Long Exercise No. 3

Answer Key

Total Score: 105

Note: Less 1 point for every wrong term

1.) $\displaystyle\int \frac{dx}{(x^2+4)^2}$

Solution: $u^2 + a^2$; use $u = a\tan\theta$

$u^2 = x^2 \qquad a^2 = 4$

$u = x \qquad a = 2$

let $u = a\tan\theta$

$x = 2\tan\theta$ 1

$dx = 2\sec^2\theta\, d\theta$ 1

So, $\displaystyle\int \frac{2\sec^2\theta\, d\theta}{[4\tan^2\theta + 4]^2}$ 1

$\displaystyle 2\int \frac{\sec^2\theta\, d\theta}{[4(\tan^2\theta+1)]^2}$

$\displaystyle 2\int \frac{\sec^2\theta\, d\theta}{(4\sec^2\theta)^2}$ 1

$\displaystyle 2\int \frac{\sec^2\theta\, d\theta}{16\sec^4\theta}$

$\displaystyle \frac{1}{8}\int \frac{d\theta}{\sec^2\theta}$ 1

$\displaystyle \frac{1}{8}\int \cos^2\theta\, d\theta$

$\displaystyle \frac{1}{8}\int \left(\frac{1+\cos 2\theta}{2}\right) d\theta$ 1

$\displaystyle \frac{1}{16}\int (1+\cos 2\theta)\, d\theta$

$\displaystyle \frac{1}{16}\int d\theta + \frac{1}{16}\int \cos 2\theta\, d\theta$

$\displaystyle \frac{1}{16}\theta + \frac{1}{16}\cdot\frac{1}{2}\sin 2\theta + C$ 1

$\displaystyle \frac{1}{16}\theta + \frac{1}{32}\sin 2\theta + C$

$\displaystyle \frac{1}{16}\theta + \frac{1}{32}\cdot 2\sin\theta\cos\theta + C$

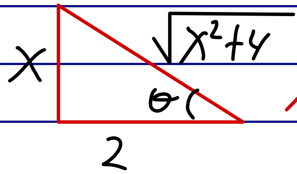
$\displaystyle \frac{1}{16}\theta + \frac{1}{16}\sin\theta\cos\theta + C$ 2

but $x = 2\tan\theta$

$2\tan\theta = x$

$\tan\theta = \dfrac{x}{2}$ 1

$\theta = \arctan\dfrac{x}{2}$ 1



1

Now, 1

$\displaystyle \frac{1}{16}\arctan\frac{x}{2} + \frac{1}{16}\cdot\frac{x}{\sqrt{x^2+4}}\cdot\frac{2}{\sqrt{x^2+4}} + C$

$\displaystyle \boxed{\frac{1}{16}\arctan\frac{x}{2} + \frac{x}{8(x^2+4)} + C}$ 2

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2.) $\displaystyle\int \frac{\sqrt{9-4x^2}}{x^2}\,dx$

Solution: a^2-u^2; use $u = a\sin\theta$

$u^2 = 4x^2 \qquad a^2 = 9$

$u = 2x \qquad a = 3$

let $u = a\sin\theta$

$2x = 3\sin\theta$

$x = \frac{3}{2}\sin\theta$ 1

$dx = \frac{3}{2}\cos\theta\,d\theta$ 1

So, $\displaystyle\int \frac{\sqrt{9-4\left(\frac{9}{4}\sin^2\theta\right)}}{\frac{9}{4}\sin^2\theta}\cdot\frac{3}{2}\cos\theta\,d\theta$ 1

$\displaystyle\frac{4}{9}\cdot\frac{3}{2}\int\frac{\sqrt{9-9\sin^2\theta}}{\sin^2\theta}\cos\theta\,d\theta$

$\displaystyle\frac{2}{3}\int\frac{\sqrt{9(1-\sin^2\theta)}}{\sin^2\theta}\cos\theta\,d\theta$ 1

$\displaystyle\frac{2}{3}\int\frac{3\sqrt{\cos^2\theta}}{\sin^2\theta}\cos\theta\,d\theta$

$\displaystyle 2\int\frac{\cos\theta}{\sin^2\theta}\cdot\cos\theta\,d\theta$ 1

$\displaystyle 2\int\frac{\cos^2\theta}{\sin^2\theta}\,d\theta$

$2\int\cot^2\theta\,d\theta$ 1

$2\int(\csc^2\theta-1)\,d\theta$

$2\int\csc^2\theta\,d\theta - 2\int d\theta$ 1

$-2\cot\theta - 2\theta + c$

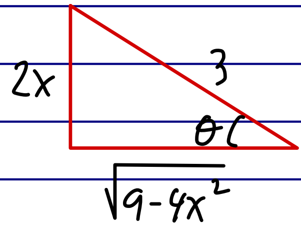
1 1

but $x = \frac{3}{2}\sin\theta$

$\frac{3}{2}\sin\theta = x$

$\sin\theta = \frac{2x}{3}$ 1

$\theta = \arcsin\frac{2x}{3}$ 1



1

Now

$$-2\left(\frac{\sqrt{9-4x^2}}{2x}\right) - 2\arcsin\frac{2x}{3} + c$$ 1

$$\boxed{-\frac{\sqrt{9-4x^2}}{x} - 2\arcsin\frac{2x}{3} + c}$$

2

3.) $\displaystyle\int \frac{(x^2-16)^{3/2}}{x^3}\,dx$

Solution: u^2-a^2; use $u = a\sec\theta$

$u^2 = x^2 \qquad a^2 = 16$

$u = x \qquad a = 4$

let $u = a\sec\theta$

$x = 4\sec\theta$

$dx = 4\sec\theta\tan\theta\,d\theta$

So, $\displaystyle\int \frac{(16\sec^2\theta-16)^{3/2}}{4^3\sec^3\theta}\cdot 4\sec\theta\tan\theta\,d\theta$

$\displaystyle\int \frac{[16(\sec^2\theta-1)]^{3/2}}{16\sec^2\theta}\tan\theta\,d\theta$

$\displaystyle\frac{1}{16}\int \frac{(16\tan^2\theta)^{3/2}}{\sec^2\theta}\tan\theta\,d\theta$

$\displaystyle\frac{1}{16}\int \frac{(4^2)^{3/2}(\tan^2\theta)^{3/2}}{\sec^2\theta}\tan\theta\,d\theta$

$\displaystyle\frac{1}{16}\int \frac{64\tan^3\theta}{\sec^2\theta}\tan\theta\,d\theta$

$\displaystyle 4\int \frac{\tan^4\theta}{\sec^2\theta}\,d\theta$

$\displaystyle 4\int \frac{(\tan^2\theta)^2}{\sec^2\theta}\,d\theta$

$\displaystyle 4\int \frac{(\sec^2\theta-1)^2}{\sec^2\theta}\,d\theta$

$\displaystyle 4\int \frac{\sec^4\theta-2\sec^2\theta+1}{\sec^2\theta}\,d\theta$

$\displaystyle 4\int \left(\sec^2\theta-2+\frac{1}{\sec^2\theta}\right)d\theta$

$\displaystyle 4\int\sec^2\theta\,d\theta-8\int d\theta+4\int\cos^2\theta\,d\theta$

$\displaystyle 4\tan\theta-8\theta+4\int\frac{1+\cos 2\theta}{2}\,d\theta$

$\displaystyle 4\tan\theta-8\theta+2\int(1+\cos 2\theta)\,d\theta$

$\displaystyle 4\tan\theta-8\theta+2\int d\theta+2\int\cos 2\theta\,d\theta$

$4\tan\theta-8\theta+2\theta+\sin 2\theta+C$

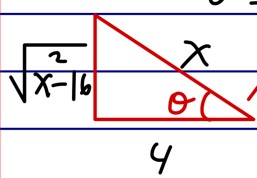
$4\tan\theta-6\theta+2\sin\theta\cos\theta+C$

but $x = 4\sec\theta$

$4\sec\theta = x$

$\sec\theta = \dfrac{x}{4}$

$\theta = \operatorname{arcsec}\dfrac{x}{4}$



Now,

$\displaystyle 4\left(\frac{\sqrt{x^2-16}}{4}\right)-6\operatorname{arcsec}\frac{x}{4}$

$\displaystyle +2\left(\frac{\sqrt{x^2-16}}{x}\right)\left(\frac{4}{x}\right)+C$

$\therefore$

$$\boxed{\sqrt{x^2-16}-6\operatorname{arcsec}\frac{x}{4}+\frac{8\sqrt{x^2-16}}{x^2}+C}$$

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4.) $\int \frac{x^2+9x+2}{(x-1)(x+1)(x+2)}\,dx$

Solution: Case I

so, $\frac{x^2+9x+2}{(x-1)(x+1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2}$ 2 Find A, B and C

$$(x-1)(x+1)(x+2)\left[\frac{x^2+9x+2}{(x-1)(x+1)(x+2)}\right] = \left[\frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2}\right](x-1)(x+1)(x+2)$$

$$x^2+9x+2 = A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1)$$ 2

let $x=1$

$(1)^2+9(1)+2 = A(1+1)(1+2)+0+0$

$1+9+2 = A(2)(3)$

$12 = 6A$

$A = 2$ 2

let $x=-1$

$(-1)^2+9(-1)+2 = 0+B(-1-1)(-1+2)+0$

$1-9+2 = B(-2)(1)$

$-6 = -2B$

$B = 3$ 2

let $x=-2$

$(-2)^2+9(-2)+2 = 0+0+C(-2-1)(-2+1)$

$4-18+2 = C(-3)(-1)$

$-12 = 3C$

$C = -4$ 2

Hence, $\int \frac{x^2+9x+2}{(x-1)(x+1)(x+2)}\,dx = \int \left(\frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2}\right)dx$

$$= \int \left(\frac{2}{x-1} + \frac{3}{x+1} + \frac{-4}{x+2}\right)dx$$ 2

$$= 2\int \frac{dx}{x-1} + 3\int \frac{dx}{x+1} - 4\int \frac{dx}{x+2}$$

$$= \boxed{2\ln|x-1| + 3\ln|x+1| - 4\ln|x+2| + C}$$

1 1 1

5.) $\int \frac{(5y-4)\,dy}{(y^3+4y^2)}$

Solution: $\int \frac{(5y-4)\,dy}{y^2(y+4)}$ Case 2

So, $\frac{5y-4}{y^2(y+4)} = \frac{A}{y} + \frac{B}{y^2} + \frac{C}{y+4}$ 2 Find A, B and C

$$\cancel{y^2}\cancel{(y+4)}\left[\frac{5y-4}{\cancel{y^2}\cancel{(y+4)}}\right] = \left[\frac{A}{y} + \frac{B}{y^2} + \frac{C}{y+4}\right] y^2(y+4)$$

$$5y-4 = A(y)(y+4) + B(y+4) + C(y^2)$$ 2

$$5y-4 = Ay^2 + 4Ay + By + 4B + Cy^2$$

Equating coefficient in Like Powers of y (ECILPOY)

y^2 : $0 = A + C \Rightarrow 0 = 3/2 + C \Rightarrow \underline{C = -3/2}$ 2

y : $5 = 4A + B \Rightarrow 5 = 4A + (-1) \Rightarrow 6 = 4A \Rightarrow \underline{A = 3/2}$ 2

constant : $-4 = 4B \Rightarrow \underline{B = -1}$ 2

using Backward substitution

Now, $\int \frac{(5y-4)}{y^2(y+4)}\,dy = \int \left(\frac{A}{y} + \frac{B}{y^2} + \frac{C}{y+4}\right) dy$

$$= \int \left(\frac{3/2}{y} + \frac{-1}{y^2} + \frac{-3/2}{y+4}\right) dy$$ 2

$$= \frac{3}{2}\int \frac{dy}{y} - \int y^{-2}\,dy - \frac{3}{2}\int \frac{dy}{y+4}$$

$$= \boxed{\frac{3}{2}\ln|y| + \frac{1}{y} - \frac{3}{2}\ln|y+4| + C}$$

1 1 1

6.) $\int \frac{6x^2+3x-2}{x^3+2x^2+2x+1}\,dx$

Solution: $\int \frac{6x^2+3x-2}{(x+1)(x^2+x+1)}\,dx$ Case 3

So, $\frac{6x^2+3x-2}{(x+1)(x^2+x+1)} = \frac{A}{x+1} + \frac{B(2x+1)+C}{x^2+x+1}$ 2

$$\cancel{(x+1)}\cancel{(x^2+x+1)}\left[\frac{6x^2+3x-2}{\cancel{(x+1)}\cancel{(x^2+x+1)}}\right] = \left[\frac{A}{x+1} + \frac{B(2x+1)+C}{x^2+x+1}\right](x+1)(x^2+x+1)$$

$6x^2+3x-2 = A(x^2+x+1) + B(2x+1)(x+1) + C(x+1)$ 2

$6x^2+3x-2 = Ax^2+Ax+A+2Bx^2+3Bx+B+Cx+C$

by ECILPOX: x^2: $6 = A+2B \rightarrow$ Eq. 1

x: $3 = A+3B+C \rightarrow$ Eq. 2

x^0 or constant: $-2 = A+B+C \rightarrow$ Eq. 3

let $x=-1$

$6(-1)^2+3(-1)-2 = A[(-1)^2+(-1)+1]+0+0$

$6-3-2 = A$

$A=1$ 2

Substitute the value of A in Eq. 1

$6 = A+2B$

$6 = 1+2B$

$5 = 2B$

$B = 5/2$ 2

Substitute the values of A and B in Eq. 3

$-2 = A+B+C$

$-2 = 1+5/2+C$

$2(-2) = (1+5/2+C)(2)$

$-4 = 2+5+2C$

$-11 = 2C \Rightarrow C = -11/2$ 2

Now, $\int \frac{6x^2+3x-2}{(x+1)(x^2+x+1)}\,dx = \int\left(\frac{A}{x+1} + \frac{B(2x+1)+C}{x^2+x+1}\right)dx$

$= \int\left(\frac{1}{x+1} + \frac{5/2(2x+1)-11/2}{x^2+x+1}\right)dx$ 2

$$= \int \frac{dx}{x+1} + \frac{5}{2}\int \frac{2x+1}{x^2+x+1}\,dx - \frac{11}{2}\int \frac{dx}{x^2+x+1}$$

$$= \ln|x+1| + \frac{5}{2}\ln|x^2+x+1| - \frac{11}{2}\int \frac{dx}{(x^2+x+1/4)+3/4}$$

$$= \ln|x+1| + \frac{5}{2}\ln|x^2+x+1| - \frac{11}{2}\int \frac{dx}{(x+1/2)^2+3/4}$$

let $u^2 = (x+1/2)^2$ $\qquad a^2 = 3/4$

$u = x + 1/2$ $\qquad a = \frac{\sqrt{3}}{2}$

$du = dx$

$$= \ln|x+1| + \frac{5}{2}\ln|x^2+x+1| - \frac{11}{2}\left[\frac{1}{\sqrt{3}/2}\arctan\frac{x+1/2}{\sqrt{3}/2}\right] + C$$

$$= \ln|x+1| + \frac{5}{2}\ln|x^2+x+1| - \frac{11}{\sqrt{3}}\arctan\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right) + C$$

$$\therefore \boxed{\ln|x+1| + \frac{5}{2}\ln|x^2+x+1| - \frac{11\sqrt{3}}{3}\arctan\frac{\sqrt{3}}{3}(2x+1) + C}$$

1 1 1

7.) $\int \frac{x^5+2x^3-3x}{(x^2+1)^3}\,dx$

Solution: Case 4

So, $\frac{x^5+2x^3-3x}{(x^2+1)^3} = \frac{A(2x)+B}{x^2+1} + \frac{C(2x)+D}{(x^2+1)^2} + \frac{E(2x)+F}{(x^2+1)^3}$ 2

$\cancel{(x^2+1)^3}\left[\frac{x^5+2x^3-3x}{\cancel{(x^2+1)^3}}\right] = \left[\frac{A(2x)+B}{x^2+1} + \frac{C(2x)+D}{(x^2+1)^2} + \frac{E(2x)+F}{(x^2+1)^3}\right](x^2+1)^3$

$x^5+2x^3-3x = A(2x)(x^2+1)^2+B(x^2+1)^2+C(2x)(x^2+1)+D(x^2+1)+E(2x)+F$ 2

$x^5+2x^3-3x = A(2x)(x^4+2x^2+1)+B(x^4+2x^2+1)+2Cx^3+2Cx+Dx^2+D+2Ex+F$

$x^5+2x^3-3x = 2Ax^5+4Ax^3+2Ax+Bx^4+2Bx^3+B+2Cx^3+2Cx+Dx^2+D+2Ex+F$

by ECILPOX: $x^5: 1=2A \Rightarrow \underline{A=1/2}$ 1

$x^4: 0=B \Rightarrow \underline{B=0}$ 1

$x^3: 2=4A+2C \Rightarrow 2=4(1/2)+2C \Rightarrow 0=2C \Rightarrow \underline{C=0}$ 1

$x^2: 0=2B+D \Rightarrow 0=2(0)+D \Rightarrow \underline{D=0}$ 1

$x: -3=2A+2C+2E \Rightarrow -3=2(1/2)+2(0)+2E \Rightarrow -3=1+2E$

$\Rightarrow -4=2E \Rightarrow \underline{E=-2}$ 1

x^0 or constant: $0=B+D+F \Rightarrow 0=0+0+F \Rightarrow \underline{F=0}$ 1

Now, $\int \frac{x^5+2x^3-3x}{(x^2+1)^3}\,dx = \int\left[\frac{A(2x)+B}{x^2+1} + \frac{C(2x)+D}{(x^2+1)^2} + \frac{E(2x)+F}{(x^2+1)^3}\right]dx$

$= \int\left[\frac{1/2(2x)+0}{x^2+1} + \frac{0(2x)+0}{(x^2+1)^2} + \frac{-2(2x)+0}{(x^2+1)^3}\right]dx$ 2

$= \frac{1}{2}\int\frac{2x\,dx}{x^2+1} - 2\int(x^2+1)^{-3}2x\,dx$ 1

by Power Formula

let $u=x^2+1$ $du=2x\,dx$

Hence, $\frac{1}{2}\ln|x^2+1| - \frac{2(x^2+1)^{-2}}{-2} + C$

$\therefore \boxed{\frac{1}{2}\ln|x^2+1| + \frac{1}{(x^2+1)^2} + C}$ 1 1