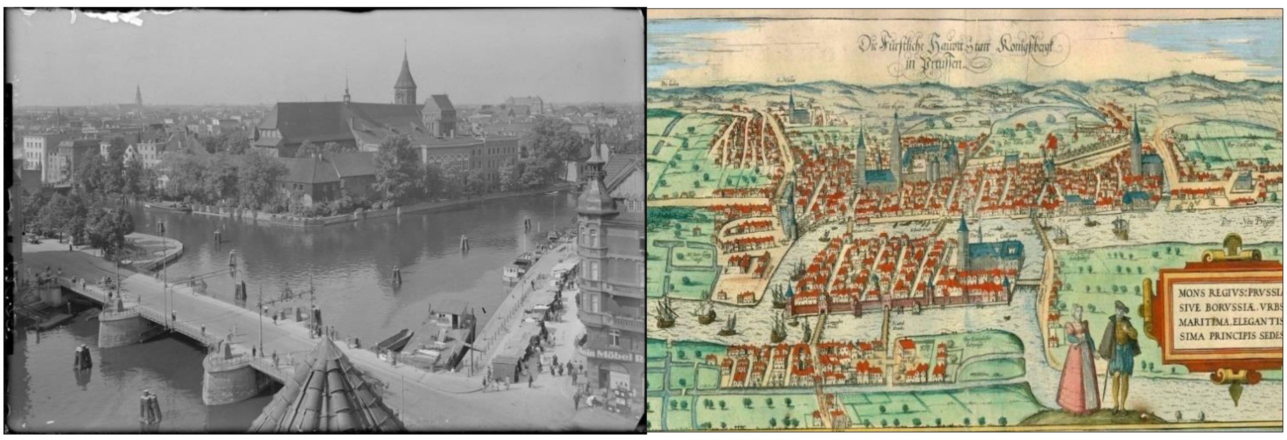
Chapter 7: GRAPH THEORY

LEARNING OBJECTIVES:

1. To understand and relate the problem about the Bridges of Konigsberg to Networking.
2. To Define and construct different types of graphs.
3. To Differentiate between paths and circuits, Euler and Hamilton Theorems.
4. To apply theorems and algorithms in the solution of problems.
5. To Solve problems about map and graph coloring.

7.1 THE BRIDGES OF KONIGSBERG

Figure 7.0



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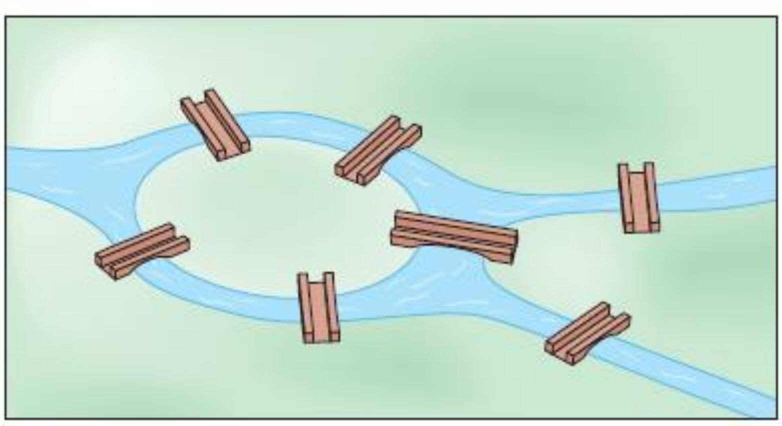
The story of graph theory starts in Konigsberg, a city on the Baltic coast (during 1700s, located in modern-day Russia and now called Kaliningrad)). This city was built around two islands in the Pregel River, all connected by seven bridges. They build the bridges so they could easily move around the city. Since people walked through there daily, they started to wonder if they can walk around the whole town, cross all the bridges only once and still walk over every bridge.

The citizens of Prusian city had an open challenge: Find a way to walk through the city in such a way as to cross every bridge exactly once. Many had a pleasant walk while attempting to solve this puzzle, but no one found a way to do it.

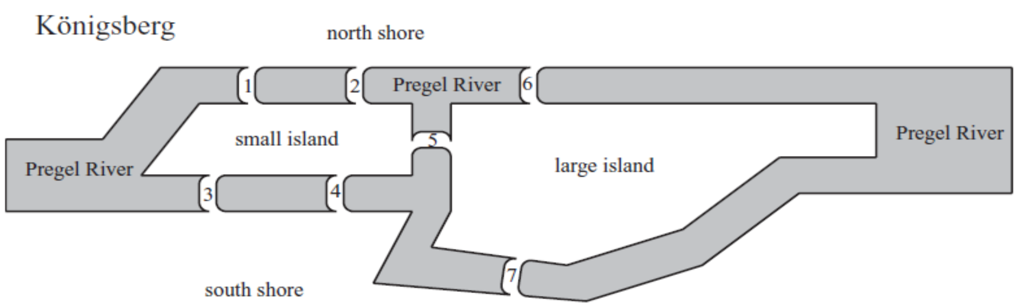
Shown in figure 7.1 is the figure of Konigsberg where all features are eliminated except the land masses and the seven bridges connecting them.

Figure 7.1



So, the problem was to devise a walk through the city that would cross each of those bridges once and only once. For example, see figure 7.2, a person who started at the north shore and walked over bridge 1, then 3, then 4, then 2, then 6 and then 7 would have missed bridge 5 and have no way to get there without crossing over 3, 4 or 7 a second time. Later, a great Swiss mathematician, **Leonard Euler proved that no such walk was possible.**

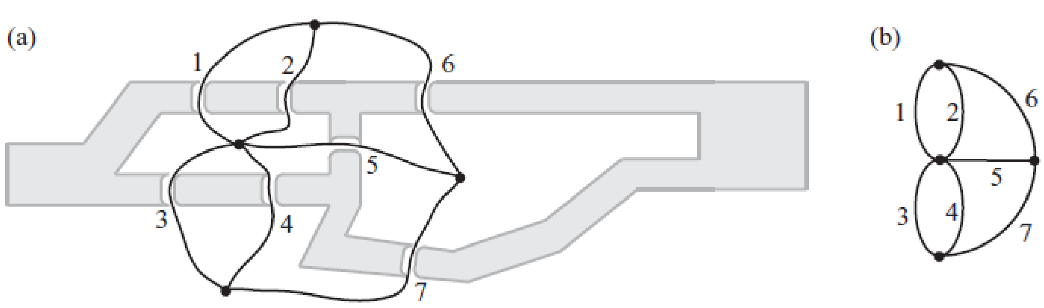
Figure 7.2



LEONARD EULER (pronounced oiler,15 April 1707 – 18 September 1783)

Swiss mathematician, physicist, astronomer, geographer, logician and engineer who made important and influential discoveries in many branches of mathematics, such as infinitesimal calculus and **graph theory**, making pioneering contributions to topology and analytic number theory. Among his many discoveries and developments, Euler is credited for introducing the Greek letter **pi** to denominate the Archimedes constant (the ratio of a circle's circumference to its diameter), and for developing a new mathematical constant, the **"e"** (also known as Euler's Number), which is equivalent to a logarithm's natural base, and has several applications such as to calculate compound interest.

- He approached this problem by imagining areas of land separated by the river into points connected with bridges or curved lines.
- His first step was to simplify the map of Konigsberg by reducing the four parts of the city to points, connected by lines representing the seven bridges, (figure a).
- He let the land area be represented as points (sometimes called vertices), and let the bridges be represented by arcs or line segments (sometimes called edges) connecting the given points.



Any journey over the bridges of the city could be drawn into graphs and described entirely as a path from point to point over the lines of this figure (a). This figure was reduced to figure in (b), where the distances between the bridges and the sizes of the islands were irrelevant.

In this chapter, you will learn how to analyze and solve a variety of problems, such as how to *find the least expensive route of travel on a vacation, how to determine the most efficient order in which to run errands, and how to schedule meetings at a conference so that no one has two required meetings at the same time.*

GRAPH is a set of all points called vertices and line segments or curves called edges that connect vertices.

- It can be used to represent many different scenarios, as shown in the figure below are the same graph used in different contexts.

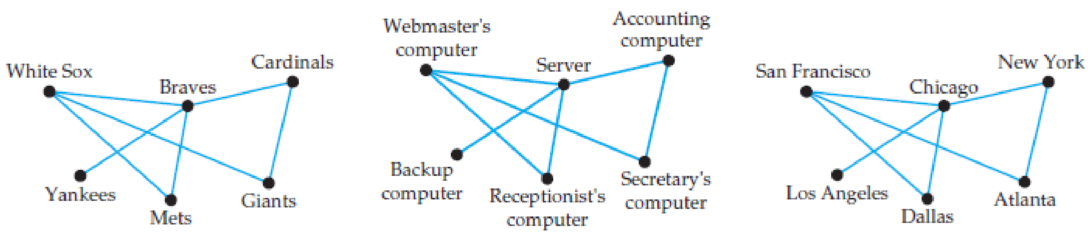


FIGURE 5.2 (a) (b) (c)

Figure 7.2 a: Each vertex represents a baseball team, an edge connecting vertices may mean two teams competing against each other.

Figure 7.2 b: Shows computer network of a small business, where each vertex represents a computer and edges are machines directly connected to each other.

Figure 7.2 c: It can be used to represent the flights available on a particular airline between a selection of cities; each vertex represents a city, and an edge connecting two cities means that there is a direct flight between the two cities.

Note: The placement of vertices has nothing to do with geographical location, the vertices can be shown in any arrangement we chose. The important information is which vertices are connected by edges.

7.2 CONSTRUCTING A GRAPH

Example 7.1: Table 7.1 lists five students at a college. An " X" indicates that the two students participate in the same study group this semester;

Table 7.1

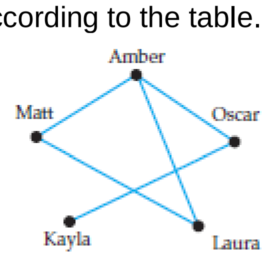
	Matt	Amber	Oscar	Laura	Kayla
Matt	—	X		X	
Amber	X	—	X	X	
Oscar		X	—		X
Laura	X	X		—	
Kayla			X		—

a. Draw a graph that represents this information where each vertex represents a student, and an edge connects two vertices if the corresponding student study together.

b. Use your graph to answer the following questions: Which student is involved in the most study groups with the others? Which student has only one study group in common with the others? How many study groups does Laura have in common with the others?

SOLUTION:

a. We draw five vertices (in any configuration we wish) to represent the five students and connect vertices with edges according to the table.



b. The vertex corresponding to Amber is connected to more edges than the others, so she is involved with more study groups (three) than the others. Kayla is the only student with one study group in common, as her vertex is the only one connected to just one edge. Laura's vertex is connected to two edges, so she shares two study groups with the others.

BRIEF SEATWORK

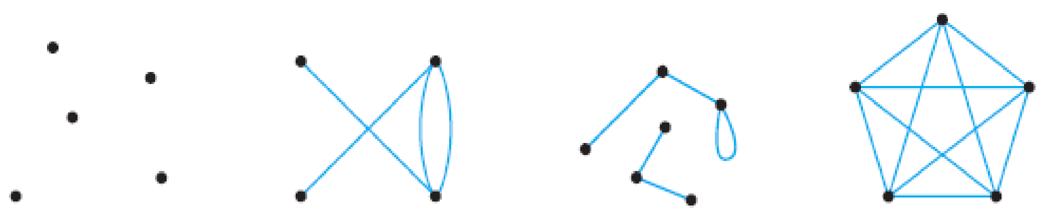
Using table 7.2, list five mobile phone companies and indicates whether they have agreements to roam onto each other's networks. Draw a graph that represent this information, where each vertex represents a phone company, and an edge connects two vertices if the corresponding companies have a roaming agreement. Which phone company has roaming agreements with the most carriers? Which company can roam with only one other network?

Table 7.2

	Mobile Plus	TalkMore	SuperCell	Airwave	Lightning
Mobile Plus		No	yes	No	Yes
TalkMore	No		yes	No	No
Supercell	Yes	yes		Yes	No
Airwave	No	No	yes		Yes
Lightning	Yes	No	No	Yes	

Example 7.2: Some graphs are shown in Figure 7.3.

Figure 7.3



This graph has five vertices but no edges and is referred to as a **null graph**. It is also an example of a *disconnected graph*.

This is a connected graph that has a pair of *multiple edges*. Note that two edges cross in the center, but there is no vertex there. Unless a dot is drawn, the edges are considered to pass over each other without touching.

This graph is not connected; it consists of two different sections. It also contains a loop.

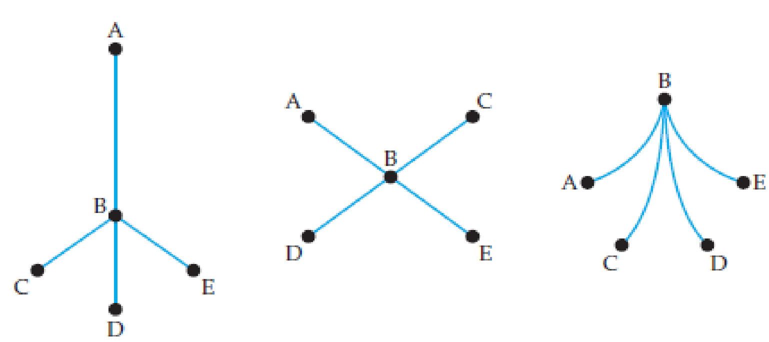
This is a connected graph in which every possible edge is drawn between vertices (without any multiple edges). Such a graph is called a **complete graph**.

Note:

- ***In general, graphs can contain vertices that are not connected to any edges, two or more edges that connect the same vertices (called multiple edges), or edges that loop back to the same vertex*.**
-
- ***That it does not matter whether the edges are drawn straight or curved, and their lengths and positions are not important. Nor is the exact placement of the vertices important.***

Equivalent graphs

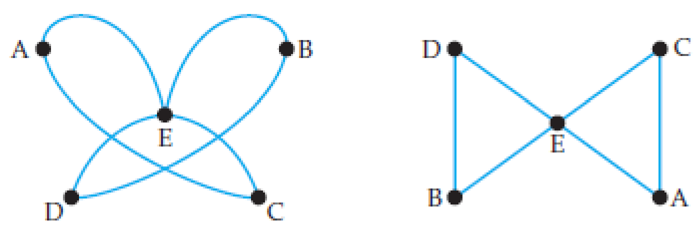
Graphs in which edges form the same connections of vertices in each graph of Figure 7.4.



Note: In each case, vertex B has an edge connecting it to each of the other four vertices, (AB, BC, BD, and BE) and no other edges exist.

Example 7.3: Determine whether the two graphs in Figure 7.5 are equivalent.

Figure 7.5



Solution:

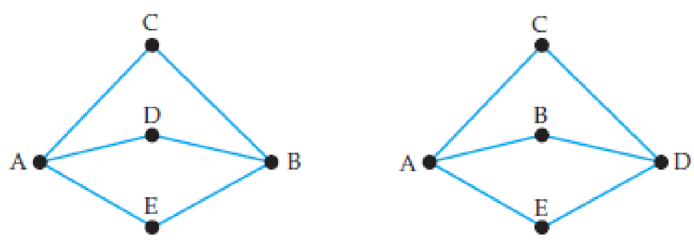
Even though the graphs above have different arrangements of vertices, they are equivalent.

Both graphs have the same edges AC, AE, BD, BE, CE and DE, they represent the same connections and therefore they are equivalent.

BRIEF SEATWORK:

Determine whether the following graphs, Figure 7.6 are equivalent.

Figure 7.6



7.3 EULER PATHS AND CIRCUITS

Path in a graph is a sequence of edges where each one begins where the last one ends.

- It can be thought of as a movement from one vertex to another by traversing edges.

Circuit or Closed Path – A path that ends at the same vertex at which it started.

In the figure, path A–D–F–G–E–B–A is a circuit because it begins and ends at the same vertex, while path A–D–F–G–E–H is not a circuit, as it does not begin and end at the same vertex.

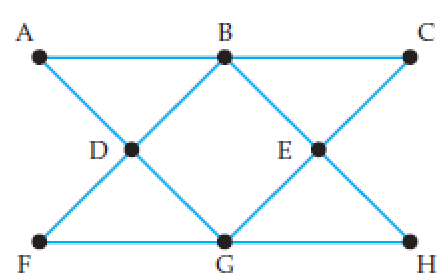


Figure 7.7

EULER PATH is a path that uses every edge but does not use any edge more than once.

EULER PATH THEOREM: A connected graph contains an Euler path **if and only if the graph has two vertices of odd degree with all other vertices of even degree**. Furthermore, every Euler path must start at one of the vertices of add degree and end at the other.

EULER CIRCUIT

A circuit that uses every edge, but never uses the same edge twice, (the path may cross though vertices more than once), and the path begins and ends at the same vertex.

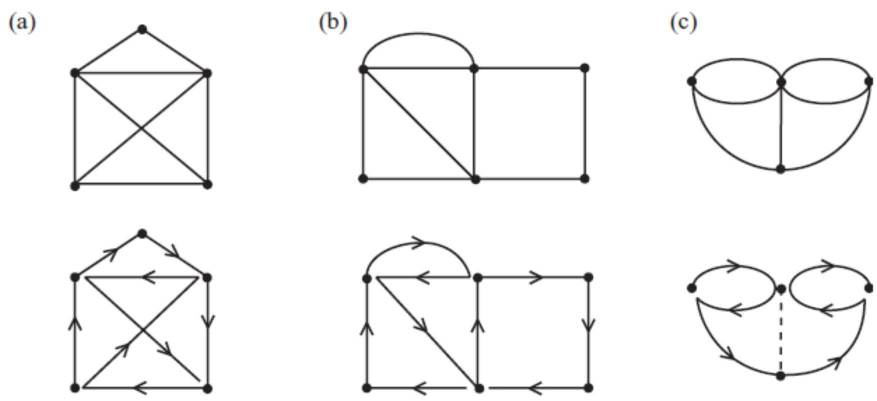
The path B–D–F–G–H– E–C–B–A–D–G–E–B in the above figure is an *Euler circuit*. It begins and ends at the same vertex and uses each edge exactly once.

The path A–B–C–E–H–G–E–B–D–A is *not an Euler circuit*. The path begins and ends at the same vertex but it does not use edges DF, DG, or FG.

The path A–B–C–E–H–G–F–D– A–B–E–G–D–A begins and ends at A but uses edges AB and AD twice so it is *not an Euler circuit*.

Example 7.4: Which is an Euler circuit?

Figure 7.8



Solution:

Fig.(a) It is a path because each edge begins where the last one ended. This is Euler because the path goes over every edge exactly once. This is not a circuit because the path begins at the lower-left vertex and ends at the lower-right vertex.

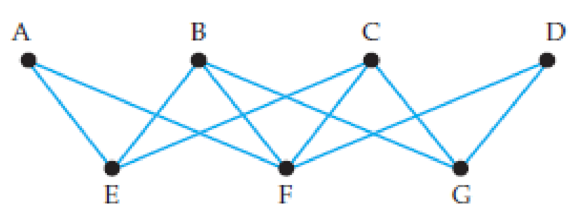
Fig. (b) It is a diagram illustrating an Euler circuit for the graph, every edge is used exactly once and the path begins and ends at the same vertex.

Fig. (c) It illustrates a path that begins at the left vertex and ends at the right vertex. This is not a circuit. This is also not Euler because the vertical edge in the middle is not used.

Note: An Euler path differs from an Euler circuit in their start and end vertices.

BRIEF SEATWORK:

Does the graph shown below have an Euler circuit?



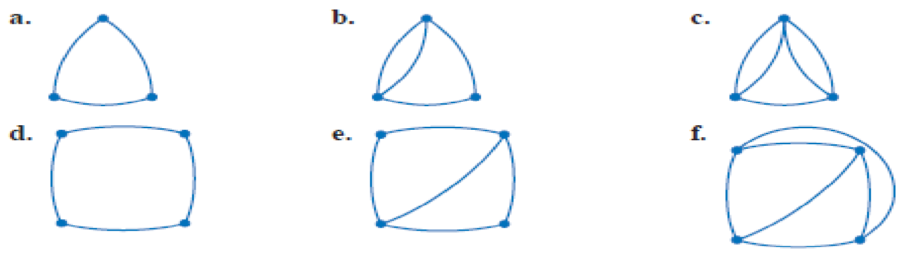
EULER'S CIRCUIT THEOREM*:* Every vertex on a graph with an Euler circuit has an **even degree**, and, conversely, if in a connected graph every vertex has an even degree, then the graph has an Euler circuit.

EULERIAN GRAPH THEOREM: A connected graph is Eulerian if and only if every vertex of the graph is of even degree.

Counting the number of edges, vertices and regions of a network.

Example 7.5: Complete table 7.2 for each of the given networks in figure 7.9.

Figure 7.9



Solution: Table 7.2 shows the no. of edges, vertices and regions of the given networks.

Table 7.2

Graph	Edges (E)	Vertices (V)	Regions (R)
a	3	3	2
b	4	3	3
c	5	3	4
d	4	4	2
e	5	4	3
f	6	4	4

Note:

- A network is said to be ***traversable*** if it can be traced in one sweep without lifting the pencil from the paper and without tracing the same edge more than once.
- Vertices may be passed through more than once.
- The degree of a vertex is the number of edges that meet at a vertex.

Count the number of odd vertices:

- If **there are no odd vertices**, the network **is traversable,** and any point may be a starting point. The point selected will also be the ending point.

- If there is one odd vertex, the network is not traversable. A network cannot have only one starting or ending point without the other.

- If **there are two odd vertices, the network is traversable**; one odd vertex must be a starting point and the other odd vertex must be the ending point.

- If there are more than two odd vertices, the network is not traversable. A network cannot have more than one starting point and one ending point.

Example 7.6: List the number of edges and the degree of each vertex, in Figure 7.9. Find the sum of the degrees of each vertex and tell whether each network is traversable.

Table 7.3

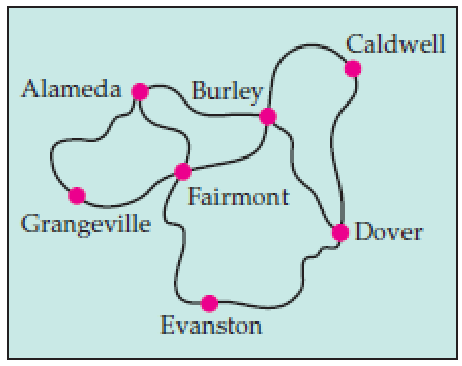
Graph	No. of Edges	Degree of Each Vertex	Sum	Traversable
a	3	2; 2; 2	6	Yes
b	4	3; 2; 3	8	Yes
c	5	4; 3; 3	10	Yes
d	4	2; 2; 2; 2	8	Yes
e	5	2; 3; 2; 3	10	Yes
f	6	3; 3; 3; 3	12	No

NOTE: The sum of the degrees of the vertices in example 7.5 equals twice the number of edges. Each edge must be connected at both ends, so the sum of all those ends must be twice the number of vertices.

7.4 AN APPLICATIONS OF EULER PATH THEOREM

Example 7.7: A photographer would like to travel across all of the roads shown on the following map. The photographer will rent a car that need not be returned to the same city, so the trip can begin in any city. Is it possible for the photographer to design a trip that traverses all the roads exactly once?

Figure 7.10

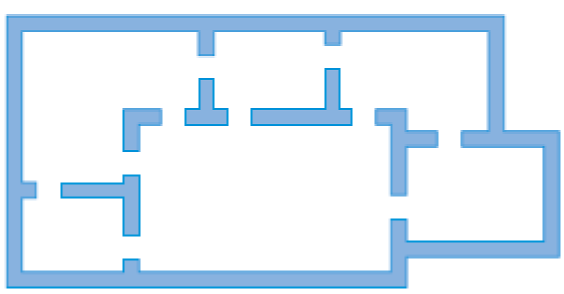


Solution:

Looking at the map of roads as a graph, we see that **a route that includes all of the roads but does not cover any road twice corresponds to an Euler path of the graph**. Notice that **only two vertices are of odd degree**, the cities Alameda and Dover. Thus, we know that an **Euler path exists**, and so it **is possible for the photographer to plan a route that travels each road once**. Because (abbreviating the cities**) A and D are vertices of odd degree,** the photographer must start at one of these cities. With a little experimentation, we find that one Euler path is **A–B–C–D–B–F–A–G–F–E–D.**

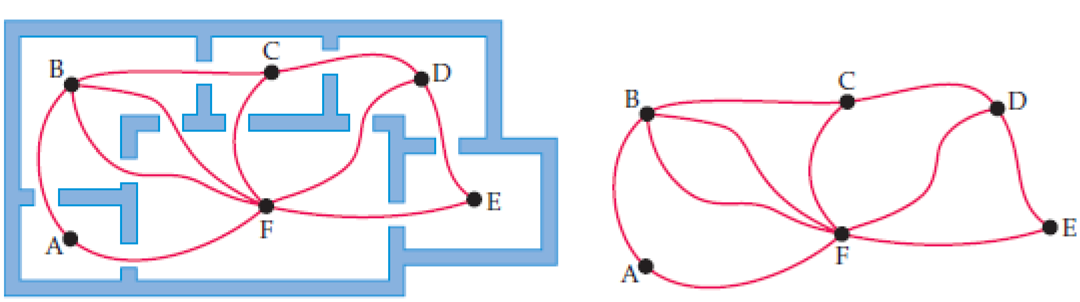
Example 7.8: The floor plan of an art gallery is pictured below. Draw a graph that represents the floor plan, where vertices correspond to rooms and edges correspond to doorways. Is it possible to take a stroll that passes through every doorway without going through the same doorway twice? If so, does it matter whether we return to the starting point?

Figure 7.11



Solution: We can represent the floor plan by a graph if we let a vertex represent each room. Draw an edge between two vertices if there is a doorway between the two rooms, as shown in the figure 7.12.

Figure 7.12

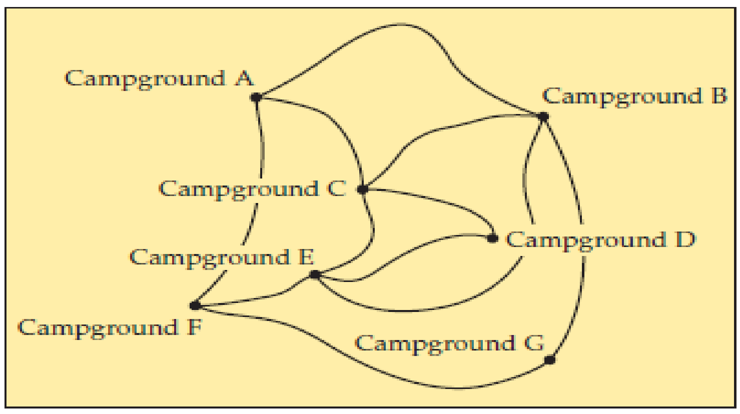


The graph above is equivalent to our floor plan. To tour the gallery and pass through every doorway once, we must find a path in our graph that uses every edge once (and no more). Thus, we are looking for an Euler path. In the graph, two vertices are of odd degree and the others are of even degree. So, we know that an Euler path exists, but not an Euler circuit. Therefore, we cannot pass through each doorway once and only once if we want to return to the starting point, but we can do it if we end up somewhere else. Furthermore, we know we must start at a vertex of odd degree either room C or room D. By trial and error, one such path is C–B–F–B–A–F–E–D–C–F–D.

BRIEF SEATWORK:

A bicyclist wants to mountain through all the trails of a national park. A map of the park is shown below. Because the bicyclist will be dropped off in the morning by friends and picked up in the evening, she does not have a preference for where she begins and ends her ride. It is possible for the cyclist to traverse all the trails without repeating any portions of her trip.

Figure 7.13



Solution:

Consider the campground map as a graph. A route through all the trails that does not repeat any trails corresponds to an Euler path. Because only two vertices (A and F) are of odd degree, we know that an Euler path exists. Furthermore, the path must begin at A and end at F or begin at F and end at A. By trial and error, one Euler path is **A–B–C–D–E–B–G–F–E–C–A–F.**

7.5 OTHER APPLICATIONS OF EULER CIRCUITS

1. Supermarket problem: Set up the shelves in a market or convenience store so that it is possible to enter the store the door and travel in each aisle exactly once (once and only once) and leave the same door.

2. Police patrol problem: Suppose a police car needs to patrol a gated subdivision and would like to enter the gate, cruise all streets exactly once and leave by the same gate.

3. Floor plan problem: Suppose you have a floor plan of a building with security guards who needs to go through the building and lock each door at the end of the day.

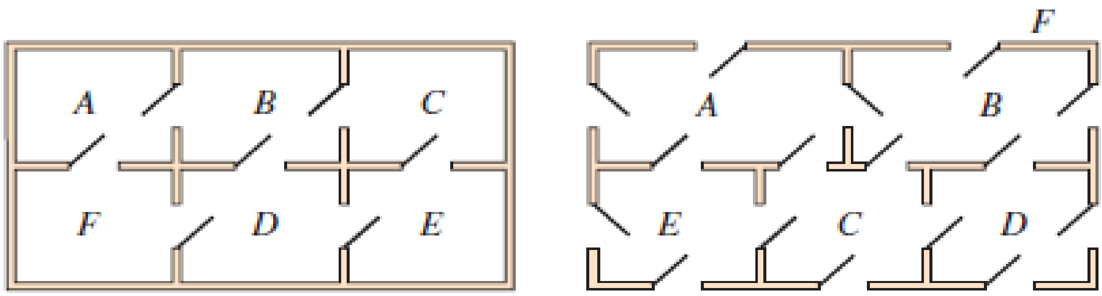
4. Water pipe problem: Suppose you have a network of water pipes, and you wish to inspect the pipeline. Can you pass your hand over each pipe exactly once without lifting your hand from a pipe and without going over some pipe a second time?

7.6 THE FLOOR PLAN PROBLEM

The floor-plan problem is related to the Konigsberg bridge problem. The floor-plan problem involves taking a trip through all the rooms and passing through each door only once. However, there is one important difference between them. The Konigsberg problem requires an Euler circuit, but the floor plan does not. The problem with bridges must end up where the trip started while in the floor-plan problems seek only traversability.

Example 7.9: Consider the floor plan in figure 7.14.

Figure 7.14



a. Six rooms, 7 doors b. Six rooms, 16 doors

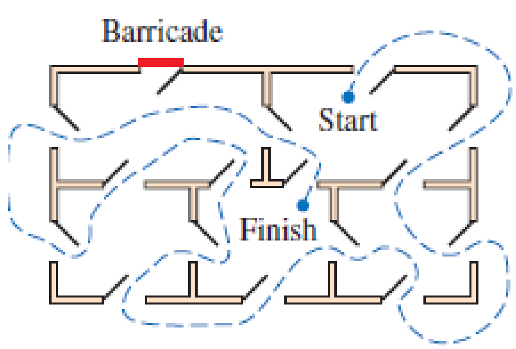
Label the rooms as A, B, C, D, E and F. Take note that in figure **a** rooms A, C, E and F have two doors and rooms B and D have three doors while in figure **b,** there are only 5 rooms, but there's a door leading outside so it must be counted as one room too, so therefore figure **b** has six rooms labelled A, B, C, D, E and F. Rooms A, B and C each have 5 doors, rooms D and E have 4 doors and room F has 9 doors.

Solution: Classifying each room as even or odd according to the number of doors in the room. A solution is possible if there are no rooms with odd number of doors, or if there are exactly two rooms with an odd number of doors.

a.) There are six rooms, and rooms B and D are odd, so this floor plan can be traversed. The solution requires that we begin in either room B or D and finish in the other.

b.) There are six rooms and rooms A, B, C and F are odd (with D and F even). Since there are more than odd rooms, this floor plan cannot be traversed. If one of the doors connecting two of the odd rooms is blocked, then the floor plan can be traversed. See the solution below:

Figure 7.15



Now here's the conclusion to THE BRIDGES OF KONIGSBERG because the graph for the bridges of Konigsberg has four vertices with odd degrees (three with degree 3 and one with degree 5), it follows that there is neither an Euler circuit nor a path. Euler thus announced to the city of Konigsberg that it was impossible to walk through town and cross over every bridge exactly once.

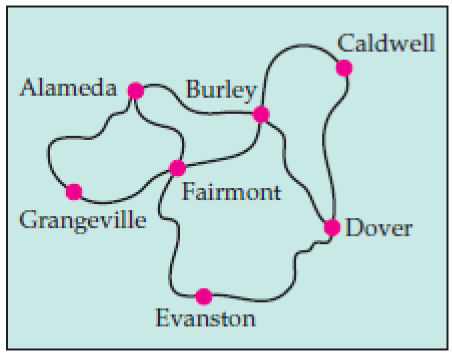
What Euler did?

Euler essentially proved that the graph could not have an Euler circuit. He accomplished this by examining the number of edges that met at each vertex. He made the observation that in order to complete the desired path, every time you approached a vertex you would then need to leave that vertex. If you traveled through that vertex again, you would again need an approaching edge and a departing edge. Thus, for an Euler circuit to exist, the degree of every vertex would have to be an even number. Furthermore, he was able to show that any graph that has even degree at every vertex must have an Euler circuit. Consequently, such graphs are called Eulerian.

7.7 HAMILTONIAN CIRCUIT

A Hamiltonian circuit is a path in a graph that uses each vertex exactly once and returns to the starting vertex. Being a circuit, it must start and end at the same vertex. If a graph has a Hamiltonian circuit, the graph is said to be Hamiltonian.

Figure 7.16



From previous example 7.7, map of cities. If our priority is to visit each city, we could travel along the route **A–B–C–D–E–F–G–A** (abbreviating the cities). This path visits each vertex once and returns to the starting vertex without visiting any vertex twice, this is called *Hamiltonian circuit.*

Note: A **Hamiltonian path** also visits every vertex once with no repeats but does not have to start and end at the same vertex.

7.8 DIRAC'S THEOREM

If **G** is a simple graph with n vertices with **n ≥ 3** such that the degree of every vertex in **G** is at least $\frac{n}{2}$ then **G** has a Hamiltonian circuit."

Example 7.10: Apply Dirac's Theorem

The graph below shows the available flights of a popular airline. (An edge between two vertices in the graph means that the airline has direct flights between the two corresponding cities.) Apply Dirac's theorem to verify that the following graph is Hamiltonian. Then find a Hamiltonian circuit. What does the Hamiltonian circuit represent in terms of flights?

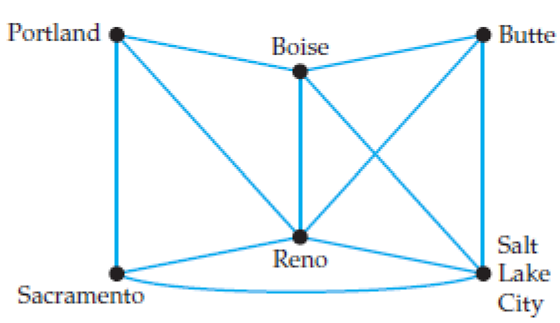


Figure 7.17

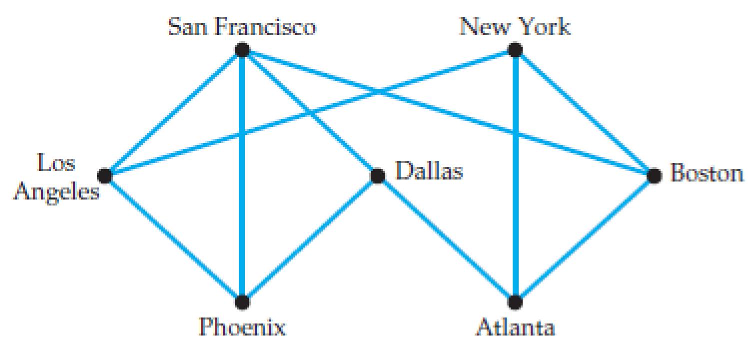
Solution:

There are six vertices in the graph, so n = 6, and every vertex has a degree of at least n/2 = 3. So, by Dirac's theorem, the graph is Hamiltonian. This means that the graph contains a circuit that visits each vertex once and returns to the starting vertex without visiting any vertex twice. By trial and error, one Hamiltonian circuit is **Portland–Boise– Butte–Salt Lake City–Reno–Sacramento–Portland**, which represents a sequence of flights that visits each city and returns to the starting city without visiting any city twice.

BRIEF SEATWORK:

A large law firm has offices in seven major cities. The firm has overnight document deliveries scheduled every day between certain offices. In the graph that follows, an edge between vertices indicates that there is delivery service between corresponding offices. Use Dirac's theorem to answer the following question. Using the firm's existing delivery service, is it possible to route a document to all the offices and return the document to its originating office without sending it through the same office twice?

Figure 7.18



Solution: For this seatwork solution is provided as follows:

The graph has seven vertices, so n = 7 and n/2 = 3.5. Several vertices are of degree less than n/2, so Dirac's theorem does not apply. Still, a routing for the document may be possible. By trial and error, one such route is Los **Angeles–New York–Boston–Atlanta–Dallas–Phoenix–San Francisco–Los Angeles.**

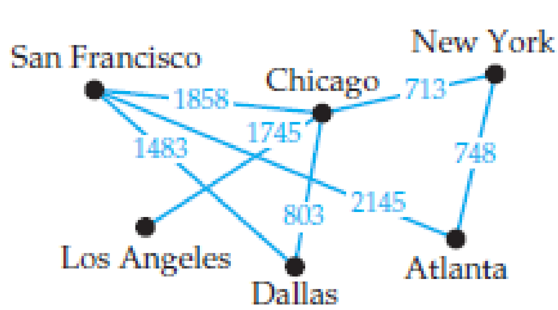
7.9 WEIGHTED GRAPHS

A weighted graph is a graph in which each edge is associated with a value, called a weight. The value can represent any quantity we desire.

A Hamiltonian circuit can identify a route that visits all of the cities represented on a graph, as in the figure below, but there are often a number of different paths we could use. If we are concerned with the distances we must travel between cities, chances are that some of the routes will involve a longer total distance than others. We might be interested in finding the route that **minimizes the total number of miles traveled.** We can represent this situation with a weighted graph.

As shown in the figure below the cities are label on each edge with the number of miles between the corresponding cities. **For each Hamiltonian circuit in the weighted graph, the sum of the weights along the edges traversed gives the total distance traveled along that route**. We can then compare different routes and find the one that requires the shortest total distance.

Figure 7.19



Note that the length of an edge does not necessarily correlate to its weight.

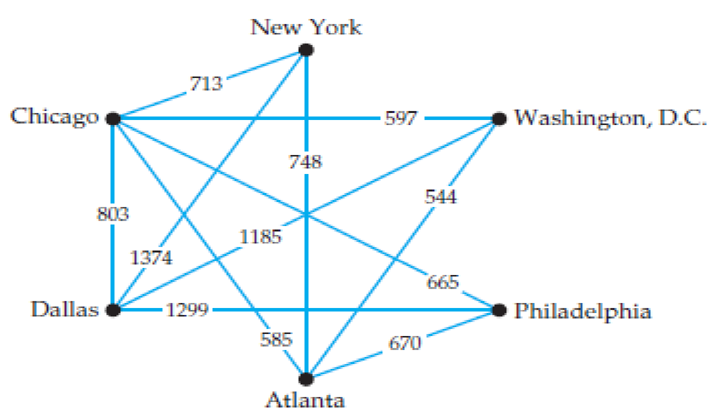
Finding Hamilton Circuits in Weighted Graphs

Example 7.11: The table 7.3 lists the distances in miles between six popular cities that a particular airline fies to. Suppose a traveler would like to start in Chicago, visit the other five cities this airline flies to, and return to Chicago. Find three different routes that the traveler could follow, and find the total flown for each route.

Table 7.3

	Chicago	New York	Washington, D.C.	Philadelphia	Atlanta	Dallas
Chicago	—	713	597	665	585	803
New York	713	—	No flights	No flights	748	1374
Washington, D.C.	597	No flights	—	No flights	544	1185
Philadelphia	665	No flights	No flights	—	670	1299
Atlanta	585	748	544	670	—	No flights
Dallas	803	1374	1185	1299	No flights	—

Solution: The various options will be simpler to analyze if we first organize the information in a graph. Begin by letting each city be represented by a vertex. Draw an edge between two vertices if there is a flight between the corresponding cities and label each edge with a weight that represents the number of miles between the two cities.



A route that visits each city just once corresponds to a Hamiltonian circuit. Beginning at Chicago, one such circuit is Chicago–New York–Dallas–Philadelphia–Atlanta–Washington, D.C.–Chicago. By adding the weights of each edge in the circuit, we see that the total number of miles traveled is

713 + 1374 + 1299 + 670 + 544 + 597 = 5,197 miles

By trial and error, we can identify two additional routes. One is Chicago–Philadelphia–Dallas–Washington, D.C.–Atlanta–New York–Chicago. The total weight of the circuit is

665 + 1299 + 1185 + 54 + 748 + 713 = 5,154 miles

A third route is Chicago–Washington, D.C.–Dallas–New York–Atlanta–Philadelphia–Chicago. The total mileage is

597 + 1185 + 1374 + 748 + 670 + 665 = 5,239 miles

Therefore, the second route has the shortest distance to travel from Chicago to the different cities and back to Chicago.

BRIEF SEATWORK:

A tourist visiting San Francisco is staying at a hotel near Moscone Center. The tourist would like to visit five locations by bus tomorrow and then return to the hotel. The number of minutes spent traveling by bus between locations is given in the table below. (N/A indicates that no convenient bus route is available). Find two different routes for the tourist to follow and compare the total travel times.

Table 7.4

	Moscone Center	Civic Center	Union Square	Embarcadero Plaza	Fisherman's Wharf	Colt Tower
Moscone Center	-	18	6	22	N/A	N/A
Civic Center	18	-	14	N/A	33	N/A
Union Square	6	14	-	24	28	36
Embarcadero Plaza	22	N/A	24	-	N/A	18
Fisherman's Wharf	N/A	33	28	N/A	-	14
Colt Tower	N/A	N/A	36	18	14	-

7.10 THE CHEESY ALGORITHM METHOD

1. Choose a vertex to start at, then travel along the connected edge that has the smallest weight. (If two or more edges have the same weight, pick anyone.)

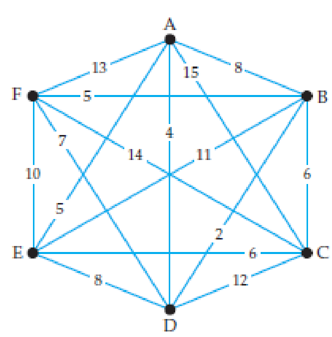
2. After arriving at the next vertex, travel along the edge of smallest weight that connects to a vertex not yet visited. Continue this process until you have visited all vertices.

3. Return to the starting vertex.

Also called greedy algorithm because it allows us to choose the "cheapest" option at every chance we get.

Example 7.12: Use the **Cheesy Algorithm Method** to find a Hamiltonian circuit in the weighted graph shown in figure 7.20

Figure 7.20



Solution: Start at vertex A. The weights of the edges from A are 13, 5, 4, 15, and 8, **the smallest is 4. Connect A to D.**

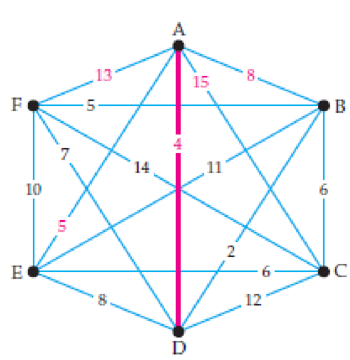


Figure 7.21

At D, the edge with the smallest weight is DB. Connect D to B.

At B, the edge with the smallest weight is BF. Connect B to F.

At F, the edge with the smallest weight, 7, is FD. However, D has already been visited. Choose the next smallest weight, edge FE. Connect F to E.

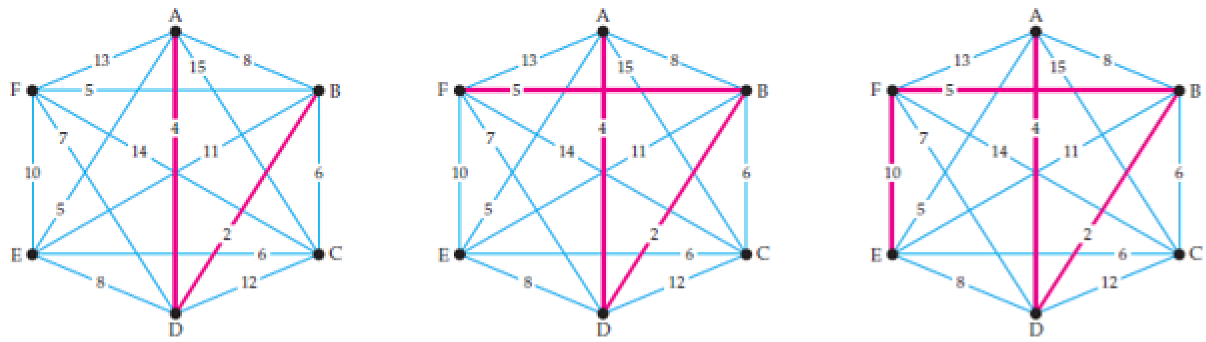
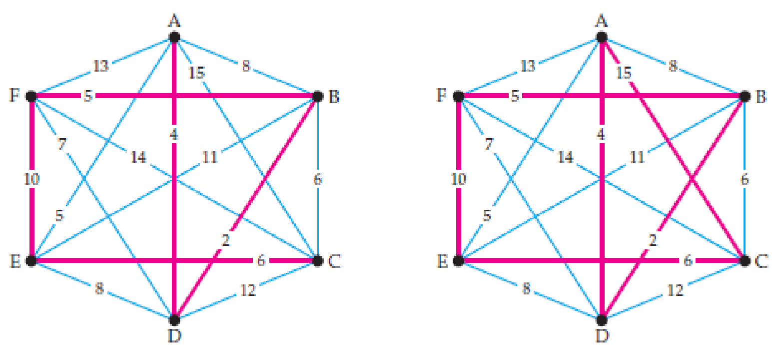


Figure 7.22

At E, the edge with the smallest weight whose vertex has not been visited is C. Connect E to C.

All vertices have been visited, so we are at step 3 of the algorithm. We return to the starting vertex by connecting C to A.

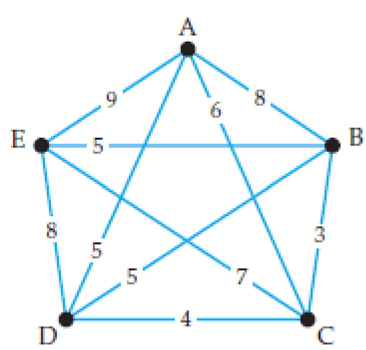


The Hamiltonian circuit is A–D–B–F–E–C–A. The weight of the circuit is

$$4 + 2 + 5 + 10 + 6 + 15 = 42$$

Example 7.13: Use the **greedy algorithm** to find a Hamiltonian circuit starting at vertex A in the weighted graph shown below:

Figure 7.23



Solution:

Starting at vertex A, the edge of smallest weight is the edge to D, with weight 5. From D, take the edge of weight 4 to C, and then the edge of weight 3 to B. From B, the edge of least weight to a vertex not yet visited is the edge to vertex E (with weight 5). This is the last vertex, so we return to A along the edge of weight 9. Thus, the Hamiltonian circuit is A–D–C– B–E–A, with a total weight of 26.

7.11 THE EDGE PICKING ALGORITHM

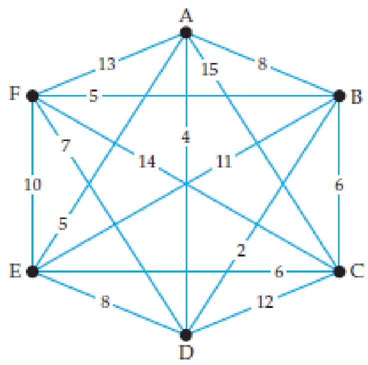
1. Mark the edge of smallest weight in the graph. (If two or more edges have the same weight, pick anyone).

2. Mark the edge of next smallest weight in the graph, as long as it does not complete a circuit and does not add a third marked edge to a single vertex.

3. Continue this process until you can no longer mark any edges. Then mark the final edge that completes the Hamiltonian circuit.

Example 7.14: Use the **edge-picking algorithm** to find a Hamiltonian circuit in the figure below:

Figure 7.24



Solution: We first highlight the edge of smallest weight, namely BD with weight 2.

Figure 7.25

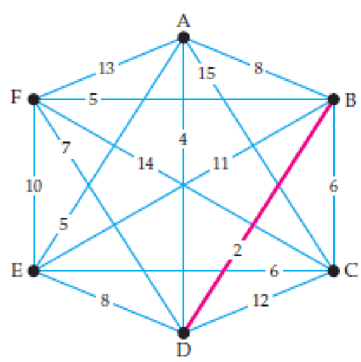
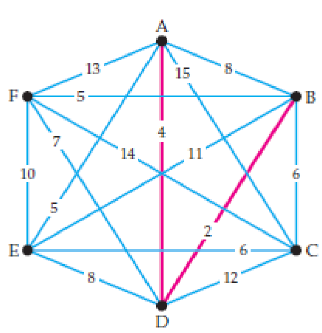


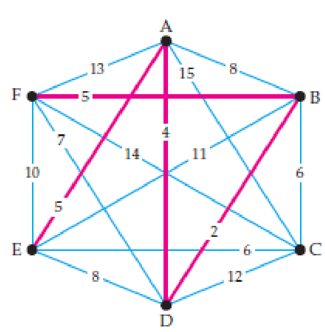
Figure 7.26

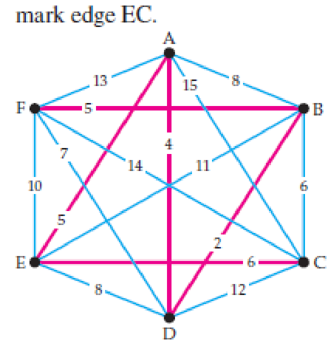
The edge of next smallest weight is AD with weight 4.

The next smallest weight is 5, which appears twice, with edges AE and FB. We can mark both of them.

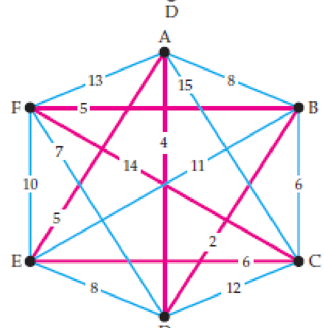
There are two edges of weight 6 (the next smallest weight), BC and EC. We cannot use BC because it would add a third marked edge to vertex B. We mark edge EC.







We are now at step 3 of the algorithm; any edge we mark will either complete a circuit or add a third edge to a vertex. So we mark the final edge to complete the Hamiltonian circuit, edge FC.



Beginning at vertex A, the Hamiltonian circuit is A–D–B–F–C–E–A. (In the reverse direction, an equivalent circuit is A–E–C–F–B–D–A.) The total weight of the circuit is

$$4 + 2 + 5 + 14 + 6 + 5 = 36$$

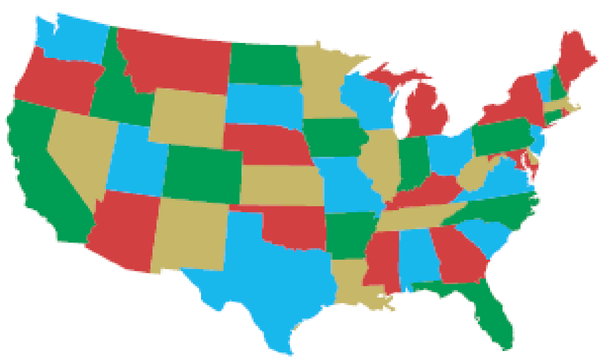
Note in previous examples the two algorithms gave different Hamiltonian circuits, and in this case the edge-picking algorithm gave the more efficient route. Is this the best route? We mentioned before that the algorithms are helpful but there is no known efficient method for finding the very best circuit. In fact, a third Hamiltonian circuit, A–D–F–B–C–E–A in the last problem has a total weight of 33, which is smaller than the weights of both routes given by the algorithms.

7.12 GRAPH COLORING

In the mid-1800s, Francis Guthrie was trying to color a map of the counties of England. So that it would be easy to distinguish the counties, he wanted counties sharing a common border to have different colors. After several attempts, he noticed that four colors were required to color the map, but not more. This observation became known as the four-color problem. (It was not proved until over 100 years later)

Here is a map of the contiguous states of the United States colored similarly. Note that the map has only four colors and that no two states that share a common border have the same color.

Figure 7.27



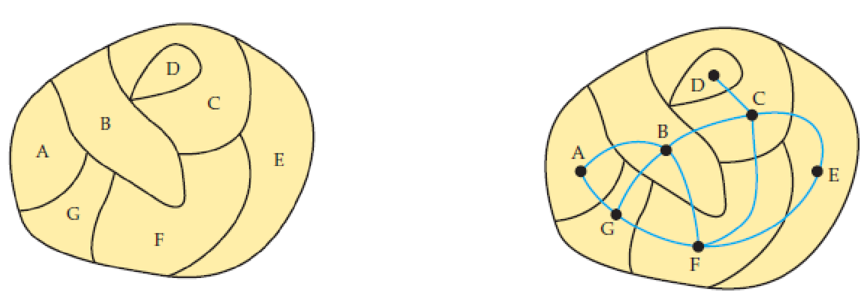
There is a connection between **coloring maps and graph theory**. This connection has many practical applications, from scheduling tasks, to designing computers, to playing Sudoku.

General Guide to Map Coloring:

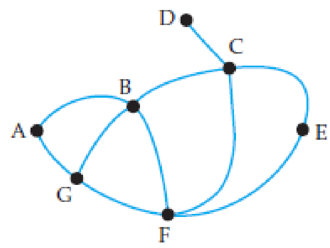
- First, draw a vertex in each country and then connect two vertices with an edge if the corresponding countries are neighbors.
- Now, try to color the vertices of the resulting graph so that no edge connects two vertices of the same color.
- We will need at least two colors, so one strategy is simply to pick a starting vertex, give it a color, and then assign colors to the connected vertices one by one.
- Try to reuse the same colors and use a new color only when there is no other option.

Example 7.15: The map in below shows the countries, labeled as letters, of a continent. Assume that no country is split into more than one piece and countries that touch at just a corner point will not be considered neighbors. Represent each country by a vertex, placed anywhere within the boundary of that country. Then connect two vertices with an edge if the two corresponding countries are neighbors—that is, if they share a common boundary. The result is as follows:

Figure 7.28



Erasing the boundaries of the countries, we are left with the graph in the figure below. The resulting graph is a planar graph, because the edges simply connect neighboring countries.



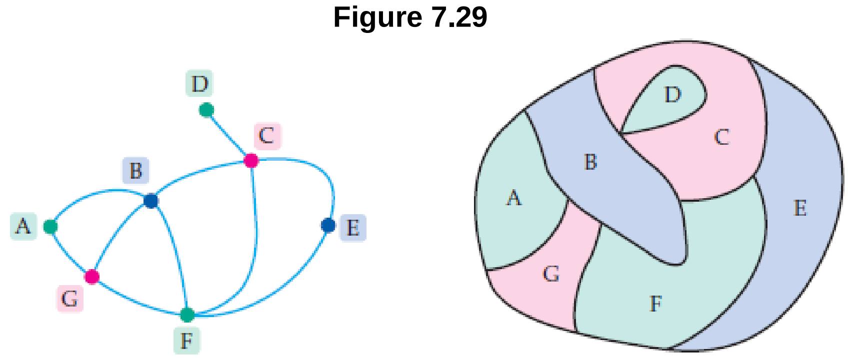
a. Can we give each vertex of the graph a color such that no two vertices connected by an edge share the same color?

b. How many different colors will be required?

Note 1: If a map can be colored by using 4 colors, then the graph is called 4-colorable.

Note 2: If a map can be colored by using 3 colors, then the graph is called 3-colorable.

Figure 7.29

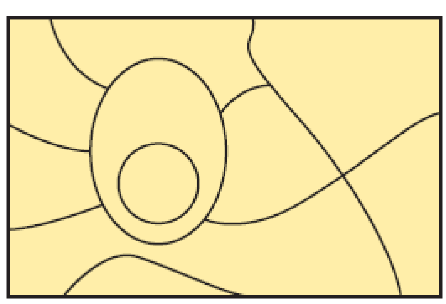


Four-Color Theorem: Every planar graph is 4-colorable.

Example 7.16: Using a Graph to Color a Map

The fictional map below shows the boundaries of countries on a rectangular continent. Represent the map as a graph and find a coloring of the graph using the fewest possible number of colors. Then color the map according to the graph coloring.

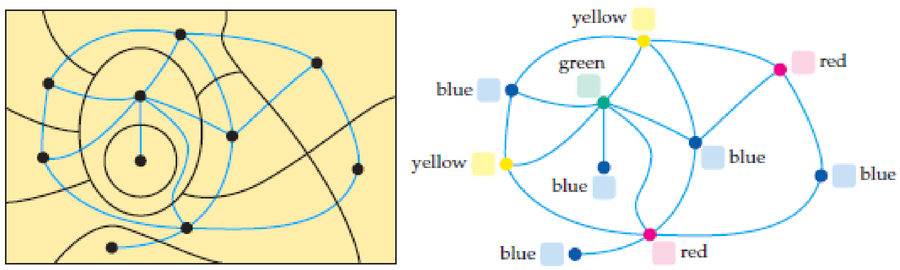
Figure 7.30



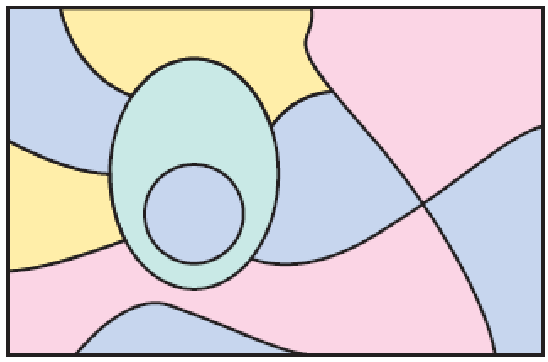
Solution: For this graph we will need four colors. (The four-color theorem guarantees that we will not need more than that.)

To see why we will need four colors, notice that the one vertex colored green in the following figure connects to a ring of five vertices. Three different colors are required to color the five-vertex ring, and the green vertex connects to all these, so it requires a fourth color.

Figure 7.31



Now we color each country the same color as the coresponding vertex.



Applications of Graph Coloring

Determining the chromatic number of a graph and finding a corresponding coloring of the graph can solve a wide assortment of practical problems. One common application is in scheduling meetings or events.

Example 7.17: A Scheduling Application of Graph Coloring

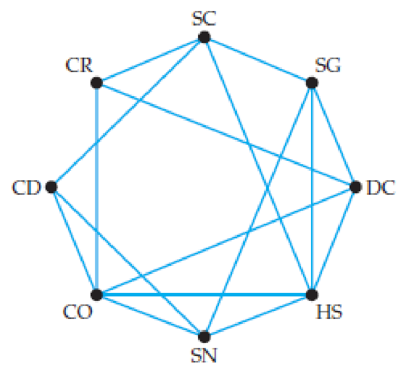
Eight different school clubs want to schedule meetings on the last day of the semester. Some club members, however, belong to more than one of these clubs, so clubs that share members cannot meet at the same time. How many different time slots are required so that all members can attend all meetings? Clubs that have a member in common are indicated with an "X" in the table below.

Table 7.5

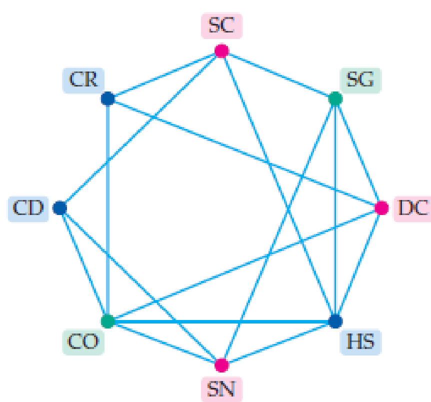
	Ski club	Student government	Debate club	Honor society	Student newspaper	Community outreach	Campus Democrats	Campus Republicans
Ski club	—	X		X			X	X
Student government	X	—	X	X	X			
Debate club		X	—	X		X		X
Honor society	X	X	X	—	X	X		
Student newspaper		X		X	—	X	X	
Community outreach			X	X	X	—	X	X
Campus Democrats	X				X	X	—	
Campus Republicans	X		X			X		—

Solution:

We can represent the given information by a graph. Each club is represented by a vertex, and an edge connects two vertices if the corresponding clubs have at least one common member.



Two clubs that are connected by an edge cannot meet simultaneously. If we let a color correspond to a time slot, then we need to find a coloring of the graph that uses the fewest possible number of colors. The graph is not 2-colorable, because we can find circuits of odd length. However, by trial and error, we can find a 3-coloring. One example is shown below. Thus, the chromatic number of the graph is 3, so we need three different time slots.



Each color corresponds to a time slot, so one scheduling is

First time slot: ski club, debate club, student newspaper

Second time slot: student government, community outreach

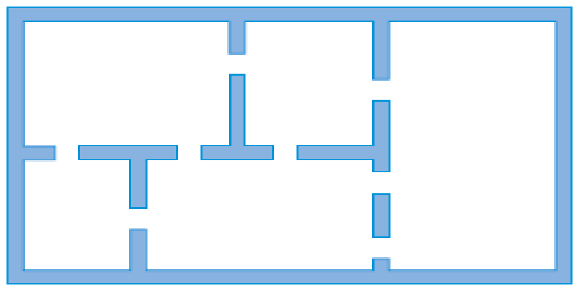
Third time slot: honor society, campus Democrats, campus Republicans

REVIEW EXERCISES 7.0

NAME___________________________________ Score______________ Date ____________
Course, Year & Section_______________Student no. ___________Professor _____________

Solve the following problems as indicated.

1. The floor plan of a warehouse is illustrated below. Use the graph to represent the floor plan and answer the following questions:



a. Is it possible to walk through the warehouse so that you pass through every doorway once but not twice?

b. Does it matter whether you return to the starting point?

2. Six friends are taking a film history course and, because they have procrastinated, need to view several films the night before the final exam. They have rented a copy of each film on DVD, and they have a total of three DVD players in different dorm rooms. If each film is two hours long and they start watching at 8:00 p.m., how soon can they all be finished watching the required films? Create a viewing schedule for the friends.

Film A needs to be viewed by Brian, Chris, and Damon.

Film B needs to be viewed by Allison and Fernando.

Film C needs to be viewed by Damon, Erin, and Fernando.

Film D needs to be viewed by Brian and Erin.

Film E needs to be viewed by Brian, Chris, and Erin.

3. The cost of flying between various European cities is shown in the table below. Use both the greedy algorithm and the edge-picking algorithm to find a low-cot route that visits each city just once and starts and ends in London. Which route is more economical?

	London, England	Berlin, Germany	Paris, France	Rome, Italy	Madrid, Spain	Vienna, Austria
London, England	-	$325	$160	$280	$250	$425
Berlin, Germany	$325	-	$415	$550	$250	$375
Paris, France	$160	$415	=	$495	$215	$545
Rome, Italy	$280	$550	$495	-	$380	$480
Madrid, Spain	$250	$675	$215	$380	-	$730
Vienna, Austria	$425	$375	$545	$480	$480	-

4. Five classes at an elementary school have arranged a tour at a zoo where the students get to feed the animals.

Class 1 wants to feed the elephants, giraffes and hippos.

Class 2 wants to feed the elephants, monkeys and hippos.

Class 3 wants to feed the monkeys, deer and sea lions.

Class 4 wants to feed the parrots, giraffes and polar bears.

Class 5 wants to feed the sea lions, polar bears and hippos.

If the zoo allows animals to be fed only once a day by one class of students, can the tour be accomplished in two days? (Assume that each class will visit the zoo on one day) If not, how many days will be required?

5. Several delis in New York city have arranged deliveries to various buildings at lunch time. The buildings' managements do not want more than one deli showing up at a building in one day, but the delis would like to deliver as often as possible. If they decide to agree on a delivery schedule, how many days will be required before each deli can return to the same building?

Deli A delivers to Empire State building, the Statue of Liberty and Rockefeller Center.

Deli B delivers to the Chrysler building, the Empire State Building and the New York Stock Exchange.

Deli C delivers to the New York Stock Exchange, the American Stock Exchange, and the United Nations building.

Deli D delivers to the New York City Hall, the Chrysler building and Rockefeller Center.

Deli E delivers to Rockefeller Center, New York City Hall Center and the United Nations Building.

6. A toolmaker needs to use one machine to create four different tools. The toolmaker needs to adjust the machine before starting each different tool. However, since the tools have parts in common, the amount of adjustment time required depends on which tool the machine was previously used to create. The table below lists the estimated time (in minutes) required to adjust the machine from making one tool to another. The machine is currently configured for tool A and should be returned to that state when all the tools are finished.

	Tool A	Tool B	Tool C	Tool D
Tool A		25	6	32
Tool B	25		18	9
Tool C	6	18		15
Tool D	32	9	15	

Use the edge-picking algorithm to determine a sequence for creating the tools.

7. Draw graphs (answers will vary) with the following properties:

a. Five edges and three vertices

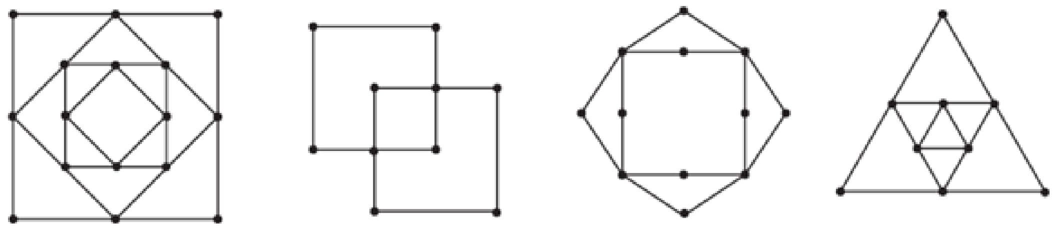
b. Two vertices, each of degree 4

c. Three vertices, each of degree 2

d. Three vertices, with degrees 1, 2, and 3

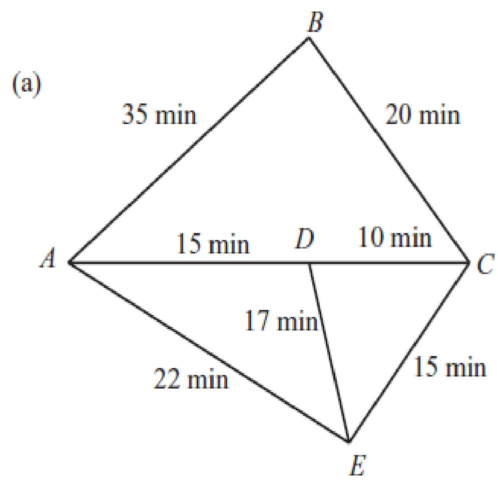
e. Four vertices, with degrees 1, 2, 3, and 4

8. Find a Hamiltonian circuit in the following graphs. If there is no Hamiltonian circuit, find a Hamiltonian path. Say so, if there is no Hamiltonian path.



a. b. c. d.

9. For the weighted graph shown, name all of the Hamiltonian circuits that begin and end at A, and find the total weight.



10. A public works department wants to design a route for their sand truck to drive through the streets of the town during snowstorms. Ideally, the route would go over every street once and end up back at the city garage. In graph theory terms, what are they looking for?

11. Six friends are taking a film history course and because they have procrastinated, need to view several films the night before the final exam. They have rented a copy of each film on DVD and they have a total of three DVD players in different dorm rooms. If each film is two hours long and they start watching at 8:00pm., how soon can they all be finished watching the required films? Create a viewing schedule for the friends.

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Film B needs to be viewed by Allison and Fernando

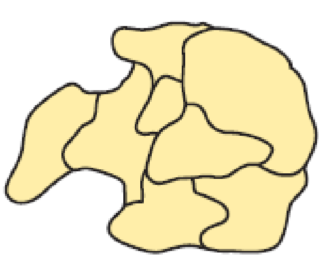
Film C needs to be viewed by Damon, Erin and Fernando

Film D needs to be viewed by Brian and Erin

Film E needs to be viewed by Brian, Chris and Erin

12. Shown below are a fictional map of a province. Represent the map by a graph and find a coloring of the graph that uses the fewest possible number of colors. Then color the map according to the graph coloring you found.

a.



b.