is called **pivoting.**

Since the pivot is 1, no need to reduce further, its row is now called $\mathbf{R_0}$, the **reduced pivotal row.**

R_0: -1 1 1 0 0 0 11

Other entries in the pivotal column may be reduced as follows:

Multiply R_0 by -1 (additive inverse of 1 in R_1 of pivotal column) then add to all entries of R_2,

$-R_0$	:	1	-	1	-	1	+	0	+	0	+	0	-	11
R_2	:	1	+	1	+	0	+	1	+	0	+	0	+	27
		2	**+**	**0**	**-**	**1**	**+**	**1**	**+**	**0**	**+**	**0**	**+**	**16**

Now, reducing R_3, multiply R_0 by -5, then add to all entries of R_3

$-5R_0$	:	5	-	5	-	5	+	0	+	0	+	0	-	55
R_3	:	2	+	5	+	0	+	0	+	1	+	0	+	90
		7	**+**	**0**	**-**	**5**	**+**	**0**	**+**	**1**	**+**	**0**	**+**	**35**

Reducing the objective row, multiply R_0 by 6, then add to all entries of the objective row,

$6R_0$	:	-6	+	6	+	6	+	0	+	0	+	0	+	66
R_4	:	-4	-	6	+	0	+	0	+	0	+	1	+	0
		-10	**+**	**0**	**+**	**6**	**+**	**0**	**+**	**0**	**+**	**1**	**+**	**66**

The reduced entries are reflected in the table below, **the pivotal row S_1 is replaced by y,**

Before Pivoting

Basic	x	y	S_1	S_2	S_3	P	Q
S_1	-1	1	1	0	0	0	11
S_2	1	1	0	1	0	0	27
S_3	2	5	0	0	1	0	90
	-4	-6	0	0	0	1	0

$\Longrightarrow$

After Pivoting

Basic	x	y	S_1	S_2	S_3	P	Q
y	-1	1	1	0	0	0	11
S_2	2	0	-1	1	0	0	16
S_3	7	0	-5	0	1	0	35
	-10	0	6	0	0	1	66

Note that the basic variables are y, S_2, S_3 and P and non-basic variables are x and S_1

Basic	y	S_2	S_3	P	Q
y	1	0	0	0	11
S_2	0	1	0	0	16
S_3	0	0	1	0	35
	0	0	0	1	66

Summarizing new values,

Y = 11 x = 0

S_2 = 16 S_1 = 0

S_3 = 35

P = 66

But since the objective column still has a negative number in its row, this means that the optimum solution is not reached yet, this means we need to continue with the next step.

Step 5: Iteration of steps 3 and 4 until the optimum solution is obtained.

Repeat doing steps 3 and 4 using the new entries listed in the table, the purpose is to reduce the entries in the objective row to positive numbers and zeros. Zeros and positive numbers in the objective rows indicate that the optimum solution is reached.

From the table, the most negative value is - 10 which is in column x, this means that column x is the new pivotal column, and to determine the pivotal row,

Basic	x	y	S_1	S_2	S_3	P	Q
y	-1	1	1	0	0	0	11
S_2	2	0	-1	1	0	0	16
S_3	7	0	-5	0	1	0	35
	-10	0	6	0	0	1	66

To determine the pivotal row,

R_1: $11 \div -1 = -11$

R_2: $16 \div 2 = 8$

R_3: $35 \div 7 = 5$

The smallest positive ratio is 5, therefore the pivotal row is S_3 row,

Basic	x	y	S_1	S_2	S_3	P	Q
y	-1	1	1	0	0	0	11
S_2	2	0	-1	1	0	0	16
S_3	7	0	-5	0	1	0	35
	-10	0	6	0	0	1	66

The pivot element is 7, so multiply the entries of the pivotal row by $\frac{1}{7}$, the new reduced pivotal row is as follows.

$$R_0 \;:\; 1 + 0 - \tfrac{5}{7} + 0 + \tfrac{1}{7} + 0 + 5$$

Then, we add R_0 to the entries in row R_1,

$$\begin{array}{lccccccccccccccc}
R_0 & : & 1 & + & 0 & - & \frac{5}{7} & + & 0 & + & \frac{1}{7} & + & 0 & + & 5 \\
R_1 & : & -1 & + & 1 & + & 1 & + & 0 & + & 0 & + & 0 & + & 11 \\
\hline
 & & \mathbf{0} & + & \mathbf{1} & + & \mathbf{\frac{2}{7}} & + & \mathbf{0} & + & \mathbf{\frac{1}{7}} & + & \mathbf{0} & + & \mathbf{16}
\end{array}$$

The next row to be reduced is R_2, multiply R_0 by -2, then add all entries of R_2,

$$\begin{array}{lccccccccccccccc}
-2R_0 & : & -2 & + & 0 & + & \frac{10}{7} & + & 0 & - & \frac{2}{7} & + & 0 & - & 10 \\
R_2 & : & 2 & + & 0 & - & 1 & + & 1 & + & 0 & + & 0 & + & 16 \\
\hline
 & & \mathbf{0} & + & \mathbf{0} & + & \mathbf{\frac{3}{7}} & + & \mathbf{1} & - & \mathbf{\frac{2}{7}} & + & \mathbf{0} & + & \mathbf{6}
\end{array}$$

Next, reducing the objective row, multiply R_0 by 10, then add R_4,

$$\begin{array}{lccccccccccccccc}
10R_0 & : & 10 & + & 0 & - & \frac{50}{7} & + & 0 & + & \frac{10}{7} & + & 0 & + & 50 \\
R_4 & : & -10 & + & 0 & + & 6 & + & 0 & + & 0 & + & 1 & + & 66 \\
\hline
 & & \mathbf{0} & + & \mathbf{0} & - & \mathbf{\frac{8}{7}} & + & \mathbf{0} & + & \mathbf{\frac{10}{7}} & + & \mathbf{1} & + & \mathbf{116}
\end{array}$$

The result of the second pivoting is as follows, S_3 is now replaced by x,

First Pivoting

Basic	x	y	S_1	S_2	S_3	P	Q
y	-1	1	1	0	0	0	11
S_2	2	0	-1	1	0	0	16
S_3	7	0	-5	0	1	0	35
	-10	0	6	0	0	1	66

$\Rightarrow$

Second Pivoting

Basic	x	y	S_1	S_2	S_3	P	Q
y	0	1	$\frac{2}{7}$	0	$\frac{1}{7}$	0	16
S_2	0	0	$\frac{3}{7}$	1	$-\frac{2}{7}$	0	6
x	1	0	$-\frac{5}{7}$	0	$\frac{1}{7}$	0	5
	0	0	$-\frac{8}{7}$	0	$\frac{10}{7}$	1	116

The tableau shows that the basic variables and their values are x = 5, y = 16, S_2 = 6 and P = 116, while non-basic variables are S_1 = 0 and S_3 = 0 but there is still a negative entry in the bottom row, so we continue the steps 3 and 4, this time the pivoting column is S_1 and the pivoting row is S_2,

Basic	x	y	S_1	S_2	S_3	P	Q
y	0	1	$\frac{2}{7}$	0	$\frac{1}{7}$	0	16
S_2	0	0	$\frac{3}{7}$	1	$-\frac{2}{7}$	0	6
x	1	0	$-\frac{5}{7}$	0	$\frac{1}{7}$	0	5
	0	0	$-\frac{8}{7}$	0	$\frac{10}{7}$	1	116

Since the pivot is $\frac{3}{7}$, multiply R_2 by $\frac{7}{3}$ in order to produce $\mathbf{R_0}$, the **reduced pivotal row,**

$$R_0 \;:\; 0 + 0 + 1 + \tfrac{7}{3} - \tfrac{2}{3} + 0 + 14$$

To reduce R_1, multiply R_0 by $-\frac{7}{2}$ and add R_1,

$$\begin{array}{rccccccccccccccc}
-\frac{2}{7}R_0 & : & 0 & + & 0 & - & \frac{2}{7} & - & \frac{2}{3} & + & \frac{4}{21} & + & 0 & - & 4 \\
R_1 & : & 0 & + & 1 & + & \frac{2}{7} & + & 0 & + & \frac{1}{7} & + & 0 & + & 16 \\
\hline
 & & \mathbf{0} & \mathbf{+} & \mathbf{1} & \mathbf{+} & \mathbf{0} & \mathbf{-} & \frac{2}{3} & \mathbf{+} & \frac{1}{3} & \mathbf{+} & \mathbf{0} & \mathbf{+} & \mathbf{12}
\end{array}$$

To reduce R_3, multiply R_0 by $\frac{5}{7}$ and add R_3,

$$\begin{array}{rccccccccccccccc}
\frac{5}{7}R_0 & : & 0 & + & 0 & + & \frac{5}{7} & + & \frac{5}{3} & - & \frac{10}{21} & + & 0 & + & 10 \\
R_3 & : & 1 & + & 0 & - & \frac{5}{7} & + & 0 & + & \frac{1}{7} & + & 0 & + & 5 \\
\hline
 & & \mathbf{1} & \mathbf{+} & \mathbf{0} & \mathbf{+} & \mathbf{0} & \mathbf{+} & \frac{5}{3} & \mathbf{-} & \frac{1}{3} & \mathbf{+} & \mathbf{0} & \mathbf{+} & \mathbf{15}
\end{array}$$

To reduce the objective row, multiply R_0 by $\frac{8}{7}$ and add R_4,

$$\begin{array}{rccccccccccccccc}
\frac{8}{7}R_0 & : & 0 & + & 0 & + & \frac{8}{7} & + & \frac{8}{3} & - & \frac{16}{21} & + & 0 & + & 16 \\
R_4 & : & 0 & + & 0 & - & \frac{8}{7} & + & 0 & + & \frac{10}{7} & + & 1 & + & 116 \\
\hline
 & & \mathbf{0} & \mathbf{+} & \mathbf{0} & \mathbf{+} & \mathbf{0} & \mathbf{+} & \frac{8}{3} & \mathbf{+} & \frac{2}{3} & \mathbf{+} & \mathbf{1} & \mathbf{+} & \mathbf{132}
\end{array}$$

Now, shown below is the new tableau,

Basic	x	y	S_1	S_2	S_3	P	Q
y	0	1	0	$-\frac{2}{3}$	$\frac{1}{3}$	0	12
S_1	0	0	1	$\frac{7}{3}$	$-\frac{2}{3}$	0	14
x	1	0	0	$\frac{5}{3}$	$-\frac{1}{3}$	0	15
	0	0	0	$\frac{8}{3}$	$\frac{2}{3}$	1	**132**

There is no more negative element at the bottom row, this indicates that the optimum solution is determined**: x = 15, y = 12, S_1 = 14 and P =132**

Checking with the objective function: $P = 4x + 6y$

$$\mathbf{P = 4\,(14) + 6(12) = 132}$$

The Simplex Method: Minimization

$$a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n \geq b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots a_{mn}x_n \geq b_2$$

where $x_i \geq 0$ and $b_1 \geq 0$

To solve this linear programming problem is to convert it to a maximization problem in standard form, and then apply the simplex method discussed previously.

Example 5.6: Find the minimum value of Objective function $z = 3x_1 + 8x_2$, subject to the following constraints,

$$2x_1 + 7x_2 \geq 9$$
$$x_1 + 2x_2 \geq 4$$
$$x_1, x_2 \geq 0$$

where $x_1 \geq 0$ and $x_2 \geq 0$.

Solution: We first convert the given problem to a maximization problem by forming augmented matrix, we transpose of matrix by interchanging its rows and columns

$$\begin{bmatrix} 2 & 7 & \vdots & 9 \\ 1 & 2 & \vdots & 4 \\ \cdots & \cdots & \vdots & \cdots \\ 3 & 8 & \vdots & 0 \end{bmatrix} \quad \text{then transposing} \quad \begin{bmatrix} 2 & 1 & \vdots & 3 \\ 7 & 2 & \vdots & 8 \\ \cdots & \cdots & \vdots & \cdots \\ 9 & 4 & \vdots & 0 \end{bmatrix}$$

Take note that the rows of matrix become the columns of first matrix and vice versa. The new matrix can be interpreted as maximization problem, but we need to introduce new variables such as y_1, y_2, y_3, etc. This maximization problem is called the **dual** of the original minimization problem.

Dual Objective Function: $w = 9y_1 + 4y_2$

Subject to the Constraints:
$$2y_1 + y_2 \leq 3$$
$$7y_1 + 2y_2 \leq 8$$
Where $y_1, y_2 \geq 0$

Adding slack variables to constraints:

$$2y_1 + y_2 + S_1 = 3$$
$$7y_1 + 2y_2 + S_2 = 8$$

Also, with the objective function:

$$-9y_1 - 4y_2 + W = 0$$

Now, creating the tableau and applying the simplex method. Initially the pivotal column is y_1, the pivotal row is R_2 and pivotal element is 7, as shown in the tableaus below:

Basic	Y_1	Y_2	S_1	S_2	P	Q
S_1	2	1	1	0	0	3
S_2	7	2	0	1	0	8
	-9	-4	0	0	1	**0**

Basic	Y_1	Y_2	S_1	S_2	P	Q
S_1	2	1	1	0	0	3
S_2	7	2	0	1	0	8
	-9	-4	0	0	1	**0**

Applying elementary operations, $R_0 = \frac{1}{7}R_2$,

$$\mathbf{R_0} \;:\; \mathbf{1} + \mathbf{\frac{2}{7}} \quad \mathbf{0} + \mathbf{\frac{1}{7}} + \mathbf{0} + \mathbf{\frac{8}{7}}$$

Multiply R_0 by (-2), then add to R_1, the result is the new R_1.

$$\begin{array}{lcccccccccccc} \mathbf{-2R_0} & : & -2 & - & \frac{4}{7} & + & 0 & - & \frac{2}{7} & + & 0 & - & \frac{16}{7} \\ \mathbf{R_1} & : & 2 & + & 1 & + & 1 & + & 0 & + & 0 & + & 3 \\ \hline & & \mathbf{0} & \mathbf{+} & \mathbf{\frac{3}{7}} & \mathbf{+} & \mathbf{1} & \mathbf{-} & \mathbf{\frac{2}{7}} & \mathbf{+} & \mathbf{0} & \mathbf{+} & \mathbf{\frac{5}{7}} \end{array}$$

Next is to multiply R_0 by 9, then add to R_3, the result is the new R_3.

$$\begin{array}{lcccccccccccc} \mathbf{9R_0} & : & 9 & + & \frac{18}{7} & + & 0 & + & \frac{9}{7} & + & 0 & + & \frac{72}{7} \\ \mathbf{R_3} & : & -9 & - & 4 & + & 0 & + & 0 & + & 1 & + & 0 \\ \hline & & \mathbf{0} & \mathbf{-} & \mathbf{\frac{10}{7}} & \mathbf{+} & \mathbf{0} & \mathbf{-} & \mathbf{\frac{9}{7}} & \mathbf{+} & \mathbf{1} & \mathbf{+} & \mathbf{\frac{72}{7}} \end{array}$$

In the tableau that follows, note that there is still a negative entry at the objective function row, so there is still a need continue the previous steps.

Basic	Y_1	Y_2	S_1	S_2	P	Q
S_1	0	$\frac{3}{7}$	1	$-\frac{2}{7}$	0	$\frac{5}{7}$
Y_1	1	$\frac{2}{7}$	0	$\frac{1}{7}$	0	$\frac{8}{7}$
	0	$-\frac{10}{7}$	0	$-\frac{9}{7}$	1	$\frac{72}{7}$

Basic	Y_1	Y_2	S_1	S_2	P	Q
S_1	0	$\frac{3}{7}$	1	$-\frac{2}{7}$	0	$\frac{5}{7}$
y_1	1	$\frac{2}{7}$	0	$\frac{1}{7}$	0	$\frac{8}{7}$
	0	$-\frac{10}{7}$	0	$-\frac{9}{7}$	1	$\frac{72}{7}$

The new pivotal column is Y_2 column, the pivotal row is R_1 and the pivotal element is $\frac{3}{7}$ and the result of applying elementary operations is shown below:

Basic	Y_1	Y_2	S_1	S_2	P	Q
Y_2	0	1	$\frac{7}{3}$	$-\frac{2}{3}$	0	$\frac{5}{3}$
y_1	1	0	$-\frac{2}{3}$	$\frac{1}{3}$	0	$\frac{2}{3}$
	0	0	$\frac{10}{3}$	$\frac{1}{3}$	1	$\mathbf{\frac{38}{3}}$
			$\uparrow$ x_1	$\uparrow$ x_2		

The above tableau shows that optimal value is $\mathbf{\frac{38}{3}}$ and this value occurs when the value of $y_1 = \mathbf{\frac{2}{3}}$ and $y_2 = \mathbf{\frac{5}{3}}$**.** Take note that this dual maximization problem has the same solution as its original minimization problem.

Original minimization problem: $z = 3x_1 + 8x_2 = 3\left(\frac{10}{3}\right) + 8\left(\frac{1}{3}\right) = 10 + \frac{8}{3} = \mathbf{\frac{38}{3}}$

of the original minimization problem: $w = 9y_1 + 4y_2 = 9\left(\frac{2}{3}\right) + 4\left(\frac{5}{3}\right) = 6 + \frac{20}{3} = \mathbf{\frac{38}{3}}$

Theorem: The Von Neumann Duality Principle

The objective value **P** of a minimization problem in standard form has a minimum value if and only if the objective value **w** of the dual maximization problem has a maximum value. Moreover, the minimum value of **w** is equal to the maximum value of **P.**

5.4 The Simplex Method: Mixed Constraints

Mixed-constraint problems is a type of linear programming problem in which constraints involve both types of inequalities, $\leq, \geq$ or $=$.

Maximization Problem with Mixed Constraints

A linear programming in maximization with maximization constraints form if the following conditions are met.

a. The objective function is to be maximized.
b. All variables are non-negative.
c. All explicated constraints involve $\leq, \geq$ or $=$.
d. All constants in the constraints are non-negative.

Example 5.7: Find the maximum value of $w = 3x_1 + 2x_2 + 4x_3$

Subject to the following constraints

$$3x_1 + 2x_2 + 5x_3 \leq 18$$
$$4x_1 + 2x_2 + 3x_3 \leq 16$$
$$2x_1 + x_2 + x_3 \geq 4$$

Where $x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$

Solution: Add the slack variables S_1 and S_2 to the first two constraints and subtract a **surplus variable, S_3** from the third constraint as shown below:

$$3x_1 + 2x_2 + 5x_3 + S_1 = 18$$
$$4x_1 + 2x_2 + 3x_3 + S_2 = 16$$
$$2x_1 + x_2 + x_3 - S_3 = 4$$

The initial tableau is shown below, take note that the tableau does not have a feasible solution because **S_3 is negative**. By trial and error, column x_2 can be chosen as the entering variable and S_3 as the departing variable,

Now, because this simplex tableau does represent a feasible solution, we proceed as usual, choosing the most negative entry in the bottom row to be the entering variable. (In this case, we have a tie, so we arbitrarily choose to be the entering variable.)

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Q
S_1	3	2	5	1	0	0	18
S_2	4	2	3	0	1	0	16
S_3	2	1	1	0	0	-1	4
	-3	-2	-4	0	0	0	**0**

By elementary row operations, the following tableau is obtained, there are still negative values at the bottom row so we proceed as usual, x_3 would be the entering variable while S_1 as the departing variable

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Q
S_1	-1	0	3	1	0	2	10
S_2	0	0	1	0	1	2	8
X_2	2	1	1	0	0	-1	4
	1	0	-2	0	0	-2	**8**

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Q
S_1	-1	0	3	1	0	2	10
S_2	0	0	1	0	1	2	8
X_2	2	1	1	0	0	-1	4
	1	0	-2	0	0	-2	**8**

Repeat the procedure by elementary row operations, the new tableau is shown below.

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Q
X_3	$-\frac{1}{3}$	0	1	$\frac{1}{3}$	0	$\frac{2}{3}$	$\frac{10}{3}$
S_2	$\frac{1}{3}$	0	0	$-\frac{1}{3}$	1	$\frac{4}{3}$	$\frac{14}{3}$
X_2	$\frac{7}{3}$	1	0	$-\frac{1}{3}$	0	$-\frac{5}{3}$	$\frac{2}{3}$
	$\frac{1}{3}$	0	0	$\frac{2}{3}$	0	$-\frac{2}{3}$	$\frac{44}{3}$

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Q
X_3	$-\frac{1}{2}$	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	1
S_2	$\frac{1}{4}$	0	0	$-\frac{1}{4}$	$\frac{3}{4}$	1	$\frac{7}{2}$
X_2	$\frac{11}{4}$	1	0	$-\frac{3}{4}$	$\frac{5}{4}$	0	$\frac{13}{2}$
	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	**17**

Finally, the last tableau shows the maximum value of the objective function to be w = 17 and this value occurs when $x_1 = 0$, $x_2 = \frac{13}{2}$ and $x_3 = 1$.

Source: https://college.cengage.com/mathematics/larson/elementary_linear/4e/shared/downloads/c09s5.pdf

REVIEW EXERCISES 5.1

Name ____________________________________ Score ___________ Date ______________
Course, Year & Section _______________Student no. __________ Professor______________

Solve the following problems:

A. Determine which of the constraints below are not on the form appropriate for a standard maximum-type problem. If the constraint is not on the appropriate form, rewrite it if possible, to reduce to standard maximum type problem.

1. $5x - y \leq 4$
2. $-6x + 5y + z \geq -3$
3. $7x - 4y \leq -4$
4. $x + y - 11z \leq 0$
5. $7x - 5y \geq 2$
6. $4x - 5z \leq -10$

B. Solve the following linear program by Simplex algorithm.

1. Objective: $P = 8x_1 + x_2$
   Constraints: $2x_1 + 3x_2 \leq 6$
   $x_1 + 2x_2 \leq 6$
   $2x_1 \geq 0$
   $X_2 \geq 0$

2. Objective: $P = 2x_1 + 4x_2 - 4x_3$
   Constraints: $3x_1 + 2x_2 + 4x_3 \geq 1$
   $4x_1 - 3x_2 \geq 2$
   $2x_1 + x_2 + 5x_3 \leq 3$
   $x_1, x_2 \geq 0$
   x_3 : Unrestricted:

3. Objective: $P = 3x_1 - 7x_2 + 5x_3$
   Constraints: $x_2 - x_3 \leq -9$
   $-x_1 - 2x_3 \geq 5$
   $4x_1 - x_2 = 6$
   $x_1 \leq 0$
   $X_2 \geq 0$

4. Objective: $P = 2x_1 - 3x_2 + 4x_3$
   Constraints: $x_1 - 5x_2 + 6x_3 \geq 8$
   $x_1 - 4x_2 \leq -12$
   $2x_1 - x_2 + 4x_3 = 5$
   $x_1, x, x_3 \geq 0$

5. Objective: $P = 3x_1 + 2x_2 + x_3$
   Constraints: $2x_1 + 3x_2 + x_3 \leq 60$
   $4x_1 + x_2 + x_3 = 30$
   $x_1 + 2x_2 + 3x_3 \leq 40$
   $x_1, x_2, x_3 \geq 0$

6. Objective: $P = - x_1 + 3x_2 - 3x_3$
   Constraints: $3x_1 - x_2 - 2x_3 \leq 7$
   $x_1 - 2x_3 \leq 4$
   $-2x_1 + 2x_2 + x_3 \leq 8$
   $3x_1 \leq 5$
   $x_1, x_2, x_3 \geq 0$

7. Objective: $P = x_2$
   Constraints: $x_1 + 5x_2 \geq 10$
   $-6x_1 + 5x_2 \geq 3$
   $x_1, x_2 \geq 0$
   $x_1, x_2, x_3 \geq 0$
   $x_1, x_2, x_3 \geq 0$

6. Objective: $P = 2x_1 + 10x_2 + 8x_3$
   Constraints: $x_1 + x_2 + x_3 \geq 6$
   $x_2 + 2x_3 \geq 8$
   $-x_1 + 2x_2 + 2x_3 \geq 4$
   $x_1, x_2, x_3 \geq 0$

C. Solve the following problems:

7. A secondhand dealer of appliances is selling used refrigerators that were sold to them. They sell two popular brands of refrigerators Panasonic and Kelvinator. The 7.5 cubic feet, two door Panasonic would give the dealer a profit of ₱3,500 and the 8 cubic feet, Kelvinator would give ₱3,000 profit. Before the dealer could sell the used refrigerators, they need to evaluate about their working condition, some may need parts replacement or repairs. The Panasonic two door refrigerators would take about 8 hours to be repaired and must pass for 4 hours of quality control process, while the Kelvinator would only take 6 hours of repairs and 2 hours of quality control.

The technician has 3 days available for repairs per period of production and 5 days spend for quality control job. How many Panasonic and Kelvinator of the same specification should be undergone repairs and quality control per production period so that the dealer could get a maximum profit?

8. A multinational manufacturing company has three production plants, each of which produces three different models of a certain product. The daily capacities (in thousand units) of the three plants are given in the following table:

	Model A	Model B	Model C
Plant 1	8	4	8
Plant 2	6	6	3
Plant 3	12	4	8

The total demand for Model A is 300,000 units, for Model 2 is 172,000 units and for model 3 is 249,500 units. The daily operating cost for Plant 1 is ₱55,000, for Plant 2 is ₱60,000 and Plant 3 is ₱60,000. How many days should each plant be operated in order to fill the total demand and keep the operating cost at a minimum?

9. In the problem no. 8 the multinational manufacturing company has lowered the daily operating cost for Plant 3 to ₱50,000. How many days should each plant be operated in order to fill the total demand and keep the operating cost at a minimum?

10. A big motorcycle company has two warehouses. One warehouse located in the city has 400 units of motorcycles (2020 model) and the other warehouse located in nearby province has 300 units of motorcycles (2019 model) in stock. There are two clients of the motorcycles. The first client needs 200 units of the motorcycles and the second customer needs 300 units. The cost of freight is shown in the table below:

	Client 1	Client 2
Warehouse 1	30	25
Warehouse 2	36	30

How should the company ship the motorcycles in order to minimize the freight cost?