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Instructional Materials in

MATH 20014

Mathematical Analysis 1

compiled by

DMS Faculty

College of Science

Polytechnic University of the Philippines

2020

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Republic of the Philippines
POLYTECHNIC UNIVERSITY OF THE PHILIPPINES
COLLEGE OF SCIENCE
Department of Mathematics and Statistics

Course Title	:	**MATHEMATICAL ANALYSIS 1**
Course Code	:	**MATH 20014**
Course Credit	:	**4 units**
Pre-Requisite	:	**PRE CALCULUS, BASIC CALCULUS (SHS)**

Course Description :

This is a first course calculus. It covers limits, continuity, derivatives of algebraic and transcendental functions (exponential, logarithmic, trigonometric, hyperbolic and their inverses), applications of derivatives, differentials; antiderivatives, definite integrals, Fundamental Theorem of Calculus, and applications of definite integrals.

Course Outcomes :

1. Define limit of a function and evaluate limit of a function using the different theorems on limits.
2. Define continuity of a function at a point and on an interval.
3. Determine if a function is continuous or discontinuous at a point and on an interval.
4. Find the equation of a tangent line to the curve of a function.
5. Define derivative using the concept of limits.
6. Understand the relationship between differentiability and continuity of a function.
7. Apply the different techniques on differentiation on different types of functions.
8. Apply chain rule in differentiation, differentiating implicit functions and higher-order derivatives.
9. Demonstrate skills in differentiating transcendental functions such as exponential, logarithmic, trigonometric, hyperbolic and their inverses.
10. Solve some applications of derivatives such as equation of a tangent line and normal to a curve, rates of change, velocity and acceleration in rectilinear motion.
11. Apply the derivative tests to determine the intervals where a function is increasing or decreasing, to find the relative extrema, to sketch its graph, and to solve optimization problems.

12. Find higher-order derivatives, point of inflection, and concavity of a function.

13. Apply the derivative test to find maxima/minima of a function, graph functions and solve optimization problems.

14. Define differential and solve problems related to differentials.

15. Evaluate limits that lead to indeterminate forms.

16. Compute antiderivatives of various functions and definite integrals.

17. Solve problems involving areas of regions, volumes of solids of revolution, arc lengths of curve and differential equations.

COURSE CALENDAR/ SCHEDULE

Week	Date	Unit/ Lesson Topic to Study	Activity
1 - 2	Sept. 14, 2020 to Sept. 26, 2020	**Unit 1: Limits and Continuity** Lesson 1.1: Intuitive Idea of a Limit Lesson 1.2: Formal Definition of Limits Lesson 1.3: Limit Theorems Lesson 1.4: One-Sided Limits Lesson 1.5: Limits Involving Infinity Lesson 1.6: Continuity of a Function Lesson 1.7: The Extreme Value and Intermediate Value Theorem Lesson 1.8: The Squeeze Theorem Lesson 1.9: Limits Involving Trigonometric and Exponential Functions	After studying the course material on Unit 1, answer Unit Test 1 **(see pages 46-49)**
3 - 4	Sept. 28, 2020 to Oct. 10, 2020	**Unit 2: Derivatives and Differentiation** Lesson 2.1: The Tangent Line Problem Lesson 2.2: Rate of Change Lesson 2.3: Derivatives and Differentiability of Functions Lesson 2.4: Rules of Differentiation Lesson 2.5: Derivatives of Trigonometric Functions Lesson 2.6: The Chain Rule Lesson 2.7: Derivatives of Other Transcendental Function Lesson 2.8: Implicit Differentiation Lesson 2.9: Higher Order Derivatives Lesson 2.10: L'Hopital's Rule	After studying the course material on Unit 2, answer Unit Test 2 **(see pages 93-97)**
5	Oct. 12, 2020 to Oct. 17, 2020		Answer the Midterm Examination **(See Appendix A)**

6 - 8	Oct. 19, 2020 to Nov. 7, 2020	**Unit 3: Applications of Differentiation** Lesson 3.1: Relative Extrema Lesson 3.2: Finding Extrema of Continuous Functions on a Closed Interval Lesson 3.3: Increasing and Decreasing Functions and the First Derivative Test Lesson 3.4: Concavity and the Second Derivative Test Lesson 3.5: The Rolle's Theorem and the Mean Value Theorem Lesson 3.6: Local Linear Approximation and Differentials Lesson 3.7: Optimization Problems Lesson 3.8: Rectilinear Motions Lesson 3.9: Related Rates	After studying the course material on Unit 3, answer Unit Test 3 **(see pages 152-155)**
9 - 10	Nov. 9, 2020 to Nov. 21, 2020	**Unit 4: Antiderivatives and Indefinite Integrals** Lesson 4.1: Antiderivatives Lesson 4.2: Integration by Substitution Lesson 4.3: Integration Leading to Logarithm Lesson 4.4: Integrals of Trigonometric Functions Lesson 4.5: Integrals Leading to Inverse Trigonometric Functions Lesson 4.6: Integrals of Exponential Functions Lesson 4.7: Integrals of Hyperbolic Functions Lesson 4.8: Integrals Yielding Inverse Hyperbolic Functions	After studying the course material on Unit 4, answer Unit Test 4 **(see pages 181-185)**
11-13	Nov. 23, 2020 to Dec. 12, 2020	**Unit 5: Definite Integrals and Its Applications** Lesson 5.1: Area of the Plane Region using Rectangular Method Lesson 5.2: Fundamental Theorem of Calculus Lesson 5.3: Applications of the Definite Integral	After studying the course material on Unit 5, answer Unit Test 5 **(see pages 228-233)**
14	Dec. 14, 2020 to Dec. 19, 2020		Answer the Final Examination **(see Appendix B)**

COURSE GRADING SYSTEM

Class Standing refers to the average of the assessment task per unit.

Midterm Grade = 70%(Class Standing) + 30%(Midterm Examination)
Final Term Grade = 70%(Class Standing) + 30%(Final Examination)

Final Grade = (Midterm Grade + Final Term Grade) ÷ 2

The **Final SIS Grade** equivalent will be based on the following table according to the approved University Student Handbook.

SIS Grade	Percentage/Equivalent	Description
1.00	97.00 - 100	Excellent
1.25	94.00-96.99	Excellent
1.50	91.00-93.99	Very Good
1.75	88.00-90.99	Very Good
2.00	85.00-87.99	Good
2.25	82.00-84.99	Good
2.50	79.00-81.99	Satisfactory
2.75	77.00-78.99	Satisfactory
3.00	75.00-76.99	Passing
5.00	65.00-74.99	Failure
Inc		Incomplete
W		Withdrawn

Final grades are rounded off to 2 decimal places.

Reference Materials:

• Anton, H., Bivens, I. & Stephen, D., Calculus: Early Transcendentals (10th Edition), John Wiley & Sons, Inc. 2012
• Ayres, Frank Jr. & Mendelson, Elliot, Schaum's Outline of Theory and Problems of Differential and Integral Calculus (3rd Edition)
• Larson, R. & Edwards, B., Calculus (9th Edition), Brooks/Cole, Cengage Learning, 2010
• Leithold, L., The Calculus 7, Harper Collins, 1997
• Peterson, T., Calculus and Analytic Geometry, KEN, Inc., 1968
• Stein, S.K., Calculus and Analytic Geometry (4th Edition), Mc-Graw Hill Book Company, 1987
• Stewart, J., Calculus: Early Transcendentals (8th Edition), Cengage Learning, 2016
• UP Institute of Mathematics, Mathematics 53 Elementary Analysis I (Course Module), UP Diliman – Institute of Mathematics, 2014

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Contents

Unit 1: Limits and Continuity

This unit introduces the concept of limits which is fundamental in studying calculus. We aim to give an appropriate mathematical lnguage to formalize the concept of being 'near' and being 'far'. In particular. we will be interested with limits of functions. Given a function $f(x)$ we ask: How near must x be to a number a so that $f(x)$ is close enough to a number L?

Lesson 1.1: Intuitive Idea of a Limit

Learning Outcomes

At the end of this lesson, you should be able to:

1. Define the limit of a function in an intuitive approach; and
2. Interpret the limit of a function through graphs and table of values.

This section allows us to have an intuitive and a concrete idea of limits of functions which will help us understand the formal definition of limits which is discussed in the section that follows.

We start by considering the function defined by

$$f(x) = \frac{x^2 - 5x + 6}{x - 3}.$$

Clearly, f is not defined at $x = 3$. The domain of f is the set of all real numbers except 3, $\{x \in \mathbb{R} | x \neq 3\}$. We investigate the value of $f(x)$ for values of x which are near 3.

Consider the following table of values.

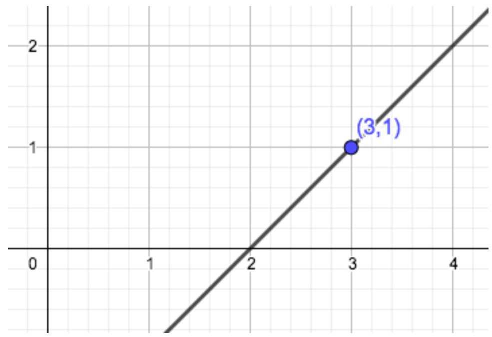
x	2.9	2.99	2.999	2.9999	3	3.0001	3.001	3.01	3.1
$f(x)$	0.9	0.99	0.999	0.9999	undefined	1.0001	1.001	1.01	1.1

The table of values indicates that if x tends closer and closer to 3, the value of $f(x)$ approaches 1.

Also, observe that if $x \neq 3$,

$$f(x) = \frac{x^2 - 5x + 6}{x - 3} = \frac{(x-2)(x-3)}{x-3} = x - 2.$$

Hence, the graph of f is the straight line $y = x - 2$ with the point $(3,\ 1)$ removed from the graph.



We see that the value of $f(x)$ gets closer and closer to 1 as x approaches to 3 even though $f(x)$ is not defined at $x = 3$. This motivates the following definition.

Definition 1: Limit of a Function

Let f be a real-valued function, and, $a, L \in \mathbb{R}$ such that f is defined on some open interval I containing a, except possibly at a. The **limit of** $f(x)$ **as** x **approaches** a **is** L, denoted by

$$\lim_{x \to a} f(x) = L,$$

if the values of $f(x)$ get closer and closer to L as the values of x, not equal to a, get closer and closer to a. If no such $L \in \mathbb{R}$ exists, then we say that the **limit does not exist**.

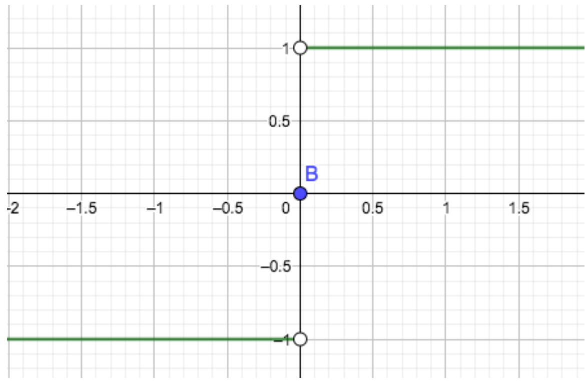
We remark that f need not be defined at $x = a$ for $\lim_{x \to a} f(x)$ to exist, that is, the existence of a limit of a function at a point does not depend whether the function is defined or not defined at that number.

Referring to the previous example, we may write

$$\lim_{x \to 3} \left(\frac{x^2 - 5x + 6}{x - 3} \right) = 1.$$

Example 1. (Limits may not exist). Consider the **signum function** defined by

$$sgn(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$



We investigate the values of $sgn(x)$ near 0. Note that if $0 < x < 0.1$, then $sgn(x) = 1$. On the other hand, if $-0.1 < x < 0$, then $sgn(x) = -1$. Hence, $sgn(x)$ does not approach a single value as x approaches 0. Therefore, the limit of $sgn(x)$ as x approaches 0 does not exist.

Example 2. Let the function $g(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ be given. Consider the following table of values.

x	$\frac{2}{3\pi}$	$\frac{2}{5\pi}$	$\frac{2}{7\pi}$	$\frac{2}{9\pi}$	$\frac{2}{11\pi}$	$\rightarrow$	0
$g(x)$	-1	1	-1	1	-1	$\rightarrow$	does not exist

Since the values of $g(x)$ 'oscillates' to 1 and -1 as values of x approaches 0, we conclude that the limit

$$\lim_{x \to 0} g(x) \text{ does not exist.}$$

Example 3. Let $h(x) = \dfrac{1}{x^2}$. Observe that the value of $\dfrac{1}{x^2}$ increases without bound as x approaches zero. In particular, if we consider the following table of values,

x	-1	-0.1	-0.01	$\rightarrow$	0	$\leftarrow$	0.01	0.1	1
$h(x)$	1	100	10,000	$\rightarrow$	∞	$\leftarrow$	10,000	100	1

we see that the values of $h(x)$ do not tend to any 'finite' number as $x \to 0$. Hence, $\lim_{x \to 0} h(x)$ does not exist.

Lesson 1.2: Formal Definition of Limits

Learning Outcomes

At the end of this lesson, you should be able to:

1. State the formal definition of limits;
2. Verify determined limits using its formal definition; and
3. Obtain precise values suitable for δ given any ϵ for verifying limits.

Definition 2: Formal Definition of Limits

Let f be a real-valued function, a be a real number and I be an open interval containing a. Suppose that f is defined on I except possibly at a. The **limit of $f(x)$ as x approaches a is L**, written

$$\lim_{x\to a} f(x) = L,$$

if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

In other words, for us to say that $f(x)$ has a limit L as x approaches a, we need to make sure that no matter what positive number ϵ we propose, there must correspond a positive number δ such that whenever $0 < |x - a| < \delta$ holds. Observe that

$$0 < |x - a| < \delta \Longleftrightarrow x \in (a - \delta,\, a + \delta) \text{ but } x \neq a$$

and

$$|f(x) - L| < \epsilon \Longleftrightarrow f(x) \in (L - \epsilon,\, L + \epsilon).$$

Hence, the definition implies that for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever x is in the open interval $(a - \delta, a + \delta)$ but $x \neq a$, then the number $f(x)$ can be found in the open interval $(L - \epsilon, L + \epsilon)$.

Thus, if one is tasked to prove the $\lim\limits_{x\to a} f(x) = L$, the proof starts with "Let $\epsilon > 0$ be given." Our goal is to show that there exists $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$.

Example 4. Prove that $\lim\limits_{x\to 2}(5x - 7) = 3$.

Proof. Let $\epsilon > 0$ be given. We need to show that there exists $\delta > 0$ such that whenever $0 < |x - 2| < \delta$, then $|(5x - 7) - 3| < \epsilon$. To find such δ, we observe that

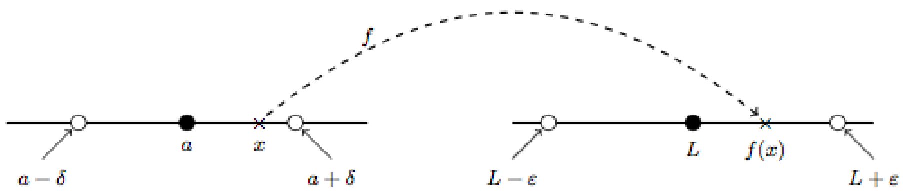
$$|(5x-7)-3| = |5x-10| = 5|x-2|.$$

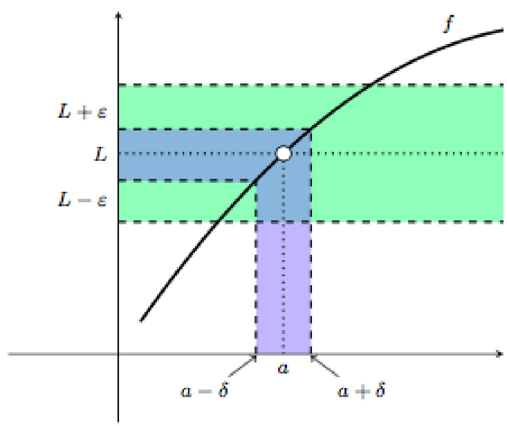
We see that if we take $\delta = \frac{\epsilon}{5}$ and assume that $0 < |x-2| < \delta$, then

$$|(5x-7)-3| = 5|x-2| < 5\delta = 5\left(\tfrac{\epsilon}{5}\right) = \epsilon.$$

Therefore, $\lim\limits_{x\to 2}(5x-7) = 3$. □

The proof tells us that in order to ensure that $|(5x-7)-3|$ is less than ϵ, we have to make $0 < |x-2| < \frac{\epsilon}{5}$. For example, we can make $|(5x-7)-3| < 1$ whenever $0 < |x-2| < 0.2$.





We remark that once a $\delta > 0$ has been found for a given $\epsilon > 0$, any other positive number less than or equal to δ will work, i.e. if $0 < \delta_0 \le \delta$ and $0 < |x-a| < \delta_0 \le \delta$, then $|f(x)-L| < \epsilon$.

Example 5. Show that $\lim\limits_{x\to 1}(x^2+2) = 3$.

Proof. Let $\epsilon > 0$ be given. Observe that

$$|(x^2+2)-3| = |x^2-1| = |(x+1)(x-1)| = |x+1||x-1|.$$

Note that if $|x-1| < 1$, then

$$|x+1| = |(x-1)+2| \le |x-1| + 2 < 1 + 2 = 3.$$

Hence, if we take $\delta = min\{1, \epsilon/3\}$ (this means δ is the smaller between 1 and $\epsilon/3$) and if we assume that $0 < |x-1| < \delta$, then

$$|(x^2+2)-3| = |x+1||x-1| \le 3|x-1| < 3\delta \le 3(\epsilon/3) = \epsilon.$$

Hence,

$$0 < |x-1| < \delta \Longrightarrow |(x^2+2)-3| < \epsilon.$$

Therefore, $\lim_{x\to 1}(x^2+2) = 3$. □

Theorem 1

If $f(x)$ has a limit L as $x \to a$, then it is unique.

Proof. Let ϵ be given. Suppose that $f(x)$ has limits L_1 and L_2 as $x \to a$. Since $\lim_{x\to a} f(x) = L_1$, then there exists $\delta_1 > 0$ such that

$$0 < |x-a| < \delta_1 \Longrightarrow |f(x) - L_1| < \epsilon/2.$$

Also, since $\lim_{x\to a} f(x) = L_2$, then there exists $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \Longrightarrow |f(x) - L_2| < \epsilon/2.$$

Hence, if $\delta = min\{\delta_1, \delta_2\}$ and $0 < |x-a| < \delta$, then

$$|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq |f(x) - L_1| + |f(x) - L_2| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Since ϵ is arbitrarily chosen, it follows that $|L_1 - L_2| = 0$, that is $L_1 = L_2$. □

Lesson 1.3: Limit Theorems

Learning Outcomes

At the end of this lesson, you should be able to:

1. Apply the limit theorems on evaluating limits of polynomial and rational functions; and
2. Investigate on the limit of indeterminate form of type $\frac{0}{0}$.

Theorem 2: Limit Theorem

Let f and g be functions defined on some open interval I containing a, except possibly at a.

1. If $\lim_{x\to a} f(x)$ exists, then it is unique.

2. If $c \in \mathbb{R}$, then $\lim_{x\to c} c = c$.

3. $\lim_{x\to a} x = a$

4. If $c \in \mathbb{R}$ and $n \in \mathbb{N}$, then

a. $\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

b. $\lim_{x\to a}(c \cdot f(x)) = c \cdot \lim_{x\to a} f(x)$

c. $\lim_{x\to a}(f(x) \cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

d. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, where $\lim_{x\to a} g(x) \neq 0$

e. $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$

f. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, where $n > 1$, and, $\lim_{x\to a} f(x) > 0$ if n is even.

The above properties of limits can be proven using the $\epsilon - \delta$ definition of limits. However, we omit the details here and prove only limit theorem (4.a).

Proof. Let $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. let $\epsilon > 0$ be given. The first equality implies that there exists $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \Longrightarrow |f(x) - L| < \epsilon/2.$$

The second equality on the other hand implies that there exists $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \Longrightarrow |g(x) - M| < \epsilon/2.$$

If we set $\delta = min\{\delta_1, \delta_2\}$ and $0 < |x - a| < \delta$, then

$$\begin{aligned}
|[f(x) + g(x)] - [L + M]| &= |[f(x) - L] + [g(x) - M]| \\
&\leq |f(x) - L| + |g(x) - M| \\
&\text{¡}\ \epsilon/2 + \epsilon/2 \\
&= \epsilon
\end{aligned}$$

Hence,

$$\lim_{x \to a}[f(x) + g(x)] = L + M = \lim_{x \to a} f(x) + \lim_{x \to a} g(x).$$

$\square$

Theorem 3

Let f be a either a polynomial or a rational function. If $a \in domf$, then $\lim\limits_{x \to a} f(x) = f(a)$.

Example 6. Evaluate the following limits.

1. $\lim\limits_{x \to -1}(x^3 - 3x^2 + 2x + 1)$
Solution:

$$\begin{aligned}
\lim_{x \to -1}(x^3 - 3x^2 + 2x + 1) &= \lim_{x \to -1} x^3 - \lim_{x \to -1} 3x^2 + \lim_{x \to -1} 2x + \lim_{x \to -1} 1 \\
&= \left(\lim_{x \to -1} x\right)^3 - 3\left(\lim_{x \to -1} x\right)^2 + 2\lim_{x \to -1} x + \lim_{x \to -1} 1 \\
&= (-1)^3 - 3(-1)^2 + 2(-1) + 1 \\
&= -5
\end{aligned}$$

2. $\lim\limits_{x \to 3} \dfrac{x^2 + 3x}{x + 2}$
Solution:

$$\begin{aligned}
\lim_{x \to 3} \frac{x^2 + 3x}{x + 2} &= \frac{\lim\limits_{x \to 3}(x^2 + 3x)}{\lim\limits_{x \to 3}(x + 2)} \\
&= \frac{\lim\limits_{x \to 3} x^2 + \lim\limits_{x \to 3} 3x}{\lim\limits_{x \to 3} x + \lim\limits_{x \to 3} 2} \\
&= \frac{\left(\lim\limits_{x \to 3} x\right)^2 + 3\lim\limits_{x \to 3} x}{\lim\limits_{x \to 3} x + \lim\limits_{x \to 3} 2} \\
&= \frac{(3)^2 + 3(3)}{3 + 2} \\
&= \frac{18}{5}
\end{aligned}$$

3. $\lim_{x\to 3} \sqrt{x^3 - 3x^2 + 9}$

Solution:

$$\begin{aligned}
\lim_{x\to 3} \sqrt{x^3 - 3x^2 + 9} &= \sqrt{\lim_{x\to 3}(x^3 - 3x^2 + 9)} \\
&= \sqrt{\left(\lim_{x\to 3} x\right)^3 - 3\left(\lim_{x\to 3} x\right)^2 + \lim_{x\to 3} 9} \\
&= \sqrt{(3)^3 - 3(3)^2 + 9} \\
&= 3
\end{aligned}$$

Definition 3: Indeterminate Form of Type $\frac{0}{0}$

Let f and g be functions defined on some open interval I containing a, except possibly at a. If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ is called an **indeterminate form of type** $\frac{0}{0}$.

Remark 1

If the $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then to solve the $\lim_{x\to a} \frac{f(x)}{g(x)}$ we need to use the definition that $x \to a$ implies that $x \neq a$ and $x - a \neq 0$.

Example 7. Evaluate the following limits.

1. $\lim_{x\to 3} \frac{x^2 - 9}{x - 3}$

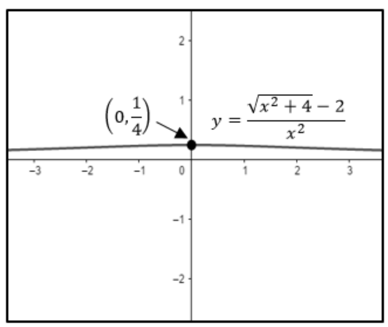
Solution: Since $\lim_{x\to 3}(x^2 - 9) = 0$ and $\lim_{x\to 3}(x - 3) = 0$ we will use the fact that $x \neq 3$ and $x - 3 \neq 0$. Also, $\frac{x-3}{x-3} = 1$. Thus,

$$\begin{aligned}
\lim_{x\to 3} \frac{x^2 - 9}{x - 3} &= \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} \\
&= \lim_{x\to 3}(x + 3) \\
&= \lim_{x\to 3} x + \lim_{x\to 3} 3 \\
&= 6
\end{aligned}$$

2. $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+4}-2}{x^2}$

Solution: Since $\lim\limits_{x\to 0}(\sqrt{x^2+4}-2)=0$ and $\lim\limits_{x\to 0}x^2=0$ we will the fact that $x\neq 0$ and $\dfrac{x}{x}=1$. Thus,

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{x^2+4}-2}{x^2} &= \lim_{x\to 0}\frac{\sqrt{x^2+4}-2}{x^2}\cdot\frac{\sqrt{x^2+4}+2}{\sqrt{x^2+4}+2}\\
&= \frac{(x^2+4)-4}{x^2(\sqrt{x^2+4}+2)}\\
&= \lim_{x\to 0}\frac{x^2}{x^2(\sqrt{x^2+4}+2)}\\
&= \lim_{x\to 0}\frac{1}{\sqrt{x^2+4}+2}\\
&= \frac{\lim\limits_{x\to 0}1}{\sqrt{\lim\limits_{x\to 0}(x^2+4)}+\lim\limits_{x\to 0}2}\\
&= \frac{1}{2+2}\\
&= \frac{1}{4}
\end{aligned}$$



Lesson 1.4: One-Sided Limits

Learning Outcomes

At the end of this lesson, you should be able to:

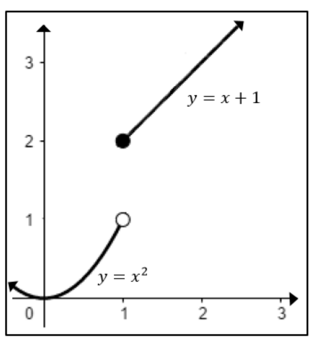
1. Interpret the one-sided limit of a function through graphs and tables of values;
2. Evaluate one-sided limits of functions; and
3. Determine the limit of piecewise functions using one sided limit.

Suppose you are walking in the middle of nowhere and you are looking for a specific location. On your side, as you keep walking, you were arrived to a wrong location and decided to go to the other direction. Eventually, on that way, as time goes by you found your desired place and felt happy! This section will provide us the other concept of the limit of a functions.

Q. Is the limit *always* exist for all functions? **No!**

Consider the piecewise function,

$$f(x) = \begin{cases} x^2 & ; \text{if } x < 1 \\ x+1 & ; \text{if } x \geq 1 \end{cases}$$



By computing its limit as x approaches to 1 i.e.,

$$\lim_{x \to 1} f(x)$$

we will do a special case. Observe that if $x < 1$ all the values of x are came from the left of 1 and is defined only for $y = x^2$. On the other side, that is $x \geq 1$ are all values from the right of 1 and allowed only for $y = x + 1$.

At $x < 1$,

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} x^2 = 1^2 = 1 \tag{1.1}$$

At $x \geq 1$,

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} (x+1) = 1 + 1 = 2 \tag{1.2}$$

Equations (1.1) and (1.2) are called ***left-hand limit*** and ***right-hand limit***, respectively.

Definition 4: One-Sided Limits

et f be a real-valued function, and, $a, L \in \mathbb{R}$ such that f is defined on some open interval I containing a, except possibly at a.
The **limit of** $f(x)$ **as** x **approaches** a **from the left is** L, denoted by

$$\lim_{x \to a^-} f(x) = L,$$

if the values of $f(x)$ get closer and closer to L as the values of x, that are less than a, get closer and closer to a.
The **limit of** $f(x)$ **as** x **approaches** a **from the right is** L, denoted by

$$\lim_{x \to a^+} f(x) = L,$$

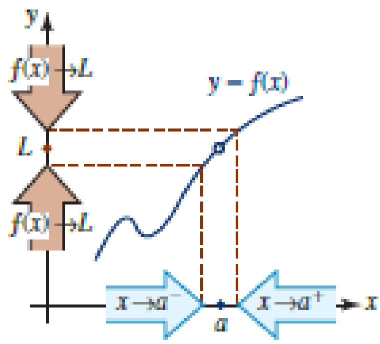
if the values of $f(x)$ get closer and closer to L as the values of x, that are greater than a, get closer and closer to a.

Definition 5

Let f be a real-valued function such that f is defined on some open interval I containing a, except possibly at a. Suppose that $\lim_{x \to a} f(x) = 0$.
If $f(x) > 0$ for all x approaching a, then we say that f **approaches 0 through positive values** and write $f(x) \to 0^+$.
If $f(x) < 0$ for all x approaching a, then we say that f **approaches 0 through negative values** and write $f(x) \to 0^-$.

Theorem 4

Let f be a real-valued function such that f is defined on some open interval I containing a, except possibly at a. Then, $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$.



Example 8. Evaluate the following limits of the indicated function.

1. Let $f(x) = \begin{cases} 2x+1 & ; \text{if } x < 3 \\ 10-x & ; \text{if } x \geq 3 \end{cases}$

   Determine

   1. $\lim\limits_{x\to 3^+} f(x)$
      Solution:
$$\begin{aligned} \lim_{x\to 3^+} f(x) &= \lim_{x\to 3^+} (10-x) \\ &= 10-3 \\ &= 7 \end{aligned}$$

   2. $\lim\limits_{x\to 3^-} f(x)$
      Solution:
$$\begin{aligned} \lim_{x\to 3^-} f(x) &= \lim_{x\to 3^+} (2x+1) \\ &= 2(3)+1 \\ &= 7 \end{aligned}$$

   3. $\lim\limits_{x\to 3} f(x)$
      Solution:
$$\lim_{x\to 3} f(x) = 7$$

2. Let $f(x) = \dfrac{x-5}{|x-5|}$. Determine

   (a) $\lim\limits_{x\to 5^+} f(x)$
   Solution: If $x \to 5^+$, then $5 < x < 5.1$. Thus, $0 < x-5 < 0.1$ and $|x-5| = x-5$
$$\begin{aligned} \lim_{x\to 5^+} f(x) &= \lim_{x\to 5^+} \frac{x-5}{x-5} \\ &= \lim_{x\to 5^+} 1 \\ &= 1 \end{aligned}$$

   (b) $\lim\limits_{x\to 5^-} f(x)$
   Solution: If $x \to 5^-$, then $4.9 < x < 5$. Thus, $-0.1 < x-5 < 0$ and $|x-5| = -(x-5)$

$$\begin{aligned}\lim_{x\to5^+} f(x) &= \lim_{x\to5^+} \frac{x-5}{-(x-5)}\\ &= \lim_{x\to5^+}(-1)\\ &= \text{-1}\end{aligned}$$

(c) $\lim_{x\to5} f(x)$
Solution: Since $\lim_{x\to5^+} f(x) \neq \lim_{x\to5^-} f(x)$, then $\lim_{x\to5} f(x)$ does not exist.

Remark 2

If $f(x)\to 0^+$ as $x\to a$, and n is even, then $\lim_{x\to a}\sqrt[n]{f(x)} = 0$.
If $f(x)\to 0^-$ as $x\to a$, and n is even, then $\lim_{x\to a}\sqrt[n]{f(x)}$ does not exist.

Example 9. Let $f(x) = \sqrt{3-x} = \sqrt{g(x)}$. Note that $g(x)\to 0^+$ as $x\to 3^-$, so

$$\lim_{x\to3^-}\sqrt{3-x} = 0.$$

On the other hand, $g(x)\to 0^-$ as $x\to 3^+$, so

$$\lim_{x\to3^+}\sqrt{3-x} \text{ does not exist.}$$

Example 10. Consider $g(x) = \sqrt{2x^2+x-1}$. Determine

1. $\lim_{x\to-1^-}\sqrt{2x^2+x-1}$
   Solution: Since $2x^2+x-1 = (2x-1)(x+1)$, and as $x\to-1^-$ implies that $-1.1 < x < -1$. Also, $x\to-1^-$ implies that $2x-1\to-3$ and $x+1\to0^-$, so $2x^2+x-1\to0^+$. By previous remark,
   $$\lim_{x\to-1^-}\sqrt{2x^2+x-1} = 0.$$

2. $\lim_{x\to-1^+}\sqrt{2x^2+x-1}$
   Solution: Since $2x^2+x-1 = (2x-1)(x+1)$, and as $x\to-1^+$ implies that $-1 < x < -0.9$. Also, $x\to-1^+$ implies that $2x-1\to-3$ and $x+1\to0^+$, so $2x^2+x-1\to0^-$. By previous remark,
   $$\lim_{x\to-1^+}\sqrt{2x^2+x-1} \text{ does not exist.}$$

3. $\lim_{x\to\frac{1}{2}^-}\sqrt{2x^2+x-1}$
   Solution: Since $2x^2+x-1 = (2x-1)(x+1)$, and as $x\to\frac{1}{2}^-$ implies that $0.4 < x < 0.5$. Also, $x\to\frac{1}{2}^-$ implies that $2x-1\to0^-$ and $x+1\to\frac{3}{2}$, so $2x^2+x-1\to0^-$. By previous remark,

$$\lim_{x \to \frac{1}{2}^-} \sqrt{2x^2 + x - 1} \text{ does not exist.}$$

4. $\displaystyle\lim_{x \to \frac{1}{2}^+} \sqrt{2x^2 + x - 1}$

   Solution: Since $2x^2 + x - 1 = (2x - 1)(x + 1)$, and as $x \to \frac{1}{2}^+$ implies that $0.5 < x < 0.6$. Also, $x \to \frac{1}{2}^+$ implies that $2x - 1 \to 0^+$ and $x + 1 \to \frac{3}{2}$, so $2x^2 + x - 1 \to 0^+$. By previous remark,

$$\lim_{x \to \frac{1}{2}^+} \sqrt{2x^2 + x - 1} = 0.$$

Example 11. Let

$$f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x + 1}, & \text{if } x < 0 \\ \sqrt{4 - x}, & \text{if } 0 \le x \le 4 \\ x^2 - 5x + 4, & \text{if } x \ge 4 \end{cases}$$

Find

1. $\displaystyle\lim_{x \to -1^-} f(x)$

   Solution: Since $x \to -1^-$ implies that $-1.1 < x < -1$. So, $f(x) = \dfrac{x^2 - x - 2}{x + 1}$. Thus,

$$\begin{aligned} \lim_{x \to -1^-} f(x) &= \lim_{x \to -1^-} \frac{x^2 - x - 2}{x + 1} \\ &= \lim_{x \to -1^-} \frac{(x + 1)(x - 2)}{x + 1} \\ &\quad \text{by previous remark} \\ &= \lim_{x \to -1^-} (x - 2) \\ &= -1 - 2 \\ &= -3 \end{aligned}$$

2. $\displaystyle\lim_{x \to 0^-} f(x)$

   Solution: Since $x \to 0^-$ implies that $-0.1 < x < 0$. So, $f(x) = \dfrac{x^2 - x - 2}{x + 1}$. Thus,

$$\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} \frac{x^2 - x - 2}{x + 1} \\ &= \frac{(0)^2 - 0 - 2}{0 + 1} \\ &= \frac{-2}{1} \\ &= -2 \end{aligned}$$

3. $\lim_{x\to 0^+} f(x)$

   Solution: Since $x \to 0^+$ implies that $0 < x < 0.1$. So, $f(x) = \sqrt{4-x}$. Thus,

$$\begin{aligned} \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} \sqrt{4-x} \\ &= \sqrt{\lim_{x\to 0^+}(4-x)} \\ &= \sqrt{4-0} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

4. $\lim_{x\to 4^+} f(x)$

   Solution: Since $x \to 4^+$ implies that $4 < x < 4.1$. So, $f(x) = x^2 - 5x + 4$. Thus,

$$\begin{aligned} \lim_{x\to 4^+} f(x) &= \lim_{x\to 4^+} (x^2 - 5x + 4) \\ &= (4)^2 - 5(4) + 4 \\ &= 0 \end{aligned}$$

Lesson 1.5: Limits Involving Infinity

Learning Outcomes

At the end of this lesson, you should be able to:

1. Interpret infinite limits and limits at infinity of a function through graphs and table of values; and
2. Evaluate infinite limits and limits at infinity of functions.

We now take a look at the function $f(x) = \dfrac{1}{x^2}$. Note that as x approaches 0 through the positive values, the value of $f(x)$ increases without bound. Also, as x approaches 0 through the negative values, the value of $f(x)$ increases without bound as well. In fact,

$$\text{if } 0 < |x| < 0.1, \text{ then } f(x) > 100.$$

In general, if $\delta > 0$ is given, then

$$0 < |x| < \delta \Longrightarrow f(x) > \frac{1}{\delta^2}.$$

Observe that, if δ is very small, then $\dfrac{1}{\delta^2}$ is very large. Hence, $\lim\limits_{x\to 0} f(x)$ does not exist. We say that $f(x)$ approaches infinity as x approaches 0. This motivates the following definitions.

Definition 6: Infinite Limits

Let f be a real-valued function defined on some open interval I containing a, except possibly at a.

The **limit of $f(x)$ as x approaches a is positive infinity**, denoted by

$$\lim_{x\to a} f(x) = +\infty,$$

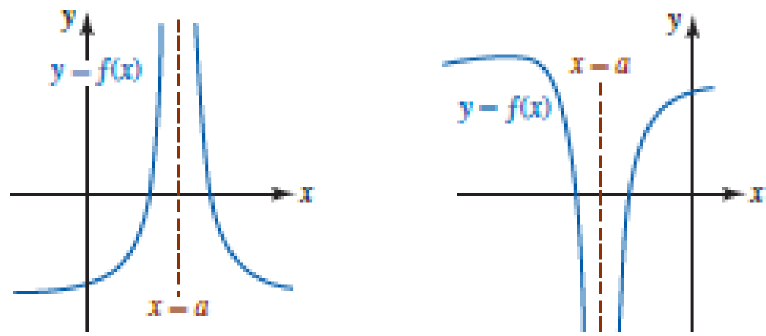
if the value of $f(x)$ increases without bound as the values of x, not equal to a, get closer and closer to a.

The **limit of $f(x)$ as x approaches a is negative infinity**, denoted by

$$\lim_{x\to a} f(x) = -\infty,$$

if the value of $f(x)$ decreases without bound as the values of x, not equal to a, get closer and closer to a.

The figure below shows the illustration of $\lim\limits_{x \to a} f(x) = +\infty$ and $\lim\limits_{x \to a} f(x) = -\infty$.



The figure above shows the unbounded behavior of a function f as x approaches a from one side.

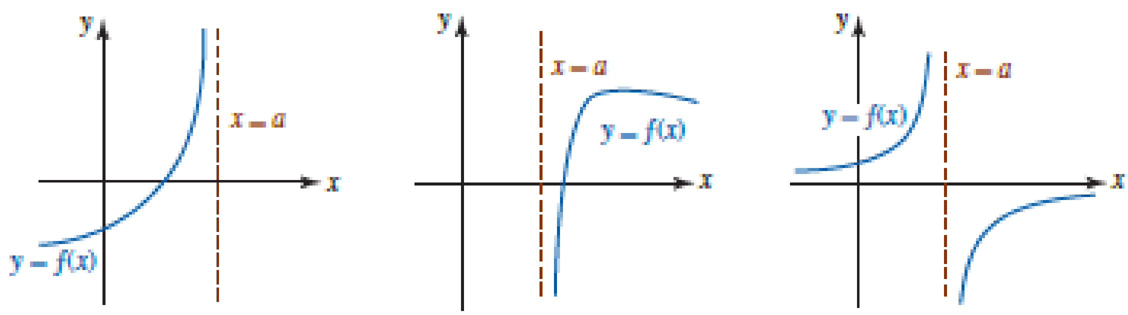


Figure 1.1: Three more types of limit. (a) $\lim\limits_{x \to a^-} f(x) = +\infty$ (b) $\lim\limits_{x \to a^+} f(x) = -\infty$ (c) $\lim\limits_{x \to a^-} f(x) = +\infty$ and $\lim\limits_{x \to a^+} f(x) = -\infty$

In general, any limit of the six types

$$
\begin{aligned}
\lim_{x \to a^-} f(x) &= -\infty, & \lim_{x \to a^-} f(x) &= \infty \\
\lim_{x \to a^+} f(x) &= -\infty, & \lim_{x \to a^+} f(x) &= \infty \\
\lim_{x \to a} f(x) &= -\infty, & \lim_{x \to a} f(x) &= \infty
\end{aligned}
\tag{1.3}
$$

is called an **infinite limit**.

Theorem 5

Let f and g be real-valued functions defined on some open interval I containing a, except possibly at a. Suppose that $c \in \mathbb{R}^*$, $\lim\limits_{x\to a} f(x) = c$ and $\lim\limits_{x\to a} g(x) = 0$.

1. If $c > 0$,

- **i**. and $g(x) \to 0^+$ as $x \to a$, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = +\infty$.
- **ii**. and $g(x) \to 0^-$ as $x \to a$, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = -\infty$.

2. If $c < 0$.

- **i**. and $g(x) \to 0^+$ as $x \to a$, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = -\infty$.
- **ii**. and $g(x) \to 0^-$ as $x \to a$, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = +\infty$.

Example 12. Evaluate $\lim\limits_{x\to -2} \dfrac{5x-7}{x^2+4x+4}$.

Solution: Since $\lim\limits_{x\to -2}(x^2+4x+4) = 0$ and $\lim\limits_{x\to -2}(5x-7) = -17$. Also, as $x \to -2$, the value of $(x^2+4x+4) \to 0^+$. Then,

$$\begin{aligned} \lim_{x\to -2} \frac{5x-7}{x^2+4x+4} &= \lim_{x\to -2} \frac{5x-7}{(x+2)^2} \\ &= -\infty \end{aligned}$$

Example 13. Let $f(x) = \dfrac{2}{x^2+5x+6}$. Determine

1. $\lim\limits_{x\to -2^+} f(x)$

Solution: Since $x^2+5x+6 = (x+2)(x+3)$, $\lim\limits_{x\to -2^+}(x^2+5x+6) = 0$ and $\lim\limits_{x\to -2^+} 2 = 2$, we will apply **Theorem 5**. Hence, $x \to -2^+$ implies that $-2 < x < -1.9$. Also, as $x \to -2^+$ the value of $x+2 \to 0^+$ and $x+3 \to 0^+$. Thus, $x^2+5x+6 \to 0^+$ and

$$\begin{aligned} \lim_{x\to -2^+} f(x) &= \lim_{x\to -2^+} \frac{2}{(x+2)(x+3)} \\ &= +\infty \end{aligned}$$

2. $\lim\limits_{x\to -2^-} f(x)$

Solution: Since $x^2+5x+6 = (x+2)(x+3)$, $\lim\limits_{x\to -2^+}(x^2+5x+6) = 0$ and $\lim\limits_{x\to -2^-} 2 = 2$, we will apply **Theorem 5**. Hence, $x \to -2^-$ implies that $-2.1 < x < -2$. Also, as $x \to -2^-$ the value of $x+2 \to 0^-$ and $x+3 \to 0^+$. Thus, $x^2+5x+6 \to 0^-$ and

$$\begin{aligned} \lim_{x\to -2^-} f(x) &= \lim_{x\to -2^-} \frac{2}{(x+2)(x+3)} \\ &= -\infty \end{aligned}$$

3. $\lim\limits_{x\to -3^+} f(x)$

Solution: Since $x^2+5x+6=(x+2)(x+3)$, $\lim\limits_{x\to -3^+}(x^2+5x+6)=0$ and $\lim\limits_{x\to -3^+} 2=2$, we will apply **Theorem 5**. Hence, $x\to -3^+$ implies that $-3<x<-2.9$. Also, as $x\to -3^+$ the value of $x+2\to 0^-$ and $x+3\to 0^+$. Thus, $x^2+5x+6\to 0^-$ and

$$\begin{aligned}\lim_{x\to -3^+} f(x) &= \lim_{x\to -3^+}\frac{2}{(x+2)(x+3)}\\ &= -\infty\end{aligned}$$

4. $\lim\limits_{x\to -3^-} f(x)$

Solution: Since $x^2+5x+6=(x+2)(x+3)$, $\lim\limits_{x\to -3^-}(x^2+5x+6)=0$ and $\lim\limits_{x\to -3^-} 2=2$, we will apply **Theorem 5**. Hence, $x\to -3^-$ implies that $-3.1<x<-3$. Also, as $x\to -3^-$ the value of $x+2\to 0^-$ and $x+3\to 0^-$. Thus, $x^2+5x+6\to 0^+$ and

$$\begin{aligned}\lim_{x\to -3^-} f(x) &= \lim_{x\to -3^-}\frac{2}{(x+2)(x+3)}\\ &= +\infty\end{aligned}$$

Example 14. Let $g(x)=\dfrac{4x-7}{x^3-1}$. Determine

1. $\lim\limits_{x\to 1^+} g(x)$

Solution: Since $x^3-1=(x-1)(x^2+x+1)$, $\lim\limits_{x\to 1^+}(x^3-1)=0$ and $\lim\limits_{x\to 1^+}(4x-7)=-3$, we will apply **Theorem 5**. Hence, $x\to 1^+$ implies that $1<x<1.1$. Also, as $x\to 1^+$ the value of $x-1\to 0^+$ and $x^2+x+1\to 3$. Thus, $x^3-1\to 0^+$ and

$$\begin{aligned}\lim_{x\to 1^+} g(x) &= \lim_{x\to 1^+}\frac{4x-7}{(x-1)(x^2+x+1)}\\ &= -\infty\end{aligned}$$

2. $\lim\limits_{x\to 1^-} g(x)$

Solution: Since $x^3-1=(x-1)(x^2+x+1)$, $\lim\limits_{x\to 1^-}(x^3-1)=0$ and $\lim\limits_{x\to 1^-}(4x-7)=-3$, we will apply **Theorem 5**. Hence, $x\to 1^-$ implies that $0.9<x<1$. Also, as $x\to 1^-$ the value of $x-1\to 0^-$ and $x^2+x+1\to 3$. Thus, $x^3-1\to 0^-$ and

$$\begin{aligned}\lim_{x\to 1^-} g(x) &= \lim_{x\to 1^-}\frac{4x-7}{(x-1)(x^2+x+1)}\\ &= +\infty\end{aligned}$$

Theorem 6

Let f and g be real-valued functions defined on some open interval I containing a, except possibly at a.

1. If $\lim\limits_{x\to a} f(x)$ exists and $\lim\limits_{x\to a} g(x) = \pm\infty$, then

- **i**. $\lim\limits_{x\to a}(f(x) + g(x)) = \pm\infty$.
- **ii**. $\lim\limits_{x\to a}(f(x) - g(x)) = \mp\infty$,

2. If $\lim\limits_{x\to a} f(x) = +\infty$ and $\lim\limits_{x\to a} g(x) = +\infty$, then $\lim\limits_{x\to a}(f(x) + g(x)) = +\infty$.

3. If $\lim\limits_{x\to a} f(x) = +\infty$ and $\lim\limits_{x\to a} g(x) = -\infty$, then

- **i**. $\lim\limits_{x\to a}(f(x) - g(x)) = +\infty$.
- **ii**. $\lim\limits_{x\to a}(g(x) - f(x)) = -\infty$.

4. Let $c \in \mathbb{R}^*$. Suppose $\lim\limits_{x\to a} f(x) = c$ and $\lim\limits_{x\to a} g(x) = \pm\infty$. Then

- **i**. if $c > 0$, then $\lim\limits_{x\to a}(f(x) \cdot g(x)) = \pm\infty$.
- **ii**. if $c < 0$, then $\lim\limits_{x\to a}(f(x) \cdot g(x)) = \mp\infty$.

Example 15. Determine

1. $\lim\limits_{x\to 2^+}\left(x^2 - \dfrac{3}{x-2}\right)$

Solution: Using the above theorem, since $\lim\limits_{x\to 2^+} x^2 = 4$ and $\lim\limits_{x\to 2^+} \dfrac{3}{x-2} = +\infty$, we have

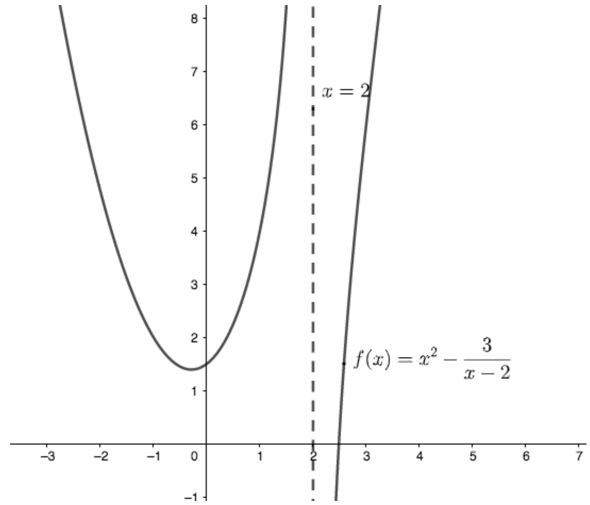
$$\lim_{x\to 2^+}\left(x^2 - \frac{3}{x-2}\right) = -\infty$$

2. $\lim\limits_{x\to 2^-}\left(x^2 - \dfrac{3}{x-2}\right)$

Solution: Using the above theorem, since $\lim\limits_{x\to 2^-} x^2 = 4$ and $\lim\limits_{x\to 2^-} \dfrac{3}{x-2} = -\infty$, we have

$$\lim_{x\to 2^-}\left(x^2 - \frac{3}{x-2}\right) = +\infty$$

Consider the graph of the function $f(x) = x^2 - \dfrac{3}{x-2}$.



Definition 7: Vertical Asymptotes

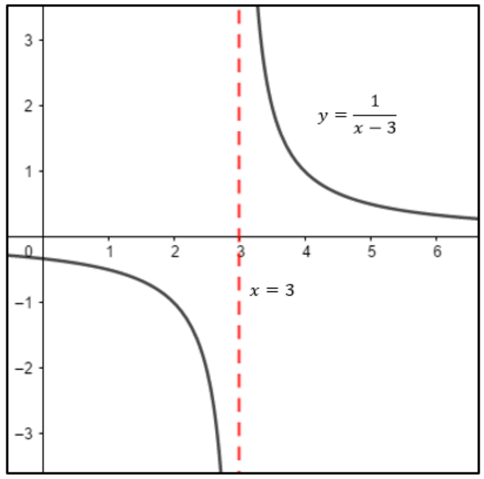
Let f be a real-valued function defined on some open interval I containing a, except possibly at a. The line $x = a$ is called a **vertical asymptote** of the graph of $y = f(x)$ if it satisfies any of the following:

1. $\lim\limits_{x \to a^+} f(x) = +\infty$,
2. $\lim\limits_{x \to a^-} f(x) = +\infty$,
3. $\lim\limits_{x \to a^+} f(x) = -\infty$, or
4. $\lim\limits_{x \to a^-} f(x) = -\infty$.

Example 16. From the previous example, since $\lim\limits_{x \to 2^+} \left(x^2 - \dfrac{3}{x-2} \right) = -\infty$ and $\lim\limits_{x \to 2^-} \left(x^2 - \dfrac{3}{x-2} \right) = +\infty$, the line $x = 2$ is a vertical asymptote of the graph of $f(x) = x^2 - \dfrac{3}{x-2}$.

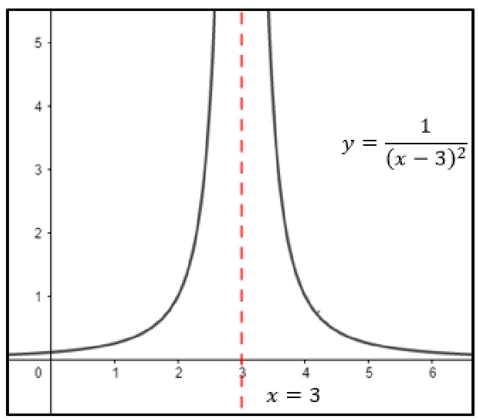
Example 17. Find the vertical asymptote of the rational function $f(x) = \dfrac{1}{x-3}$.

Solution: Since $\lim\limits_{x \to 3^+} \dfrac{1}{x-3} = +\infty$ and $\lim\limits_{x \to 3^-} \dfrac{1}{x-3} = -\infty$, the line $x = 3$ is a vertical asymptote of $f(x) = \dfrac{1}{x-3}$.



Example 18. Determine where the vertical asymptote of the rational function $f(x) = \dfrac{1}{(x-3)^2}$ located.

Solution: Since $\displaystyle\lim_{x\to 3^+} \frac{1}{(x-3)^2} = +\infty$ and $\displaystyle\lim_{x\to 3^-} \frac{1}{(x-3)^2} = +\infty$, the line $x = 3$ is a vertical asymptote of $f(x) = \dfrac{1}{x-3}$.



Remark 3

The infinite limits of the last two examples are just a special case of the following general result:

$$\lim_{x\to a^-} \frac{1}{(x-a)^n} = -\infty \text{ and } \lim_{x\to a^+} \frac{1}{(x-a)^n} = +\infty$$

for a positive odd integer n, and

$$\lim_{x\to a} \frac{1}{(x-a)^n} = +\infty$$

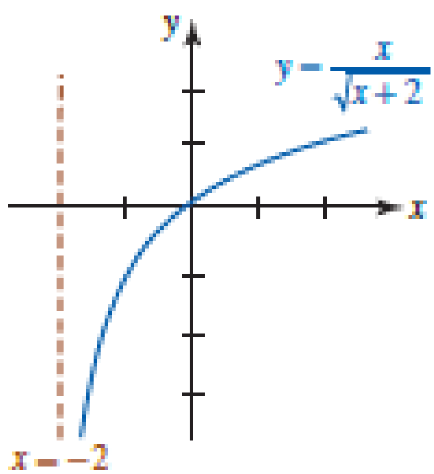
for a positive even integer n.

Example 19. Evaluate $\displaystyle\lim_{x\to -2^+} \frac{x}{\sqrt{x+2}}$.

Solution: The domain of f is $(-2, +\infty)$. Using a table of values, we can see that the value of $f(x)$ decreases without bound as the value of x approaches -2 from the right.

$x \to -2^+$	-1.9	-1.99	-1.999	-1.9999
$f(x)$	-6.01	-19.90	-63.21	-199.90

Thus, $\displaystyle\lim_{x\to -2^+} \frac{x}{\sqrt{x+2}} = -\infty$. Also, the vertical asymptote of $f(x) = \dfrac{x}{\sqrt{x+2}}$ is the line $x = -2$.



Note: The left-hand limit is not applicable to our previous example since the domain of the function is not defined to the left of $x = -2$.

Definition 8: Indeterminate Forms of Types $\infty - \infty$ and $0 \cdot \infty$

Let f and g be real-valued functions defined on some open interval I containing a, except possibly at a.

1. If $\lim_{x\to a} f(x) = +\infty$ and $\lim_{x\to a} g(x) = +\infty$, then $\lim_{x\to a} (f(x) - g(x))$ is called an **indeterminate form of type** $\infty - \infty$.

2. If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \pm\infty$, then $\lim_{x\to a} (f(x) \cdot g(x))$ is called an **indeterminate form of type** $0 \cdot \infty$.

Example 20. Evaluate $\displaystyle\lim_{x\to -1^-} \left(\frac{1}{x+1} + \frac{3}{2x^2+x-1}\right)$

Solution: Since the $\displaystyle\lim_{x\to -1^-} \frac{1}{x+1} = -\infty$ and $\displaystyle\lim_{x\to -1^-} \frac{3}{(2x-1)(x+1)} = +\infty$, the limit is indeterminate of type $\infty - \infty$. To compute , we combine the expressions into one:

$$\begin{aligned}
\lim_{x\to -1^-}\left(\frac{1}{x+1}+\frac{3}{2x^2+x-1}\right) &= \lim_{x\to -1^-}\frac{(2x-1)+3}{(2x-1)(x+1)} \\
&= \lim_{x\to -1^-}\frac{2x+2}{(2x-1)(x+1)} \\
&= \lim_{x\to -1^-}\frac{2(x+1)}{(2x-1)(x+1)} \\
&\quad \text{since } x\to -1 \text{ imples } x\neq -1, \text{ hence } \frac{x+1}{x+1}=1 \\
&= \lim_{x\to -1^-}\frac{2}{2x-1} = -\frac{2}{3}
\end{aligned}$$

Example 21. Evaluate $\lim\limits_{x\to 4^+}\left(\frac{1}{4-x}\right)\left(\frac{x}{x-1}-\frac{8}{x+2}\right)$

Solution:

Since $\lim\limits_{x\to 4^+}\frac{1}{4-x}=-\infty$ and $\lim\limits_{x\to 4^+}\left(\frac{x}{x-1}-\frac{8}{x+2}\right)=0$, the limit is indeterminate of type $0\cdot\infty$.
To compute, we rewrite the expression as a quotient:

$$\begin{aligned}
\lim_{x\to 4^+}\left(\frac{1}{4-x}\right)\left(\frac{x}{x-1}-\frac{8}{x+2}\right) &= \lim_{x\to 4^+}\left(\frac{1}{4+x}\right)\left(\frac{x(x+2)-8(x-1)}{(x-1)(x+2)}\right) \\
&= \lim_{x\to 4^+}\left(\frac{x^2-6x+8}{(4-x)(x-1)(x+2)}\right) \\
&= \lim_{x\to 4^+}\left(\frac{-(x-2)(4-x)}{(4-x)(x-1)(x+2)}\right) \\
&\quad \text{since } x\to 4^+ \text{ imples } x\neq 4, \text{ hence } \frac{x-4}{x-4}=\frac{4-x}{4-x}=1 \\
&= \lim_{x\to 4^+}\frac{-(x-2)}{(x-1)(x+2)} = -\frac{1}{9}
\end{aligned}$$

Definition 9: Limits at Infinity

Let f be a real-valued function defined on some open interval $I=(a,+\infty)$. We say that the **limit of $f(x)$ as x approaches positive infinity is L**, denoted by

$$\lim_{x\to +\infty} f(x) = L,$$

if the values of $f(x)$ get closer and closer to L as the values of x increase without bound.
Let f be a real-valued function defined on some open interval $I=(-\infty,a)$. We say that the **limit of $f(x)$ as x approaches negative infinity is L**, denoted by

$$\lim_{x\to -\infty} f(x) = L,$$

if the values of $f(x)$ get closer and closer to L as the values of x decrease without bound.

Theorem 7

1. Let $n \in \mathbb{N}$.

i. If n is even, then $\lim\limits_{x\to\pm\infty} x^n = +\infty$.

ii. If n is odd, then $\lim\limits_{x\to\pm\infty} x^n = \pm\infty$.

iii. $\lim\limits_{x\to\pm\infty} \dfrac{1}{x^n} = 0$.

2. Let $c \in \mathbb{R}$. Suppose $\lim\limits_{x\to+\infty} f(x) = c$ and $\lim\limits_{x\to+\infty} g(x) = \pm\infty$. Then $\lim\limits_{x\to+\infty} \dfrac{f(x)}{g(x)} = 0$.

Example 22. Evaluate $\lim\limits_{x\to+\infty}(4x^5 - 7x^4 + 3x^3 - 9x + 2)$

Solution:

Since that we are letting x increase without bound, we have $x \neq 0$. We may then write $4x^5 - 7x^4 + 3x^3 - 9x + 2 = x^5 \cdot \left(4 - \dfrac{7}{x} + \dfrac{3}{x^2} - \dfrac{9}{x^4} + \dfrac{2}{x^5}\right)$. By the previous theorem, $\lim\limits_{x\to+\infty} x^5 = +\infty$, while each $\dfrac{7}{x}$, $\dfrac{3}{x^2}$, $\dfrac{9}{x^4}$ and $\dfrac{2}{x^5}$ approach 0, as $x \to +\infty$. Thus,

$$\begin{aligned} \lim_{x\to+\infty} x^5 \cdot \left(4 - \frac{7}{x} + \frac{3}{x^2} - \frac{9}{x^4} + \frac{2}{x^5}\right) &= (+\infty)(5) \\ &= +\infty \end{aligned}$$

Example 23. Evaluate $\lim\limits_{x\to+\infty} \dfrac{3x-1}{9x+3}$

Solution:

Note that $\lim\limits_{x\to+\infty}(3x-1) = +\infty$ and $\lim\limits_{x\to+\infty}(9x+3) = +\infty$. Thus, the limit has the form $\dfrac{\infty}{\infty}$. Does this mean that the limit does not exist? Consider the table of values below.

x	$\frac{3x-1}{9x+3}$
1	≈ 0.1666667
10	≈ 0.311828
$100,000$	≈ 0.3333311
$1,000,000,000$	≈ 0.3333333

It seems that as $x \to +\infty$, the quotient $\dfrac{3x-1}{9x+3}$ approaches a particular value: 0.3333333. Let us verify this. Using long division, we can write $\dfrac{3x-1}{9x+3} = \dfrac{1}{3} - \dfrac{\frac{2}{3}}{3x+1}$. Therefore,

$$\lim_{x\to+\infty} \frac{3x-1}{9x+3} = \lim_{x\to+\infty} \left(\frac{1}{3} - \frac{\frac{2}{3}}{3x+1}\right) = \frac{1}{3} - 0 = \frac{1}{3}.$$

So the limit exists, but it is not equal to 0.3333333 as we initially guessed, it is equal to $\frac{1}{3}$.

Definition 10: Indeterminate Form of Type $\dfrac{\infty}{\infty}$

Let f and g be real-valued functions defined on some open interval I containing a, except possibly at a. If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ is called an **indeterminate form of type** $\dfrac{\infty}{\infty}$.

Example 24. Evaluate $\lim_{x\to+\infty} \dfrac{3x-1}{9x+3}$

Solution:

Note that $\lim_{x\to+\infty} \dfrac{3x-1}{9x+3}$ is an indeterminate form of type $\dfrac{\infty}{\infty}$. Since we are letting x approach $+\infty$, we have $x \neq 0$. Thus, we may divide both numerator and denominator by the highest power of x in the denominator which is x:

$$\begin{aligned}
\lim_{x\to+\infty} \frac{3x-1}{9x+3} &= \lim_{x\to+\infty} \frac{3x-1}{9x+3} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \\
&= \lim_{x\to+\infty} \frac{3-\frac{1}{x}}{9+\frac{3}{x}} \\
&= \frac{3-0}{9+0} \\
&= \frac{3}{9} \\
&= \frac{1}{3}
\end{aligned}$$

Example 25. Evaluate $\lim_{x\to-\infty} \dfrac{\sqrt{x^2-3}}{x+2}$

Solution:

This is an indeterminate form of type $\dfrac{\infty}{\infty}$. To evaluate this we use the fact that

$$\sqrt{x^2} = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

Since we are letting $x \to -\infty$, we have $x < 0$ and so, $\sqrt{x^2} = -x$ or $x = -\sqrt{x^2}$. Hence,

$$\begin{aligned}
\lim_{x\to-\infty} \frac{\sqrt{x^2-3}}{x+2} &= \lim_{x\to-\infty} \frac{\sqrt{x^2-3}}{x+2} \cdot \frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{\sqrt{x^2}}} \\
&= \lim_{x\to-\infty} \frac{\sqrt{x^2-3}}{x+2} \cdot \frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{-x}} \\
&= \lim_{x\to-\infty} \frac{\sqrt{1-\frac{3}{x^2}}}{-1-\frac{2}{x}} \\
&= \frac{\sqrt{1-0}}{-1-0} \\
&= -1
\end{aligned}$$

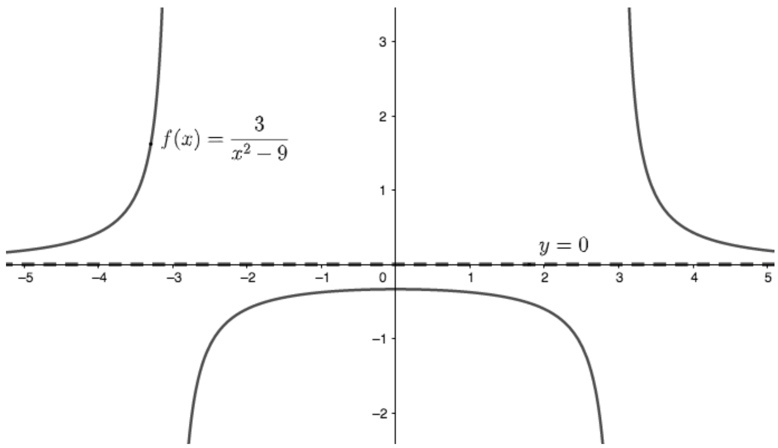
Definition 11: Horizontal Asymptotes

Let f be a real-valued function defined on some open interval I containing a, except possibly at a. If either

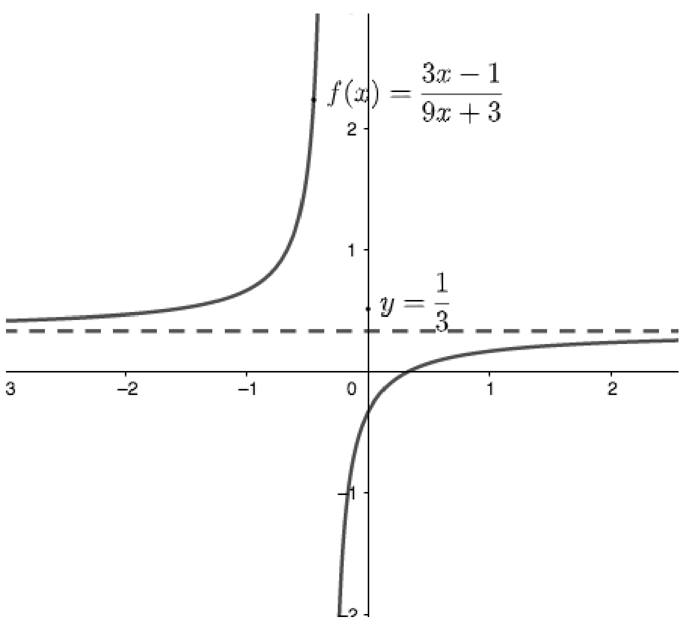
$$\lim_{x\to+\infty} f(x) = L \text{ or } \lim_{x\to-\infty} f(x) = L,$$

then the line $y = L$ is called a **horizontal asymptote** of the graph of $y = f(x)$.

Example 26. Consider $\displaystyle\lim_{x\to+\infty} \frac{3}{x^2-9} = 0$. Here, the line $y = 0$ is a horizontal asymptote of the graph of $y = \dfrac{3}{x^2-9}$.



Example 27. From the previous example, $\displaystyle\lim_{x\to-\infty} \frac{3x-1}{9x+3} = \frac{1}{3}$. The line $y = \dfrac{1}{3}$ is a horizontal asymptote of the graph of $y = \dfrac{3x-1}{9x+3}$.



Lesson 1.6: Continuity of a Function

Learning Outcomes

At the end of this lesson, you should be able to:

1. Define continuity of a function at a point and on an interval;
2. Apply the definition and some special properties on continuity to determine whether a function is continuous or discontinuous at a point or on an interval; and
3. Classify the type of discontinuity of a function at a point.

First we will discuss the continuity at a particular point with abscissa $x = a$.

Definition 12: Continuity at a Point

Let f be a real-valued function and $a \in \mathbb{R}$. We say that f **is continuous at** $x = a$ iff the following are satisfied:

i. $f(a)$ exists

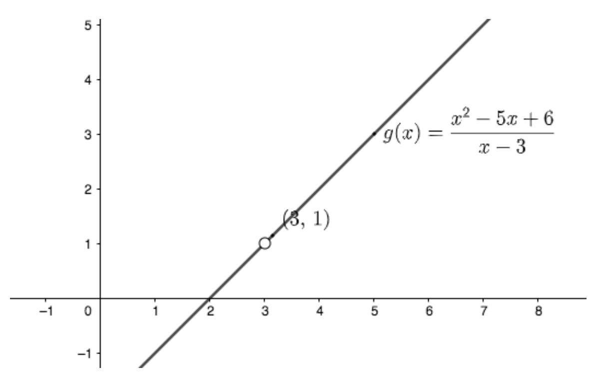
ii. $\lim\limits_{x \to a} f(x)$ exists

iii. $f(a) = \lim\limits_{x \to a} f(x)$.

Otherwise, f is said to be **discontinuous** at $x = a$.

Example 28. Let $f(x) = 3x^4 - 5x^2 - 5$. Let us examine the continuity of f at $x = 2$. First, $f(2) = 23$. Moreover, $\lim\limits_{x \to 1} f(x) = 23$. Therefore, f is continuous at $x = 1$.

Example 29. Let $g(x) = \dfrac{x^2 - 5x + 6}{x - 3}$. Since g is not defined at $x = 3$, g is discontinuous at $x = 3$. However, it is continuous at every other $a \in \mathbb{R}\backslash\{2\}$.



Remark 4

If f is a polynomial or a rational function and a is any element in the domain of f, then f is continuous at $x = a$.

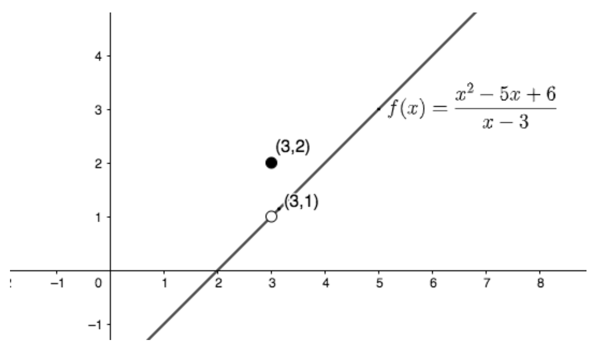
Example 30. Let f be defined by

$$f(x) = \begin{cases} \dfrac{x^2 - 5x + 6}{x - 3}, & \text{if } x \neq 3 \\ 2, & \text{if } x = 3 \end{cases}.$$

Note that f is defined at $x = 3$ with $f(3) = 2$. However, $x \to 3$ implies $x \neq 3$. Thus,

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2 - 5x + 6}{x - 3} = \lim_{x \to 3} (x - 2) = 1.$$

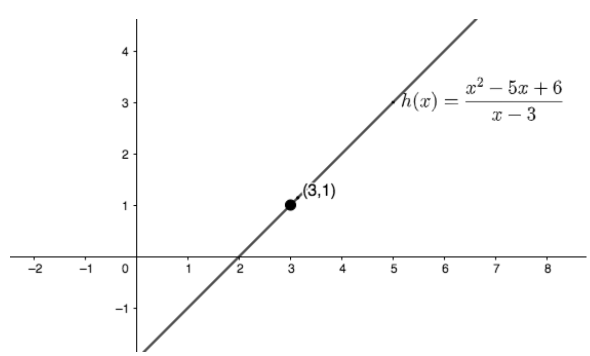
Since $f(3)$ is not equal to $\lim\limits_{x \to 3} f(x)$. f is discontinuous at $x = 3$.



Example 31. Let h be defined by

$$h(x) = \begin{cases} \dfrac{x^2 - 5x + 6}{x - 3}, & \text{if } x \neq 3 \\ 1, & \text{if } x = 3 \end{cases}.$$

For this function, we have $h(3) = 1 = \lim\limits_{x \to 3} h(x)$. Therefore, $h(x)$ is continuous at $x = 3$.



Definition 13: Types of Discontinuity

Let f be a discontinuous function at $x = a$.

1. If $\lim_{x\to a} f(x)$ exists but, either $f(a)$ does not exist or $f(a) \neq \lim_{x\to a} f(x)$, then f is said to have a **removable discontinuity at** $x = a$.

2. If $\lim_{x\to a} f(x)$ does not exist, then f is said to have an **essential discontinuity at** $x = a$.

   i. If $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist but are not equal, then f is said to have a **jump essential discontinuity at** $x = a$.

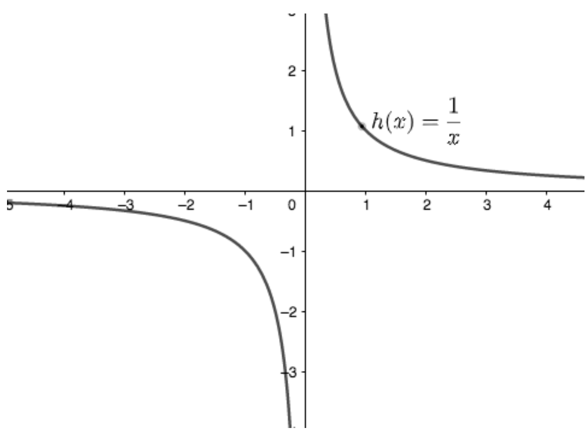
   ii. If either $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$, then f is said to have an **infinite essential discontinuity at** $x = a$.

Example 32. The function f defined by

$$f(x) = \begin{cases} \dfrac{x^2 - 5x + 6}{x - 3}, & \text{if } x \neq 3 \\ 2, & \text{if } x = 3 \end{cases}.$$

has a removable discontinuity at $x = 3$.

Example 33. The function h defined by $h(x) = \dfrac{1}{x}$ is undefined at $x = 0$, while $\lim_{x\to 0^-} h(x) = -\infty$ and $\lim_{x\to 0^+} h(x) = +\infty$. Thus, h has an infinite essential discontinuity at $x = 0$.



Remark 5

The discontinuity in statement 1 of Definition 13 is called removable, because the discontinuity can be "removed" by reducing the value of f at a so that $f(a) = \lim_{x\to a} f(x)$, resulting in a function that is now continuous at $x = a$. On the other hand, it is not possible to do this for essential discontinuities.

Theorem 8

Let f and g be real-valued functions at $x = a$, and, $c \in \mathbb{R}$. Then, the following are also continuous at $x = a$:

1. $f + g$
2. $f - g$
3. $f \cdot g$
4. $\frac{f}{g}$, where $g(a) \neq 0$
5. $c \cdot f$.

Example 34. Let

$$g(x) = \begin{cases} x^2 + 2x + 2 & ; \text{if } x < -1 \\ \dfrac{2}{x-1} & ; \text{if } x \geq -1. \end{cases}$$

Determine if g is continuous at $a = -1$ and $a = 1$.

Solution:

a. $a = -1$

   i. $g(-1) = \dfrac{2}{(-1) - 1} = -1 \in \mathbb{R}$.
   $\therefore g(-1)$ exists.

   ii. $\lim\limits_{x \to -1^-} g(x) = \lim\limits_{x \to -1^-} (x^2 + 2x + 2) = (-1)^2 + 2(-1) + 2 = 1$.
   $\lim\limits_{x \to -1^+} g(x) = \lim\limits_{x \to -1^-} \dfrac{2}{x-1} = \dfrac{2}{(-1) - 1} = -1$.
   $\lim\limits_{x \to -1^-} g(x) = 1 \neq -1 = \lim\limits_{x \to -1^+} g(x)$.
   $\therefore \lim\limits_{x \to -1} g(x)$ does not exist.
   $\therefore$ g is discontinuous at $a = -1$. Moreover, g has a jump essential discontinuity at $a = -1$.

b. $a = 1$

   i. $g(1) = \dfrac{2}{(1) - 1} = \dfrac{2}{0} \notin \mathbb{R}$.
   $\therefore g(1)$ does not exist.

   ii. $\lim\limits_{x \to 1^-} g(x) = \lim\limits_{x \to 1^-} \dfrac{2}{x-1} = -\infty$.
   $\lim\limits_{x \to 1^+} g(x) = \lim\limits_{x \to 1^+} \dfrac{2}{x-1} = +\infty$.
   $\therefore \lim\limits_{x \to 11} g(x)$ does not exist.
   $\therefore$ g is discontinuous at $a = 1$. Moreover, g has an infinite essential discontinuity at $a = 1$.

Example 35. Let

$$f(x) = \begin{cases} |x-4| & ; \text{ if } x \neq 4 \\ -2 & ; \text{ if } x = 4. \end{cases}$$

Determine if f is continuous at $a = 4$.
Solution:

i. $f(4) = -2 \in \mathbb{R}$.
$\therefore f(4)$ exists.

ii. $\lim\limits_{x\to 4^+} f(x) = \lim\limits_{x\to 4^+} |x-4| = \lim\limits_{x\to 4^+} (x-4) = (4-4) = 0.$
$\lim\limits_{x\to 4^-} f(x) = \lim\limits_{x\to 4^-} |x-4| = \lim\limits_{x\to 4^-} -(x-4) = -(4-4) = 0.$
$\therefore \lim\limits_{x\to 4} f(x) = 0$ and it exists.

iii. But $f(4) = -2 \neq 0 = \lim\limits_{x\to 4} f(x)$.

$\therefore f$ is discontinuous at $a = 4$. Moreover, f has a removable discontinuity at $a = 4$.

Definition 14

Let f be a real-valued function and $a \in \mathbb{R}$. We say that f is

1. **continuous from the left at** $x = a$ if $f(a) = \lim\limits_{x\to a^-} f(x)$.

2. **continuous from the right at** $x = a$ if $f(a) = \lim\limits_{x\to a^+} f(x)$.

Example 36. The function f defined by $f(x) = \sqrt{3-x}$ is continuous from the left at $x = 3$ but not from the right at $x = 3$, since f is undefined for $x > 3$.

Example 37. Let $g(x) = [\![x]\!]$ be the greatest integer function. Then g is continuous from the right at $x = 1$, but not continuous from the left at $x = 1$. In general, if n is any integer, then g is continuous from the right at $x = n$, but not continuous from the left at $x = n$.

Definition 15: Continuity on an Interval

Let f be a real-valued function and $a \in \mathbb{R}$. We say that f is continuous

1. everywhere if f is continuous at every real number.

2. on (a, b) if f is continuous at every real number in (a, b).

3. on $[a, b)$ if f is continuous on (a, b) and from the right at a.

4. on $(a, b]$ if f is continuous on (a, b) and from the left at b.

5. on $[a,b]$ if f is continuous on $[a,b)$ and on $(a,b]$.

6. on $(a,+\infty)$ is continuous at every real number greater than a.

7. on $[a,+\infty)$ is continuous on $(a,+\infty)$ and from the right of a.

8. on $(-\infty,b)$ is continuous at every real number less than b.

9. on $(-\infty,b]$ is continuous on $(-\infty,b)$ and from the left of b.

Remark 6

1. Polynomial functions are continuous everywhere.

2. The absolute value function $f(x)=|x|$ is continuous everywhere.

3. Rational functions are continuous on their respective domains.

4. The square root function $f(x)=\sqrt{x}$ is continuous on $[0,\,+\infty)$.

5. The greatest integer function $f(x)=[\![x]\!]$ is continuous on $[n,\,n+1)$, where n is any integer.

Theorem 9

Let f and g be real-valued functions and $a\in\mathbb{R}$. If f is continuous at $\lim\limits_{x\to a} g(x)$, then $\lim\limits_{x\to a} f(g(x)) = f\big(\lim\limits_{x\to a} g(x)\big)$.

Theorem 10

Let f and g be real-valued functions and $a\in\mathbb{R}$. If g is continuous at a and f is continuous at $g(a)$, then $f\circ g$ is continuous at $x=a$.

Example 38. Evaluate $\lim\limits_{x\to\frac{5}{2}}\left|\dfrac{5-2x}{4x^2-25}\right|$.

Solution: Let $g(x)=\dfrac{5-2x}{4x^-25}$. Also, $g(x)=\dfrac{-(2x-5)}{(2x-5)(2x+5)}=\dfrac{-1}{2x+5}$.
Since, $\frac{5}{2}\in\operatorname{dom}(g)$, then g is continuous at $a=\frac{5}{2}$.

Let $f(x)=|x|$. Note that f is continuous everywhere. Thus, f is continuous at $g\left(\frac{5}{2}\right)$.

Now,

$$
\begin{aligned}
\lim_{x\to\frac{5}{2}}\left|\frac{5-2x}{4x^2-25}\right| &= \left|\lim_{x\to\frac{5}{2}}\frac{-(2x-5)}{4x^2-25}\right| \\
&= \left|\lim_{x\to\frac{5}{2}}\frac{-1}{2x+5}\right| \\
&= \left|\frac{-1}{2\left(\frac{5}{2}\right)+5}\right| \\
&= \left|\frac{-1}{10}\right| \\
&= \frac{1}{10}.
\end{aligned}
$$

Example 39. Evaluate $\lim_{x\to\frac{\sqrt{3}}{2}}\left[\!\left[\frac{4}{9}x^2\right]\!\right]$.

Solution:
Let $g(x)=\frac{4}{9}x^2$. Since, g is everywhere continuous, the g is continuous at $a=\frac{\sqrt{3}}{2}$.
Note that $g\left(\frac{\sqrt{3}}{2}\right)=\frac{1}{3}$.

Let $f(x)=[\![x]\!]$. Since, f is continuous on intervals $[n,n+1)$ for all $n\in Z$ and $\frac{1}{3}\in(0,1)$, then f is continuous at $g\left(\frac{\sqrt{3}}{2}\right)=\frac{1}{3}$.
Now,

$$
\begin{aligned}
\lim_{x\to\frac{\sqrt{3}}{2}}\left[\!\left[\frac{4}{9}x^2\right]\!\right] &= \left[\!\left[\lim_{x\to\frac{\sqrt{3}}{2}}\frac{4}{9}x^2\right]\!\right] \\
&= \left[\!\left[\tfrac{4}{9}\left(\tfrac{\sqrt{3}}{2}\right)^2\right]\!\right] \\
&= [\![\tfrac{4}{9}\left(\tfrac{3}{4}\right)]\!] \\
&= [\![\tfrac{1}{3}]\!] \\
&= 0.
\end{aligned}
$$

Example 40. Determine the largest interval in which $f(x)=\dfrac{4}{x^2-16}$ is continuous.
Solution:
Since, f is rational function, then f is continuous on its domain. Thus, the largest interval in which f is continuous is $\mathbb{R}-\{-4,4\}=(-\infty,-4)\cup(-4,4)\cup(4,+\infty)$.

Example 41. Determine the largest interval in which $g(x)=\sqrt{x^2-25}$ is continuous.
Solution:
Note that dom $g=(-\infty,-5]\cup[5,+\infty)$. Since $\lim_{x\to-5^+}g(x)$ and $\lim_{x\to5^-}g(x)$ do not exist, then the largest interval in which g is continuous is $(-\infty,-5)\cup(5,+\infty)$.

Lesson 1.7: The Extreme Value and Intermediate Value Theorem

Learning Outcome

At the end of this lesson, you should be able to:

1. State and apply the Extreme Value Theorem and Intermediate Value Theorem on specified conditions.

Theorem 11: Extreme Value Theorem (EVT)

Let f be a continuous function on the closed interval $[a, b]$. Then, there exists $c \in [a, b]$ such that for all $x \in [a, b]$, f takes on an extreme value on $[a, b]$.

Theorem 12: Intermediate Value Theorem (IVT)

Let f be a continuous function on the closed interval $[a, b]$ where $f(a) \neq f(b)$. Then, for every $k \in (f(a), f(b))$, there exists $c \in (a, b)$ such that $f(c) = k$.

The Intermediate Value Theorem states that if f is continuous on $[a,\ b]$, then the graph of $y = f(x)$ intersects any horizontal line $y = k$ between $y = f(a)$ and $y = f(b)$.

Remark 7

1. The continuity of the function on $[a,\ b]$ in IVT is required. In general, the theorem is not true for functions that are discontinuous on $[a,\ b]$.

2. The c in the conclusion of IVT may not be unique.

Example 42. Let $f(x) = \sin 3x$. Show that f takes an extreme value of $[0, \pi]$.
Solution:
If $f(x) = \sin 3x$, then f is continuous everywhere. Thus, f is continuous on $[0, \pi]$.
By EVT, f takes an extreme value on $[0, \pi]$. Note that $\text{ran} f = [-1, 1]$.

Hence, if $\sin 3x = 1$ on $[0, \pi]$, then

$$\begin{aligned} \sin 3x &= 1 \\ 3x &= \text{Sin}^{-1} 1 \\ 3x &= \frac{\pi}{2} + 2k\pi \\ x &= \frac{\pi}{6} + \frac{2k\pi}{3}, \end{aligned}$$

where $k = 0$ and $k = 1$.
$\therefore f$ takes maximum values at $\frac{\pi}{6}, \frac{5\pi}{6} \in [0, \pi]$.
Also, if $\sin 3x = -1$ on $[0, \pi]$, then

$$\begin{aligned} \sin 3x &= -1 \\ 3x &= \mathrm{Sin}^{-1} - 1 \\ 3x &= \frac{-\pi}{2} + 2k\pi \\ x &= \frac{-\pi}{6} + \frac{2k\pi}{3}, \end{aligned}$$

where $k = 1$.
$\therefore f$ takes a minimum values at $\frac{\pi}{2} \in [0, \pi]$.

Example 43. Let $g(x) = \frac{2x-1}{x+5}$. Show that IVT holds over the interval $[0, 6]$. Be able to determine $c \in (0, 6)$ so that $g(c) = \frac{1}{2}$.
Solution:
If $g(x) = \frac{2x-1}{x+5}$, then $\mathrm{dom}\,(g) = \mathbb{R} - \{-5\}$. Hence, $[0, 6] \subseteq \mathrm{dom}\,(g)$. Thus, g is continuous on $[0, 6]$. We have $g(0) = -\frac{1}{5}$ and $g(6) = 1$. By IVT, for all $k \in \left(-\frac{1}{5}, 1\right)$, there exist $c \in (0, 6)$ such that $g(c) = \frac{1}{2}$.
Now,

$$\begin{aligned} \frac{2c-1}{c+5} &= \frac{1}{2} \\ 2(2c-1) &= 1(c+5) \\ 4c - 2 &= c + 5 \\ 3c &= 7 \\ c &= \frac{7}{3}. \end{aligned}$$

Note, $\frac{7}{3} \in (0, 6)$.

Lesson 1.8: The Squeeze Theorem

Learning Outcomes

At the end of this lesson, you should be able to:

1. State and illustrate the conditions of the Squeeze Theorem;
2. Apply the Squeeze Theorem on evaluating concerned limits; and
3. Recall properties of specific functions relevant in the application of the Squeeze Theorem.

Theorem 13: Squeeze Theorem

Let I be an open interval containing a. Suppose that f, g and h are functions defined on I except possibly at $x = a$, where $f(x) \leq g(x) \leq h(x)$, for all $x \in I - \{a\}$. If $\lim\limits_{x \to a} f(x)$ and If $\lim\limits_{x \to a} h(x)$ exist and are both equal to L, then $\lim\limits_{x \to a} g(x) = L$.

Example 44. Evaluate $\lim\limits_{x \to 0} x^2 \cos\left(\frac{1}{x}\right)$.
Solution:
Note that for all $x \in \mathbb{R} - \{0\}$,

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1.$$

Since $x^2 \geq 0$, then

$$-x^2 \leq x^2 \cos\left(\frac{1}{x}\right) \leq x^2.$$

Evaluating the limit as $x \to 0$, we have

$$\begin{array}{rcccl} \lim\limits_{x \to 0} (1x^2) & \leq & \lim\limits_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) & \leq & \lim\limits_{x \to 0} x^2 \\ 0 & \leq & \lim\limits_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) & \leq & 0. \end{array}$$

$\therefore \lim\limits_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0$ by the Squeeze Theorem.

Example 45. Evaluate $\lim\limits_{x \to +\infty} \frac{2x + \sin x}{x}$.
Solution:
Note that for all $x \in \mathbb{R}$,

$$-1 \leq \sin x \leq 1.$$

Also,

$$2x - 1 \leq 2x + \sin x \leq 2x + 1.$$

Now, for all $x > 0$, we have

$$\frac{2x-1}{x} \leq \frac{2x+\sin x}{x} \leq \frac{2x+1}{x}.$$

Evaluating the limit as $x \to \infty$, we have

$$\begin{aligned}
\lim_{x\to+\infty} \frac{2x-1}{x} &\leq \lim_{x\to+\infty} \frac{2x+\sin x}{x} \leq \lim_{x\to+\infty} \frac{2x+1}{x} \\
\lim_{x\to+\infty} \left(2 - \frac{1}{x}\right) &\leq \lim_{x\to+\infty} \frac{2x+\sin x}{x} \leq \lim_{x\to+\infty} \left(2 - \frac{1}{x}\right) \\
2 &\leq \lim_{x\to+\infty} \frac{2x+\sin x}{x} \leq 2.
\end{aligned}$$

$\therefore \lim\limits_{x\to+\infty} \dfrac{2x+\sin x}{x} = 2$ by the Squeeze Theorem.

Example 46. Evaluate $\lim\limits_{x\to-\infty} \dfrac{4[\![x]\!]+1}{2x}$.
Solution:
Note that for all $x \in \mathbb{R}$,

$$x - 1 \leq [\![x]\!] \leq x.$$

Since $4 > 0$, then

$$4x - 4 \leq 4[\![x]\!] \leq 4x.$$

Thus,

$$4x - 3 \leq 4[\![x]\!] + 1 \leq 4x + 1.$$

Now, for all $x < 0$, we have

$$\frac{4x+1}{x} \leq \frac{4[\![x]\!]+1}{x} \leq \frac{4x-3}{x},$$

and

$$\frac{4x+1}{2x} \leq \frac{4[\![x]\!]+1}{2x} \leq \frac{4x-3}{2x},$$

Evaluating the limit as $x \to -\infty$, we have

$$\begin{aligned}
\lim_{x\to-\infty} \frac{4x+1}{2x} &\leq \lim_{x\to-\infty} \frac{4[\![x]\!]+1}{2x} \leq \lim_{x\to-\infty} \frac{4x-3}{2x} \\
\lim_{x\to-\infty} \left(2 + \frac{1}{2x}\right) &\leq \lim_{x\to-\infty} \frac{4[\![x]\!]+1}{2x} \leq \lim_{x\to-\infty} \left(2 + \frac{1}{2x}\right) \\
2 &\leq \lim_{x\to-\infty} \frac{4[\![x]\!]+1}{2x} \leq 2.
\end{aligned}$$

$\therefore \lim\limits_{x\to-\infty} \dfrac{4[\![x]\!]+1}{2x} = 2$ by the Squeeze Theorem.

Lesson 1.9: Limits Involving Trigonometric and Exponential Functions

Learning Outcomes

At the end of this lesson, you should be able to:

1. Evaluate limits of $\dfrac{\sin x}{x}$ and $\dfrac{1-\cos x}{x}$ as x approaches 0;
2. Evaluate limits of trigonometric functions on its respective domain; and
3. Evaluate limit of $\dfrac{e^x - 1}{x}$ as x approaches 0.

Theorem 14

1. $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$

2. $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

3. $\displaystyle\lim_{x\to 0} \sin x = 0$

4. $\displaystyle\lim_{x\to 0} \cos x = 1$

Theorem 15

1. Any trigonometric function is continuous on its respective domain.

2. If f is a trigonometric function and $a \in dom f$, then $\displaystyle\lim_{x\to a} f(x) = f(a)$.

Theorem 16

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1.$$

Example 47. Evaluate $\displaystyle\lim_{x\to 0} \frac{\sin 4x}{3x}$.
Solution:

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin 4x}{3x} &= \frac{1}{3} \cdot \lim_{x\to 0} \frac{\sin 4x}{x} \\
&= \frac{1}{3} \cdot \lim_{x\to 0} \frac{\sin 4x}{x} \cdot \frac{4}{4} \\
&= \frac{4}{3} \cdot \lim_{x\to 0} \frac{\sin 4x}{4x} \\
&= \frac{4}{3} \cdot 1 \\
&= \frac{4}{3}.
\end{aligned}$$

Example 48. Evaluate $\lim\limits_{x\to 2} \dfrac{\sin(x^3-8)}{x-2}$.
Solution:

$$\begin{aligned}
\lim_{x\to 2} \frac{\sin(x^3-8)}{x-2} &= \lim_{x\to 2} \frac{\sin(x^3-8)}{x-2} \cdot \frac{x^2+2x+4}{x^2+2x+4} \\
&= \lim_{x\to 2} \frac{\sin(x^3-8)\cdot(x^2+2x+4)}{x^3-8} \\
&= \lim_{x\to 2} \frac{\sin(x^3-8)}{x^3-8} \cdot \lim_{x\to 2} (x^2+2x+4) \\
&= 1\cdot(2^2+2(2)+4) \\
&= 12.
\end{aligned}$$

Example 49. Evaluate $\lim\limits_{x\to 0} \dfrac{3-3\cos 4x}{5x}$.
Solution:

$$\begin{aligned}
\lim_{x\to 0} \frac{3-3\cos 4x}{5x} &= \frac{3}{5}\cdot \lim_{x\to 0} \frac{1-\cos 4x}{x} \\
&= \frac{3}{5}\cdot \lim_{x\to 0} \frac{1-\cos 4x}{x}\cdot\frac{4}{4} \\
&= \frac{12}{5}\cdot \lim_{x\to 0} \frac{1-\cos 4x}{4x} \\
&= \frac{12}{5}\cdot 0 \\
&= 0.
\end{aligned}$$

Example 50. Evaluate $\lim\limits_{x\to 0^-} \dfrac{1-\cos 2x}{x^5}$.
Solution:

$$\begin{aligned}
\lim_{x\to 0^-} \frac{1-\cos 2x}{x^5} &= \lim_{x\to 0^-} \frac{1-\cos 2x}{x^5}\cdot\frac{1+\cos 2x}{1+\cos 2x} \\
&= \lim_{x\to 0^-} \frac{1-\cos^2 2x}{x^5(1+\cos 2x)} \\
&= \lim_{x\to 0^-} \frac{\sin^2 2x}{x^5(1+\cos 2x)} \\
&= \left(\lim_{x\to 0^-} \frac{\sin^2 2x}{x^2}\right)\cdot\left(\lim_{x\to 0^-} \frac{1}{x^3(1+\cos 2x)}\right) \\
&= \left(\lim_{x\to 0^-} \frac{\sin^2 2x}{x^2}\cdot\frac{4}{4}\right)\cdot\left(\lim_{x\to 0^-} \frac{1}{x^3(1+\cos 2x)}\right) \\
&= \left(4\lim_{x\to 0^-} \frac{\sin^2 2x}{4x^2}\right)\cdot\left(\lim_{x\to 0^-} \frac{1}{x^3(1+\cos 2x)}\right)
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to 0^-} \frac{1-\cos 2x}{x^5} &= \left(2 \lim_{x\to 0^-} \frac{\sin 2x}{2x}\right)^2 \cdot \left(\lim_{x\to 0^-} \frac{1}{x^3(1+\cos 2x)}\right) \\
&= (2\cdot 1)^2 \cdot \frac{1}{(0^-)(1+1)} \\
&= 4 \cdot \frac{1}{0^-} \\
&= -\infty.
\end{aligned}$$

Example 51. Evaluate $\lim\limits_{x\to\frac{\pi}{8}} 2\sin 4x$.

Solution: Note that sin is continuous everywhere. Thus,

$$\begin{aligned}
\lim_{x\to\frac{\pi}{8}} 2\sin 4x &= 2 \lim_{x\to\frac{\pi}{8}} \sin 4x \\
&= 2\sin\left(\lim_{x\to\frac{\pi}{8}} 4x\right) \\
&= 2\sin\left(4\left(\frac{\pi}{8}\right)\right) \\
&= 2\sin\left(\frac{\pi}{2}\right) \\
&= 2(1) \\
&= 2.
\end{aligned}$$

Example 52. Evaluate $\lim\limits_{x\to-\frac{\pi}{8}} \frac{3}{5}\tan 2x$.

Solution:

If $f(x) = \tan 2x$, then $2x \in \operatorname{dom}\tan = \mathbb{R} - \left\{\frac{k\pi}{2} | k \text{ is odd}\right\}$.

Thus, $x \in \mathbb{R} - \left\{\frac{k\pi}{4} | k \text{ is odd}\right\}$.

Note that $-\frac{\pi}{8} \in \mathbb{R} - \left\{\frac{k\pi}{4} | k \text{ is odd}\right\}$.

Thus,

$$\begin{aligned}
\lim_{x\to-\frac{\pi}{8}} \frac{3}{5}\tan 2x &= \frac{3}{5} \cdot \lim_{x\to-\frac{\pi}{8}} \tan 2x \\
&= \frac{3}{5} \cdot \tan\left(\lim_{x\to-\frac{\pi}{8}} 2x\right) \\
&= \frac{3}{5} \cdot \tan\left(-\frac{\pi}{4}\right) \\
&= \frac{3}{5} \cdot (-1) \\
&= -\frac{3}{5}.
\end{aligned}$$

Example 53. Evaluate $\lim\limits_{x\to\frac{\pi}{3}^-} \csc 3x$.

Solution:

If $f(x) = \csc 3x$, then $3x \in \operatorname{dom}\csc = \mathbb{R} - \{k\pi | k \in Z\}$

Thus, $x \in \mathbb{R} - \left\{\dfrac{k\pi}{3} | k \in Z\right\}$.

But, $\dfrac{\pi}{3} \notin \mathbb{R} - \left\{\dfrac{k\pi}{3} | k \in Z\right\}$.

$\therefore \lim\limits_{x\to\frac{\pi}{3}^-} \csc 3x = +\infty$, based on the range of values.

Example 54. Evaluate $\lim\limits_{x\to 0} \dfrac{e^{3x} - 1}{5x}$.

Solution:

$$\begin{aligned}
\lim_{x\to 0} \frac{e^{3x} - 1}{5x} &= \frac{1}{5}\lim_{x\to 0} \frac{e^{3x} - 1}{x} \\
&= \frac{1}{5}\lim_{x\to 0} \frac{e^{3x} - 1}{x} \cdot \frac{3}{3} \\
&= \frac{3}{5}\lim_{x\to 0} \frac{e^{3x} - 1}{3x} \\
&= \frac{3}{5}(1) \\
&= \frac{3}{5}.
\end{aligned}$$

Example 55. Evaluate $\lim\limits_{x\to 0} \dfrac{e^{6x} - e^{4x}}{7x}$.

Solution:

$$\begin{aligned}
\lim_{x\to 0} \frac{e^{6x} - e^{4x}}{7x} &= \frac{1}{7} \cdot \lim_{x\to 0} \frac{e^{6x} - e^{4x}}{x} \\
&= \frac{1}{7} \cdot \lim_{x\to 0} \frac{(e^{6x} - e^{4x}) + (1 - 1)}{x} \\
&= \frac{1}{7} \cdot \lim_{x\to 0} \frac{(e^{6x} - 1) - (e^{4x} - 1)}{x} \\
&= \frac{1}{7} \cdot \lim_{x\to 0} \left(\frac{e^{6x} - 1}{x} - \frac{e^{4x} - 1}{x}\right) \\
&= \frac{1}{7} \cdot \left[\lim_{x\to 0} \frac{e^{6x} - 1}{x} - \lim_{x\to 0} \frac{e^{4x} - 1}{x}\right] \\
&= \frac{1}{7} \cdot \left[\lim_{x\to 0} \frac{e^{6x} - 1}{x} \cdot \frac{6}{6} - \lim_{x\to 0} \frac{e^{4x} - 1}{x} \cdot \frac{4}{4}\right] \\
&= \frac{1}{7} \cdot \left[6\lim_{x\to 0} \frac{e^{6x} - 1}{6x} - 4\lim_{x\to 0} \frac{e^{4x} - 1}{4x}\right] \\
&= \frac{1}{7} \cdot [6(1) - 4(1)] \\
&= \frac{2}{7}.
\end{aligned}$$

1.10 Unit Test 1

Limits and Continuity

Instruction: Write all your official answers and solutions on sheets of yellow pad paper, using only either black or blue pens.

I. **Write T if the statement is true and F if otherwise.**

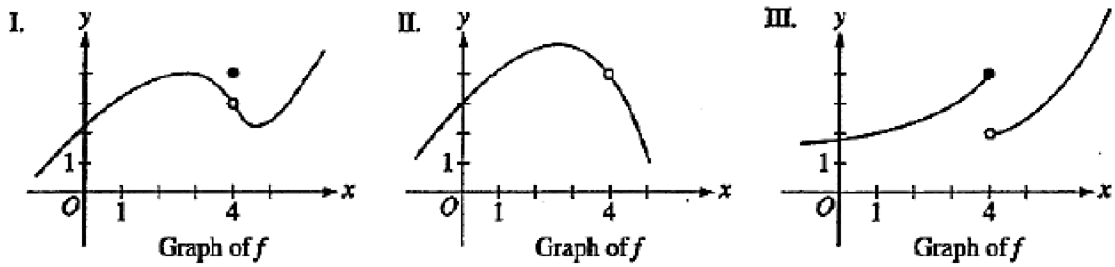
1. Let C be any real number and $\lim_{x\to a} f(x) = M$ then the $\lim_{x\to a} C[f(x)]^n = (CM)^n$.
2. If $\lim_{x\to a} f(x) = M$ then the $\lim_{x\to a^+} f(x) = M$.
3. Let M be a positive number. If $\lim_{x\to a} f(x) = M$ and $\lim_{x\to a} g(x) = 0^-$ then $\lim_{x\to a} \dfrac{f(x)}{g(x)} = -\infty$.
4. If f and g are polynomial functions in x then $\lim_{x\to 2} f(x)g(x) = f(2)g(2)$.
5. If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ then the $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ is indeterminate.
6. The $\lim_{x\to a} (f(x) + g(x))$ is always equal to $\lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.
7. If $f(x) = \dfrac{x}{|x|}$ then the $\lim_{x\to 1} f(x) = \underline{1}$.
8. The equation $3x^5 - 4x^2 = 3$ is solvable on the interval $[0, 2]$.
9. $\lim_{x\to 1^-} \sqrt{x-1} = 0$
10. The function $f(x) = \begin{cases} x-4, & \text{if } x < -4 \\ \sqrt{16-x^2}, & \text{if } -4 \le x \le 4 \\ 4-x, & \text{if } x > 4 \end{cases}$. is discontinuous at $x = 4$ and continuous at $x = -4$.

II. **Each question is a multiple-choice question with four answer choices. Read each question and answer choice carefully and choose the ONE best answer.**

1. Evaluate $\lim_{x\to +\infty} \dfrac{x^3 - 2x^2 + 3x - 4}{4x^3 - 3x^2 + 2x - 1}$.

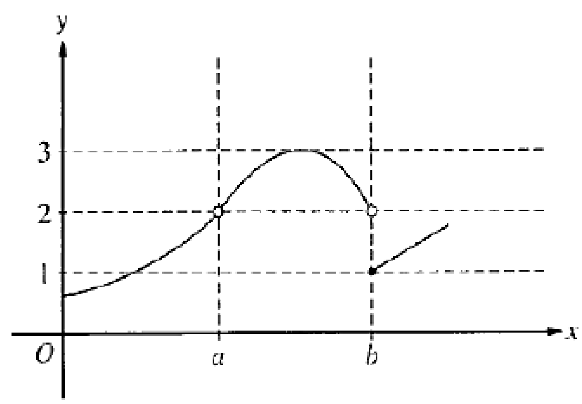
   a. 4 b. 1 c. $\dfrac{1}{4}$ d. 0

2. For which of the following does $\lim_{x\to 4} f(x)$ exists?



a. I and II b. I and III c. II and III d. I, II and III

3. The graph of the function f is shown in the figure below. Which of the following statements about f is true?



a. $\lim_{x\to a} f(x) = \lim_{x\to b} f(x)$
b. $\lim_{x\to b} f(x) = 1$
c. $\lim_{x\to a} f(x)$ does not exist
d. $\lim_{x\to a} f(x) = 2$

4. Find the interval on which the function $f(x) = \dfrac{7x-1}{x^3-4x}$ is continuous.

a. $\left(-\infty,\, \frac{1}{7}\right) \cup \left(\frac{1}{7},\, +\infty\right)$
b. $(-\infty,\, -2) \cup (-2,\, 2) \cup (2,\, +\infty)$
c. $(-\infty,\, -2) \cup (-2,\, 0) \cup (0,\, 2) \cup (2,\, +\infty)$
d. $(-\infty,\, 0) \cup (0,\, +\infty)$

5. Assume that $\lim_{x\to a} f(x) = 8$ and $\lim_{x\to a} g(x) = -9$. Evaluate $\lim_{x\to a} \dfrac{7\sqrt[3]{f(x)} - 6g(x)}{7 + g(x)}$.

a. -34 b. -55 c. 34 d. 55

For items 6 and 7: Consider the following statements below:

i. If f and g are defined at $x = a$, and $\lim_{x\to a} f(x) \neq \lim_{x\to a} g(x)$ then $f(a) \neq g(a)$.

ii. If $g(a)$ and $\lim_{x\to a} g(x)$ both exist, then g is continuous at $x = a$.

iii. If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} \frac{f(x)}{g(x)} = 2$, then $\lim_{x\to a} g(x) = 0$.

6. Which among the statements above is always true?

   a. i b. ii c. iii d. None of the above

7. Which among the statements above is/are false?

   a. iii only b. i and ii c. i and iii d. ii and iii

8. The function $g(x) = \sqrt{-x-1}$ is said to be ________ at $x = 1$.

   a. discontinuous c. continuous from the right

   b. continuous from the left d. none of the above

9. The vertical asymptote and horizontal asymptote for $f(x) = \frac{\sqrt{x}}{x+4}$ is

   a. no vertical asymptote, $y = 0$ c. $x = -4$, $y = 0$

   b. $x = -4$, no horizontal asymptote d. no vertical or horizontal asymptote

10. If $2 \leq f(x) \leq (1-x)^2 + 2$ for $x \neq 1$, then $\lim_{x\to 1} f(x) =$ ________.

    a. 1 b. 2 c. 3 d. 4

11. If $f(x) = \frac{5}{x-2}$ and $\lim_{x\to(-k+1)} f(x)$ does not exist, then $k =$ ________.

    a. 2 b. 1 c. -2 d. -1

12. If $f(x) = \begin{cases} x+2, & \text{if } x \leq -2 \\ \sqrt{4-x^2}, & \text{if } -2 < x < 2 \\ -x, & \text{if } x \geq 2 \end{cases}$, then f is discontinuous on what interval?

    a. $(-2, 2)$ b. $[-2, 2)$ c. $(-2, 2]$ d. None

13. Find a value for the constant k, that will make the function $f(x) = \begin{cases} \frac{5x+3}{x-5}, & \text{if } x \leq 1 \\ kx^2+1, & \text{if } x > 1 \end{cases}$ continuous everywhere.

    a. -2 b. -3 c. 1 d. 4

14. The limit of the following is 1 except

    a. $\lim_{x\to -2}(x^2+2x+1)$ c. $\lim_{x\to 1}\sqrt{5x-4}$

    b. $\lim_{x\to 3}\frac{2x^2-11x+15}{x-3}$ d. $\lim_{x\to -1}\frac{3x^2+5x+2}{x+1}$

15. The function $g(x) = \dfrac{3x^2+3x-12}{x-3}$ has a/an __________ at $x = 3$.

    a. continuity
    b. removable discontinuity
    c. infinite essential discontinuity
    d. jump essential discontinuity

16. The function $f(x) = \dfrac{1}{x}$ has a/an __________ at $x = 3$.

    a. continuity
    b. removable discontinuity
    c. infinite essential discontinuity
    d. jump essential discontinuity

17. Evaluate $\lim\limits_{x\to -2^+} \dfrac{x}{x^2-3x-10}$.

    a. 0 b. $+\infty$ c. $-\infty$ d. does not exist

18. Evaluate $\lim\limits_{x\to 5^-} \dfrac{x}{x^2-3x-10}$.

    a. 0 b. $+\infty$ c. $-\infty$ d. does not exist

19. Evaluate $\lim\limits_{x\to 2} \dfrac{\sin\left(x^2+5x-14\right)}{x-2}$

    a. 20 b. 5 c. 14 d. 9

20. Evaluate $\lim\limits_{x\to 0} \dfrac{e^{7x}-e^{5x}}{2x}$.

    a. 1 b. -2 c. -1 d. 2

Unit 2: Derivatives and Differentiation

This unit will cover an important class of functions called *differentiable functions*. Briefly, the study of differentiable functions and their *derivatives* is motivated by investigating rates of changes. For example, one may want to calculate the rate in which the distance travelled by a car moving along a straight path is changing with respect to time. This rate of change is commonly known in physics as *speed*. We can use calculus to find the velocity of an olympic diver the moment he hits the water in the swimming pool. All of these problems will be taken in consideration in this chapter.

Lesson 2.1: The Tangent Line Problem

Learning Outcomes

At the end of this lesson, you should be able to:

1. Derive the formula of the slope the tangent line to a given curve; and
2. Compute the slope of the tangent line to a given curve.

Consider a plane curve given by the equation $y = f(x)$ and let P be the point with coordinate $(c, f(c))$. In this section, we will give a precise definition for a *tangent line* to the curve $y = f(x)$ at the point $(c, f(c))$.

Let $\Delta x \neq 0$ and let Q be the point with coordinate $(c + \Delta x, f(c + \Delta x))$. Draw the line $\overleftrightarrow{PQ}$. Then the line $\overleftrightarrow{PQ}$ is a secant line to the curve. Observe that the slope of $\overleftrightarrow{PQ}$ is

$$m_{\overleftrightarrow{PQ}} = \frac{f(c + \Delta x) - f(c)}{\Delta x}.$$

We investigate the behavior of $\overleftrightarrow{PQ}$ and $m_{\overleftrightarrow{PQ}}$ as $\Delta x \longrightarrow 0$. We see that the secant line approaches a line ℓ. This line is called the **tangent line** to the curve $y = f(x)$ at the point $(c, f(c))$.

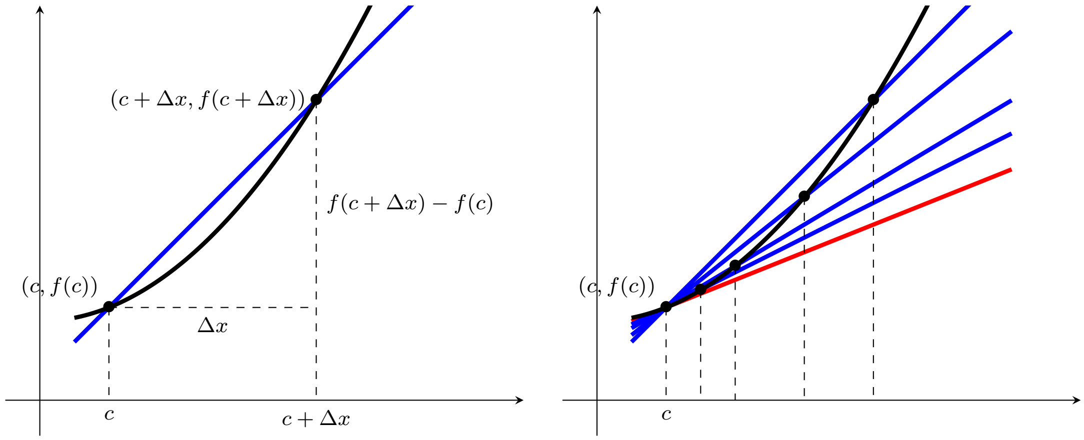


Figure 2.1: Secants Approximate the Tangent

Definition 16: Tangent Line

The **tangent line** to the curve $y = f(x)$ at $(c, f(c))$ is the line whose slope is

$$m_{\tan} = \lim_{\Delta x \longrightarrow 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

through the point $(c, f(c))$, provided that this limit exists. The equation of this tangent line is

$$y - f(c) = m_{\tan}(x - c).$$

Example 56. Find the equation of the line tangent to the graph of $y = x^2+2x+1$ at the point $(0, 1)$.

Solution. Let $f(x) = x^2 + 2x + 1$. The slope of the tangent line at $(0, 1)$ is then

$$\begin{aligned} m_{\tan} &= \lim_{\Delta x \longrightarrow 0} \frac{f(0+\Delta x) - f(0)}{\Delta x} \\ &= \lim_{\Delta x \longrightarrow 0} \frac{(\Delta x)^2 + 2\Delta x + 1 - 1}{\Delta x} \\ &= \lim_{\Delta x \longrightarrow 0} (\Delta x + 2) \\ &= 2. \end{aligned}$$

Therefore, the equation of the tangent line is $y - 1 = 2(x - 0)$ or $y = 2x + 1$.

Example 57. Find the slope of the tangent line to the graph of $y = \sqrt{x}$ at the point where $x = 4$.

Solution. Let $f(x) = \sqrt{x}$. The slope of the tangent line is

$$\begin{aligned}
m_{\tan} &= \lim_{\Delta x \longrightarrow 0} \frac{f(4+\Delta x) - f(4)}{\Delta x} \\
&= \lim_{\Delta x \longrightarrow 0} \frac{\sqrt{4+\Delta x} - 2}{\Delta x} \\
&= \lim_{\Delta x \longrightarrow 0} \frac{\sqrt{4+\Delta x} - 2}{\Delta x} \cdot \frac{\sqrt{4+\Delta x} + 2}{\sqrt{4+\Delta x} + 2} \\
m_{\tan} &= \lim_{\Delta x \longrightarrow 0} \frac{(4+\Delta x) - 4}{\Delta x(\sqrt{4+\Delta x} + 2)} \\
&= \lim_{\Delta x \longrightarrow 0} \frac{1}{\sqrt{4+\Delta x} + 2} \\
&= \frac{1}{4}.
\end{aligned}$$

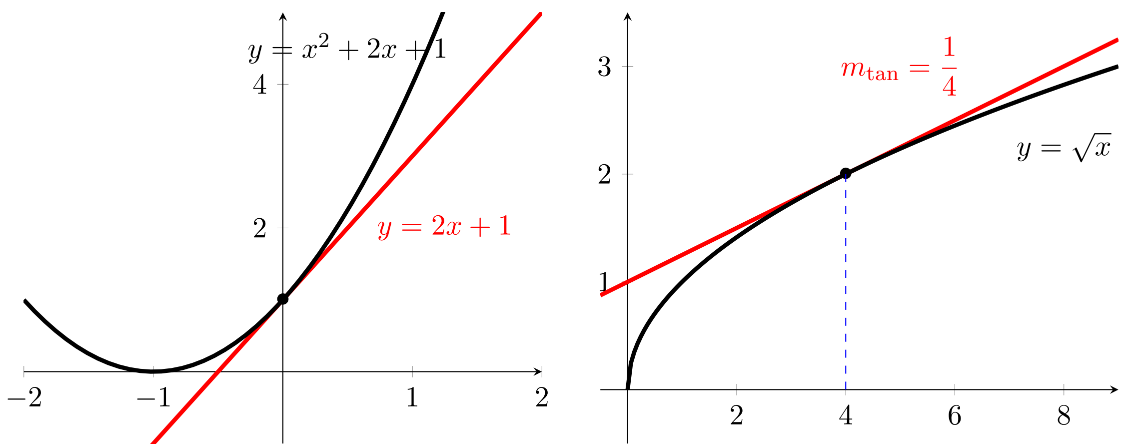


Figure 2.2: Tangent Lines

Definition 17

If the limit $m_{\tan}$ is either ∞ or $-\infty$ at $(c, f(c))$, we say that the curve $y = f(x)$ has a **vertical tangent line** at $(c, f(c))$.

Example 58. Show that $y = 2 + (x-3)^{1/3}$ has a vertical tangent line at $(3, 2)$.

Solution. Let $f(x) = 2 + (x-3)^{1/3}$. Then

$$\begin{aligned}
m_{\tan} &= \lim_{\Delta x \longrightarrow 0} \frac{f(3+\Delta x) - f(3)}{\Delta x} \\
&= \lim_{\Delta x \longrightarrow 0} \frac{[2 + (\Delta x)^{1/3}] - 2}{\Delta x} \\
&= \lim_{\Delta x \longrightarrow 0} \frac{1}{(\Delta x)^{2/3}} \\
&= \infty.
\end{aligned}$$

Therefore, $y = f(x)$ has a vertical tangent line $(3, 2)$.

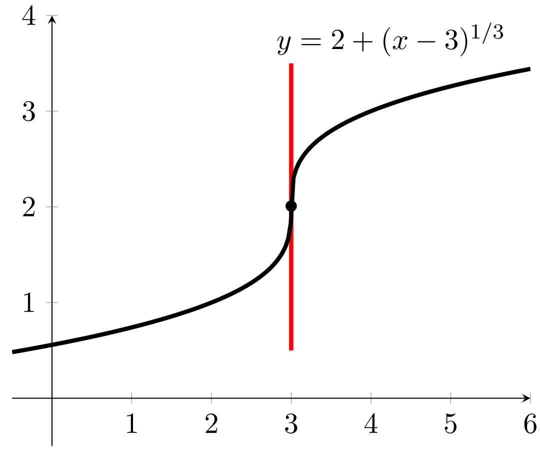


Figure 2.3: Vertical Tangent Line

Lesson 2.2: Rate of Change

Learning Outcomes

At the end of this lesson, you should be able to:

1. Use the concepts of derivatives in different scenarios; and
2. Compute problems involving rate of change.

In the beginning of this chapter, the definition of the derivative of a function was motivated by the tangent line problem. The slope of the tangent line at a point on the curve represents the derivative of the function at that point.

If a function is used to model the relationship between two variables, the derivative has a useful interpretation.

Suppose that the values of a variable y is a function of a variable x, that is $y = f(x)$ for some function f. If f is a non-constant function, the values of the *dependent variable* y varies as the values of the *independent variable* x varies. Suppose that the value of x changes from x to $x+\Delta x$ for some $\Delta x \neq 0$. Consequently, the value of y changes from y to $y+\Delta y$. Thus, we have $y+\Delta y = f(x+\Delta x)$. It follows that

$$\Delta y = f(x + \Delta x) - y = f(x + \Delta x) - f(x).$$

Dividing both sides by Δx, we get

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Note that the quantity $\dfrac{\Delta y}{\Delta x}$ can be interpreted as the *net change in y per unit change in x*. This motivates us in the following definition.

Definition 18

Suppose that the variables x and y are related as $y = f(x)$, for some function f. The **average rate of change of y with respect to x over the interval** $[x_1, x_2]$ is given by

$$r_{\text{ave}} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

If we let $\Delta x = x_2 - x_1$, then we may write

$$r_{\text{ave}} = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}.$$

Example 59. Suppose that $s = 16 - t^2$. Find the average rate of change of s with respect to t over the interval $[0, 3]$.

Solution. Let $f(t) = 16 - t^2$. Then the average rate of change of s with respect to t over $[0, 3]$ is

$$r_{ave} = \frac{f(3) - f(0)}{3 - 0} = \frac{7 - 16}{3 - 0} = -3.$$

This means that on the interval $[0, 3]$, per 1 unit change in the value of t, on the average s changes -3 units on the average.

Example 60. Find the average rate the area of a circle is changing if its radius is increased from 3 cm to 5 cm.

Solution. Note that the area of a circle is a function of its radius. In fact, for a circle of radius r, the area is $A(r) = \pi r^2$. The average rate the area of a circle with respect to its radius when the radius is increased from 3 cm to 5 cm is

$$r_{ave} = \frac{A(5) - A(3)}{5 - 3} = \frac{\pi(5^2) - \pi(3^2)}{2} = 8\pi.$$

Hence, the area of the circle changed at an average rate of 8π cm^2 per 1 cm change in its radius.

In practical applications, if a variable y is a function of another variable x, we are also interested to know the rate in which y changes with respect to x at the 'instant' when x takes a value on a particular point.

Definition 19

Suppose that the variables x and y are related as $y = f(x)$, for some function f. The **instantaneous rate of change of y with respect to x at $x = x_0$** is defined as

$$r_{\text{inst}} = f'(x_0) = \lim_{\Delta x \longrightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

The instantaneous rate of change of a function at a point is precisely the limit of the average rate of change of the function over an interval of length Δx as $\Delta x \longrightarrow 0$. This means that, if Δx is small, $r_{\text{inst}} \approx r_{\text{ave}}$, that is the difference $\left| r_{\text{inst}} - r_{\text{ave}} \right|$ is negligible.

Example 61. Suppose that $y = 3x \cos x$. Calculate the instantaneous rate of change of y with respect to x at $x = 0$. Compare the resulting value to the average rate of change of y with respect to x over the interval $[0, \pi/6]$

Solution. Let $f(x) = 3x \cos x$. By definition the instantaneous rate of change of y with respect to x at $x = 0$ is

$$r_{\text{inst}} = f'(x_0) = \left. \frac{dy}{dx} \right|_{x=0}.$$

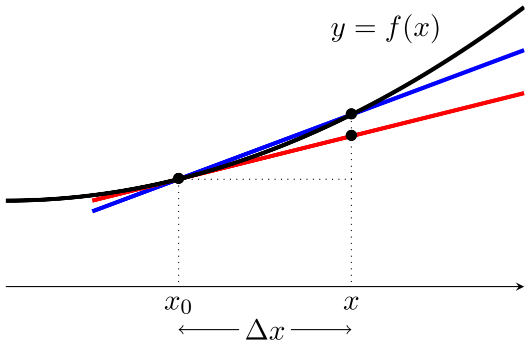


Figure 2.4: $r_{\text{ave}} \approx r_{\text{inst}}$

We calculate $\dfrac{dy}{dx}$ first.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(3x\cos x) \\
&= 3x \cdot \frac{d}{dx}(\cos x) + \cos x \cdot \frac{d}{dx}(3x) \\
&= 3x(-\sin x) + \cos x \cdot 3 \\
&= -3x\sin x + 3\cos x.
\end{aligned}$$

Therefore,

$$r_{\text{inst}} = \left.\frac{dy}{dx}\right|_{x=0} = -3(0)\sin 0 + 3\cos 0 = 3.$$

On the other hand, the average rate of change of y with respect to x over the interval $[0, \pi/6]$ is

$$r_{\text{ave}} = \frac{f(\pi/6) - f(0)}{(\pi/6) - 0} = \frac{3(\pi/6)\cos(\pi/6) - 0}{\pi/6} = \frac{3\sqrt{3}}{2} \approx 2.5981.$$

which is reasonably close to r_{inst}.

To investigate closeness of r_{ave} to r_{inst} further, we use computer aid to calculate its values as $\Delta x \longrightarrow 0$. The result is as follows.

Δx	0.250	0.200	0.150	0.100	0.010	0.001
r_{ave}	2.1951	2.4760	2.7013	2.8660	2.9987	2.9999

Lesson 2.3: Derivatives and Differentiability of Functions

Learning Outcomes

At the end of this lesson, you should be able to:

1. Define the derivative of a function;
2. Apply the definition of derivative in determining the derivative of a given function; and
3. Identify when the function is differentiable or not.

The tangent line problem motivates the following definition.

Definition 20: Differentiability at a Point

Let f be a function defined in an open interval containing c. The **derivative** of f at $x = c$ is defined as

$$f'(c) = \lim_{\Delta x \longrightarrow 0} \frac{f(c + \Delta x) - f(c)}{\Delta x},$$

provided that this limit exists. We say that f is **differentiable at** $x = c$ if $f'(c)$ exists.

Definition 21: Differentiability on an Open Interval

If $f'(x)$ exists for all $x \in (a, b)$, we say that f is **differentiable** on (a, b). If we write $y = f(x)$, then we use the notation

$$\frac{dy}{dx} = \frac{d}{dx}[f(x)] = f'(x) = \lim_{\Delta x \longrightarrow 0} \frac{\Delta y}{\Delta x},$$

where

$$\Delta y = f(x + \Delta x) - f(x).$$

We note that the expression $\frac{dy}{dx}$ is not to be interpreted as a fraction. It is however a limit of fractions. The expression $\frac{dy}{dx}$ is read as the **derivative of y with respect to x.**

Example 62 (Derivative of Constants)**.** Let $f(x) = C$ be a constant function. Then f is differentiable at every point in the real number line. In fact, for all real number x,

$$f'(x) = \lim_{\Delta x \longrightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \longrightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \longrightarrow 0} \frac{C - C}{\Delta x} = 0.$$

We express this fact by writing

$$\frac{d}{dx}(C) = 0,$$

that is, the derivative of a constant is 0.

Example 63 (Derivative of the Identity Function)**.** Let $f(x) = x$. If x is a real number, then

$$f'(x) = \lim_{\Delta x \longrightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \longrightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \longrightarrow 0} \frac{(x+\Delta x) - x}{\Delta x} = 1.$$

Therefore, we have the following.

$$\frac{d}{dx}(x) = 1.$$

Example 64. Find $\dfrac{dy}{dx}$ if $y = x^2 - 4x + 1$.

Solution. This can be done by setting $f(x) = x^2 - 4x + 1$. Then

$$\begin{aligned}
\frac{dy}{dx} &= f'(x) \\
&= \lim_{\Delta x \longrightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \longrightarrow 0} \frac{[(x+\Delta x)^2 - 4(x+\Delta x) + 1] - (x^2 - 4x + 1)}{\Delta x} \\
&= \lim_{\Delta x \longrightarrow 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - 4x - 4\Delta x + 1 - x^2 + 4x - 1}{\Delta x} \\
&= \lim_{\Delta x \longrightarrow 0} \frac{2x\Delta x + (\Delta x)^2 - 4\Delta x}{\Delta x} \\
&= \lim_{\Delta x \longrightarrow 0} \frac{\Delta x(2x + \Delta x - 4)}{\Delta x} \\
&= \lim_{\Delta x \longrightarrow 0} 2x + \Delta x - 4 \\
&= 2x - 4.
\end{aligned}$$

Therefore, $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^2 - 4x + 1) = 2x - 4$. We remark that the derivative $2x - 4$ is the slope of the tangent line to $y = x^2 - 4x + 1$ at the point $(x, f(x))$. In particular, the slope of the tangent line to $y = x^2 - 4x + 1$ at the point $(2, -3)$ is

$$\left.\frac{dy}{dx}\right|_{x=2} = (2x - 4)\Big|_{x=2} = 2(2) - 4 = 0.$$

Note. The expression $\left.\dfrac{dy}{dx}\right|_{x=c}$ means to evaluate the derivative at $x = c$.

Example 65. Let $s = \sqrt{t+2}$. Evaluate $\left.\dfrac{ds}{dt}\right|_{t=7}$.

Solution. Let $s = f(t) = \sqrt{t+2}$. Then

$$\begin{aligned}
\frac{ds}{dt} &= \lim_{\Delta t \longrightarrow 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} \\
&= \lim_{\Delta t \longrightarrow 0} \frac{\sqrt{(t+\Delta t)+2} - \sqrt{t+2}}{\Delta t} \\
&= \lim_{\Delta t \longrightarrow 0} \frac{(t+\Delta t+2) - (t+2)}{\Delta t(\sqrt{t+\Delta t+2} + \sqrt{t+2})} \\
&= \lim_{\Delta t \longrightarrow 0} \frac{1}{\sqrt{t+\Delta t+2} + \sqrt{t+2}} \\
&= \frac{1}{2\sqrt{t+2}}.
\end{aligned}$$

Therefore,

$$\left.\frac{ds}{dt}\right|_{t=7} = \frac{1}{2\sqrt{7+2}} = \frac{1}{6}.$$

Theorem 17: Continuity Implies Differentiability

If a function f is differentiable at c, then f is continuous at c.

Proof. Let f be differentiable at c. Then

$$f'(c) = \lim_{\Delta x \longrightarrow 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

exists. Hence,

$$\begin{aligned}
\lim_{\Delta x \longrightarrow 0} [f(c+\Delta x) - f(c)] &= \lim_{\Delta x \longrightarrow 0} \left[\Delta x \cdot \frac{f(c+\Delta x) - f(c)}{\Delta x}\right] \\
&= \lim_{\Delta x \longrightarrow 0} \Delta x \cdot \lim_{\Delta x \longrightarrow 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} \\
&= 0 \cdot f'(c) \\
&= 0.
\end{aligned}$$

Hence, it follows that $\lim_{\Delta x \longrightarrow 0} f(c+\Delta x) = f(c)$. Letting $\Delta x = x - c$, we see that $x \longrightarrow c$ as $\Delta x \longrightarrow 0$. Therefore,

$$\lim_{x \longrightarrow c} f(x) = \lim_{\Delta x \longrightarrow 0} f(c+\Delta x) = f(c)$$

which proves that f is continuous at c. $\square$

The previous theorem implies that the set of all differentiable functions is a subset of the set of all continuous functions. However, its converse is not true. There exists a function continuous at a point but not differentiable there.

Example 66 (Functions With Sharp Turns)**.** Consider the absolute value function

$$f(x) = |x|.$$

We look at the differentiability of f at $x = 0$. Let $\Delta x \neq 0$. Then

$$\frac{f(0+\Delta x) - f(0)}{\Delta x} = \frac{|\Delta x| - 0}{\Delta x} = \frac{|\Delta x|}{\Delta x}.$$

Thus,

$$\lim_{\Delta x \longrightarrow 0^+} \frac{f(0+\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \longrightarrow 0^+} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x \longrightarrow 0^+} \frac{\Delta x}{\Delta x} = 1.$$

On the other hand,

$$\lim_{\Delta x \longrightarrow 0^-} \frac{f(0+\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \longrightarrow 0^-} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x \longrightarrow 0^-} \frac{-\Delta x}{\Delta x} = -1.$$

Therefore, $f'(0) = \lim_{\Delta x \longrightarrow 0} \frac{f(0+\Delta x) - f(0)}{\Delta x}$ does not exist and that f is not differentiable at $x = 0$ although f is continuous at 0.

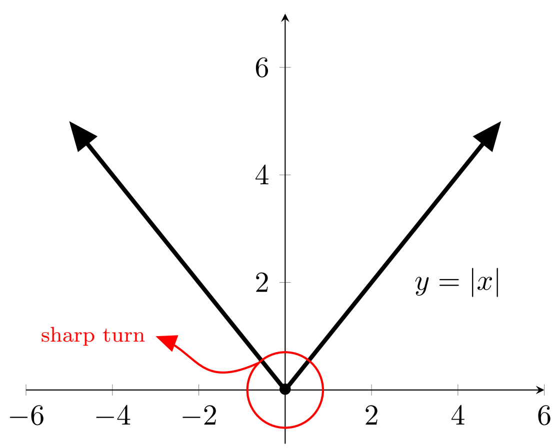


Figure 2.5: a non-differentiable function

The previous example motivates us in the following definition.

Definition 22

If a function f is defined on an open interval (a, c), we define the **left-derivative** of f at c as

$$f'_-(c) = \lim_{\Delta x \longrightarrow 0^-} \frac{f(c + \Delta x) - f(c)}{\Delta x},$$

provided the limit exists. On the other hand, if f is defined on an open interval (c, b), we define the **right-derivative** of f at c as

$$f'_+(c) = \lim_{\Delta x \longrightarrow 0^+} \frac{f(c + \Delta x) - f(c)}{\Delta x},$$

provided that this limit exists.

Immediately, we have the following result.

Theorem 18:

Let a function f be continuous at c. Then f is differentiable at c if and only if $f'_-(c)$ and $f'_+(c)$ both exist and $f'_-(c) = f'_+(c)$. In this case, the common value is $f'(c)$.

Example 67. Consider the function $f(x) = \begin{cases} 1 + x^2 & \text{if } x \geq 0; \\ 1 & \text{if } x < 0. \end{cases}$ We investigate the differentiability of f at 0. Since f is piecewise-defined, we make use of Theorem 18. We first check for the continuity of f at 0 by looking at the one-sided limits.

- $\lim\limits_{x \longrightarrow 0^-} f(x) = \lim\limits_{x \longrightarrow 0^-} 1 = 1.$
- $\lim\limits_{x \longrightarrow 0^+} f(x) = \lim\limits_{x \longrightarrow 0^+} (1 + x^2) = 1.$

Thus, $\lim\limits_{x \longrightarrow 0} f(x) = 1 = f(0)$ and that f is continuous at 0. We now look at the one-sided derivatives of f at 0.

- $f'_-(0) = \lim\limits_{\Delta x \longrightarrow 0^-} \dfrac{f(0 + \Delta x) - f(0)}{\Delta x} = \lim\limits_{\Delta x \longrightarrow 0^-} \dfrac{1 - 1}{\Delta x} = 0.$
- $f'_+(0) = \lim\limits_{\Delta x \longrightarrow 0^+} \dfrac{f(0 + \Delta x) - f(0)}{\Delta x} = \lim\limits_{\Delta x \longrightarrow 0^-} \dfrac{1 + (0 + \Delta x)^2 - 1}{\Delta x} = 0.$

Since $f'_-(0) = f'_+(0)$, we can conclude from Theorem 18 that f is differentiable at 0 and $f'(0) = 0$.

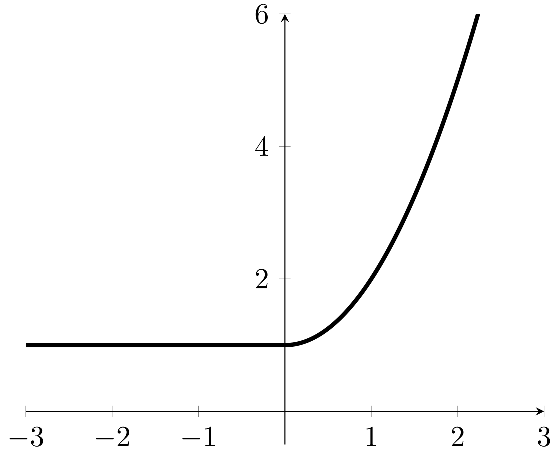


Figure 2.6: The Graph of $f(x) = 1$, if $x < 0$ and $f(x) = 1 + x^2$, if $x \geq 0$

Remark 8

If we let $x = x_0 + \Delta x$, where $\Delta x \longrightarrow 0$, then $x \longrightarrow x_0$ as $\Delta \longrightarrow 0$. Hence, we may also write

$$f'(x_0) = \lim_{\Delta x \longrightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Lesson 2.4: Rules of Differentiation

Learning Outcomes

At the end of this lesson, you should be able to:

1. Enumerate the differentiation formula; and
2. Compute the derivatives using differentiation rules.

For a differentiable function f, the process of obtaining a formula for its derivative $f'(x)$ is called **differentiation**. In this section, we present techniques in performing differentiation of well-known functions.

The following are the basic rules of differentiation.

Theorem 19: Basic Rules of Differentiation

(1) (Constant Rule) If C is a constant, then

$$\frac{d}{dx}(C) = 0.$$

(2) If f and g are differentiable functions, then so is $f+g$ and $(f+g)' = f' + g'$, that is

$$\frac{d}{dx}[f(x)+g(x)] = f'(x) + g'(x) = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)].$$

(3) If k is a constant and f is a differentiable function, then $k \cdot f$ is differentiable and $(k \cdot f)' = k \cdot f'$, that is

$$\frac{d}{dx}[k \cdot f(x)] = f \cdot f'(x) = k \cdot \frac{d}{dx}[f(x)].$$

(4) (Product Rule) If f and g are differentiable functions, then fg is differentiable and $(fg)' = fg' + gf'$, that is

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x) = f(x) \cdot \frac{d}{dx}[g(x)] + g(x) \cdot \frac{d}{dx}[f(x)].$$

(5) (Quotient Rule) If f and g are differentiable functions and $g(x) \neq 0$, then $\frac{f}{g}$ is differ-

entiable and $\left(\dfrac{f}{g}\right)' = \dfrac{gf' - fg'}{g^2}$, that is

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} = \frac{g(x) \cdot \frac{d}{dx}[f(x)] - f(x) \cdot \frac{d}{dx}[g(x)]}{[g(x)]^2}$$

We only prove rules number (2) and (4). The proof for other rules can be done similarly.

Proof. For number (2), note that

$$\begin{aligned}
\frac{d}{dx}[f(x) + g(x)] &= \lim_{\Delta x \longrightarrow 0} \frac{[f(x+\Delta x) + g(x+\Delta x)] - [f(x) + g(x)]}{\Delta x} \\
&= \lim_{\Delta x \longrightarrow 0} \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} + \frac{g(x+\Delta x) - g(x)}{\Delta x}\right] \\
&= \lim_{\Delta x \longrightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \longrightarrow 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&= f'(x) + g'(x).
\end{aligned}$$

For number (4), observe that

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{\Delta x \longrightarrow 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \longrightarrow 0} \frac{f(x+\Delta x)[g(x+\Delta x) - g(x)] + g(x)[f(x+\Delta x) - f(x)]}{\Delta x} \\
&= \lim_{\Delta x \longrightarrow 0} f(x+\Delta x)\left[\frac{g(x+\Delta x) - g(x)}{\Delta x}\right] + \lim_{\Delta x \longrightarrow 0} g(x)\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right].
\end{aligned}$$

Since f is differentiable, then Theorem 17 implies that f is continuous so

$$\lim_{\Delta x \longrightarrow 0} f(x+\Delta x) = f(x).$$

Therefore,

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x),$$

which is the desired result. □

The following is an important differentiation formula used in differentiating functions involving powers.

Theorem 20: Power Rule

Let n be an integer. Then

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}.$$

Proof. If $n = 0$, then the power rule holds using Theorem 19. Assume first that n is a positive integer. We prove the result by induction on n.

Let $n = 1$. Then
$$\frac{d}{dx}(x^n) = \frac{d}{dx}x = 1 = 1x^0 = nx^{n-1}.$$
Assume that the formula holds for $n = k$. Let $n = k + 1$. Then
$$\begin{aligned}
\frac{d}{dx}(x^n) &= \frac{d}{dx}(x^{k+1}) \\
&= \frac{d}{dx}(x^k \cdot x) \\
&= x^k \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(x^k) \\
&= x^k \cdot 1 + x \cdot kx^{k-1} \\
&= x^k + kx^k \\
&= (k+1)x^{(k+1)-1} \\
&= nx^{n-1}.
\end{aligned}$$

Thus, the formula also holds for $n = k + 1$. The principle of mathematical induction implies that the formula holds for all positive integer n.

Finally, if n is a negative integer, we let $m = -n$ which is positive. Then
$$\begin{aligned}
\frac{d}{dx}(x^n) &= \frac{d}{dx}\left(\frac{1}{x^m}\right) \\
&= \frac{x^m \cdot \frac{d}{dx}(1) - 1 \cdot \frac{d}{dx}(x^m)}{(x^m)^2} \\
&= \frac{mx^{m-1}}{x^{2m}} \\
&= -mx^{-m-1} \\
&= nx^{n-1}.
\end{aligned}$$

Thus, the formula also holds if n is negative. This completes the proof. □

We now see how we can apply these rules in the following examples. For convenience, we abbreviate the operator $\frac{d}{dx}$ by D_x.

Example 68. Find the derivative of $f(x) = -25x^3 + \frac{5}{2}x^2 - 2x + 35 - \frac{2}{x}$.

Solution.

$$\begin{aligned}
f'(x) &= D_x\left(-25x^3 + \frac{5}{2}x^2 - 2x + 35 - \frac{2}{x}\right) \\
&= D_x(-25x^3) + D_x\left(\frac{5}{2}x^2\right) + D_x(-2x) + D_x(35) + D_x\left(-\frac{2}{x}\right) \\
&= -25D_x(x^3) + \frac{5}{2}D_x(x^2) - 2D_x(x) + 0 + -2 \cdot D_x(x^{-1}) \\
&= -75x^2 + 5x - 2 + \frac{2}{x^2}
\end{aligned}$$

Example 69. Find $f'(x)$ if $f(x) = x(x+2)^2$.

Solution. We first expand $x(x+2)^2$.

$$x(x+2)^2 = x(x^2 + 4x + 4) = x^3 + 4x^2 + 4x.$$

Thus,

$$\begin{aligned}
f'(x) &= D_x(x(x+2)^2) \\
&= D_x(x^3 + 4x^2 + 4x) \\
&= D_x(x^3) + 4D_x(x^2) + 4D_x(x) \\
&= 3x^2 + 8x + 4.
\end{aligned}$$

Example 70. Find $\frac{ds}{dt}$ if $s = (3 - t^2)(4t - t^3)$.

Solution. In this case, we can apply the product rule as follows.

$$\begin{aligned}
\frac{ds}{dt} &= D_t[(3 - t^2)(4t - t^3)] \\
&= (3 - t^2) \cdot D_t(4t - t^3) + (4t - t^3) \cdot D_t(3 - t^2) \\
&= (3 - t^2)(4 - 3t^2) + (4t - t^3)(-2t) \\
&= (12 - 13t + 3t^4) + (-8t^2 + 2t^4) \\
&= 12 - 13t - 8t^2 + 5t^4.
\end{aligned}$$

Example 71. (Equation of a Tangent Line) Find the slope of the tangent line to the curve $y = \frac{x^2 - 4x + 3}{x + 5}$ at the point where $x = 0$.

Solution. Note that the slope of the tangent line to $y = \frac{x^2 - 4x + 3}{x + 5}$ at the point where $x = 0$ is the value of the derivative $\frac{dy}{dx}$ at $x = 0$. Thus, we first calculate $\frac{dy}{dx}$ using the quotient rule as

$$\begin{aligned}
\frac{dy}{dx} &= D_x\left(\frac{x^2-4x+3}{x+5}\right) \\
&= \frac{(x+5)\cdot D_x(x^2-4x+3)-(x^2-4x+3)\cdot D_x(x+5)}{(x+5)^2} \\
&= \frac{(x+5)(2x-4)-(x^2-4x+3)(1)}{(x+5)^2} \\
&= \frac{(2x^2+6x-20)-(x^2-4x+3)}{(x+5)^2} \\
&= \frac{x^2+10x-23}{(x+5)^2}.
\end{aligned}$$

Hence, the slope of the tangent line at the point where $x = 0$ is $\left.\frac{dy}{dx}\right|_{x=0} = \left.\frac{x^2+10x-23}{(x+5)^2}\right|_{x=0} = -\frac{23}{25}$. To get the equation of the line, we need to find the point of tangency. If $x = 0$, observe that $y = \frac{3}{5}$ so the point of tangency is $(0, 3/5)$. Therefore, the equation of the tangent line is $y = -\frac{23}{25}x + \frac{3}{5}$.

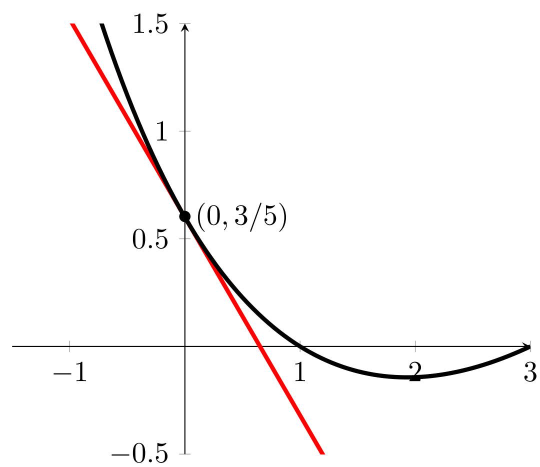


Figure 2.7: The Tangent Line (red) to $y = \frac{x^2-4x+3}{x+5}$ at $(0, 3/5)$

Example 72. (Horizontal Tangent Lines) Determine the point(s) on the graph of

$$y = \frac{x^2}{x^2+1}$$

where there is a horizontal tangent line.

Solution. Note that a tangent line is horizontal if and only if its slope is 0, that is the point where the derivative is 0. We first compute $\dfrac{dy}{dx}$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{x^2}{x^2+1}\right) \\
&= \frac{(x^2+1)\cdot D_x(x^2) - x^2\cdot D_x(x^2+1)}{(x^2+1)^2} \\
&= \frac{(x^2+1)(2x) - x^2(2x)}{(x^2+1)^2} \\
&= \frac{2x}{(x^2+1)^2}
\end{aligned}$$

Thus, $\dfrac{dy}{dx} = 0$ if and only if $\dfrac{2x}{(x^2+1)^2} = 0$ if and only if $2x = 0$ if and only if $x = 0$. Therefore, the curve has a horizontal tangent line at the point where $x = 0$, that is, at the point $(0, 0)$. (Since $y = 0$ whenever $x = 0$.)

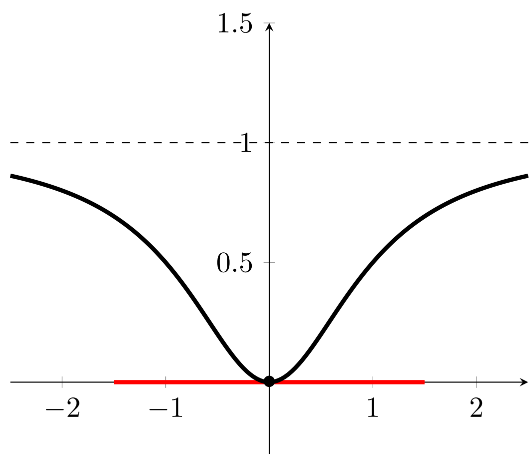


Figure 2.8: Horizontal Tangent Line to $y = \dfrac{x^2}{x^2+1}$

It is worth noting that the function $f(x) = \dfrac{x^2}{x^2+1}$ has an absolute minimum of 0 attained at $x = 0$.

Lesson 2.5: Derivatives of Trigonometric Functions

Learning Outcomes

At the end of this lesson, you should be able to:

1. Derive the differentiation formula for trigonometric functions;
2. Enumerate the formula of differentiation for trigonometric functions; and
3. Compute the derivative of trigonometric functions.

We turn now our attention to the trigonometric functions. Recall that the six trigonometric functions are continuous on their respective domains. In this section, we look at the differentiability of the trigonometric functions and we provide formulas for their derivatives.

We begin with the sine and cosine functions. Note that

$$\frac{d}{dx}\sin x = \lim_{\Delta x \longrightarrow 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x}.$$

Using the fact that $\sin(u+v) = \sin u \cos v + \cos u \sin v$, we see that

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{\Delta x \longrightarrow 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x} \\
&= \lim_{\Delta x \longrightarrow 0} \frac{\sin x \cos(\Delta x) + \cos x \sin(\Delta x) - \sin x}{\Delta x}. \\
&= \sin x \cdot \lim_{\Delta x \longrightarrow 0} \left[\frac{\cos(\Delta x) - 1}{\Delta x}\right] + \cos x \cdot \lim_{\Delta x \longrightarrow 0} \left[\frac{\sin(\Delta x)}{\Delta x}\right].
\end{aligned}$$

Recalling that $\lim_{\theta \longrightarrow 0} \dfrac{\sin\theta}{\theta} = 1$ and $\lim_{\Delta x \longrightarrow 0} \dfrac{\cos\theta - 1}{\theta} = 0$, we see that $\dfrac{d}{dx}(\sin x) = \sin x \cdot 0 + \cos x \cdot 1 = \cos x$.

A similar computation shows that $\dfrac{d}{dx}(\cos x) = -\sin x$. The student is advised to verify this an an exercise.

Theorem 21: Derivatives of Sine and Cosine Functions

The sine and cosine functions are differentiable. Moreover,

$$\frac{d}{dx}(\sin x) = \cos x \quad \text{and} \quad \frac{d}{dx}(\cos x) = -\sin x.$$

Example 73. Find the derivative of $f(x) = \sin 2x$.

Solution. Using the *double-angle identity* for the sine function, $\sin 2x = 2\sin x \cos x$. The student may refer to precalculus texts for details. Thus,

$$\begin{aligned} f'(x) &= D_x(\sin 2x) \\ &= D_x(2\sin x\cos x) \\ &= 2\cdot D_x(\sin x\cos x) \\ &= 2[\sin x\cdot D_x(\cos x)+\cos x\cdot D_x(\sin x)] \\ &= 2[\sin x(-\sin x)+\cos x(\cos x)] \\ &= 2(\cos^2 x-\sin^2 x). \end{aligned}$$

Observe that $f'(x) = 2(\cos^2 x - \sin^2 x) = 2\cos 2x$.

Example 74. Determine the slope of the tangent line to the graph of $y = \sin x(\cos x - 1)$ at the point where $x = \dfrac{\pi}{4}$.

Solution. The slope of the tangent line is just $\left.\dfrac{dy}{dx}\right|_{x=\pi/4}$. Observe that

$$\begin{aligned} \frac{dy}{dx} &= D_x[\sin x(\cos x-1)] \\ &= \sin x\cdot D_x(\cos x-1)+(\cos x-1)\cdot D_x(\sin x) \\ &= \sin x(-\sin x)+(\cos x-1)\cos x \\ &= -\sin^2 x+\cos^2 x-\cos x. \end{aligned}$$

Therefore, the slope of the tangent line at the point where $x = \dfrac{\pi}{4}$ is

$$\left.\frac{dy}{dx}\right|_{x=\pi/4} = -\sin^2(\pi/4)+\cos^2(\pi/4)-\cos(\pi/4) = -\frac{1}{2}+\frac{1}{2}-\frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}.$$

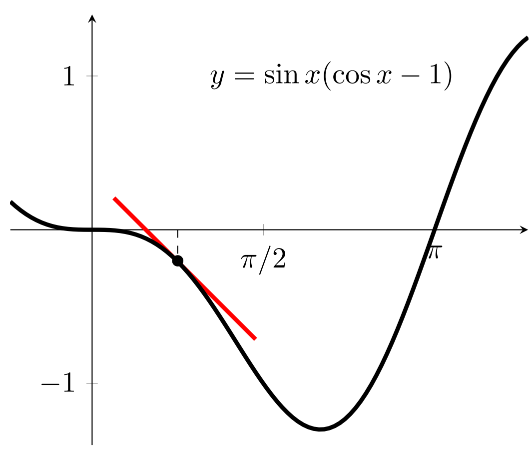


Figure 2.9: The Graph of $y = \sin x(\cos x - 1)$ and the Tangent Line at $x = \pi/4$

Using the derivatives of the sine and cosine functions we can derive the formulas for the derivatives of the other trigonometric functions. We simply note that the other four trigonometric functions can be written in terms of sine and cosine. For the tangent function, observe that

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\cos x \cdot D_x(\sin x) - \sin x \cdot D_x(\cos x)}{\cos^2 x} \\
&= \frac{\cos x \cos x + \sin x \sin x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
\frac{d}{dx}(\tan x) &= \sec^2 x.
\end{aligned}$$

For the secant function, we compute the derivative as follows.

$$\begin{aligned}
\frac{d}{dx}(\sec x) &= \frac{d}{dx}\left(\frac{1}{\cos x}\right) \\
&= \frac{\cos x \cdot D_x(1) - 1 \cdot D_x(\cos x)}{\cos^2 x} \\
&= \frac{\sin x}{\cos^2 x} \\
&= \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) \\
&= \sec x \tan x.
\end{aligned}$$

The derivatives for $\cot x$ and $\csc x$ can be derived similarly.

Theorem 22

The following functions are differentiable in their respective domains. Moreover,

(1) $\dfrac{d}{dx}(\tan x) = \sec^2 x$

(2) $\dfrac{d}{dx}(\cot x) = -\csc^2 x$

(3) $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

(4) $\dfrac{d}{dx}(\csc x) = -\csc x \cot x.$

Example 75. Find the derivative of $f(x) = \dfrac{\cot x}{1 - \sec x}$.

Solution.

$$\begin{aligned} f'(x) &= \frac{(1 - \sec x) \cdot D_x(\cot x) - \cot x \cdot D_x(1 - \sec x)}{(1 - \sec x)^2} \\ &= \frac{(1 - \sec x)(-\csc^2 x) - \cot x(-\sec x \tan x)}{1 - 2\sec x + \sec^2 x} \\ &= \frac{-\csc^2 x + \sec x \csc^2 x + \cot x \sec x \tan x}{\tan^2 x - 2\sec x} \\ &= \frac{-\csc^2 x + \sec x(\csc^2 x - 1)}{\tan^2 x - 2\sec x} \\ &= \frac{-\csc^2 x + \sec x \cot^2 x}{\tan^2 x - 2\sec x} \end{aligned}$$

Example 76. Evaluate the following limit. $\displaystyle\lim_{x \longrightarrow \pi} \frac{\sec x + 1}{x - \pi}$

Solution. First, observe that $\sec(\pi) = -1$. Thus,

$$\lim_{s \longrightarrow \pi} \frac{\sec x + 1}{x - \pi} = \lim_{x \longrightarrow \pi} \frac{\sec x - \sec \pi}{x - \pi}.$$

Let $\Delta x = x - \pi$. Observe that $\Delta x \longrightarrow 0$ as $x \longrightarrow \pi$. Thus, we have

$$\begin{aligned} \lim_{s \longrightarrow \pi} \frac{\sec x + 1}{x - \pi} &= \lim_{x \longrightarrow \pi} \frac{\sec x - \sec \pi}{x - \pi} \\ &= \lim_{\Delta x \longrightarrow 0} \frac{\sec(\pi + \Delta x) - \sec \pi}{\Delta x} \\ &= \frac{d}{dx}(\sec x)\bigg|_{x=\pi} \\ &= \sec x \tan x\bigg|_{x=\pi} \\ &= \sec \pi \tan \pi \\ &= -1 \cdot 0 \\ &= 0. \end{aligned}$$

Lesson 2.6: The Chain Rule

Learning Outcomes

At the end of this lesson, you should be able to:

1. Generalize differentiation formula using chain rule; and
2. Solve the derivative of a given function involving chain rule.

Suppose that we are tasked to take the derivative of $h(x) = (x-1)^{100}$. If we are to apply the basic rules of differentiation, then we must expand $(x-1)^{100}$ first then apply the formulas. But this is not an easy work. As one may notice, the function is just the same as the function $g(x) = x^{100}$ which is 'shifted' 1 unit to the right. Hence, $h'(x) = g'(x-1) = 100(x-1)^{99}$. Observe that $h = g \circ f$, where $f(x) = x - 1$.

The following theorem makes this observation more precise.

Theorem 23: Chain Rule

If a function f is differentiable at x and a function g is differentiable at $f(x)$, then $g \circ f$ is differentiable at x and

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x).$$

Alternatively, if we let $u = f(x)$ and $y = g(u)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

It is important to note that the expression

$$\frac{dy}{du} \cdot \frac{du}{dx}$$

is not to be interpreted as fractions to be multiplied. Although, it helps on remembering the formula whenever needed. The proof of the chain rule is not trivial and we omit it in this text. For those who are utterly curious, the proof begins by considering the following.

$$\frac{g(f(x)) - g(f(x_0))}{x - x_0} = \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \frac{f(x) - f(x_0)}{x - x_0},$$

if $x \neq x_0$. If As $x \longrightarrow x_0$, it can be shown that $f(x) \longrightarrow f(x_0)$. (Why?)

Example 77. Let $h(x) = (2x-1)^{25}$. One observes that if we let $f(x) = 2x - 1$ and $g(x) = x^{25}$, then $h(x) = g(f(x))$. Therefore, by the chain rule,

$$h'(x) = g'(f(x)) \cdot f'(x) = 25(2x-1)^{24} \cdot 2 = 50(2x-1)^{24}.$$

Example 78. Find $\dfrac{dy}{dx}$ if $y = \tan\left(\dfrac{x+1}{x-1}\right)$.

Solution. Let $u = \dfrac{x+1}{x-1}$. Then $y = \tan u$. Using the chain rule, we see that

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \sec^2 u \cdot \frac{(x-1)-(x+1)}{(x-1)^2} \\ &= \frac{-2\sec^2 u}{(x-1)^2} \\ &= \frac{-2\sec^2\left(\dfrac{x+1}{x-1}\right)}{(x-1)^2}.\end{aligned}$$

Example 79. Determine the slope of the tangent line to the curve whose equation is $y = (1+\cos x)^3$ at the point where $x = \pi/4$.

Solution. First, we calculate $\dfrac{dy}{dx}$ using the chain rule. Let $u = 1 + \cos x$. Then $y = u^3$. So,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= 3u^2 \cdot (-\sin x) \\ &= -3(1+\cos x)^2(-\sin x) \\ &= 3\sin x\,(1+\cos x)^2.\end{aligned}$$

The slope of the tangent line is then

$$\left.\frac{dy}{dx}\right|_{x=\pi/4} = 2\sin(\pi/4) \cdot (1+\cos(\pi/4))^2 = \frac{4+3\sqrt{2}}{2} \approx 4.12.$$

Example 80. The position function of an object moving along a straight line segment is given by $s(t) = 2(4t-2)^2 + 5(4t-2)$. Find its acceleration at any time t.

Solution. By taking note that the acceleration of an object is given by the function $a(t) = v'(t) = s''(t)$. We first calculate $v(t)$. Let $f(t) = 4t - 2$ and $g(t) = 2t^2 + 5t$. Then $s(t) = g(f(t))$. Hence, by the chain rule,

$$\begin{aligned} v(t) = s'(t) &= g'(f(t)) \cdot f'(t) \\ &= (4(4t-2)+5) \cdot 4 \\ &= 64t - 12. \end{aligned}$$

Therefore, $a(t) = 64$. This means that the object is *uniformly accelerated*, that is, its acceleration is constant.

The chain rule can be generalized even when a function is a composition of more than two functions.

Theorem 24: Generalized Chain Rule

Suppose that

$$y = f_1(u_1),\, u_1 = f_2(u_2), u_2 = f_3(u_3), \cdots, u_n = f_n(x),$$

where $f_1,\, f_2, \ldots, f_n$ are differentiable functions. Then

$$\frac{dy}{dx} = \frac{dy}{du_1} \cdot \frac{du_1}{du_2} \cdot \frac{du_2}{du_3} \cdot \cdots \cdot \frac{du_n}{dx}.$$

Example 81. Find the derivative of the function $f(x) = \cos^3\left(\sin(2x^2-1)\right)$.

Solution. Note that if we let $u = 2x^2 - 1$, $v = \sin u$, $w = \cos v$ and $y = w^3$, then $y = f(x)$. The chain rule implies that

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dw} \cdot \frac{dw}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} \\ &= 3w^2 \cdot (-\sin v) \cdot \cos u \cdot 4x \\ &= 3\cos^2 v \cdot (-\sin(\sin u)) \cdot \cos(2x^2-1) \cdot 4x \\ &= 3\cos^2(\sin u) \cdot (-\sin(\sin(2x^2-1))) \cdot \cos(2x^2-1) \cdot 4x \\ &= 3\cos^2(\sin(2x^2-1)) \cdot (-\sin(\sin(2x^2-1))) \cdot \cos(2x^2-1) \cdot 4x. \end{aligned}$$

Recall that for any integer n, we have $\dfrac{d}{dx}(x^n) = nx^{n-1}$. It turns out that this formula still holds even when n is any rational number. We are now ready to prove this result.

Lemma 1

Let $q \geq 2$ be an integer and let $f(x) = x^{1/q}$. Then $f'(x) = (1/q)x^{(1/q)-1}$.

Proof. We first recall from algebra that

$$(a-b)^q = (a-b)(a^{q-1} + a^{q-2}b + a^{q-3}b^2 + \cdots + ab^{q-2} + b^{q-1}),$$

for any real numbers a and b.

Let $\Delta x \neq 0$. Then

$$\begin{aligned}
\frac{f(x+\Delta x)-f(x)}{\Delta x} &= \frac{(x+\Delta x)^{1/q}-x^{1/q}}{\Delta x} \\
&= \frac{(x+\Delta x)^{1/q}-x^{1/q}}{\Delta x} \cdot \frac{(x+\Delta x)^{((q-1)/q)}+(x+\Delta x)^{((q-2)/q)}x+\cdots+x^{(q-1)/q}}{(x+\Delta x)^{((q-1)/q)}+(x+\Delta x)^{((q-2)/q)}x+\cdots+x^{(q-1)/q}} \\
&= \frac{(x+\Delta x)-x}{\Delta x[(x+\Delta x)^{((q-1)/q)}+(x+\Delta x)^{((q-2)/q)}x+\cdots+x^{(q-1)/q}]} \\
&= \frac{1}{(x+\Delta x)^{((q-1)/q)}+(x+\Delta x)^{((q-2)/q)}x+\cdots+x^{(q-1)/q}}
\end{aligned}$$

Letting $\Delta x \longrightarrow 0$, we see that the above equation implies

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \longrightarrow 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \\
&= \frac{1}{x^{((q-1)/q)}+x^{((q-1)/q)}+\cdots+x^{((q-1)/q)}} \\
&= \frac{1}{qx^{((q-1)/q)}} \\
&= \frac{1}{q}x^{(1/q)-1}.
\end{aligned}$$

□

We now have the following theorem.

Theorem 25

Let p and q be relatively prime integers, where $q \geq 2$. Then

$$\frac{d}{dx}(x^{p/q}) = \frac{p}{q}x^{(p/q)-1}.$$

Proof. To prove this, we simply apply the preceding lemma, and the chain rule. Note that

$$\begin{aligned}
\frac{d}{dx}(x^{p/q}) &= \frac{d}{dx}(x^{1/q})^p \\
&= p(x^{1/q})^{p-1} \cdot \frac{d}{dx}(x^{1/q}) \\
&= p(x^{1/q})^{p-1} \cdot \frac{1}{q}x^{(1/q)-1} \\
&= \frac{p}{q}x^{(p/q)-1}.
\end{aligned}$$

□

Corollary 1: Power Formula for Rational Exponents

For every rational number r,
$$\frac{d}{dx}(x^r) = rx^{r-1}.$$

Example 82. If $y = \sqrt[3]{x^2+1}$, then

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\sqrt[3]{x^2+1} \\
&= \frac{d}{dx}(x^2+1)^{1/3} \\
&= \frac{1}{3}(x^2+1)^{(1/3)-1}\frac{d}{dx}(x^2+1) \\
&= \frac{1}{3}(x^2+1)^{-2/3}\cdot 2x \\
&= \frac{2x}{3\sqrt[3]{(x^2+1)^2}}
\end{aligned}$$

Example 83. Let $f(x) = \dfrac{x}{\sqrt{x^4+4}}$. Find $f'(0)$.

Solution. Note that

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left[\frac{x}{\sqrt{x^4+4}}\right] \\
&= \frac{\sqrt{x^4+4}\cdot\frac{d}{dx}x - x\cdot\frac{d}{dx}\sqrt{x^4+4}}{(\sqrt{x^4+4})^2} \\
f'(x) &= \frac{\sqrt{x^4+4} - x\left[\frac{1}{2}(x^4+4)^{-1/2}\cdot 4x^3\right]}{x^4+4} \\
&= \frac{\sqrt{x^4+4} - \frac{2x^4}{\sqrt{x^4+4}}}{x^4+4} \\
&= \frac{(x^4+4)-2x^4}{(x^4+4)^{3/2}} \\
&= \frac{4-x^4}{(x^4+4)^{3/2}}.
\end{aligned}$$

Lesson 2.7: Derivatives of Other Transcendental Function

2.7.1 Derivatives of Exponential Function and Logarithmic Function

Learning Outcomes

At the end of this lesson, you should be able to:

1. Derive the differentiation formula for exponential functions and logarithmic functions; and
2. Compute the derivative of exponential functions and logarithmic function.

Another type of transcendental function is the exponential function. Recall that an exponential function is a function of the form $f(x) = a^x$ where $a > 0$ and $a \neq 1$. In this section, we look at the differentiability of the exponential functions and provide formula for their derivatives.

Let $f(x) = a^x$ where $a > 0$ and $a \neq 1$. Then, by definition of derivatives

$$f'(x) = \frac{d}{dx}(a^x) = \lim_{\Delta x \longrightarrow 0} \frac{a^{x+\Delta x} - a^x}{\Delta x} = \lim_{\Delta x \longrightarrow 0} \frac{a^x(a^{\Delta x} - 1)}{\Delta x}.$$

It can be simplified as $f'(x) = \dfrac{d}{dx}(a^x) = \lim_{\Delta x \longrightarrow 0} a^x \cdot \dfrac{a^{\Delta x} - 1}{\Delta x}$. Note that $\lim_{x \longrightarrow 0} \dfrac{a^x - 1}{x} = f'(0)$. Hence,

$$f'(x) = \frac{d}{dx}(a^x) = a^x f'(0)$$

.

Therefore, we have shown that if the exponential function $f(x) = a^x$ is differentiable at 0, then it is differentiable everywhere. Of all possible choices for the base a, the simplest differentiation formula occurs when $f'(0) = 1$. Recall that $\lim_{x \longrightarrow 0} \dfrac{e^x - 1}{x} = 1$. Hence, $f'(0) = 1$ if $a = e$.

Theorem 26: Derivative of e^x

$$\frac{d}{dx}(e^u) = e^u dx$$

The inverse function of the exponential function is called logarithmic function. This function is in the form of $f(x) = \log_a x$ where $a > 0$ and $a \neq 1$. When $a = e$, the logarithmic function is in its natural form and we write $f(x) = \log_a x = \ln x$.

To determine the derivative of logarithmic function, we use the fact the if $y = \ln x$ then $e^y = x$. Hence, $\dfrac{d}{dx}(e^y) = \dfrac{d}{dx}(x)$ implies that $e^y y' = 1$ or $y' = \dfrac{1}{e^y} = \dfrac{1}{x}$ since $x = e^y$.

Theorem 27: Derivative of $\ln x$

$$\frac{d}{dx}\ln u = \frac{1}{u}dx$$

To determine the derivative of $\log_a x$, we us the fact that $\log_a x = \dfrac{\ln x}{\ln a}$. Hence,

$$\frac{d}{dx}(\log_a x) = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{\ln a}\cdot\frac{d}{dx}(\ln x) = \frac{1}{x\ln a}$$

Theorem 28: Derivative of $\log_a x$

$$\frac{d}{dx}(\log_a U) = \frac{1}{u\ln a}dx$$

Now, let us back to the derivative of a^x. Note that if $y = a^x$, then it can be written as $\log_a y = x$. Hence, $\dfrac{d}{dx}(\log_a y) = \dfrac{d}{dx}(x)$ implies that $\dfrac{1}{y\ln a}y' = 1$. Therefore, $y' = y\ln a = a^x \ln a$ since $y = a^x$.

Theorem 29: Derivative of a^x

$$\frac{d}{dx}(a^u) = a^u \ln a dx$$

Example 84. Find the derivative of $f(x) = e^{4x}$.

Solution. Using the formula,

$$f'(x) = \frac{d}{dx}(e^{4x}) = e^{4x}\frac{d}{dx}(4x) = 4e^{4x}$$

Example 85. Find the derivative of $f(x) = 3^x$.

Solution. Using the formula,

$$f'(x) = \frac{d}{dx}(3^x) = 3^x \cdot \ln 3$$

Example 86. Find the derivative of $y = \ln 2x$.

Solution. Using the formula,

$$y' = \frac{d}{dx}(\ln 2x) = \frac{1}{2x}\cdot\frac{d}{dx}(2x) = \frac{1}{2x}\cdot 2 = \frac{1}{x}$$

Example 87. Find the derivative of $y = \log_3 2x$.

Solution. Using the formula,

$$y' = \frac{d}{dx}(\log_3 2x) = \frac{1}{2x\ln 3} \cdot \frac{d}{dx}(2x) = \frac{1}{2x\ln 3} \cdot 2 = \frac{1}{x\ln 3}$$

2.7.2 Derivative of Inverse Trigonometric Function

Learning Outcomes

At the end of this lesson, you should be able to:

1. Derive the differentiation formula for inverse trigonometric functions; and
2. Compute the derivative of inverse trigonometric functions.

Now, let us discuss the derivative of inverse trigonometric function. Consider the inverse trigonometric function of sine, $y = \text{Arcsin}x$. To determine the derivative of this function, we will use the fact that $y = \text{Arcsin}x$ is basically $\sin y = x$.

Then, let us differentiate both sides of the equation with respect x, $\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$. It implies that $\cos y y' = 1$ or $y' = \frac{1}{\cos y}$. Note that if we represent the equation $\sin y = x$ using right triangle, then the measure of opposite side of angle y is x and the measure of hypotenuse is 1. Hence, the measure of the opposite side of angle y is $\sqrt{1-x^2}$.

Therefore,

$$y' = \frac{d}{dx}(\text{Arcsin}x) = \frac{1}{\cos y} = \frac{1}{\frac{\sqrt{1-x^2}}{1}} = \frac{1}{\sqrt{1-x^2}}.$$

Similarly, other inverse trigonometric functions can be proved using this way.

Theorem 30: Derivative of Inverse Trigonometric Functions

Let u be a differentiable function. Then,

1. $\frac{d}{dx}(\text{Arcsin}u) = \frac{1}{\sqrt{1-u^2}}dx$
2. $\frac{d}{dx}(\text{Arccos}u) = -\frac{1}{\sqrt{1-u^2}}dx$
3. $\frac{d}{dx}(\text{Arctan}u) = \frac{1}{1+u^2}dx$
4. $\frac{d}{dx}(\text{Arccot}u) = -\frac{1}{1+u^2}dx$
5. $\frac{d}{dx}(\text{Arcsec}u) = \frac{1}{u\sqrt{u^2-1}}dx$
6. $\frac{d}{dx}(\text{Arccsc}u) = -\frac{1}{u\sqrt{u^2-1}}dx$

Example 88. Find the derivative of $y = \text{Arcsin}x^2$.

Solution. Using the formula,

$$y' = \frac{d}{dx}(\text{Arcsin}x^2) = \frac{1}{\sqrt{1-(x^2)^2}} \cdot \frac{d}{dx}(x^2) = \frac{1}{\sqrt{1-x^4}} \cdot 2x = \frac{2x}{\sqrt{1-x^4}}$$

Example 89. Find the derivative of $y = \text{Arccot}e^x$.

Solution. Using the formula,

$$y' = \frac{d}{dx}(\text{Arccot}e^x) = -\frac{1}{1+(e^x)^2} \cdot \frac{d}{dx}(e^x) = -\frac{1}{1+e^{2x}} \cdot e^x = -\frac{e^x}{1+e^{2x}}$$

Example 90. Find the derivative of $y = \text{Arcsec}(\ln x)$.

Solution. Using the formula,

$$y' = \frac{d}{dx}(\text{Arccot}(\ln x)) = \frac{1}{(\ln x)\sqrt{(\ln x)^2 - 1}} \cdot \frac{d}{dx}(\ln x) = \frac{1}{\ln x\sqrt{\ln^2 x - 1}} \cdot \frac{1}{x} = \frac{1}{x \ln x\sqrt{\ln^2 x - 1}}$$

2.7.3 Derivatives of Hyperbolic Functions

Learning Outcomes

At the end of this lesson, you should be able to:

1. Enumerate different hyperbolic identities;
2. Derive the differentiation formula for hyperbolic functions; and
3. Compute the derivatives of hyperbolic functions.

The last transcendental function that we will look into is the hyperbolic function. The combination of terms e^x and e^{-x} arise often in physical situation. Hence, the combination of these terms gave way to create new functions. These are called hyperbolic functions. They are defined as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x}$$

$$\text{sech}x = \frac{1}{\cosh x}$$
$$\text{csch}x = \frac{1}{\sinh x}$$

Similar to trigonometric functions, hyperbolic functions also have identities. The following are some of the basic identities of hyperbolic functions:

$$\sinh(-x) = -\sinh x$$
$$\cosh(-x) = \cosh x$$
$$\cosh^2 x - \sinh^2 x = 1$$
$$1 - \tanh^2 x = \text{sech}^2 x$$

Since the hyperbolic functions are defined in terms of exponential functions, finding their derivatives are fairly simple.

Theorem 31: Derivative of Hyperbolic Functions

Let u be a differentiable function. Then,

1. $\frac{d}{dx}\sinh u = \cosh u dx$
2. $\frac{d}{dx}\cosh u = \sinh u dx$
3. $\frac{d}{dx}\tanh u = \text{sech}^2 u dx$
4. $\frac{d}{dx}\coth u = -\text{csch}^2 u dx$
5. $\frac{d}{dx}\text{sech}u = -\text{sech}u \tanh u dx$
6. $\frac{d}{dx}\text{csch}u = -\text{csch}u \coth u dx$

Proof.

(1) $\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$

(2) $\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x$

(3) $\frac{d}{dx}(\tanh x) = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = \frac{\cosh x \frac{d}{dx}(\sinh x) - \sinh x \frac{d}{dx}(\cosh x)}{\cosh^2 x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$

$$\frac{d}{dx}(\tanh x) = \frac{1}{\cosh^2 x} = \text{sech}^2 x$$

(4), (5) and (6) can be done in the same manner. □

Example 91. Find the derivative of $y = \sinh 2x$.

Solution. Using the formula,

$$y' = \frac{d}{dx}(\sinh 2x) = \cosh 2x \frac{d}{dx}(2x) = \cosh 2x \cdot 2 = 2\cosh 2x$$

Example 92. Find the derivative of $y = \cosh(3x - 4)$.

Solution. Using the formula,

$$y' = \frac{d}{dx}(\cosh(3x-4)) = \sinh(3x-4)\frac{d}{dx}(3x-4) = \sinh(3x-4) \cdot 3 = 3\sinh(3x-4)$$

Example 93. Find the derivative of $y = \tanh(x^2)$.

Solution. Using the formula,

$$y' = \frac{d}{dx}(\tanh x^2) = \text{sech}^2(x^2)\frac{d}{dx}(x^2) = \text{sech}^2(x^2) \cdot 2x = 2x\text{sech}^2(x^2)$$

Lesson 2.8: Implicit Differentiation

Learning Outcomes

At the end of this lesson, you should be able to:

1. Differentiate explicit and implicit functions; and
2. Enumerate the steps in evaluating the derivative of an implicit functions.

In many instances, it is not possible to express the equation of a curve in the form $y = f(x)$, for some function f. For example, the circle with center at $(0, 0)$ with radius r has an equation

$$x^2 + y^2 = r.$$

Solving for y leads to

$$y^2 = r^2 - x^2$$
$$\implies y = \pm\sqrt{r^2 - x^2},$$

which gives to values for y and therefore obtaining two functions $y = \sqrt{r^2 - x^2}$ and $y = -\sqrt{r^2 - x^2}$. In this case, we say that $x^2 - y^2 = r^2$ is in *implicit form* while the equations $y = \sqrt{r^2 - x^2}$ and $y = -\sqrt{r^2 - x^2}$ are in *explicit form*. In general, an equation in x and y are said to be in **implicit form** if it takes the form

$$F(x, y) = C,$$

for some constant C. On the other hand, equations of the form $y = f(x)$ are said to be in **explicit form**.

In this section, we study how we can compute for $\frac{dy}{dx}$ given an implicit equation for x and y without explicitly solving y in terms of x. This process is called **implicit differentiation**.

Consider an implicit equation $F(x, y) = C$. The following are the guidelines in performing implicit differentiation.

Guidelines 1: Guidelines in Performing Implicit Differentiation

1. Differentiate both sides of $F(x, y) = C$ with respect to x while consider y as a function of x. Thus, $\frac{d}{dx}(y) = y'$.
2. Collect all the terms with y' on the left-hand side and factor y'.
3. Solve for y'.

We note that performing implicit differentiation will express y' in terms of both x and y.

Example 94. Find y' if $x^2y + xy^2 = 5$.

Solution. We apply $\dfrac{d}{dx}$ on both sides as follows.

$$\frac{d}{dx}(x^2y + xy^2) = \frac{d}{dx}(5)$$

$$\implies \frac{d}{dx}(x^2y) + \frac{d}{dx}(xy^2) = 0$$

$$\implies \left[x^2\frac{d}{dx}(y) + y\frac{d}{dx}(x^2)\right] + \left[x\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x)\right] = 0$$

$$\implies x^2y' + y(2x) + x(2yy') + y^2 = 0$$

$$\implies y'(x^2 + 4xy) = -y^2$$

$$\implies y' = \frac{-y^2}{x^2 + 4xy}.$$

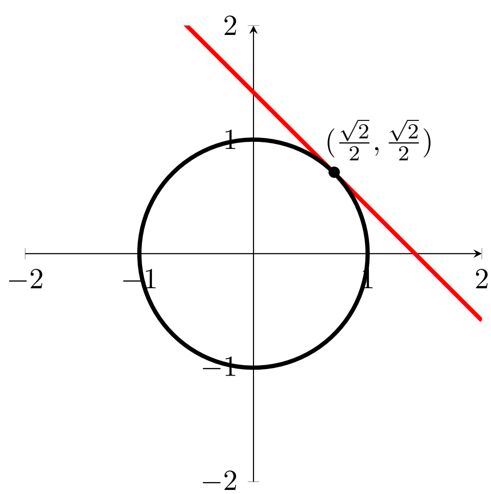
Example 95. Determine the equation of the tangent line to the unit circle $x^2 + y^2 = 1$ at the point $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

Solution. First, we solve for the slope of the tangent line by implicitly computing for y' on $x^2 + y^2 = 1$.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1)$$

$$\implies 2x + 2yy' = 0$$

$$\implies y' = -\frac{x}{y}.$$



Hence, the slope of the tangent line at $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ is the value of y' when $x = \frac{\sqrt{2}}{2}$ and $y = \frac{\sqrt{2}}{2}$. That is,

$$y' = -\frac{x}{y} = -\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = -1.$$

Furthermore, if the equation of the line is $y = -x + b$ and since the line passes through $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$,

it follows that
$$\frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2} + b \implies b = \sqrt{2}.$$
Therefore, the equation of the tangent line is $y = -x + \sqrt{2}$.

Example 96. (Finding Second Derivatives Implicitly) Find $\dfrac{d^2y}{dx^2}$ if $x^2y^2 - 2x = 3$.

Solution. We first calculate $\dfrac{dy}{dx}$ using implicit differentiation.

$$\frac{d}{dx}(x^2y^2 - 2x) = \frac{d}{dx}(3)$$

$$\implies x^2\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x^2) - 2 = 0$$

$$\implies 2x^2yy' + 2xy^2 - 2 = 0$$

$$\implies y' = \frac{1 - xy^2}{x^2y}$$

Differentiating both sides of the resulting equation once more yields

$$y'' = \frac{d}{dx}\left(\frac{1 - xy^2}{x^2y}\right)$$

$$\implies y'' = \frac{x^2y\frac{d}{dx}(1 - xy^2) - (1 - xy^2)\frac{d}{dx}(x^2y)}{x^4y^2}$$

$$\implies y'' = \frac{x^2y(-2xyy' - y^2) - (1 - xy^2)(x^2y' + 2xy)}{x^4y^2}$$

$$\implies y'' = \frac{(-2x^3y^2 - x^2 + x^3y^2)y' + (x^2y^3 - 2xy + 2x^2y^3)}{x^4y^2}$$

$$\implies y'' = \frac{(-x^3y^2 - x^2)\left(\frac{1 - xy^2}{x^2y}\right) + (3x^2y^3 - 2xy)}{x^4y^2}$$

Lesson 2.9: Higher Order Derivatives

Learning Outcomes

At the end of this lesson, you should be able to:

1. Identify the uses of some higher derivatives; and
2. Compute the higher derivatives of a given function.

In the previous section, we learned how to obtain the velocity function of an object, that is, the velocity function of an object moving along a straight line is simply the derivative of its position function. In physical applications, the velocity of a moving object is not usually constant. Thus, we are interested to know the rate in which the velocity of the object changes over time. This rate is called the object's **acceleration**. Hence, the acceleration of the object is the derivative of the velocity function, which is the derivative of its position function. This motivates us in the following definition.

Definition 23

Let f be a differentiable function and let $y = f(x)$. We define the **second derivative** of f as the function

$$\frac{d^2y}{dx^2} = y'' = f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

The **third derivative** of f is the function

$$\frac{d^3y}{dx^3} = y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right).$$

Recursively, for any integer $n \geq 2$, we define the **nth derivative** of f as the function

$$\frac{d^ny}{dx^n} = y^{(n)} = f^{(n)}(x) = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right).$$

Example 97. If $y = 3x^3 + 4x^2 - 3x + 1$, prove that $y^{(4)} = 0$.

Proof. Note that

$$y' = 9x^2 + 8x - 3,$$

from which it follows that

$$y'' = 18x + 8.$$

Differentiating the function once more, we see that

$$y''' = 18.$$

Therefore, $y^{(4)} = 0$ since y''' is constant.

FACT: Observe that if $y = f(x)$ defines a polynomial function of degree n, then

$$\frac{d^n y}{dx^n} = n!.$$

Hence, it follows that $f^{(k)}(x) = 0$, for $k > n$.

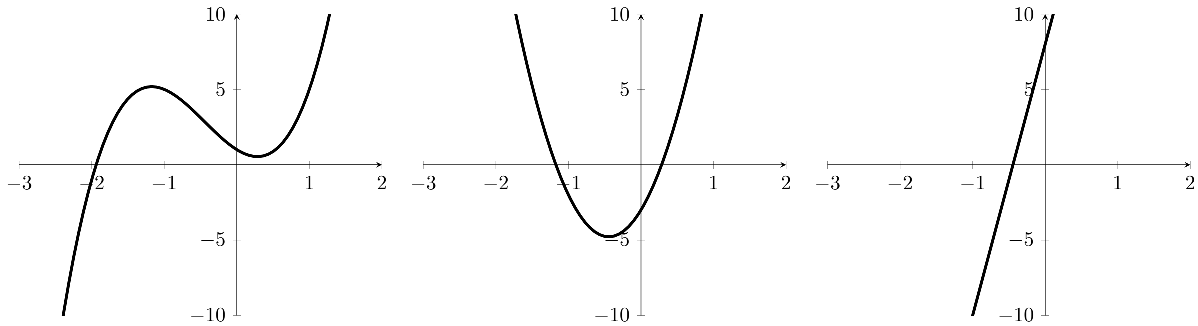


Figure 2.10: The Graphs of $y = 3x^3 + 4x^2 - 3x + 1$, y' and y''

Example 98. Let $y = x^3(x + \cos x)$. Determine $\dfrac{d^2y}{dx^2}$.

Solution. Note that

$$\begin{aligned}\frac{dy}{dx} &= x^3 \cdot D_x(x + \cos x) + (x + \cos x) \cdot D_x(x^3)\\ &= x^3(1 - \sin x) + (x + \cos x) \cdot 3x^2\\ &= 4x^3 - x^3 \sin x + 3x^2 \cos x.\end{aligned}$$

It now follows that

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left[4x^3 - x^3 \sin x + 3x^2 \cos x\right]\\ &= 12x^2 - [x^3 \cdot D_x(\sin x) + \sin x \cdot D_x(x^3)] + [3x^2 \cdot D_x(\cos x) + \cos x \cdot D_x(3x^2)]\\ &= 12x^2 - x^3 \cos x - 3x^2 \sin x - 3x^2 \sin x + 6x \cos x\\ &= 12x^2 - (x^3 - 6x) \cos x - 6x^2 \sin x.\end{aligned}$$

Example 99. Let $f(x) = |x|^3$. Prove that $f'(0)$, $f''(0)$ both exist but $f'''(0)$ does not exist.

Proof. First, observe that we can alternatively express $f(x)$ as

$$f(x) = \begin{cases} x^3 & \text{if } x \geq 0;\\ -x^3 & \text{if } x < 0.\end{cases}$$

Observe that $lim_{x\to 0} f(x) = 0 = f(0)$ and hence, f is continuous at 0. We look at the one-sided derivatives of f at 0:

$$f'_-(0) = D_x(-x^3)\Big|_{x=0} = -3x^2\Big|_{x=0} = 0,$$

and

$$f'_+(0) = D_x(x^3)\Big|_{x=0} = 3x^2\Big|_{x=0} = 0.$$

Hence, $f'(0) = 0$ and

$$f'(x) = \begin{cases} 3x^2 & \text{if } x \geq 0 \\ -3x^2 & \text{if } x < 0. \end{cases}$$

We again see that f' is continuous at $x = 0$ by noting that $lim_{x \to 0} f'(x) = 0 = f'(0)$. We look at the one-sided derivatives of f' at $x = 0$:

$$f''_-(0) = D_x(-3x^2)\Big|_{x=0} = -6x\Big|_{x=0} = 0,$$

and

$$f''_+(0) = D_x(3x^2)\Big|_{x=0} = 6x\Big|_{x=0} = 0.$$

Therefore, $f''(0) = 0$ and

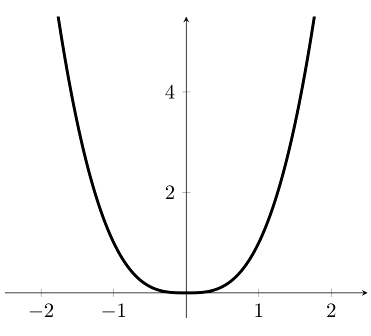
$$f''(x) = \begin{cases} 6x & \text{if } x \geq 0; \\ -6x & \text{if } x < 0. \end{cases}$$

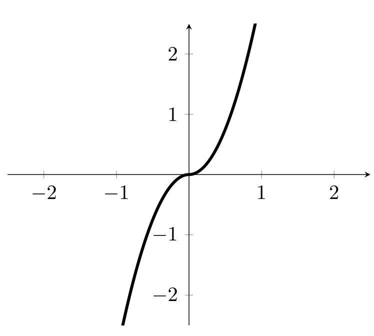
Note that f'' is still continuous at $x = 0$, however,

$$f'''_-(0) = -6 \neq 6 = f'''_+(0).$$

Therefore, $f'''(0)$ does not exist.

The previous example shows that not because the derivative of a function exists, we can be guaranteed that this derivative is differentiable.





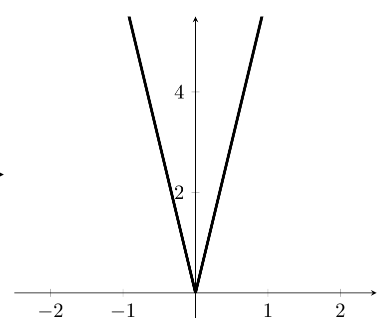


Figure 2.11: The Graphs of $f(x) = |x|^3$, $f'(x)$, and $f''(x)$

We end this section by defining what is meant by the acceleration of an object moving along a straight line.

Definition 24

Suppose that $s = s(t)$ represents the position function of a particle moving along a straight line segment. The **acceleration** of the particle is defined as the derivative of its velocity function, that is,

$$a(t) := v'(t) = s''(t).$$

Note that for a *free falling object*, that is, a falling object where the only force affecting the motion of the object is gravity, the acceleration is proven to be constant, that is $a(t) = k$, for some constant k. The value of the constant k depends on where the object is falling. Physicists have shown through their experiments that a free falling object towards the surface of the earth accelerates at a rate of 9.8 meters per second squared or equivalently, 32 feet per second squared. This value is commonly known as the **acceleration due to the gravity of the earth** and is usually denoted by $\vec{g}$.

The constant acceleration due to the gravity towards a surface of a planet or moon is affected by the body's mass. For example, astronauts have made several experiments to measure the acceleration due to gravity in the moon. Because the moon has no atmosphere, hence, no air resistance, their experiments leads to the conclusion that the acceleration due to the moon's gravity is 1.62 meters per second squared. Hence, a man weighs 6 times more on earth than on moon.

For an object moving along a straight line with a constant acceleration a, it can be shown that its position function is

$$s(t) = s_0 + v_0 t + \frac{1}{2}at^2,$$

where $s_0 = s(0)$ is the initial position of the object and $v_0 = v(0)$ is its initial velocity. One checks that

$$v(t) = s'(t) = v_0 + at$$

and

$$a(t) = v'(t) = a.$$

Lesson 2.10: L'Hôpital's Rule

Learning Outcomes

At the end of this lesson, you should be able to:

1. Enumerate different indeterminate forms; and
2. Solve limits involving indeterminate form.

Recall:

Indeterminate Forms: $\frac{0}{0},\ \frac{\pm\infty}{\pm\infty},\ \infty-\infty,\ 0\cdot\infty,\ 0^0,\ 1^\infty,\ \infty^0$

Theorem 32: L'Hôpital's Rule

Let I be an open interval and $c \in I$. Let f and g be differentiable functions on $I-\{c\}$. If $g'(x) \neq 0$, for all $x \in I-\{c\}$, and $\lim\limits_{x\to c} f(x) = 0 = \lim\limits_{x\to c} g(x)$ or $\lim\limits_{x\to c} f(x) = \pm\infty = \lim\limits_{x\to c} g(x)$, then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)},$$

provided that the right-hand side of the equation exists.

Example 100. Evaluate the following limits.

1. $\lim\limits_{x\to 0}\dfrac{e^{-3x}-1}{x}$
2. $\lim\limits_{x\to\infty}\dfrac{\ln x}{x}$
3. $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3}$
4. $\lim\limits_{x\to\infty} x^2e^{-x}$

Solutions:

1. $\lim\limits_{x\to 0}\dfrac{e^{-3x}-1}{x} = \lim\limits_{x\to 0}\dfrac{-3e^{-3x}}{1} = \dfrac{-3e^0}{1} = \dfrac{-3}{1} = -3$
2. $\lim\limits_{x\to\infty}\dfrac{\ln x}{x} = \lim\limits_{x\to\infty}\dfrac{\frac{1}{x}}{1} = \dfrac{1}{x} = 0$
3. $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3} = \lim\limits_{x\to 0}\dfrac{1-\cos x}{3x^2} = \lim\limits_{x\to 0}\dfrac{\sin x}{6x} = \lim\limits_{x\to 0}\dfrac{\cos x}{6} = \dfrac{1}{6}$
4. $\lim\limits_{x\to\infty} x^2e^{-x} = \lim\limits_{x\to\infty}\dfrac{x^2}{e^x} = \lim\limits_{x\to\infty}\dfrac{2x}{e^x} = \lim\limits_{x\to\infty}\dfrac{2}{e^x} = 0$

2.11 Unit Test 2

Derivatives and Differentiation

Instruction: Write all your official answers and solutions on sheets of yellow pad paper, using only either black or blue pens.

I. **Each question is a multiple-choice question with four answer choices. Read each question and answer choice carefully and choose the ONE best answer.**

1. Find the slope of the tangent line to the parabola $y = x^2 + 2x$ at the point $(-3, 3)$.

   a. -8
   b. -4
   c. 4
   d. 8

2. In the previous item, what is the equation of the tangent line?

   a. $4x + y + 9 = 0$
   b. $4x - y + 15 = 0$
   c. $8x + y + 21 = 0$
   d. $8x - y + 27 = 0$

3. Consider the table below for the population of a certain city from 1990 to 2020:

Year	1990	2000	2010	2020
Population	10,036	15,832	20,145	23,456

   What is the average rate of change from 1990 to 2020?

   a. 617
   b. 671
   c. 716
   d. 761

4. What is the instantaneous rate of change of $f(x) = x^3 + 2x$ at $x = -1$?

   a. -3
   b. 1
   c. 3
   d. 5

5. Which of the following is true when a function $f(x)$ is differentiable at $x = c$?

   a. $\lim_{x \to a} f(x) = c$.
   b. $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$
   c. $f(x)$ is continuous at $x = c$.
   d. The graph of $f(x)$ has a sharp turn in $x = c$.

6. Which of the following is the first derivative of $y = x\sqrt{x}$?

   a. $\frac{1}{2}x^{-1/2}$
   b. $\frac{1}{2}x^{1/2}$
   c. $\frac{3}{2}x^{-1/2}$
   d. $\frac{3}{2}x^{1/2}$

7. Find the first derivative of the the function $y = x + \sqrt{x}$.

   a. $1 + \frac{1}{2}x^{-1/2}$
   b. $1 + \frac{1}{2}x^{1/2}$
   c. 1 - $\frac{1}{2}x^{-1/2}$
   d. 1 - $\frac{1}{2}x^{1/2}$

8. Find the first derivative of $y = \dfrac{x}{x^2+1}$.

   a. $y' = -\dfrac{1}{x^2+1}$
   b. $y' = \dfrac{1}{x^2+1}$
   c. $y' = \dfrac{1-x^2}{(x^2+1)^2}$
   d. $y' = \dfrac{1+x^2}{(x^2+1)^2}$

9. What is the first derivative of $f(x) = \cos 3x^2$?

   a. $f'(x) = -6x\cos 3x^2$
   b. $f'(x) = 6x\cos 3x^2$
   c. $f'(x) = -6x\sin 3x^2$
   d. $f'(x) = 6x\sin 3x^2$

10. Which of the following function will not result to even function derivative?

    a. $y = 3\sin 2x$
    b. $f(x) = \dfrac{5}{\tan x}$
    c. $y = \csc 3x$
    d. $f(x) = \sec x$

11. What is the first derivative of the function $f(x) = \cos(\sin(2x))$?

    a. $f'(x) = -2\cos 2x\cos(\sin 2x)$
    b. $f'(x) = -2\cos 2x\sin(\sin 2x)$
    c. $f'(x) = 2\cos 2x\cos(\sin 2x)$
    d. $f'(x) = 2\cos 2x\sin(\sin 2x)$

12. Consider the function $y = (x^2+1)^3$. Which of the following expressions may equal to its derivative?

    a. $6x^4 + 12x^2 + 6$
    b. $6x^4 - 12x^2 + 6$
    c. $6x^5 + 12x^3 + 6x$
    d. $6x^5 - 12x^3 + 6x$

13. Which of the following functions has a first derivative $y' = \tan x^3 + 3x^3 \sec^2 x^3$?

a. $y = \tan x^3 + \sec x^3$
b. $y = \tan x^3 \sec x^3$
c. $y = x \sec x^3$
d. $y = x \tan x^3$

14. What is the first derivative of the function $y = \dfrac{1}{(x+1)^5}$?

a. $y' = \dfrac{-5}{(x+1)^4}$
b. $y' = \dfrac{10}{(x+1)^4}$
c. $y' = \dfrac{-5}{(x+1)^{10}}$
d. $y' = \dfrac{10}{(x+1)^{10}}$

15. Which of the following expressions is the first derivative of the function $y = 4^x$?

a. $y' = 4^x$
b. $y' = \dfrac{4^x}{\ln 4}$
c. $y' = 4^x \ln 4$
d. $y' = x4^{x-1}$

16. Find the first derivative of $y = 2^x e^x$?

a. $y' = 2^x e^x (\ln 2)$
b. $y' = 2^x e^x (1 + \ln 2)$
c. $y' = 2^x \ln e + e^x \ln 2$
d. $y' = \dfrac{2^x}{\ln 2} + \dfrac{e^x}{\ln e}$

17. Find the first derivative of the function $f(x) = \ln(\sin(x^2))$.

a. $f'(x) = 2x \cot x^2$
b. $f'(x) = 2x \tan x^2$
c. $f'(x) = \dfrac{2x}{\sin x^2}$
d. $f'(x) = \dfrac{2\cos 2x}{\sin x^2}$

18. What is the derivative of $y = \log_2 x^3$?

a. $y' = \dfrac{2}{x \ln 3}$
b. $y' = \dfrac{3}{x \ln 2}$
c. $y' = \dfrac{2}{x \log 3}$
d. $y' = \dfrac{3}{x \log 2}$

19. Which of the following expressions is the first derivative of $y = \mathrm{Arcsin}(e^{-x})$

a. $-\dfrac{e^{-x}}{\sqrt{1 - e^{-2x}}}$
b. $\dfrac{e^{-x}}{\sqrt{1 - e^{-2x}}}$
c. $-\dfrac{e^{-x}}{\sqrt{1 - e^{2x}}}$
d. $\dfrac{e^{-x}}{\sqrt{1 - e^{2x}}}$

20. Which of the following expressions is the first derivative of $y = \text{Arccot}(x^2)$?

a. -$\dfrac{2}{1+x^4}$

b. -$\dfrac{2}{x\sqrt{x^4+1}}$

c. -$\dfrac{2x}{1+x^4}$

d. -$\dfrac{2}{x\sqrt{x^4-1}}$

21. Which of the following expressions is the first derivative of $y = \text{Arccsc}(x^2)$?

a. -$\dfrac{2}{1+x^4}$

b. -$\dfrac{2}{x\sqrt{x^4+1}}$

c. -$\dfrac{2x}{1+x^4}$

d. -$\dfrac{2}{x\sqrt{x^4-1}}$

22. The expression $\sinh^2 - \cosh^2$ is equal to ______.

a. -1

b. 0

c. 1

d. $+\infty$

23. What is the first derivative of $y = \tanh(3x)$?

a. $-3\text{sech}3x$

b. $3\text{sech}3x$

c. $-3\tanh 3x$

d. $3\tanh 3x$

24. Consider the implicit function $x^2 + xy + y^2 = 5$. Determine $\dfrac{dy}{dx}$.

a. $\dfrac{dy}{dx} = \dfrac{2x+y}{x+2y}$

b. $\dfrac{dy}{dx} = \dfrac{2x+y}{x-2y}$

c. $\dfrac{dy}{dx} = \dfrac{-2x+y}{x+2y}$

d. $\dfrac{dy}{dx} = \dfrac{-2x+y}{x-2y}$

25. Find the $\dfrac{dy}{dx}$ for the implicit function $\cos(x) + xy = 1$.

a. $\dfrac{dy}{dx} = \dfrac{y}{x}$

b. $\dfrac{dy}{dx} = -\dfrac{y}{x}$

c. $\dfrac{dy}{dx} = \dfrac{y(\sin xy + 1)}{x(\sin xy - 1}$

d. $\dfrac{dy}{dx} = -\dfrac{y(\sin xy + 1)}{x(\sin xy - 1}$

26. Using the previous item, find the second derivative with respect to x.

a. $\dfrac{d^2y}{dx^2} = \dfrac{2y}{x^2}$

b. $\dfrac{d^2y}{dx^2} = -\dfrac{2y}{x^2}$

c. $\dfrac{d^2y}{dx^2} = 1$

d. $\dfrac{d^2y}{dx^2} = -1$

27. Which of the following expressions is the second derivative of the function $y = \dfrac{1}{x+1}$?

a. $\dfrac{-2}{(x+1)^3}$

b. $\dfrac{-1}{(x+1)^3}$

c. $\dfrac{1}{(x+1)^3}$

d. $\dfrac{2}{(x+1)^3}$

28. Using the function in the previous item, find the third derivative?

a. $\dfrac{-6}{(x+1)^4}$

b. $\dfrac{-2}{(x+1)^4}$

c. $\dfrac{2}{(x+1)^4}$

d. $\dfrac{6}{(x+1)^4}$

29. Evaluate the $\lim\limits_{x \to 1} \dfrac{x^2-1}{x^2+3x-4}$.

a. 0

b. 1/4

c. 2/5

d. 4/7

30. Evaluate the $\lim\limits_{x \to 0^+} \sqrt{x} \ln x$.

a. -1

b. 0

c. 1

d. $+\infty$

Unit 3: Applications of Differentiation

In this unit, we discuss properties of differentiable functions which can be used in optimization, curve sketching and approximation. Furthermore, we use calculus to analyze the behavior of graphs of differentiable functions and reveal their analytical properties.

Lesson 3.1: Relative Extrema

Learning Outcomes

At the end of this lesson, you should be able to:

1. Define relative extrema of given function; and
2. Determine the critical points of a given function.

One of the important applications of calculus in the field of applied mathematics is its ability to make the analysis of the extreme values a given real-valued function more systematic. Given a function and an interval, we are interested to know whether the function attains a maximum or minimum value in that interval. Moreover, we ask how can we determine these extreme values.

Aside from the absolute maximum and absolute minimum values of functions in an interval which are discussed in Chapter 2, there are also "localized" versions of these definition.

Definition 25: Definition of Relative Extrema

We say that a function f attains a

1. **relative maximum** at a number c if there exists an open interval containing c, on which $f(c)$ is a maximum, that is, if there exists $\delta > 0$ such that $f(x) \leq f(c)$, for all $x \in (c - \delta, c + \delta)$.
2. **relative minimum** at a number c if there exists an open interval containing c, on which $f(c)$ is a minimum, that is, if there exists $\delta > 0$ such that $f(x) \geq f(c)$, for all $x \in (c - \delta, c + \delta)$.

We use the term **relative extremum** to refer to either a relative maximum or a relative minimum. By looking closely at the definition, one can think of a relative maximum as "hill" on the graph while a relative minimum as a "valley".

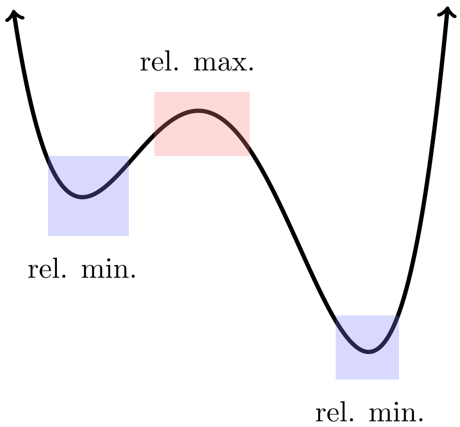
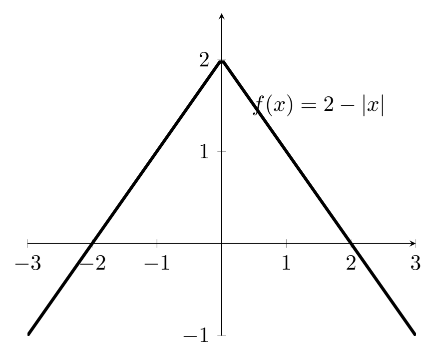


Figure 3.1: Relative Extrema

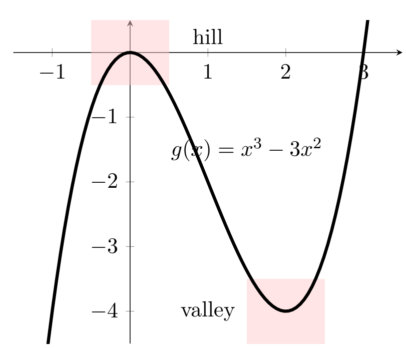
Example 101.

The function $f(x) = 2-|x|$ attains a relative maximum at $c = 0$. In this case, f also attains an absolute maximum at $c = 0$.



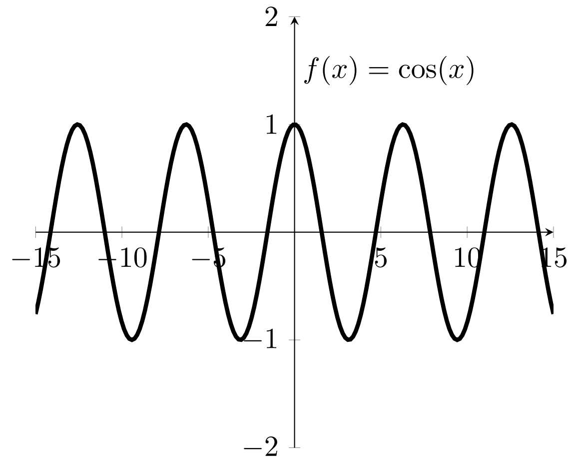
Example 102.

The function $g(x) = x^3 - 3x^2$ is graphed below. As we may observe, g attains a relative maximum at $x = 0$ while it attains a relative minimum at $x = 2$. Note that g does not attain an absolute maximum and an absolute minimum on $(-\infty, \infty)$.



Example 103.

The cosine function attains relative maximum and relative minimum at infinitely many points. The relative maxima are attained at points $x = 2\pi k$, $k \in \mathbb{Z}$ while the relative minima are attained at points $x = (2k+1)\pi$, $k \in \mathbb{Z}$.



In order to establish the main result of this section, we need to know the following lemma.

Lemma 2

Let f be defined on some open interval containing c. Suppose that $\lim\limits_{x \longrightarrow c} f(x) = L$.

1. If $L > 0$, then there exists an interval I containing c such that $f(x) > 0$, for all $x \in I, x \neq c$.
2. If $L < 0$, then there exists an interval I containing c such that $f(x) < 0$, for all $x \in I, x \neq c$.

The above lemma guarantees that if a function has a positive limit as $x \longrightarrow c$, the function is positive on some interval containing c, except possibly at c. On the other hand, if a continuous function has a negative limit as $x \longrightarrow c$, the function is negative on some interval containing c, except possibly at c.

Proof. We show that there exists an interval I containing c such that $f(x) > 0$, for all $x \in I$. Let $\varepsilon = \frac{L}{2} > 0$. Since $\lim\limits_{x \longrightarrow c} f(x) = L$, then there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$. But,

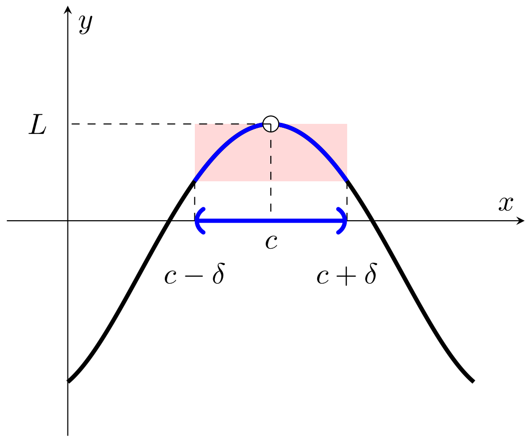
$$|f(x) - L| < \varepsilon \Longleftrightarrow |f(x) - L| < \frac{L}{2}$$

$$\Longleftrightarrow L - \frac{L}{2} < f(x) < L + \frac{L}{2}$$

$$\Longleftrightarrow \frac{L}{2} < f(x) < \frac{3L}{2}.$$

Thus, if $0 < |x - c| < \delta$, then $f(x) > \dfrac{L}{2} = \varepsilon > 0$. This means that if we take $I = (c - \delta, c + \delta)$, and $x \in I \setminus \{c\}$, then $f(x) > 0$. This completes the proof of (1).

For number (2), assume that f is continuous at c and $L < 0$. Consider now the function $-f$. Since $\lim_{x \longrightarrow c} [-f(x)] = -L > 0$, the result now follows by applying number (1) on $-f$. $\square$



We now state the main result of this section.

Theorem 33

If a function f has a relative extremum at a number c and $f'(c)$ exists, then $f'(c) = 0$.

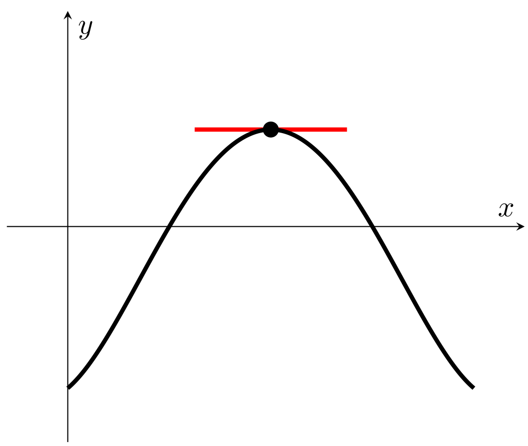


Figure 3.2: Horizontal Tangent Line at Relative Extremum

Proof. Assume that f has a relative maximum at c. Since $f'(c)$ exists, f must be continuous at c. To prove that $f'(c) = 0$, we argue indirectly. Suppose that $f'(c) \neq 0$. We consider two distinct cases: (1) $f'(c) > 0$ (2) $f'(c) < 0$.

Case (1): If $f'(c) > 0$, then from the definition of $f'(c)$,

$$\lim_{x \longrightarrow c} \left[\frac{f(x) - f(c)}{x - c} \right] > 0.$$

Lemma 2 now guarantees that there exists an interval I containing c such that

$$\frac{f(x)-f(c)}{x-c} > 0, \quad \text{for all } x \in I \setminus \{c\}.$$

This implies that if $x \in I$ and $x > c$, then $f(x) > f(c)$. Moreover, if $x \in I$ and $x < c$, then $f(x) < f(c)$. These contradict the fact that f has a relative maximum at c. Therefore, $f'(c) \not> 0$.
Case (2): If $f'(c) < 0$, then from the definition of $f'(c)$,

$$\lim_{x \longrightarrow c} \left[\frac{f(x)-f(c)}{x-c}\right] < 0.$$

Lemma 2 can be applied to conclude that there exists an interval I containing c such that

$$\frac{f(x)-f(c)}{x-c} < 0, \quad \text{for all } x \in I \setminus \{c\}.$$

This implies that if $x \in I$ and $x > c$, then $f(x) < f(c)$. Moreover, if $x \in I$ and $x < c$, then $f(x) > f(c)$. These contradict the fact that f has a relative maximum at c. Therefore, $f'(c) \not< 0$.

Both cases lead to a contradiction. Therefore, $f'(c) = 0$. The proof when f attains a relative minimum at c is similar and the student is asked to give the details. □

Theorem 33 implies that if f has a relative extremum at a point c, then either $f'(c)$ does not exist or $f'(c) = 0$. Consequently, if $f'(c) \neq 0$, then f cannot have a relative extremum at c. This motivates us in the following definition.

Definition 26: Critical Points

A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist. The ordered pair $(c, f(c))$ is called a **critical point** of f.

Theorem 3.1 now implies that *if a function has a relative extremum at a number c, then c is a critical number of the function.* In order to get the critical number(s) of a given function f, the following steps are suggested:

1. Calculate $f'(x)$.
2. Equate $f'(x)$ to 0 and solve for all possible x. All x's which lie in the domain of f are critical numbers of f.
3. Determine all x in the domain of f such that $f'(x)$ does not exist. All such values of x are critical numbers as well.

Example 104. Determine all the critical numbers of the following functions.

(1) $f(x) = 24x^5 - 45x^4 - 40x^3 + 9$

(2) $g(t) = 2\sqrt[3]{t^2 - 1}$

(3) $h(s) = \dfrac{s^2+1}{s^2-4}$

(4) $r(x) = 6x - 4\cos(3x)$, $x \in [0, 2\pi]$

Solutions.

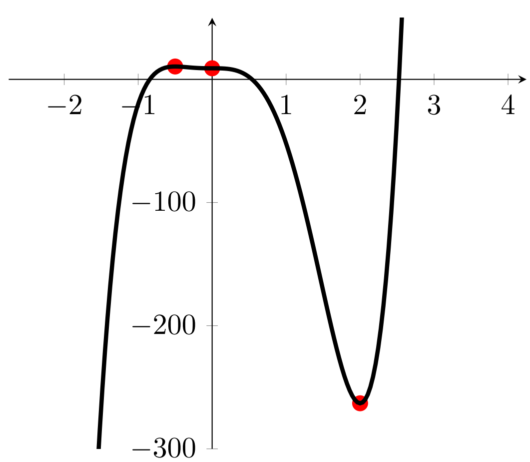
(1) Note that

$$f'(x) = 120x^4 - 180x^3 - 120x^2.$$

Since $f'(x)$ is still a polynomial, it is defined at every real number x. Equating $f'(x) = 0$ yields

$$\begin{aligned} 120x^4 - 180x^3 - 120x^2 &= 0 \\ \implies 60x^2(2x^2 - 3x - 2) &= 0 \\ \implies 60x^2(2x+1)(x-2) &= 0 \\ \implies x = 0 \text{ or } x = -\frac{1}{2} &\text{ or } x = 2. \end{aligned}$$

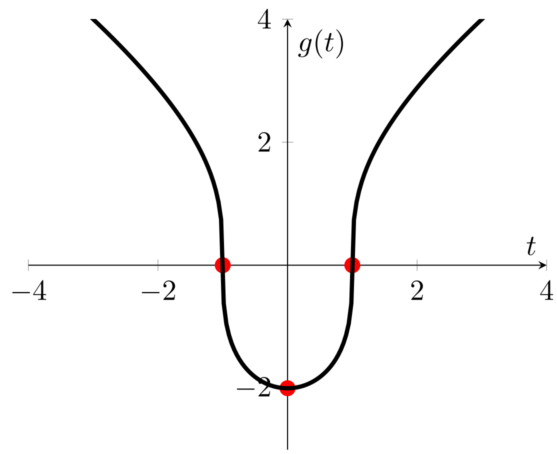
Hence, the critical numbers of f are $0, -\frac{1}{2}$ and 2. The actual graph of f is shown below. The critical points are marked red.



(2) We calculate $g'(t)$ as

$$g'(t) = 2 \cdot \frac{1}{3}(t^2-1)^{-2/3} \cdot 2t = \frac{4t}{3\sqrt[3]{(t^2-1)^2}}.$$

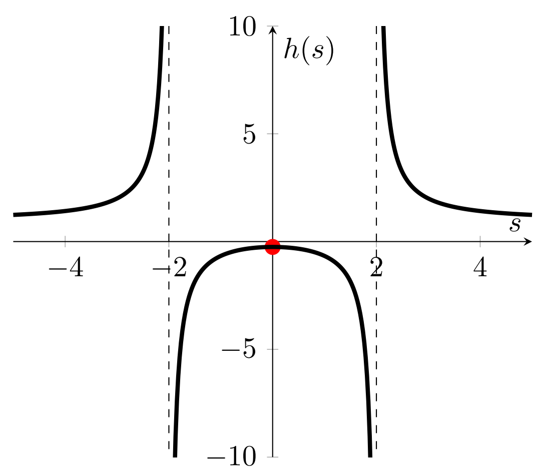
Note that $g'(t) = 0$ if and only if $4t = 0$, or when $t = 0$. Hence, 0 is a critical number of g. Moreover, $g'(t)$ does not exists if and only if $t^2 - 1 = 0$, that is, if $t = 1$ or $t = -1$. Therefore, g has exactly three critical numbers; namely $-1, 0$ and 1. The graph of g is shown below and the critical points are marked red.



(3) We calculate $h'(s)$ as follows

$$h'(s) = \frac{d}{ds}\left(\frac{s^2+1}{s^2-4}\right) = \frac{(s^2-4)(2s)-(s^2+1)(2s)}{(s^2-4)^2} = \frac{-10s}{(s^2-4)^2}.$$

Thus, $h'(s) = 0$ if and only if $s = 0$. Moreover, $h'(s)$ does not exists if and only if $s^2 - 4 = 0$, that is, if and only if $s = 2$ or $s = -2$. However, the numbers 2 and -2 are not in the domain of h so the only critical number is $s = 0$. Consequently, the function h only has one critical point $(0, -1/4)$.



This example shows that the condition that a critical number must be in the domain of the given function is important.

(4) We first calculate $r'(x)$. Note that

$$r'(x) = 6 + 12\sin(3x).$$

Since $r'(x)$ is defined for all x, to get all the critical numbers, we simply solve for all values of $x \in [0, 2\pi]$ which satisfy $r'(x) = 0$. We note that $x \in [0, 2\pi]$ if and only if $3x \in [0, 6\pi]$. So,

$$\begin{aligned}
r'(x) &= 0 \\
6 + 12\sin(3x) &= 0 \\
\sin(3x) &= -\frac{1}{2} \\
3x &= \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}, \frac{31\pi}{6}, \text{ or } \frac{35\pi}{6} \\
x &= \frac{7\pi}{18}, \frac{11\pi}{18}, \frac{19\pi}{18}, \frac{23\pi}{18}, \frac{31\pi}{18}, \text{ or } \frac{35\pi}{18}.
\end{aligned}$$

Therefore, the critical numbers of $r(x)$ in $[0, 2\pi]$ are $\frac{7\pi}{18}, \frac{11\pi}{18}, \frac{19\pi}{18}, \frac{23\pi}{18}, \frac{31\pi}{18}, \text{ or } \frac{35\pi}{18}$.

Lesson 3.2: Finding Extrema of Continuous Functions on a Closed Interval

Learning Outcomes

At the end of this lesson, you should be able to:

1. Calculate the critical numbers of a given function; and
2. Find the extrema of functions on a closed interval.

Recall that if a function f defined on a closed interval $[a, b]$ is continuous, the extreme value theorem guarantees that f attains both maximum and minimum values in the interval $[a, b]$. However, the extreme value theorem does not provide us information on where these extreme values can be found. Knowing where the extreme values occur is essential in applied mathematics.

Recall from the previous section that if f has a relative extremum at c, which lies in the interval (a, b), then either $f'(c) = 0$ or $f'(c)$ does not exist. Using this, we can give the following guidelines in looking for the extreme values of f.

Guidelines 2: Finding Extreme Values of Functions

Suppose that a function f is continuous on the interval $[a, b]$.

1. Solve for all the critical numbers of f in the open interval (a, b).
2. Evaluate f at these critical points on (a, b).
3. Evaluate f at the endpoints a and b.

The largest among these values of f is its maximum value and the smallest among these values of f is its minimum value.

Example 105. Determine the extreme values of

$$f(x) = x^3 - 12x$$

on the interval $[0, 4]$.

Solution. Observe that f is continuous on $[0, 4]$ so it attains both maximum and minimum values in $[0, 4]$. We first look at the critical numbers of f on $(0, 4)$. First, note that

$$f'(x) = 3x^2 - 12$$

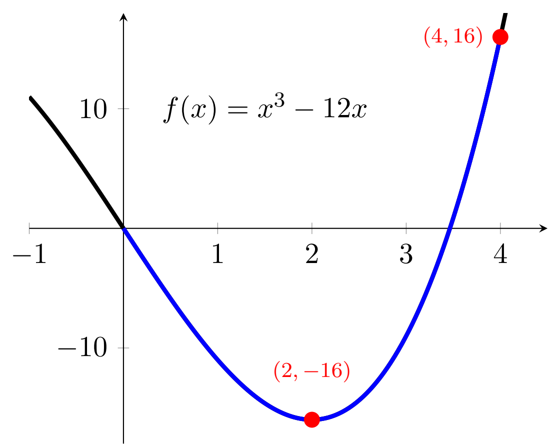
is defined for all $x \in (0, 4)$, so the only critical numbers of f in $(0, 4)$ are those $x \in (0, 4)$ such that $f'(x) = 0$. The critical numbers are then those x such that

$$3x^2 - 12 = 0 \Longleftrightarrow x^2 = 4 \Longleftrightarrow x = 2, -2.$$

Since $-2 \notin (0, 4)$, the only critical point of f in $(0, 1)$ is 2. Next, we evaluate f at the critical points and at the endpoints of the interval $[0, 4]$.

x	0	2	4
$f(x)$	0	-16	16

By looking at the table we see that in the interval $[0, 4]$, f has a maximum value of 16 attained when $x = 4$ and has a minimum value -16 which is attained when $x = 2$.



Example 106. Calculate the maximum and minimum values of the function

$$g(x) = 3 - |x - 3|$$

on the interval $[-1, 5]$.

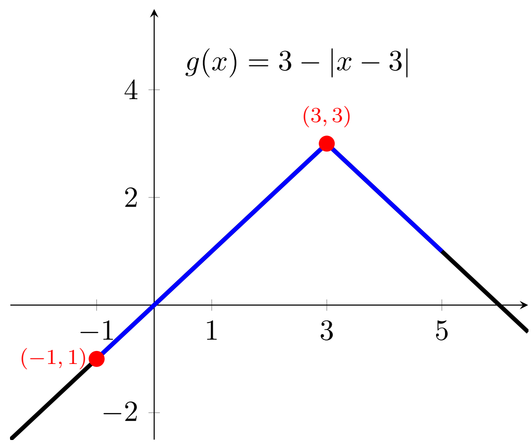
Solution. Observe that g is continuous on $[-1, 5]$. We next look for the critical points of g on $(-1, 5)$. To do this, we first rewrite the $g(x)$ as

$$g(x) = \begin{cases} 3 - (3 - x) & \text{if } -1 \leq x < 3; \\ 3 - (x - 3) & \text{if } 3 \leq x \leq 5 \end{cases} = \begin{cases} x & \text{if } -1 \leq x < 3; \\ 6 - x & \text{if } 3 \leq x \leq 5 \end{cases}$$

We see that $g'(x) = 1$ if $-1 \leq x < 3$ and $g'(x) = -1$ if $3 < x \leq 5$. This implies that $f'(3)$ does not exist. Hence, the only critical number of g in $(-1, 5)$ is 3. We now evaluate g at 3 and at the endpoints of $[-1, 5]$ as in the following table.

x	-1	3	5
$g(x)$	-1	3	1

Therefore, g attains a maximum value of 3 at $x = 3$ and a minimum value of -1 at $x = -1$.



Example 107. Find two nonnegative numbers whose sum is 50 and whose product is maximum.

Solution. Let x be one of the numbers. Then the other number must be $50 - x$. Hence, their product $P(x)$ is

$$P(x) = x(50 - x) = 50x - x^2.$$

Observe that the range of feasible values for x is $0 \le x \le 50$. Note that when the product $P(x)$ is viewed as a function of x, then it is continuous on $[0, 50]$. We see that $P'(x) = 50 - 2x$ and hence the only critical number of P in $(0, 50)$ is x, where $50 - 2x = 0$, that is, $x = 25$. We evaluate $P(x)$, for $x = 0, 25, 50$.

x	0	25	50
$P(x)$	0	625	0

We see that $P(x)$ reaches its maximum value of 625 when $x = 25$. Therefore, the two numbers must be 25 and 25.

Example 108. A stone is thrown vertically upward in a controlled environment so that its height $s(t)$, measured in feet, after t seconds is given by $s(t) = 64t - 4t^3$, $\quad 0 \le t \le 4$. What is the maximum height reached by the stone?

Solution. We want to find the maximum value of $s(t)$ on the interval $[0, 4]$. Thus, we look at the critical numbers of s first. Note that

$$s'(t) = 64 - 12t^2, \text{ for } t \in (0, 4).$$

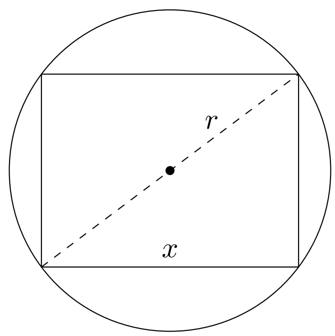
Setting $s'(t) = 0$ and solving for t, we see that the critical numbers of $s(t)$ in $(0, 4)$ is $t = \dfrac{4\sqrt{3}}{3}$. We evaluate $s(t)$ for $t = 0, \frac{4\sqrt{3}}{3}, 4$ and we see that

t	0	$\frac{4\sqrt{3}}{3}$	4
$s(t)$	0	$\frac{512\sqrt{3}}{9}$	0

Therefore, the stone reaches its maximum height of $\frac{512\sqrt{3}}{9}$ (≈ 98.53) feet after $\frac{4\sqrt{3}}{3}$ (≈ 2.31) seconds.

Example 109. Prove that the largest rectangle which can be inscribed in a given circle is a square.

Solution. Consider a circle with radius r and an arbitrary rectangle inscribed in the circle as shown in the following figure.



Note that the diagonal of the rectangle must be of length $2r$ which serves as a hypotenuse of a right triangle with two legs being the two consecutive sides of the rectangle. If x is the length of one of its sides then the Pythagorean theorem implies that the other side must be $\sqrt{4r^2 - x^2}$. Hence, the area $A(x)$ of the rectangle is

$$A(x) = x\sqrt{4r^2 - x^2}, \text{ where } 0 \leq x \leq 2r.$$

Thus, we can consider the area of the rectangle as a function of one of its sides that is continuous on the closed interval $[0, 2r]$. Our goal is to maximize the value of $A(x)$. Hence, we first seek for the critical numbers of A on $(0, 2r)$. Note that

$$A'(x) = \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}}.$$

Observe that $A'(x)$ exists for all $x \in (0, 2r)$. So the only critical numbers of A in $(0, 2r)$ are those x for which $A'(x) = 0$. But,

$$A'(x) = 0 \Longleftrightarrow 4r^2 - 2x^2 = 0 \Longleftrightarrow x = \sqrt{2}r,$$

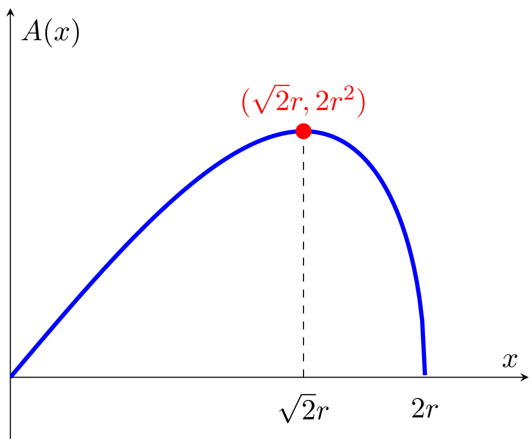
since $-\sqrt{2}r \notin (0, 2r)$. Computing the values of $A(x)$ for $x = \sqrt{2}r, 0, 2r$ we see that

x	0	$\sqrt{2}r$	$2r$
$A(x)$	0	$2r^2$	0

Hence, $A(x)$ attains its maximum value $2r^2$ when $x = \sqrt{2}r$. In this case, the length of the other side of the rectangle is

$$\sqrt{4r^2 - (\sqrt{2}r)^2} = \sqrt{2}r$$

which shows that, indeed, the rectangle is a square with area $2r^2$.



Lesson 3.3: Increasing and Decreasing Functions and the First Derivative Test

Learning Outcomes

At the end of this lesson, you should be able to:

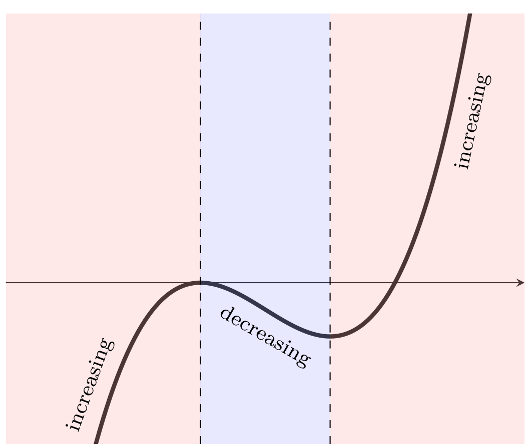
1. Check the monotonicity of a given function;
2. Apply the first derivative test; and
3. Determine the relative extrema of a given function.

In this section, we learn how to classify relative extrema as to whether they are relative maxima or relative minima using derivatives. We first define when is a function *increasing* and when is it *decreasing*.

Definition 27

A function f is said to be **increasing** on an interval I if whenever $x_1, x_2 \in I$ and $x_1 < x_2$ imply $f(x_1) < f(x_2)$. On the other hand, we say f is **decreasing** on an interval I if whenever $x_1, x_2 \in I$ and $x_1 < x_2$ imply $f(x_1) > f(x_2)$.

We also say f is **monotone** on I if f is either increasing or decreasing on I. Intuitively, if f is increasing on I, as x moves from left to right, then $f(x)$ moves upward. On the other hand, if f is decreasing on I, as x moves from left to right, then $f(x)$ moves downward.



The following theorem gives the intertwine between differentiability and monotonicity of functions.

Theorem 34

Let a function f is differentiable on an open interval I.

(1) If $f'(x) > 0$, for all $x \in I$, then f is increasing on I.

(2) If $f'(x) < 0$, for all $x \in I$, then f is decreasing on I.

Proof. We only prove (1). The proof of (2) is similar. Suppose that $f'(x) > 0$ for all $x \in I$. Let $x_1, x_2 \in I$ such that $x_1 < x_2$. Since f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2), by the mean value theorem, there exists $c \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

Since $f'(c) > 0$ and $x_2 - x_1 > 0$, it follows that $f(x_2) - f(x_1) > 0$ so $f(x_1) < f(x_2)$. Therefore, f is increasing on I. □

The above theorem says that if the derivative of a function doesn't change its sign in an open interval, then it is monotone in that interval. It is worth to note that if a function has a zero derivative at every point in an open interval, then it is constant there. We set up the following guidelines to determine the open intervals where a given function is increasing or decreasing.

Guidelines 3

To determine the open intervals, where a given function f is increasing or decreasing,

(1) determine all the critical numbers of f and use these to determine the test intervals;

(2) make a table of signs for f' at one test value for each test intervals;

(3) use Theorem 34 to decide whether f is increasing or decreasing on each intervals.

We illustrate these steps in the following example.

Example 110. Determine the intervals where the function $f(x) = x^4 - 8x^2$ is increasing or decreasing.

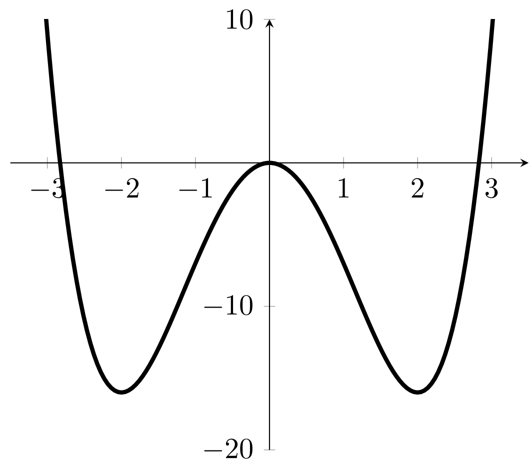
Solution. We first calculate the critical numbers of f. Note that

$$f'(x) = 4x^3 - 16x = 4x(x-2)(x+2).$$

This implies that the critical numbers of f are $0, 2$ and -2. We now consider the following table of signs.

test intervals	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
test value (x)	-3	-1	1	3
$f'(x) = 4x(x-2)(x+2)$	$-$	$+$	$-$	$+$

Therefore, f is increasing on the open intervals $(-2, 0)$ and $(2, \infty)$; and f is decreasing on $(-\infty, -2)$ and $(0, 2)$. The actual graph of f is shown below.



Example 111. Consider the function $f(x) = x^5$. Observe that $f'(x) = 5x^4$ and so, the only critical number of f is 0. Note that if $x < 0$, then $f'(x) > 0$ and if $x > 0$, $f'(x) > 0$. Hence, f is increasing on $(-\infty, 0)$ and $(0, \infty)$. Since f is differentiable and continuous at 0, we conclude that f is increasing on $(-\infty, \infty)$. **Remark:** Note that $f'(0) = 0$ but still, f is increasing on any open interval containing 0.

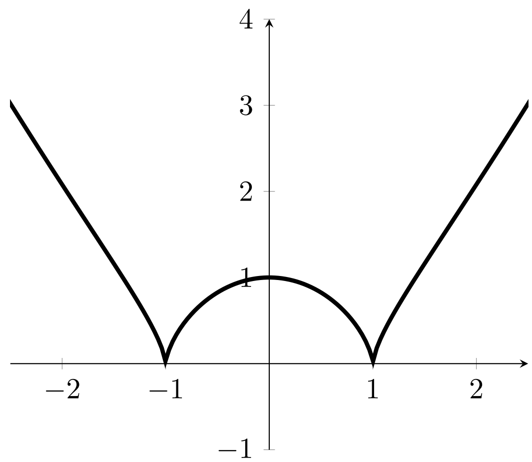
Example 112. Consider now the function $f(x) = (x^2 - 1)^{2/3}$. Observe that

$$f'(x) = \frac{2}{3}(x^2 - 1)^{-1/3} \cdot (2x) = \frac{4x}{3(x^2 - 1)^{1/3}}.$$

Observe that $f'(x) = 0$ if and only if $x = 0$; $f'(-1)$ and $f'(1)$ do not exist. Hence, $-1, 0$ and 1 are the critical numbers of f. We now look at the test intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$ and $(1, \infty)$.

test intervals	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
test value (x)	-2	-0.5	0.5	2
$f'(x) = \dfrac{4x}{3(x^2 - 1)^{1/3}}$	$-$	$+$	$-$	$+$

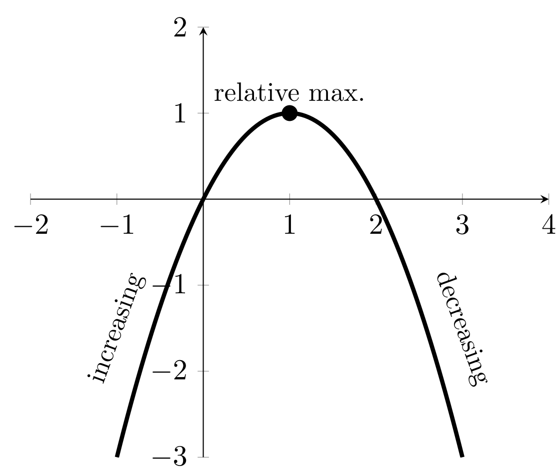
Therefore, f in decreasing on the intervals $(-\infty, -1)$ and $(0, 1)$; and f is increasing on $(-1, 0)$ and $(1, \infty)$. The actual graph of f is shown below.



Now that we have learned to determine the intervals where a given function is increasing or decreasing, it is not difficult to classify relative extrema of a function. As an illustration, consider the function

$$f(x) = 2x - x^2.$$

Since $f'(x) = 2 - 2x$, then the only critical number of f is 1. Observe that if $x < 1$, then $f'(x) > 0$ and if $x > 1$, then $f'(x) < 0$. Thus, the function is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$. Note that the monotonicity of f changes from being increasing to being decreasing at $x = 1$ and since $f'(1) = 0$, we conclude that f has a relative (absolute) maximum at $x = 1$.



We generalize this observation in the following called the *first derivative test* the proof of which is left for the student.

Theorem 35: First Derivative Test

Let c be a critical number of f such that f is continuous in an open interval I containing c. If f is differentiable at every point in $I \setminus \{c\}$, then $f(c)$ can be classified as follows.

(1) If $f'(x) > 0$, for all $x \in I$, $x < c$ and $f'(x) < 0$, for all $x \in I$, $x > c$, then f has a relative maximum $f(c)$ attained at c.

(2) If $f'(x) < 0$, for all $x \in I$, $x < c$ and $f'(x) > 0$, for all $x \in I$, $x > c$, then f has a relative minimum $f(c)$ attained at c.

(3) If $f'(x) > 0$, for all $x \in I$, $x \neq c$, then f doesn't have a relative extremum at c.

(4) If $f'(x) < 0$, for all $x \in I$, $x \neq c$, then f doesn't have a relative extremum at c.

In words, if the sign of $f'(x)$ changes from positive to negative at c, then f has a relative maximum at c. On the other hand, if the sign of $f'(x)$ changes from negative to positive at c, then f has a relative minimum at c

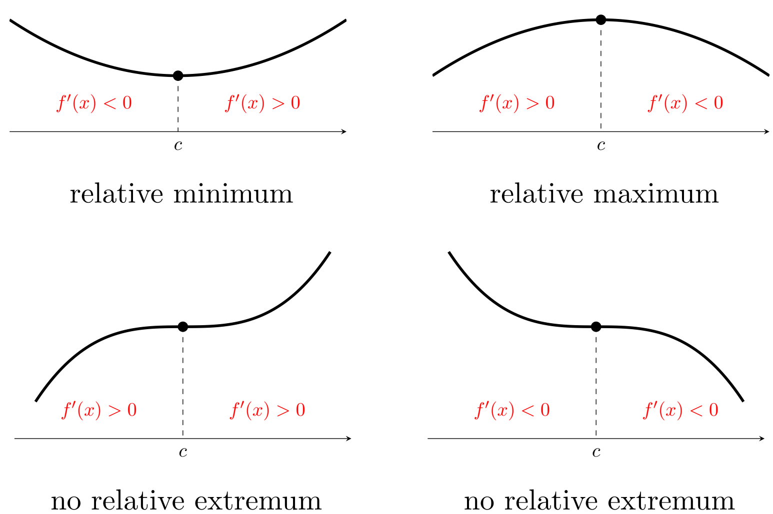


Figure 3.3: First Derivative Test

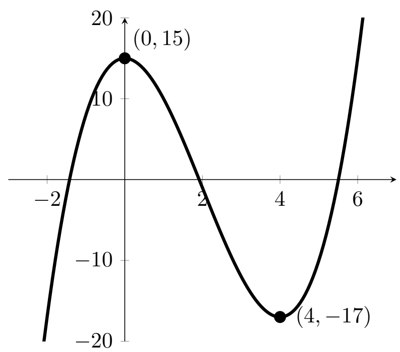
We now apply the first derivative test.

Example 113. Determine the relative extrema of the function defined by $f(x) = x^3 - 6x^2 + 15$.

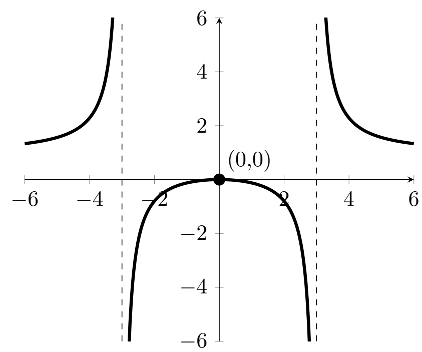
Solution. The first step is to determine the critical numbers of f. Note that $f'(x) = 3x^2 - 12x = 3x(x-4)$. Hence, the critical numbers of f are 0 and 4. We now look at the following table of signs.

test interval	$(-\infty,0)$	$(0,4)$	$(4,\infty)$
test value (x)	-1	1	5
$f'(x)=3x(x-4)$	$+$	$-$	$+$

Since $f'(x)$ changes from positive to negative at $x=0$, we conclude that f has a relative maximum $f(0)=15$ at $x=0$. On the other hand, since $f'(x)$ changes from negative to positive at $x=4$, then f has a relative minimum $f(4)=-17$ at $x=4$, by the first derivative test. The actual graph of f is shown below.



$f(x)=x^3-6x^2+15$



$f(x)=\dfrac{x^2}{x^2-9}$

It is important to note that critical numbers of function to be considered is necessarily *in the domain* of the given function.

Example 114. Use the first derivative test to classify the relative extrema of the function

$$f(x)=\frac{x^2}{x^2-9}.$$

Solution. First, note that

$$f'(x)=\frac{(x^2-9)(2x)-(x^2)(2x)}{(x^2-9)^2}=\frac{-18x}{(x^2-9)^2}.$$

Since 3 and -3 are not, in the first place, in the domain of f, we conclude that f only has one critical number, namely 0. We now look at the following table of signs.

test intervals	$(-3,0)$	$(0,3)$
test value (x)	-1	1
$f'(x)=\dfrac{-18x}{(x^2-9)^2}$	$+$	$-$

Hence, the first derivative test guarantees that f has a relative maximum at $x=0$. The relative maximum is $f(0)=0$. The actual graph of f is shown above.

Lesson 3.4: Concavity and the Second Derivative Test

Learning Outcomes

At the end of this lesson, you should be able to:

1. Check the concavity of a given function;
2. Apply the second derivative test; and
3. Determine the point of inflection/s of a given function.

Definition 28

Let f be differentiable on an open interval I. We say that the graph of f is **concave upward** if $f'(x)$ is increasing on I, and **concave downward** if $f'(x)$ is decreasing on I.

Theorem 36: Concavity Test

Let f be a function such that $f''(x)$ exists on an open interval I.

1. If $f''(x) > 0$, for all $x \in I$, then f is concave upward on I.
2. If $f''(x) < 0$, for all $x \in I$, then f is concave downward on I.

Example 115.

1. Determine the intervals where the graph of $f(x) = \dfrac{x^2+1}{x-1}$ is concave upward or concave downward.

   Solution:

$$f(x) = \frac{x^2+1}{x-1}$$

$$f'(x) = \frac{d}{dx}\left(\frac{x^2+1}{x-1}\right) = \frac{(x-1)(2x)-(x^2+1)(1)}{(x-1)^2} = \frac{x^2-2x-1}{(x-1)^2}$$

$$f\text{"}(x) = \frac{d}{dx}\left(\frac{x^2-2x-1}{(x-1)^2}\right) = \frac{(x-1)^2(2x-2)-(x^2-2x-1)[2(x-1)]}{(x-1)^4}$$

$$f\text{"}(x) = \frac{(x^2-2x+1)(2x-2)-(x^2-2x-1)(2x-2)}{(x-1)^4} = \frac{4(x-1)}{(x-1)^4} = \frac{4}{(x-1)^3}$$

$f"(x)$ does not exist when $x = 1$
Table of Signs for $f"(x)$

	$f"(x)$	Concavity
$x < 1$	-	Concave Downward
$x > 1$	$+$	Concave Upward

Therefore, $f(x) = \dfrac{x^2+1}{x-1}$ is concave downward on $(-\infty, 1)$ and concave upward on $(1, \infty)$

2. Find the intervals where $f(x) = 2x - \tan x$ is concave upward or concave downward.

   Solution

   $f(x) = 2x - \tan x$
   $f'(x) = \dfrac{d}{dx}(2x - \tan x) = 2 - \sec^2 x$
   $f"(x) = -2\sec x \cdot \sec x \tan x = -2\sec^2 x \tan x$
   $f"(x) = 0$ iff $-2\sec^2 x \tan x = 0$ iff $\tan x = 0$ iff $x = 0$.
   $f"(x)$ exists for all $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

   Table of Signs for $f"(x)$:

	$-2\sec^2 x$	$\tan x$	$f"(x)$	
$-\dfrac{\pi}{2} < x < 0$	-	-	$+$	Concave Upward
$0 < x < \dfrac{\pi}{2}$	-	$+$	-	Concave Downward

   Therefore, $f(x)$ is concave upward on $\left(-\dfrac{\pi}{2}, 0\right)$, and concave downward on $\left(0, \dfrac{\pi}{2}\right)$

Definition 29

Let f be continuous on an open interval I containing c. If the concavity of f changes across a number c, then the point $(c, f(c))$ is called a **point of inflection** of f.

Theorem 37

If $(c, f(c))$ is a point of inflection of the graph of f, and if $f''(c)$ exists, then $f''(c) = 0$.

Example 116. Determine the points of inflection of $f(x) = x^4 - 4x^3$.
Solution:
$f(x) = x^4 - 4x^3$

$f'(x) = 4x^3 - 12x^2$
$f"(x) = 12x^2 - 24x = 12x(x-2)$ $f"(x) = 0$ iff $12x(x-2) = 0$ iff $12x = 0$ or $x - 2 = 0$. Therefore, $f"(x) = 0$ iff $x = 0, 2$. Therefore, the point of inflections are $(0, 0)$ and $(2, -16)$.

Theorem 38: Second Derivative Test (SDT)

Let f be a function such that $f'(c) = 0$ and that $f''(x)$ exists on some open interval containing c.

1. If $f''(c) > 0$, then f has a relative minimum at $x = c$.
2. If $f''(c) < 0$, then f has a relative maximum at $x = c$.

Remark: If $f''(c) = 0$, then SDT is inclusive. In this case, use the FDT.

Example 117. Find the relative extrema of $f(x) = -3x^5 + 5x^3$.
Solution:
$f(x) = -3x^5 + 5x^3$
$f'(x) = -15x^4 + 15x^2 = -15x^2(x^2 - 1) = -15x^2(x+1)(x-1)$
$f'(x) = 0$ iff $-15x^2(x+1)(x-1) = 0$ iff $x = 0, 1, -1$

The critical numbers are 0, 1, -1.

$f"(x) = -60x^3 + 30x = -30x(2x^2 - 1)$

	$f"(x)$	
$x = 0$	0	Neither
$x = 1$	-	Maximum
$x = -1$	+	Minimum

Therefore, $f(x)$ has relative maximum at $(1, 2)$, and has relative minimum at $(-1, -2)$.

3.4.1 Graph Sketching

Knowing the monotonicity, relative extrema, concavity, and points of inflection, can be extremely useful in sketching graphs of functions. In sketching graphs of functions, we consider the following:

1. domain (and range)
2. x and y intercepts
3. continuity (where is f is continuous or discontinuous? what type of discontinuity?)
4. asymptotes (vertical, horizontal, others)

5. differentiability
6. monotonicity
7. relative extrema
8. concavity
9. points of inflection

Example 118.

1. Sketch the graph of $f(x) = \dfrac{x^2}{x^2+3}$

   Solution:

   a. Domain: $(-\infty, \infty)$
   b. x intercept: 0
   y intercept: 0, passes through (0,0).
   c. Continuous everywhere
   d. Vertical Asymptote: None
   Horizontal Asymptote:

$$\lim_{x\to\pm\infty} \frac{x^2}{x^2+3} = \lim_{x\to\pm\infty} \frac{1}{1+\dfrac{3}{x^2}}$$

   Therefore, $y = 1$ is the horizontal asymptote.
   e. Differentiability

$$f'(x) = \frac{(x^2+3)(2x) - (x^2)(2x)}{(x^2+3)^2} = \frac{6x}{(x^2+3)^2}$$

   Therefore, the critical number is 0.
   f. Monotonicity

	$x < 0$	$x > 0$
$6x$	-	+
$(x^2+3)^2$	+	+
$f'(x)$	-	+
	Increasing	Decreasing

   Therefore, f is increasing on $(0, \infty)$; and f is decreasing on $(-\infty, 0)$.
   g. Relative Extrema
   By First Derivative Test, (0,0) is the relative minimum point.
   h. Concavity

$$f"(x) = \frac{d}{dx}\left(\frac{6x}{(x^2+3)^2}\right) = \frac{(x^2+3)(6) - (6x)[2(x^2+3)(2x)]}{(x^2+3)^4}$$

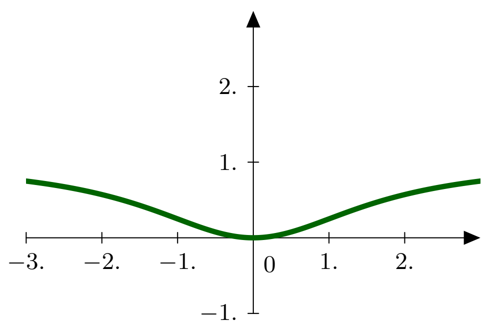
$$f"(x) = \frac{-18(x+1)(x-1)}{(x^2+3)^3}$$

$f"(x) = 0$ iff $x \in \{-1, 1\}$.
Table of Signs for $f"(x)$:

	$x < -1$	$-1 < x < 1$	$x > 1$
$-18(x+1)$	+	-	-
$(x-1)$	-	-	+
$(x^2+3)^3$	+	+	+
$f"(x)$	-	+	-

Therefore, f is concave upward on (-1,1) and concave downward on $(-\infty, -1)$ and $(1, \infty)$.

i. Point of inflection/s: $(-1, \frac{1}{3})$ and $(1, \frac{1}{3})$



2. Sketch the graph of $f(x) = \dfrac{x^3}{x^2 - 9}$

Solution:

a. Domain: $\mathbb{R} \setminus \{3, -3\}$

b. x intercept is 0 and y intercept is 0. The graph passes through (0,0).

c. Continuity: f is continuous everywhere except $x = 3$ and $x = -3$

d. Vertical Asymptotes: $x = 3$ and $x = -3$

$$\lim_{x \to -3^-} \frac{x^3}{(x+3)(x-3)} = \frac{-27}{(0^-)(-6)} = -\infty$$

$$\lim_{x \to -3^+} \frac{x^3}{(x+3)(x-3)} = \frac{-27}{(0^+)(-6)} = \infty$$

$$\lim_{x \to 3^-} \frac{x^3}{(x+3)(x-3)} = \frac{-27}{(6)(0^-)} = \infty$$

$$\lim_{x \to 3^+} \frac{x^3}{(x+3)(x-3)} = \frac{-27}{(6)(0^-)} = -\infty$$

Horizontal Asymptotes:

$$\lim_{x\to\pm\infty}\frac{x^3}{x^2-9}=\lim_{x\to\pm\infty}\frac{x}{1-\dfrac{9}{x^2}}=\pm\infty$$

Therefore, no horizontal asymptotes.

Other Asymptotes:

$$\frac{x^3}{x^2-9}=x+\frac{9x}{x^2-9}$$

$$\lim_{x\to\pm\infty}\left(\frac{x^3}{x^2-9}-x\right)=\lim_{x\to\pm\infty}\frac{9x}{x^2-9}=\lim_{x\to\pm\infty}\frac{9}{x\left(1+\dfrac{9}{x^2}\right)}=0$$

Therefore, $y=x$ is an asymptote.

e. Differentiability

$$f'(x)=\frac{d}{dx}\left(\frac{x^3}{x^2-9}\right)$$

$$f'(x)=\frac{(x^2-9)(3x^2)-(x^3)(2x)}{(x^2-9)^2}$$

$$f'(x)=\frac{x^2(x^2-27)}{(x^2-9)^2}$$

If $f'(x)=0$ then $x=0,3\sqrt{3},-3\sqrt{3}$ (Note: They are critical numbers.)

If $f"(x)$ does not exist then $x=3,-3$. (Note: 3 and -3 are not a critical numbers)

f. Monotonicity

Table of Signs

	x^2	$x-3\sqrt{3}$	$x+3\sqrt{3}$	$(x^2-9)^2$	f'(x)	
$x<-3\sqrt{3}$	+	-	-	+	+	increasing
$-3\sqrt{3}<x<-3$	+	-	+	+	-	decreasing
$-3<x<0$	+	-	+	+	-	decreasing
$0<x<3$	+	-	+	+	-	decreasing
$3<x<3\sqrt{3}$	+	-	+	+	-	decreasing
$x>3\sqrt{3}$	+	+	+	+	+	increasing

g. Relative extrema

Therefore, f has relative maximum at $x=-3\sqrt{3}$ and relative minimum at $x=3\sqrt{3}$.
(0,0) is neither a relative maximum nor a relative minimum.

h. Concavity

$$f"(x)=\frac{d}{dx}\left(\frac{x^2(x^2-27)}{(x^2-9)^2}\right)=\frac{d}{dx}\left(\frac{x^4-27x^2}{(x^2-9)}\right)$$

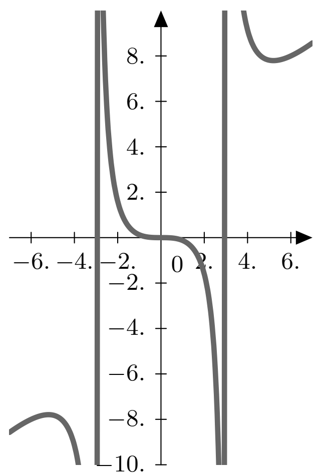
$$f"(x) = \frac{(x^2-9)^2(4x^3-54x)-(x^4-27x^2)[2(x^2-9)(2x)]}{(x^2-9)} = \frac{18x(x^2+27)}{(x^2-9)^3}$$

Therefore, $f"(x) = 0$ iff $x = 0$ and $f"(x)$ does not exist at $x = \pm 3$. (Note: not part of the domain)

Table of Signs for $f"(x)$

	$18x$	x^2+27	$x+3$	$x-3$	$(x^2-9)^3$	$f"(x)$	
$x < -3$	-	+	-	-	+	-	Concave Downward
$-3 < x < 0$	-	+	+	-	-	+	Concave Upward
$0 < x < 3$	+	+	+	-	-	-	Concave Downward
$x > 3$	+	+	+	+	+	+	Concave Upward

i. Points of Inflection: (0,0).



3. Sketch the graph of $f(x) = 2x - \tan x$ over the interval $\left(-\frac{\pi}{2}, -\frac{\pi}{2}\right)$

Solution:

a. Domain: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

b. Asymptotes:

$$\lim_{x \to \frac{\pi}{2}^-} f(x) = \pi - (+\infty) = -\infty$$

$$\lim_{x \to \frac{\pi}{2}^+} f(x) = -\pi - (+-\infty) = \infty$$

c. Continuity: f is continuous on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

d. Differentiability

$$f'(x) = 2 - \sec^2 x$$

If $f'(x) = 0$ then $2 - \sec^2 x = 0$. It implies that $\sec x = \sqrt{2}$ or $\sec x = -\sqrt{2}$. Reject $\sec x = -\sqrt{2}$ since it can't happen in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Therefore, $x \in \left\{\frac{\pi}{4}, -\frac{\pi}{4}\right\}$.
The critical numbers are $\frac{\pi}{4}$ and $-\frac{\pi}{4}$ and critical points are $\left(\frac{\pi}{4}, \frac{\pi}{2} - 1\right)$ and $\left(-\frac{\pi}{4}, 1 - \frac{\pi}{2}\right)$.
e. Relative Extremum
Table of Signs for $f'(x)$

	Test Value	$f'(x)$
$-\frac{\pi}{2} < x < -\frac{\pi}{4}$	$x = -\frac{\pi}{3}$	-
$-\frac{\pi}{4} < x < \frac{\pi}{4}$	$x = 0$	$+$
$\frac{\pi}{4} < x < \frac{\pi}{2}$	$\frac{\pi}{3}$	-

Therefore, f has a relative minimum at $x = -\frac{\pi}{4}$, and relative maximum at $x = \frac{\pi}{4}$.
f. Concavity

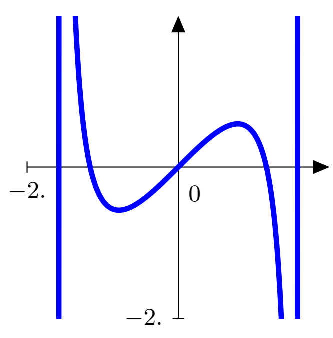
$$f"(x) = -2\sec x \cdot \sec x \tan x = -2\sec^2 x \tan x$$

$f"(x) = 0$ iff $-2\sec^2 x \tan x = 0$ iff $\tan x = 0$ iff $x = 0$.
$f"(x)$ exists for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Table of Values for $f"(x)$:

	$-2\sec^2 x$	$\tan x$	$f"(x)$	
$-\frac{\pi}{2} < x < 0$	-	-	$+$	Concave Upward
$0 < x < \frac{\pi}{2}$	-	$+$	-	Concave Downward

g. Point of Inflection: (0,0).



4. Sketch the graph of $f(x) = 3x^4 + 4x^3$.
Solution;
a. Domain: $(-\infty, \infty)$

b. y intercept: 0

x intercept:

$$3x^4 + 4x^3 = 0$$

$$x^3(3x+4) = 0$$

$$x = 0 \text{ or } x = -\frac{4}{3}$$

Therefore, f passes through $(0,0)$ and $\left(-\frac{4}{3}, 0\right)$.

c. Asymptotes: No vertical, horizontal and other asymptotes.

d. Continuity: f is continuous everywhere.

e. Differentiability:

$$f'(x) = \frac{d}{dx}(3x^4 + 4x^3) = 12x^3 + 12x^2$$

f. Monotonicity and Relative Maximum

$f'(x) = 0$ iff $12x^2(x+1) = 0$ iff $x \in \{0, -1\}$.

Table of Signs for $f'(x)$

	$12x^2$	$x+1$	$f'(x)$	
$x < -1$	$+$	-	-	decreasing
$-1 < x < 0$	$+$	$+$	$+$	increasing
$x > 0$	$+$	$+$	$+$	increasing

Therefore, the critical numbers are 0 and 1. The critical points are (0,0) and (-1,-1). Therefore, the relative minimum is (-1,-1).

g. Concavity and Point of Inflection

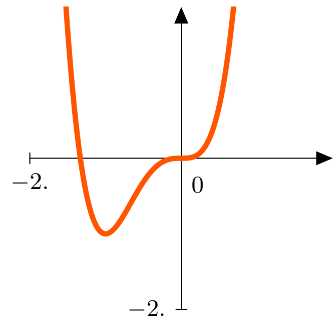
$$f"(x) = \frac{d}{dx}(12x^3 + 12x^2) = 36x^2 + 24x$$

$f"(x) = 0$ iff $12x(3x+2) = 0$ iff $x = 0$, $x = -\frac{2}{3}$

Table of Signs for $f"(x)$

	$12x$	$3x+2$	$f"(x)$	
$x < -\frac{2}{3}$	-	-	$+$	Concave Upward
$-\frac{2}{3} < x < 0$	-	$+$	-	Concave Downward
$x > 0$	$+$	$+$	$+$	Concave Upward

Points of Inflection: $\left(-\frac{2}{3},-\frac{16}{27}\right)$ and (0,0).



Lesson 3.5: The Rolle's Theorem and the Mean Value Theorem

Learning Outcomes

At the end of this lesson, you should be able to:

1. State the Rolle's Theorem and Mean Value Theorem;
2. Prove the Rolle's Theorem and Mean Value Theorem; and
3. Solve problems involving the Rolle's Theorem and Mean Value Theorem.

In the previous section, we see that if a differentiable function attains a relative extremum inside the open interval, then the function has a derivative 0 at the point where there is a relative extremum. This means that the function has a horizontal tangent line at the point where there is a relative extremum.

It is now natural to ask the following question: When does a differentiable function attain a derivative 0 inside an open interval?

Theorem 39: Rolle's Theorem

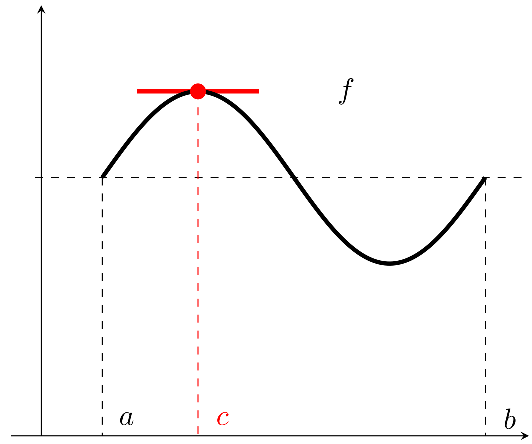
Let f be a function such that

1. f is continuous on $[a, b]$;
2. f is differentiable on (a, b);
3. $f(a) = f(b)$.

Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

The theorem is saying that if f has the three properties listed above, then the graph of f must have a horizontal tangent line at some point between a and b.

Proof. If f is a constant function, then we may take any c between a and b and $f'(c) = 0$, hence, the theorem. So, assume that f is not constant. Since f is continuous on $[a, b]$, the extreme value theorem guarantees that f attains its maximum and minimum values in $[a, b]$. Since $f(a) = f(b)$ and f is not constant, either the minimum or the maximum value of f is attained at some $c \in (a, b)$. In this case, f has a relative extremum at c and therefore, $f'(c) = 0$. □



Example 119. Consider the function $f(x) = x^2 - 6x + 5$. Observe that $f(1) = 0 = f(5)$. Since f is a polynomial function, then it is continuous on $[1, 5]$ and differentiable on $(1, 5)$. The Rolle's theorem now guarantees that there exists $c \in (1, 5)$ such that $f'(c) = 0$. In fact, $f'(x) = 0$ if and only if $2x - 6 = 0$ if and only if $x = 3$. This implies that, in fact, $c = 3$.

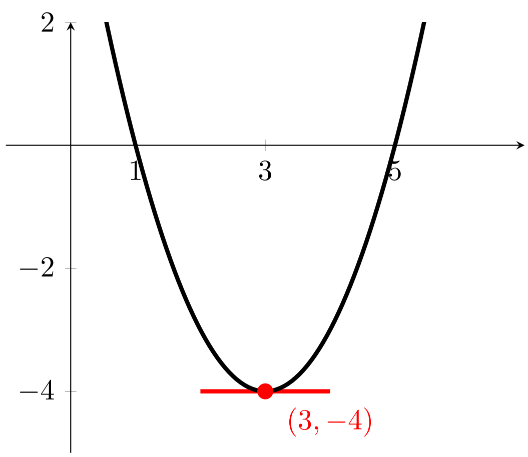


Figure 3.4: $f(x) = x^2 - 6x + 5$ on $[1, 5]$

Example 120. Let $f(x) = x^4 - x^2$. Observe that f is an even function. In particular, $f(-2) = f(2)$. Find $c \in (-2, 2)$ such that $f'(c) = 0$.

Solution. Note that $f'(x) = 4x^3 - 2x = 2x(2x^2 - 1)$. Thus,

$$\begin{aligned} f'(x) = 0 &\iff 2x(2x^2 - 1) = 0 \\ &\iff 2x = 0 \text{ or } 2x^2 - 1 = 0 \\ &\iff x = 0, \frac{\sqrt{2}}{2} \text{ or, } -\frac{\sqrt{2}}{2}. \end{aligned}$$

Therefore, $c = 0, \dfrac{\sqrt{2}}{2}$ or $-\dfrac{\sqrt{2}}{2}$.

The previous example shows that the Rolle's theorem only guarantees the existence of c and it doesn't say that such c is unique. For many cases, there could be more than one c that can satisfy the conclusion of the Rolle's theorem.

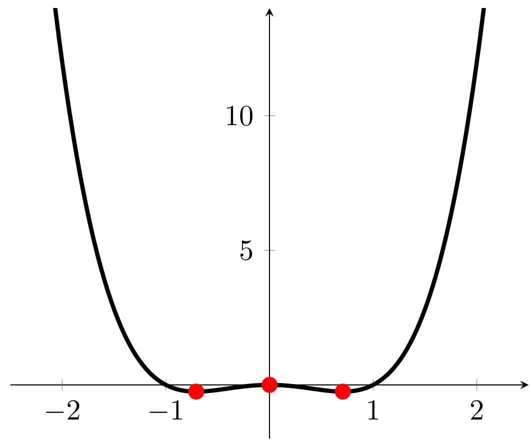


Figure 3.5: $f(x) = x^4 - x^2$ on $[-2, 2]$

Example 121. Prove that the equation $x^3 + 4x + 1 = 0$ has exactly one real solution.

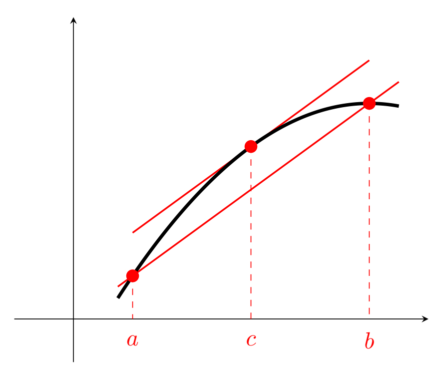
Solution. Consider the function $f(x) = x^3 + 4x + 1$ which is everywhere differentiable (hence, continuous). Note that $f(-1) = -4$ and $f(1) = 6$. Thus, f changes its sign from -1 to 1. The intermediate value theorem guarantees that f has at least one zero between -1 and 1. To prove that f cannot have more than one zeroes, note that $f'(x) = 3x^2 + 4$ assume that it has two distinct zeroes r_1 and r_2, where $r_1 < r_2$. Since $f(r_1) = f(r_2) = 0$, the Rolle's theorem applies to f on the closed interval $[r_1, r_2]$. There exists c between r_1 and r_2 such that $f'(c) = 0$. This implies that $3c^2 + 4 = 0$ implying that $c^2 = -4/3$ which is impossible. Therefore, f cannot have more than one zeroes. Consequently, f has exactly one zero between -1 and 1.

We now state the mean value theorem which is just a consequence of the Rolle's Theorem.

Theorem 40: The Mean Value Theorem

Let f be a function continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



Geometrically, the mean value theorem states that if a function f is continuous on $[a, b]$ and differentiable on (a, b), then at some point c that is between a and b, the line connecting $(a, f(a))$ and $(b, f(b))$ is parallel to the tangent line to the graph of f at the point $(c, f(c))$. This means that the average rate of change of f over $[a, b]$ is attained instantaneously at some point between a and b.

For example, if a car travelled a total distance of 200 kilometers from $1:00$ PM to $5:00$ PM, then the car averaged a speed of

$$\frac{200 \text{ km}}{4 \text{ hrs}} = 50 \text{ km/hr}.$$

Hence, the mean value theorem guarantees that at some time between $1:00$ PM and $5:00$ PM, the speedometer of the car hits exactly 50 km/hr at least once.

Proof. Consider the function

$$\varepsilon(x) := f(a) + \frac{f(b) - f(a)}{b - a}(x - a) - f(x),$$

defined for all $x \in [a, b]$. Observe that $\varepsilon(x)$ defines a continuous function on $[a, b]$ and is differentiable on (a, b). Observe that

$$\varepsilon(a) = f(a) + \frac{f(b) - f(a)}{b - a}(a - a) - f(a) = 0$$

and

$$\varepsilon(b) = f(a) + \frac{f(b) - f(a)}{b - a}(b - a) - f(b) = 0.$$

Thus, $\varepsilon(a) = \varepsilon(b)$. The Rolle's Theorem can then be applied to $\varepsilon(x)$ to conclude that there exists $c \in (a, b)$ such that $\varepsilon'(c) = 0$. But,

$$\varepsilon'(x) = \frac{f(b) - f(a)}{b - a} - f'(x), \text{ for all } x \in (a, b).$$

Since, $\varepsilon'(c) = 0$, it follows that

$$\frac{f(b) - f(a)}{b - a} - f'(c) = 0$$

implying that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

□

Although the mean value theorem can be used to solved problems in calculus, it is oftentimes used to prove more results. In fact, the mean value theorem is considered as one of the most important theorems in calculus.

Example 122. Consider the graph of the function $f(x) = x^2 - 1$.

1. Find the equation of the secant line joining the points $(-1, 0)$ and $(2, 3)$ in the graph of f.

2. Use the mean value theorem to determine a number c between -1 and 2 such that the tangent line to the graph of f at $(c, f(c))$ is parallel to the secant line through $(-1, 0)$ and $(2, 3)$.

Solution.

1. The slope of the line connecting $(-1, 0)$ and $(2, 3)$ is

$$m = \frac{3-0}{2-(-1)} = 1.$$

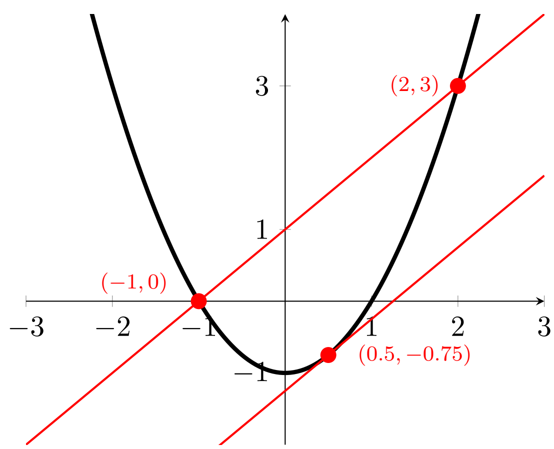
Since the line passes through $(-1, 0)$, using the *point-slope form of a line*, its equation is $y - 0 = 1 \cdot [x - (-1)]$, that is, $y = x + 1$.

2. Since $f(x)$ is a polynomial function, then f is continuous on $[-1, 2]$ and differentiable on $(-1, 2)$. Hence, the mean value theorem applies. So, there exists $c \in (-1, 2)$ such that

$$f'(c) = \frac{f(2) - f(-1)}{2-(-1)} = \frac{3-0}{2-(-1)} = 1.$$

But, $f'(x) = 2x$. So, $f'(c) = 1$ implies that $2c = 1$, therefore, $c = 1/2$. The tangent line that we need is the tangent to the graph of $f(x) = x^2 - 1$ at the point $(c, f(c)) = (1/2, -3/4)$ whose slope is 1. Therefore, the equation of this tangent line is

$$y - (-3/4) = 1 \cdot (x - 1/2) \quad \text{or} \quad y = x - (5/4).$$



Example 123. Prove that for all $x, y \in \mathbb{R}$, we have $|\sin x - \sin y| \leq |x - y|$.

Solution. Let $x, y \in \mathbb{R}$. If $x = y$, then there is nothing to prove as both sides of the inequality are equal to 0. Assume now that $x \neq y$, and for convenience, let us just assume that $y < x$. We apply the mean value theorem to $\sin t$ on the interval $[y, x]$. Since $\dfrac{d}{dt}(\sin t) = \cos t$, there exists c in the open interval (x, y) such that

$$\sin x - \sin y = \cos c(x - y).$$

Taking absolute values and using the fact that $|\cos c| \leq 1$, we therefore have

$$|\sin x - \sin y| = |\cos c|\, |x - y| \leq 1 \cdot |x - y| = |x - y|\,,$$

as we wished.

Recall that if a function is constant in an interval, then its derivative is zero in that interval. The following theorem discusses the converse if this fact, which is a consequence of the mean value theorem.

Theorem 41

If $f'(x) = 0$ for all x in an open interval I, then f is constant on I.

Proof. To show that f is constant on I, we pick two arbitrary points $a, b \in I$ such that $a < b$. We claim to prove that $f(a) = f(b)$.

Since f is continuous on $[a, b]$ and differentiable on (a, b), the mean value theorem implies that there exists $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a).$$

But, $f'(c) = 0$. Hence,

$$f(b) - f(a) = 0 \implies f(a) = f(b).$$

Therefore, f is constant on I. □

Lesson 3.6: Local Linear Approximation and Differentials

Learning Outcomes

At the end of this lesson, you should be able to:

1. Define and apply differentials in approximations;
2. Set-up Local Linear Approximations to estimate values; and
3. Apply theorems on computing differentials.

Recall that given a function f and a fixed position at $(x_0, f(x_0))$, then we have the derivative at x_0 as

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Also, if we let $x = x_0 + \Delta x$, then as $\Delta x \to 0$ we have $x \to x_0$ and

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

Lastly, letting $\Delta y = f(x_0 + \Delta x) - f(x_0)$, then

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

Hence, with sufficiently small enough Δx, then

$$f'(x_0) \approx \frac{\Delta y}{\Delta x}$$

and so

$$\Delta y \approx f'(x_0)\Delta x.$$

Definition 30: Differentials

Let f be a differentiable function over some open interval I and $y = f(x)$.

1. The **differential** $\boldsymbol{dx}$ of the independent variable x denotes an arbitrary increment in x.
2. The **differential** $\boldsymbol{dy}$ of the dependent variable y associated with x is given by $dy = f'(x)dx$.

Remark 9

1. If $dx \neq 0$, then $\dfrac{dy}{dx} = f'(x)$.
2. For sufficiently small dx, then $dy \approx \Delta y$.

Theorem 42

Let $c \in \mathbb{R}$, u and v be differentiable functions of x over some open interval I. Then

1. $d(c) = 0$
2. $d(x^n) = nx^{n-1}dx$, where $n \in \mathbb{Q}$
3. $d(c \cdot u) = c \cdot du$
4. $d(u+v) = du + dv$
5. $d(uv) = u \cdot dv + v \cdot du$
6. $d\left(\dfrac{u}{v}\right) = \dfrac{v \cdot du - u \cdot dv}{v^2}$.

Example 124. 1. If $y = 5x^3 - 14x^2 + 7x - 10$, then

$$\begin{aligned}
dy &= d(5x^3 - 14x^2 + 7x - 10) \\
dy &= d(5x^3) - d(14x^2) + d(7x) - d(10) \\
dy &= 5 \cdot d(x^3) - 14 \cdot d(x^2) + 7 \cdot d(x) - 0 \\
dy &= 5 \cdot 3x^2dx - 14 \cdot 2xdx + 7 \cdot dx \\
dy &= (15x^2 - 28x + 7)dx.
\end{aligned}$$

2. Let $y = \dfrac{x^2+3}{x^2-2}$. Then

$$\begin{aligned}
dy &= d\left(\frac{x^2+3}{x^2-2}\right). \\
dy &= \frac{(x^2-2) \cdot d(x^2+3) - (x^2+3) \cdot d(x^2-2)}{(x^2-2)^2} \\
dy &= \frac{(x^2-2) \cdot 2xdx - (x^2+3) \cdot 2xdx}{(x^2-2)^2} \\
dy &= \frac{-10x}{(x^2-2)^2}dx.
\end{aligned}$$

3. Solve $\dfrac{dy}{dx}$ in $xy^2 = 3x + 5y$.

   Solution:

   We have

$$\begin{aligned} d(xy^2) &= d(3x+5y) \\ x \cdot d(y^2) + y^2 \cdot dx &= 3dx + 5dy \\ x \cdot 2ydy + y^2 \cdot dx &= 3dx + 5dy \\ 2xydy + y^2dx &= 3dx + 5dy \\ (2xy-5)dy &= (3-y^2)dx \\ \frac{dy}{dx} &= \frac{3-y^2}{2xy-5}. \end{aligned}$$

4. A ball 10 inches in diameter is to be covered by a rubber material which is $\frac{1}{16}$ inch thick. Use differentials to estimate the volume of the rubber material that will be used.

   Solution:

   Let $V(r)$ be the volume of the ball of radius r. Thus,

$$V\left(5 + \tfrac{1}{16}\right) - V(5) = \Delta V \approx dV.$$

   Now, if $V(r) = \frac{4\pi}{3}r^3$, then $dV = 4\pi r^2 dr$. Thus, at $r = 5$ in and $dr = \frac{1}{16}$ in, then

$$dV = 4\pi(5)^2\left(\tfrac{1}{16}\right) = \tfrac{25\pi}{4}.$$

   Therefore, the estimated volume of the rubber material is $\frac{25\pi}{4}$ in^3.

5. A metal rod 15 cm long and 8 cm in diameter is to be insulated, except for the ends, with a material 0.001 cm thick. Use differentials to estimate the volume of the insulation.

   Solution:

   Let $V(r)$ be the volume of the metal rod of radius r. Thus, $V(r) = 15\pi r^2$ and

$$V(4+0.001) - V(4) = \Delta V = dV.$$

   Hence, $dV = 30\pi r dr$. At $r = 4$ cm and $dr = 0.001$ cm, then

$$dV = 30\pi(4)(0.001) = \tfrac{3\pi}{25}.$$

   Therefore, the estimated volume of the insulation is $\frac{3\pi}{25}$ in^3.

6. Suppose that the side of a square is measured with a ruler to be 8 inches with a measurement error of at most $\pm\frac{1}{64}$ of an inch. Estimate the error in the computed area of the square.

   Solution:

   Let $A(x)$ be the area of the square of side with length x. Thus, if $A(x) = x^2$, $|\Delta A| \approx |dA| = 2x|dx|$, then at $x = 8$in and $|dx| = \frac{1}{64}$ we have

   $$|dA| = 2(8)\left(\tfrac{1}{64}\right) = 0.25.$$

   Hence, the estimated error in the computed area is 0.25 in^2.

Remark 10

Since,

$$\Delta y \approx f'(x_0)\Delta x$$

then

$$f(x_0 + \Delta x) - f(x_0) \approx f'(x_0)(x - x_0).$$

Therefore,

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)(x - x_0).$$

Moreover, if $x_0 + \Delta x = x$, then

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

Note that the equation of the tangent line to the curve of f at $(x_0, f(x_0))$ is

$$y - f(x_0) = f'(x_0)(x - x_0),$$

or as in terms of x

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Henceforth, an arbitrary value for $f(x)$ can be approximated by the tangent line at that point.

Definition 31: Local Linear Approximates

Let f be a differentiable function at x_0. The function L given by

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

is called the **local linear approximation of f at x_0**.

Example 125. Approximate the following:

1. $\sqrt[3]{8.03}$

2. $\sqrt[3]{27.027}$

3. $\sqrt{15.96}$

Solutions:

1. Let $f(x) = \sqrt[3]{x}$ and express $x = 8.03$ as $x = 8 + 0.03$. Hence, we set $x_0 = 8$ and $\Delta x = 0.03$. Also, $f'(x) = \dfrac{1}{3x^{\frac{2}{3}}}$. Thus, $f'(8) = \dfrac{1}{12}$. So, using the local linear approximate L of f at $x_0 = 8$

$$\begin{aligned} L(8.03) &= f(8) + f'(8)(8.03 - 8) \\ L(8.03) &= \sqrt[3]{8} + \frac{1}{12}(0.03) \\ L(8.03) &= 2 + 0.0025 \\ L(8.03) &= 2.0025. \end{aligned}$$

Therefore, $\sqrt[3]{8} \approx 2.0025$.

2. If $f(x) = \sqrt[3]{x}$ and $x = 27.027$, then $f'(x) = \dfrac{1}{3x^{\frac{2}{3}}}, x_0 = 27, \Delta x = 0.027$ and $f'(27) = \dfrac{1}{27}$. Hence, with its local linear approximate L of f at $x_0 = 27$

$$\begin{aligned} L(27.027) &= f(27) + f'(27)(27.027 - 27) \\ L(27.027) &= \sqrt[3]{27} + \frac{1}{27}(0.027) \\ L(27.027) &= 3 + 0.001 \\ L(27.027) &= 3.001. \end{aligned}$$

So, $\sqrt[3]{27.027} \approx 3.001$.

3. With $x = 15.96$ and $f(x) = \sqrt{x}$, then let $x_0 = 16$ and $\Delta x = -0.04$. Hence, $f'(x) = \dfrac{1}{2\sqrt{x}}$ and $f'(16) = \dfrac{1}{8}$. Thus, through its local linear approximate L of f at $x_0 = 16$

$$\begin{aligned} L(15.96) &= f(16) + f'(16)(15.96 - 16) \\ L(15.96) &= \sqrt{16} + \frac{1}{8}(-0.04) \\ L(15.96) &= 4 - 0.005 \\ L(15.96) &= 3.995, \end{aligned}$$

and $\sqrt{15.96} \approx 3.995$.

Lesson 3.7: Optimization Problems

Learning Outcomes

At the end of this lesson, you should be able to:

1. Apply the concept of derivatives and relative extrema in worded problem; and
2. Compute efficiently the solution of a given optimization problem

Objective: Maximize or minimize a given function $f(x)$ where

1. x may be unrestricted, that is, $x \in \mathbb{R}$;
2. x may be restricted to lie on some interval, say $x \in [a, b]$.

Guidelines:

1. Identify all given **quantities** and all quantities **to be determined**. If possible, make a sketch or a diagram.
2. Write an equation which expresses the quantity to be maximized/minimized as a function of the given quantity.
3. Determine the **feasible domain** of the equation. (The set where the equation makes sense.)
4. Use calculus techniques to find the maximum or minimum.

Example 126.

1. Find a positive number such that the sum of the number and its reciprocal is the maximum.

   Solutions:

   Let x be the number and let the sum be represented by $S(x) = x + \dfrac{1}{x}$
   Clearly, $\mathrm{dom}(S) = \{x|x > 0\}$.
   Differentiating the both sides of equation,

   $$S'(x) = 1 - \frac{1}{x^2}$$

   .
   Set $S'(x) = 0$. Hence, $1 - \dfrac{1}{x^2} = 0$.

It implies that $\frac{x^2-1}{x^2} = 0$ or $x^2 - 1 = 0$.
Then, it shows that $(x+1)(x-1) = 0$ or $x = 1, -1$. We need to reject $x = -1$ as possible solution since $-1 \notin \text{dom}(S)$.

To check the maximality of 1, determine the second derivative of $S(x)$.

$$S\text{"}(x) = 0 - \frac{(1)(2x) - (x^2)(0)}{x^4} = -\frac{2x}{x^3} = -\frac{2}{x^3}$$

.

Then, substitute $x = 1$ in $S\text{"}(x)$.

$$S\text{"}(1) = -\frac{2}{1^3} = -2$$

Since, $S\text{"}(1) < 0$ then the sum is maximum when $x = 1$. Hence, the number we are looking for is 1.

2. A rectangular page is to contain 30 sq. inches of print. The margins on each side are 1 inch. Find the dimensions of the page such that the least amount of paper is used.

   Solution: Let x and y be the width and length of the printable rectangle. Then $xy = 30(in^2)$ so $y = \frac{30}{x}$. If A is the area of the page, then

$$A = (x+2)(y+2)$$

$$A = 30 + 2x + 2(\frac{30}{x}) + 4 = 34 + 2x + \frac{60}{x}$$

   We calculate $\frac{dA}{dx}$.

$$\frac{dA}{dx} = 2 - \frac{60}{x^2}$$

   and $\frac{dA}{dx} = 0$. It implies that $x^2 - 30 = 0$. Hence, $x = \sqrt{30}$ because $x > 0$.
   Hence, the critical number is $\sqrt{30}$.
   Since $\frac{d^2A}{dx^2} = -60(-2)(x^{-3}) = \frac{120}{x^3}$, and if $x = \sqrt{30}$, $\frac{d^2A}{dx^2} = \frac{120}{(\sqrt{30})^3} > 0$, by Second Derivative Test, A has a relative minimum when $x = \sqrt{30}$. Since there is no other relative extrema on $(0, \infty)$, then A is minimized, when $x - \sqrt{30}$.
   In this case, $y = \frac{30}{x} = \frac{30}{\sqrt{30}} = \sqrt{30}$.
   Therefore, the required dimension: $\sqrt{30}$in $\times$ $\sqrt{30}$in.

3. Show that the rectangle with largest possible area which can be inscribed in a given circle is a square.

Solution: Let r be the radius of the circle, and let x denote the length of the rectangle. Since the diagonal of the rectangle is a diameter of the circle, then by Pythagorean Theorem, the width w of the rectangle satisfies

$$w^2 = (2r)^2 - x^2$$

or

$$w = \sqrt{4r^2 - x^2}$$

.

Hence, if A is the area of the rectangle, then $A = xw = x\sqrt{4r^2 - x^2}$.

The feasible domain is $0 < x < 2r$.

We calculate $\frac{dA}{dx}$,

$$\frac{dA}{dx} = x \cdot \frac{-x}{\sqrt{4r^2 - x^2}} + \sqrt{4r^2 - x^2} \cdot 1$$

$$\frac{dA}{dx} = \frac{-x^2 + 4r^2 - x^2}{\sqrt{4r^2 - x^2}}$$

$$\frac{dA}{dx} = \frac{2(2r^2 - x^2)}{\sqrt{4r^2 - x^2}}$$

Therefore, if $\frac{dA}{dx} = 0$ then $2r^2 - x^2 = 0$ or $x = \pm\sqrt{2}r$.

Since $0 < x < 2r$, we disregard $x = -\sqrt{2}r$. And since $\frac{dA}{dx}$ exist in $(0, 2r)$, then the only critical number is $x = \sqrt{2}r$.

	$\sqrt{2}r - x$	$\sqrt{2}r + x$	f"(x)
$0 < x < \sqrt{2}r$	+	+	+
$\sqrt{2}r < x2r$	-	+	-

The first derivative test confirms that A is maximum when $x = \sqrt{2}r$. In this case, $w = \sqrt{4r^2 - (\sqrt{2}r)^2} = \sqrt{2}r$.

Therefore, the rectangle is a square.

4. A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible. He could proceed in any of the three ways:

   (a) Row his boat across the river to point C, then run to B.

   (b) Row directly to B.

   (c) Row to some point D between C and B and then run to B.

If he can row 6 km/hr and run 8km/hr, where should he land to reach B as soon as possible?

Solution:

If we let x denote CD, then the running distance is $DB = 8-x$ and by Pythagorean theorem, the rowing distance is $AD = \sqrt{x^2+9}$. Let T denote the time needed. Since time $= \dfrac{\text{distance}}{\text{rate}}$, then

$$\text{Rowing Time} = \frac{\sqrt{x^2+9}}{6}$$

and

$$\text{Running Time} = \frac{8-x}{8}$$

Thus, $T = \dfrac{\sqrt{x^2+9}}{6} + \dfrac{8-x}{8}$, where $x \in [0,8]$. Note that $\dfrac{dT}{dx} = \dfrac{x}{6\sqrt{x^2+9}} - \dfrac{1}{8}$.

Since, $x \geq 0$. $\dfrac{dT}{dx} = 0$ implies that $\dfrac{x}{6\sqrt{x^2+4}} = \dfrac{1}{8}$ or $7x^2 = 81$ or $x = \dfrac{9}{\sqrt{7}}$.

We evaluate T at $x = 0$ and $x = 8$. We see that

$$T(0) = 1.5, T(\frac{9}{\sqrt{7}}) = 1 + \frac{\sqrt{7}}{8}, T(8) = \frac{\sqrt{73}}{6}$$

.

Therefore, the smallest value of T occurs when $x = \dfrac{9}{\sqrt{7}}$.

The man should land the boat at a point $\dfrac{9}{\sqrt{7}}$ km downstream from his starting point.

Lesson 3.8: Rectilinear Motions

Learning Outcomes

At the end of this lesson, you should be able to:

1. Define different terminologies involving rectilinear motion; and
2. Compute and solve problems involving rectilinear motion.

One of the most important contributions of calculus in physics is the understanding of how objects move in space. If a particle is moving along a straight line, then this motion is called a **rectilinear motion**.

Suppose that a particle is moving along a straight line ℓ. Without loss of generality, assume that ℓ is the real number line. Suppose that at time $t \geq 0$, the coordinate of the particle is $s(t)$. The function s is called the **position function** of the particle.

We now define the following important terminologies.

Definition 32

Suppose that the position of a particle moving along a straight line is described by the position function $s(t)$.

1. The **average velocity** of the particle over the time interval $[t_0, t]$ is
$$v_{\text{ave}} = \frac{s(t) - s(t_0)}{t - t_0}.$$
2. The **average speed** of the particle over the time interval $[t_0, t]$ is
$$s_{\text{ave}} = \left| \frac{s(t) - s(t_0)}{t - t_0} \right|.$$
3. The **instantaneous velocity** of the particle at time t is
$$v(t) = \frac{ds}{dt} = s'(t) = \lim_{t \longrightarrow t_0} \frac{s(t) - s(t_0)}{t - t_0} = \lim_{\Delta t \longrightarrow 0} \frac{s(t + \Delta t) - s(t)}{\Delta t}.$$

Example 127. A particle is moving along a straight line according to the position function $s(t) = 2 + 3t - t^2$, t measured in seconds.

1. What is the initial position of the particle?

2. What is the position of the particle after 5 seconds?

3. What is the average velocity of the particle over the first 5 seconds?

4. What is the instantaneous velocity of the particle after 5 seconds?

Solution.

1. The initial position of the particle is the position or coordinate of the particle at $t = 0$. If $t = 0$, we see that $s = 2$.

2. At $t = 5$, $s = 2 + 3(5) - (5)^2 = -8$.

3. The average velocity of the particle over $[0, 5]$ is

$$v_{\text{ave}} = \frac{s(5) - s(0)}{5 - 0} = \frac{-8 - 2}{5 - 0} = -2 \text{ units/sec.}$$

   The negative sign indicates that the direction of the velocity is towards the negative values.

4. The instantaneous velocity at $t = 5$ is

$$v(5) = s'(5) = \frac{d}{dt}(2 + 3t - t^2)\bigg|_{t=5} = (3 - 2t)\bigg|_{t=5} = 3 - 2(5) = -7 \text{ units/sec.}$$

Example 128. If a stone is dropped from a top of a building of height 800 feet, it can be shown that the height of the stone t seconds after its release is $s(t) = 800 - 16t^2$ feet.

1. How high is the stone one second after its release?

2. How many seconds will it take before it reaches the ground?

3. What is its terminal velocity? (terminal velocity is the velocity of the stone the moment it reaches the ground)

Solution.

1. After one second the stone is $s(1) = 800 - 16(1)^2 = 784$ ft. high.

2. The stone reaches the ground the moment $s(t) = 0$. Hence, we solve for t in the equation

$$800 - 16t^2 = 0.$$

   It follows that $t^2 = 50$, therefore $t = \sqrt{50} \approx 7.07$ seconds. ($-\sqrt{50}$ is disregarded)

3. Observe that the velocity function of the stone is

$$v(t) = s'(t) = -32t.$$

Hence, the terminal velocity of the stone is

$$v(\sqrt{50}) = -32\sqrt{50} \approx -226.27 \text{ ft/sec}.$$

Note that the position function of the stone holds whenever we disregard *air resistance*, that is the only force acting on the stone is gravity.

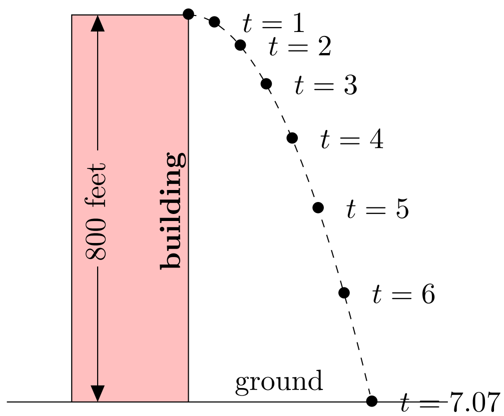


Figure 3.6: A Free Falling Object

Lesson 3.9: Related Rates

Learning Outcome

At the end of this lesson, you should be able to:

1. Apply the concept of derivative of implicit function in solving related rates problem.

Suppose that a balloon is being inflated at a constant rate of 8 cm^3/sec. Assuming that the balloon is spherical in shape, our problem is to know the rate in which the radius of the balloon changes at the moment the balloon has a radius 5 cm. This type of problem is called a **related rates problem**.

In a related rates problem, quantities involved change with respect to time. To solve this, we introduce the time variable t (which will be measured in seconds, in this case). Going back to the problem, note that the volume V of the balloon and its radius r is related by the equation $V = \dfrac{4}{3}\pi r^3$. We differentiate implicitly both sides with respect to t to get

$$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2) \cdot \frac{dr}{dt} = 4\pi r^2 \cdot \frac{dr}{dt},$$

thinking of r as a function of t. In this problem, the volume is increasing at a constant rate $8\text{cm}^3/\text{sec}$, so $\frac{dV}{dt} = 8\text{cm}^3/\text{sec}$ and since we are interested to know the rate in which the radius r changes when the radius of the balloon is exactly 5 cm, we solve for $\frac{dr}{dt}$ given that $r = 5$ cm. Thus,

$$(8\text{cm}^3/\text{sec}) = 4\pi(5\text{cm})^2 \cdot \frac{dr}{dt} \implies \frac{dr}{dt} = \frac{8\text{cm}^3/\text{sec}}{4\pi(5\text{cm})^2} = \frac{2}{25\pi}\text{cm/sec} \approx 0.0255\text{cm/sec}.$$

We set up the following guidelines in solving a related rates problem.

Guidelines 4: Guidelines in Solving a Related Rates Problem

1. Read carefully and understand the problem. Identify what are the given data and what is needed in the problem.

2. Draw and label a diagram, if possible, and find a working equation involving the variables.

3. Differentiate both sides of the working equation with respect to the time variable t implicitly.

4. Substitute known values and rates and solve for the unknown quantity. Be watchful of the units.

Example 129. Suppose that x and y are quantities which are differentiable functions of t and are related by the equation $y = \sqrt{x^2+1}$. Find $\dfrac{dy}{dt}$ given that $\dfrac{dx}{dt} = 3$ when $x = 1$.

Solution. We use the chain rule and implicit differentiation to differentiate both sides of the equation with respect to t.

$$y = \sqrt{x^2+1}$$

$$\frac{d}{dt}(y) = \frac{d}{dt}(\sqrt{x^2+1})$$

$$\frac{dy}{dt} = \frac{1}{2\sqrt{x^2+1}} \cdot 2x \cdot \frac{dx}{dt}$$

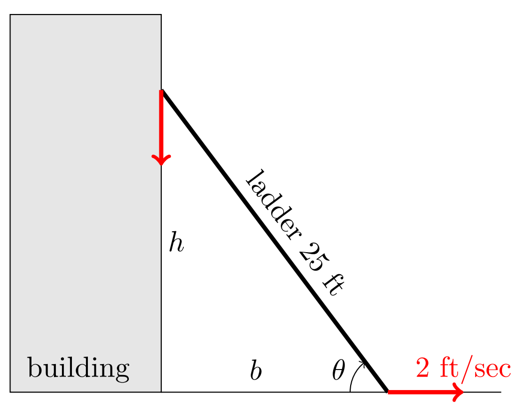
$$\frac{dy}{dt} = \frac{x}{\sqrt{x^2+1}} \cdot \frac{dx}{dt}.$$

Subsituting $x = 1$ and $\dfrac{dx}{dt} = 3$ gives

$$\frac{dy}{dt} = \frac{1}{\sqrt{1^2+1}} \cdot (3) = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}.$$

Example 130. (Sliding Ladder Problem) A ladder 25 ft. long is leaning against the wall of a building. The foot of the ladder is pulled away from the wall at a rate of 2 feet per second.

1. How fast is the top of the ladder moving down the wall when its foot is exactly 7 feet from the wall?
2. How fast is the the angle formed by the ladder and the ground changing (in radians per second) at the moment when the top of the ladder is 24 feet from the ground?



Solution. We are given that the ladder is 25 feet long and that the foot of the ladder is pulled away from the building at a rate 2 ft/sec. At any given time t, denote by h the distance of the top of the ladder from the ground and b be the distance of the foot of the ladder from the building. See accompanying figure.

1. In this item, we are asked about the rate in which the top of the ladder is moving down so we need an equation relating h and b. By pythagorean theorem, we see that

$$h^2 + b^2 = 25^2. \tag{3.1}$$

Differentiating both sides with respect to t yields

$$2h\frac{dh}{dt} + 2b\frac{db}{dt} = 0$$

implying that

$$\frac{dh}{dt} = -\frac{b}{h} \cdot \frac{db}{dt}.$$

At the moment when the foot of the ladder is 7 ft from the wall, $b = 7$ ft, $\frac{db}{dt} = 2$ ft/sec and from equation 3.1, $h = \sqrt{25^2 - 7^2}$ ft $= 24$ ft. Therefore, at this moment,

$$\frac{dh}{dt} = -\frac{7 \text{ ft}}{24 \text{ ft}} \cdot (2 \text{ ft/sec}) = -\frac{7}{12} \text{ ft/sec}.$$

Therefore, the top of the ladder is sliding down at a rate $\frac{7}{12}$ ft/sec the moment when the foot of the ladder is 7 ft from the wall. Note that the negative sign in the final answer indicates that the height h is decreasing.

2. In this item, we are asked about the rate in which the angle θ changes the moment when $h = 24$ ft. At the moment when $h = 24$ ft, equation 3.1 implies that $b = \sqrt{25^2 - 24^2}$ ft or 7 ft. We then need to find a working equation involving b and θ. Observe that

$$\cos\theta = \frac{b}{25}.$$

Differentiating both sides with respect to t yields,

$$-\sin\theta \cdot \frac{d\theta}{dt} = \frac{1}{25} \cdot \frac{db}{dt} \implies \frac{d\theta}{dt} = -\frac{\csc\theta}{25} \cdot \frac{db}{dt}.$$

At the moment when $h = 24$ ft, we have

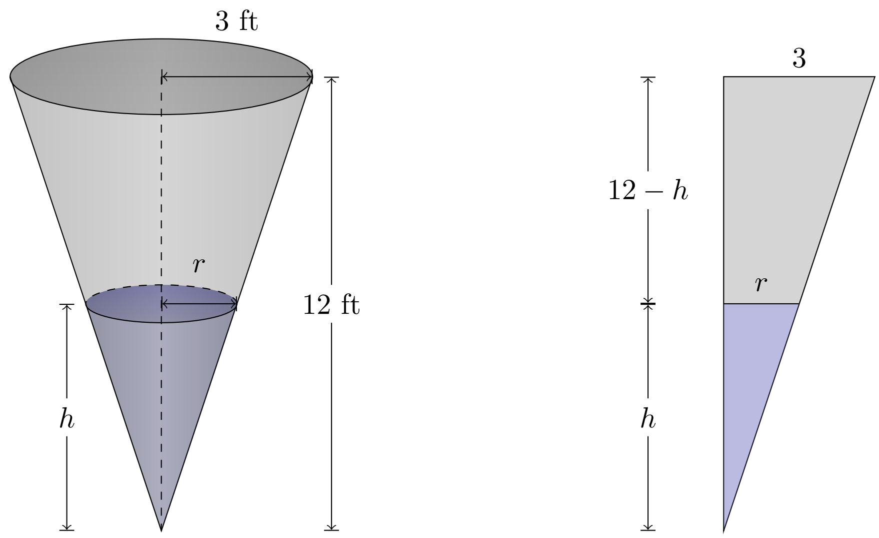
$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite side}} = \frac{25}{24}.$$

Hence,

$$\frac{d\theta}{dt} = -\frac{\csc\theta}{25}\frac{db}{dt} = -\frac{25/24}{24 \text{ ft}} \cdot (2 \text{ ft/sec}) = -\frac{25}{576} \text{ radians/sec.}$$

Therefore, the angle made by the ladder and the ground is decreasing in a rate of $-25/576$ radians per second at the moment when the top of the ladder is 24 feet from the ground.

Example 131. (Pouring Water in a Conical Tank) A water tank in a shape of a right circular cone has a height 12 feet and radius 3 feet. Water is pumped into the tank at a rate of 4 cubic feet per minute. How fast is the water level rising when the water level is 6 feet?



Solution. Let h be the water level in feet in the conical tank and r be the radius as in the figure. Using similar triangles, we see that

$$\frac{r}{3} = \frac{h}{12} \implies r = \frac{h}{4}.$$

Therefore, the volume V of the water is

$$V = \frac{1}{3}\pi(\text{radius})^2(\text{height}) = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{4}\right)^2 h$$

which implies that

$$V = \frac{\pi}{48}h^3.$$

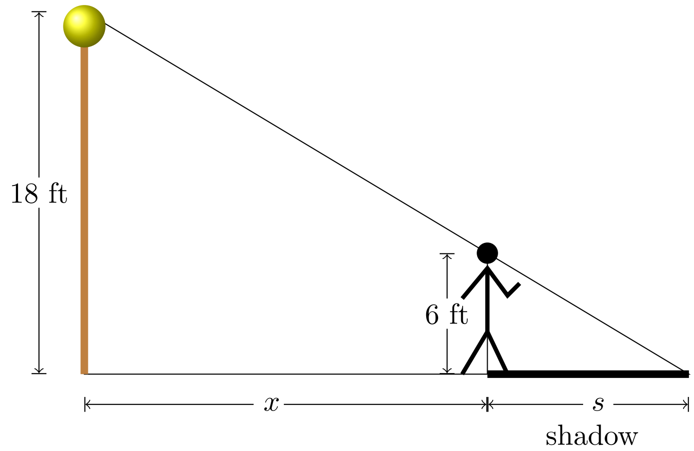
Differentiating both sides with respect to t yields

$$\frac{dV}{dt} = \frac{\pi}{16}h^2 \cdot \frac{dh}{dt} \implies \frac{dh}{dt} = \frac{16}{\pi h^2} \cdot \frac{dV}{dt}.$$

Since the water is pouring at a rate 4 ft^3/min, it follows that $\frac{dV}{dt} = 4\ \text{ft}^3/\text{min}$. Hence, at the moment the water level is 6 ft, $h = 6$ ft and the rate in which the water level rises is

$$\frac{dh}{dt} = \frac{16}{\pi h^2} \cdot \frac{dV}{dt} = \frac{16}{\pi(6\ \text{ft})^2}(4\ \text{ft}^3/\text{sec}) = \frac{16}{9\pi}\ \text{ft/sec} \approx 0.5659\ \text{ft/sec}.$$

Example 132. (Length of a Shadow at Night) One night, a man 6 feet tall walks away from a light post 18 feet high along a straight path at a speed of 4 feet per second. At what rate is the length of his shadow changing the moment he is exactly 20 feet from the light post?



Solution. Let s be the length of the shadow and x be the distance of the man from the light post as in the figure. Using similar triangles, we see that

$$\frac{s}{x+s} = \frac{6}{18} \implies s = \frac{x}{2}.$$

Differentiating both sides with respect to t gives

$$\frac{ds}{dt} = \frac{1}{2} \cdot \frac{dx}{dt}.$$

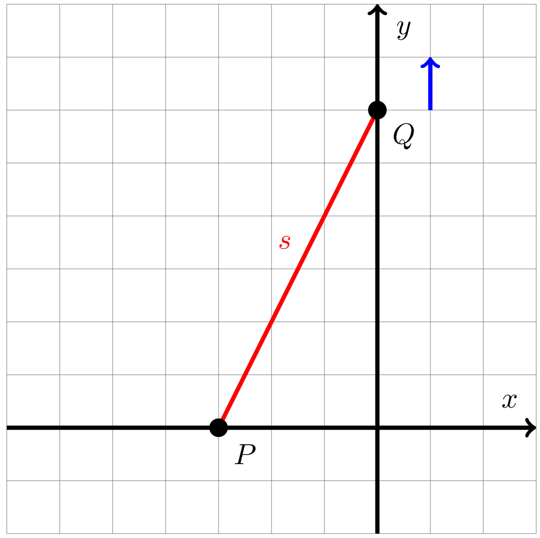
Since the man is walking at a constant rate of 4 feet per second, it follows that $\frac{dx}{dt} = 4$ ft/sec and thus,

$$\frac{ds}{dt} = \frac{1}{2} \cdot \frac{dx}{dt} = \frac{1}{2}(4 \text{ ft/sec}) = 2 \text{ ft/sec}.$$

Therefore, the shadow is increasing its length at a rate of 2 feet per second.

Note that $\dfrac{ds}{dt}$ depends on $\dfrac{dx}{dt}$ only. Hence, no matter what the distance of the man from the light post, the rate in which his shadow is increasing will depend only on how fast he walks.

Example 133. A point Q is moving along the y-axis in the positive direction at a rate of 2 units per second. At what rate is the distance of Q from $P(-3, 0)$ changing the moment when Q is located at $(0, 4)$?



Solution. At a given time t, suppose that the point Q is located at $(0, y)$. If the distance between P and Q is s, then by the distance formula,

$$s = \sqrt{(0 - (-3))^2 + (y - 0)^2} = \sqrt{9 + y^2}.$$

Differentiating both sides with respect to t yields

$$\frac{ds}{dt} = \frac{y}{\sqrt{9 + y^2}} \cdot \frac{dy}{dt}.$$

Since y is changing at a rate of 2 units per second, $\frac{dy}{dt} = 2$ so, at the moment when Q is at $(0, 5)$, $y = 5$ and

$$\frac{ds}{dt} = \frac{5}{\sqrt{9 + 4^2}} \cdot (2 \text{ units/sec}) = 2 \text{ units/sec}.$$

Therefore, at the moment when Q is at $(0, 4)$, the distance between P and Q is changing at a rate of 2 units per second.

3.10 Unit Test 3

Applications of Differentiation

Instruction: Write all your official answers and solutions on sheets of yellow pad paper, using only either black or blue pens.

I. **Write T is the statement is true with respect to the underline term(s). Otherwise, write the term(s) that will make the statement true.**

1. Let $f(x) = -\dfrac{2}{x+1}$. Then, f is $\underline{\text{increasing}}$ on $[3, +\infty]$.
2. Let $g(x) = 4 - x^2$. Then, g is $\underline{\text{decreasing}}$ on $[-100, -1]$.
3. Define f such that $f'(x) = 2x - 5$. Then, f has a relative minimum at $\underline{x = \dfrac{5}{2}}$.
4. Define g as $g(x) = x^4 - 5x^2 + 6$. Then, g is concave $\underline{\text{upward}}$ at $x = \dfrac{1}{2}$.
5. Define f such that $f''(x) = x^2 + 1$ with critical number at $x = 0$. Then, f has a relative maximum at $\underline{x = 0}$.
6. Define $g(x) = x^4 - 5x^2 + 4$ over $[1, 2]$. Then, by Rolle's Theorem, $\exists\, c \in [1, 2]$ such that $\underline{12c^2 - 10 = 4}$.
7. Define $g(x) = x^4 - 5x^2 + 4$ over $[-1, 0]$. Then, by the Mean Value Theorem, $\exists\, c \in [-1, 0]$ such that $\underline{4c^3 - 10c = 4}$.
8. Let $f(x) = 8\sqrt[4]{x}$. Then, the local linear approximate of f around $x = 8$ is $L(x) = \underline{4 + \dfrac{1}{16}(x - 4)}$.
9. Let $h(x) = \sec x$ be observed over $\left[-\dfrac{\pi}{3}, \dfrac{\pi}{3}\right]$. Then, h attains an absolute minimum at $\underline{x = 0}$.
10. A particle has a position function $s(t) = t^4 - 4t^2 + 4$. Its instantaneous speed at $t = 1$ is $\underline{\ 8\ }$ units per second.

II. **Each question is a multiple-choice question with four answer choices. Read each question and answer choice carefully and choose the ONE best answer.**

1. Let $f(x) = 2\sqrt{x}$. Which of the following is local linear approximate of f around $x = 25$?

   a. $L(x) = 5 + \frac{1}{5}(x - 10)$
   b. $L(x) = 10 + \frac{1}{5}(x - 25)$
   c. $L(x) = 15 + \frac{1}{10}(x - 5)$
   d. $L(x) = 25 + \frac{1}{10}(x - 5)$

2. A ball 6 inches in diameter is to be covered by a rubber material which is $\frac{3}{16}$ in-thick. Using differentials, which of the following is an estimate of the volume of the rubber material that will be used?

   a. $24\pi\ in^3$
   b. $25\pi\ in^3$
   c. $27\pi\ in^3$
   d. $30\pi\ in^3$

3. Suppose a metal rod has length 20 cm and diameter 6 cm. If it is to be insulated, except for the ends, with a material 0.001 cm thick, determine the estimated volume of the insulation using differentials.

   a. $0.12\pi\ in^3$
   b. $0.2\pi\ in^3$
   c. $0.35\pi\ in^3$
   d. $0.6\pi\ in^3$

For items 4 to 8, let $f(x) = -2x^3 + 6x + 4$.

4. Which of the following is the relative maximum point of the graph of f?

   a. $(-1, -4)$
   b. $(-1, 0)$
   c. $(1, 0)$
   d. $(1, 8)$

5. Which of the following is the relative minimum point of the graph of f?

   a. $(-1, -4)$
   b. $(-1, 0)$
   c. $(1, 0)$
   d. $(1, 8)$

6. Which of the following intervals have a negative trend for the graph of f?

   a. $(-3, -1)$
   b. $(-2, 0)$
   c. $(-1, 1)$
   d. $(0, 2)$

7. Which of the following is the point of inflection of the graph of f?

   a. $(-2, 2)$
   b. $(-1, 3)$
   c. $(0, 4)$
   d. $(1, 5)$

8. Which of the following intervals show a downward concavity for the graph of f?

   a. $(-3, -1)$
   b. $(-2, 0)$
   c. $(-1, 1)$
   d. $(0, 2)$

For items 9 to 13, let $g(x) = \dfrac{4x^2}{x^2 - 4}$.

9. Which of the following is a critical value of g?

   a. $x = -\sqrt{2}$
   b. $x = -2$
   c. $x = 0$
   d. $x = 2$

10. Which of the following is the relative maximum point of the graph of g?

    a. $(-1, \frac{4}{3})$
    b. $(0, 0)$
    c. $(1, \frac{4}{3})$
    d. $(3, -2)$

11. Which of the following intervals have a positive trend for the graph of g?

    a. $(-\infty, -2)$
    b. $(-4, 1)$
    c. $(0, 2)$
    d. $(2, +\infty)$

12. Which of the following intervals show an upward concavity for the graph of f?

    a. $(-\infty, -1)$
    b. $(-2, 0)$
    c. $(-1, 2)$
    d. $(4, +\infty)$

13. Which is true about its point of inflection?

    a. It inflects at $(0, 0)$.
    b. It has no point of inflection.
    c. It inflects at $(0, 0)$ and $(1, 2)$.
    d. Its only points of inflection is $(-1, 2)$.

14. Which of the following is the absolute maximum point of $f(x) = |x - 3|$ on the interval $(-1, 2)$?

    a. $(-1, 4)$
    b. $(0, 3)$
    c. $(1, 2)$
    d. $(2, 1)$

15. The sum of two numbers is 16. Find the numbers if the sum of their cubes is minimum.

    a. 5 and 11
    b. 6 and 10
    c. 7 and 9
    d. 8 and 8

16. What is the area of the largest rectangles that can inscribed in a circle of radius 8 units?

    a. 32 sq. units
    b. 64 sq. units
    c. 128 sq. units
    d. 256 sq. units

For item 17 and 18, consider the position function $\mathbf{s}$ of a particle P, defined by the function

$$\mathbf{s(t)} = -4t^3 + 33t^2 - 72t + 10,$$

moving along a horizontal line at a time t.

17. Which of the following is an instance that P has a positive velocity?

a. $t = 1$

b. $t = 3$

c. $t = 5$

d. $t = 7$

18. Which of the following is an instance that P has negative acceleration?

a. $t = 3$

b. $t = 2$

c. $t = 1$

d. $t = 0$

19. Water flows into a vertical cylindrical tank at the rate of 24 ft^3 per minute. If the radius of the tank is 6 ft, how fast is the surface rising?

a. $\dfrac{4}{3\pi}$ ft per min

b. $\dfrac{5}{6\pi}$ ft per min

c. $\dfrac{2}{3\pi}$ ft per min

d. $\dfrac{5}{3\pi}$ ft per min

20. A restaurant supplier services the restaurants in a circular area in such a way that the radius r is increasing at the rate of 3 km per year at the moment when r goes through the value $r = 5$ miles. At that moment, how fast is the area increasing?

a. 20 km^2 per year

b. 24 km^2 per year

c. 25 km^2 per year

d. 30 km^2 per year

Unit 4: Antiderivatives and Indefinite Integrals

In this unit, we will introduce the concept of antidifferentiation, the process which reverses differentiation. Given a function f, can we find a function F whose derivative is f? This chapter consists of the basic properties of integrals and some techniques on evaluating some integrals such as integration by substitution, integrals leading to logarithms, integrals of exponential functions, integrals of trigonometric functions, integrals leading to inverse trigonometric functions and integrals of hyperbolic functions.

Lesson 4.1: Antiderivatives

Learning Outcomes

At the end of this lesson, you should be able to:

1. Determine the antiderivatives of a function;
2. Discuss the integral notation; and
3. Enumerate the basic properties of indefinite integrals.

In this lesson, we will discuss the relationship between the concept of derivatives and integrals. We will introduce integrals in terms of antiderivatives, and the symbol for integration.

Definition 33: Antiderivative

A function F is called an antiderivative of a function f on a given interval I if $F'(x) = f(x)$ for all x in the interval I.

Example 134. The function $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$ on the interval $(-\infty, \infty)$ because for each x in this interval

$$F'(x) = x^2 = f(x).$$

The function $G(x) = \frac{1}{3}x^3 + 1$ is also an antiderivative of $f(x) = x^2$ on the interval $(-\infty, \infty)$ because for each x in this interval

$$G'(x) = x^2 = f(x).$$

Theorem 43

If $F(x)$ is any antiderivative of $f(x)$ on an interval I, then for any constant C the function $F(x) + C$ is also an antiderivative on that interval.

The process of finding antiderivatives is called **antidifferentiation** or **integration**. Thus, if $\frac{d}{dx}[F(x)] = f(x)$ then integrating the function $f(x)$ produces an antiderivative of the form $F(x)+C$. Which will be denoted by $\int f(x)dx$.

In $\int f(x)dx = F(x) + C$, the expression $\int f(x)dx$ is called an **indefinite integral**. The elongated s that appears on the left side is called an **integral sign**, the function $f(x)$ is called the **integrand**, and the constant C is called the **constant of integral**.

Example 135. Since $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$, then we write

$$\int x^2 dx = \frac{1}{3}x^3 + C.$$

Remark 11

1. $\int F'(x)dx = F(x) + C$
2. If $\int f(x)dx = F(x) + C$, then $\frac{d}{dx}\left[\int f(x)dx\right] = f(x)$.
3. Hence, differentiation and integration are inverses in nature.

Theorem 44: Basic Integration Formulas

Suppose that $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$, respectively, and that C is a constant. Then:

1. An antiderivative of one is the variable plus any constant; that is,

$$\int dx = x + C$$

2. A constant factor can be moved through an integral sign; that is,

$$\int cf(x)dx = cF(x) + C$$

3. An antiderivative of a sum is the sum of the antiderivatives; that is,

$$\int [f(x) + g(x)]dx = F(x) + G(x) + C$$

4. For any integer n not equal to -1,

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C$$

Example 136. Evaluate $\int (3x^4 - 5x^3 + 6x^2 - 7x + 3)dx$.

Solution:

$$\begin{aligned}
\int (3x^4 - 5x^3 + 6x^2 - 7x + 3)dx &= \int 3x^4dx - \int 5x^3dx + \int 6x^2dx - \int 7xdx + \int 3dx \\
&= 3\int x^4dx - 5\int x^3dx + 6\int x^2dx - 7\int xdx + 3\int dx \\
&= 3 \cdot \frac{1}{5}x^5 - 5 \cdot \frac{1}{4}x^4 + 6 \cdot \frac{1}{3}x^3 - 7 \cdot \frac{1}{2}x^2 + 3x + C \\
&= \frac{3}{5}x^5 - \frac{5}{4}x^4 + 2x^3 - \frac{7}{2}x^2 + 3x + C
\end{aligned}$$

Example 137. Evaluate $\int (5x^2 - 7x + 1)(7x - 3)dx$.

Solution:

$$\begin{aligned}
\int (5x^2 - 7x + 1)(7x - 3)dx &= \int (35x^3 - 64x^2 + 28x - 3)dx \\
&= \int 35x^3dx - \int 64x^2dx + \int 28xdx - \int 3dx \\
&= 35\int x^3dx - 64\int x^2dx + 28\int xdx - 3\int dx \\
&= 35 \cdot \frac{1}{4}x^4 - 64 \cdot \frac{1}{3}x^3 + 28 \cdot \frac{1}{2}x^2 - 3x + C \\
&= \frac{35}{4}x^4 - \frac{64}{3}x^3 + 14x^2 - 3x + C
\end{aligned}$$

Example 138. Evaluate $\int \left(\frac{14x^3 - 69x^2 + 41x - 63}{2x - 9}\right) dx$.

Solution:

$$\begin{aligned}
\int \left(\frac{14x^3 - 69x^2 + 41x - 63}{2x - 9}\right) dx &= \int (7x^2 - 3x + 7) dx \\
&= \int 7x^2 dx - \int 3x dx + \int 7 dx \\
&= 7\int x^2 dx - 3\int x dx + 7\int dx \\
&= 7 \cdot \frac{1}{3}x^3 - 3 \cdot \frac{1}{2}x^2 + 7x + C \\
&= \frac{7}{3}x^3 - \frac{3}{2}x^2 + 7x + C
\end{aligned}$$

Example 139. Evaluate $\int \left(\frac{8x^3 - 7x^2 + 9}{x^2}\right) dx$.

Solution:

$$\begin{aligned}
\int \left(\frac{8x^3 - 7x^2 + 9}{x^2}\right) dx &= \int x^{-2}(8x^3 - 7x^2 + 9) dx \\
&= \int (8x - 7 + 9x^{-2}) dx \\
&= \int 8x dx - \int 7 dx + \int 9x^{-2} dx \\
&= 8\int x dx - 7\int dx + 9\int x^{-2} dx \\
&= 8 \cdot \frac{1}{2}x^2 - 7x + 9 \cdot \frac{1}{-1}x^{-1} + C \\
&= 4x^2 - 7x - 9x^{-1} + C \\
&= 4x^2 - 7x - \frac{9}{x} + C
\end{aligned}$$

Definition 34: Particular Antiderivative of f

A **particular antiderivative** is an antiderivative that satisfies well specified conditions, i.e, if $\int f(x)dx = F(x) + C$ and C takes a specific value C_0 for C, then $y = F(x) + C_0$ is a **particular antiderivative of** f.

Example 140. Evaluate $\int 5\sqrt{x} dx$ and determine its particular antiderivative passing through the point (4, 3).

Solution: Let $y = F(x)$ be an equation of the curve. The slope of the tangent line at a point (4, 3) on the graph of the curve is given by $F'(x) = 5\sqrt{3}$. We have

$$F(x) = \int 5x^{\frac{1}{2}} dx = \frac{10}{3}x^{\frac{3}{2}} + C.$$

The initial condition that (4, 3) is on the curve implies that

$$F(4) = \frac{10}{3}(4)^{\frac{3}{2}} + C = 3.$$

We obtain $C = -\frac{71}{3}$. Thus, an equation of the curve is

$$y = \frac{10}{3}x^{\frac{3}{2}} - \frac{71}{3}.$$

Example 141. Evaluate $\int \sin\left(\frac{2}{3}x\right)dx$ and determine its particular antiderivative whose y-intercept is 4.

Solution: Let $y = F(x)$ be the equation of the curve. The slope of the tangent line at a point (0, 4) on the graph of the curve is given by $F'(x) = \sin\left(\frac{2}{3}x\right)$. We have

$$F(x) = \int \sin\left(\frac{2}{3}x\right)dx = \frac{3}{2}\cos\left(\frac{2}{3}x\right) + C.$$

Note that $\frac{d}{dx}\left[\frac{3}{2}\cos\left(\frac{2}{3}x\right)\right] = \sin\left(\frac{2}{3}x\right)$. The initial condition that (0, 4) is on the curve implies that

$$F(0) = \frac{3}{2}\cos\left(\frac{2}{3}(0)\right) + C = \frac{3}{2}\cdot 1 + C = \frac{3}{2} + C = 4.$$

We obtain $C = \frac{5}{2}$. Thus, an equation of the curve is

$$y = \frac{3}{2}\cos\left(\frac{2}{3}x\right) + \frac{5}{2}.$$

Lesson 4.2: Integration by Substitution

Learning Outcomes

At the end of this lesson, you should be able to:

1. Apply the concept of u-substitution; and
2. Evaluate integrals of some functions using u-substitution.

In this lesson, we integrate functions by applying the concept of differential of composite functions also known as Chain Rule.

Recall:

Given a differentiable functions $y = F(u)$ and $u = g(x)$, the chain rule states that

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x).$$

Hence, integrating both sides

$$\int F'(g(x))g'(x)dx = F(g(x)) + C.$$

Theorem 45: Antidifferentiation of a Composite Function

Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f(g(x))g'(x)dx = F(g(x)) + C.$$

Letting $u = g(x)$, gives $du = g'(x)dx$ and

$$\int f(u)du = F(u) + C.$$

Antidifferentiation of a Composite Function

$$\int f(u)du = F(u) + C$$

Example 142. Evaluate $\int 2x(x^2 + 1)^4 dx$.

Solution:

$$\int 2x(x^2 + 1)^4 dx = \int (x^2 + 1)^4 (2xdx).$$

Let $u = x^2 + 1$. Then $du = 2xdx$. Thus,

$$\begin{aligned}\int 2x(x^2+1)^4dx &= \int u^4du \\ &= \frac{u^5}{5}+C \\ &= \frac{(x^2+1)^5}{5}+C.\end{aligned}$$

Example 143. Evaluate $\int \frac{x^2}{\sqrt[3]{(x^3+3)^2}}dx$.

Solution:

$$\int \frac{x^2}{\sqrt[3]{(x^3+3)^2}}dx = \frac{1}{3}\int (x^3+1)^{-2/3}(3x^2dx).$$

Let $u = x^3+3$. Then $du = 3x^2dx$. Thus,

$$\begin{aligned}\int \frac{x^2}{\sqrt[3]{(x^3+3)^2}}dx &= \frac{1}{3}\int u^{-2/3}du \\ &= \frac{1}{3}\cdot\frac{u^{1/3}}{\frac{1}{3}}+C \\ &= u^{1/3}+C \\ &= \sqrt[3]{x^3+3}+C.\end{aligned}$$

Example 144. Evaluate $\int \left(x+\frac{1}{x}\right)^{3/2}\left(\frac{x^2-1}{x^2}\right)dx$

Solution:

$$\begin{aligned}\int \left(x+\frac{1}{x}\right)^{3/2}\left(\frac{x^2-1}{x^2}\right)dx &= \int \left(\frac{x^2+1}{x}\right)^{3/2}\left(\frac{x^2-1}{x^2}\right)dx \\ &\quad \text{let } u=\frac{x^2+1}{x}\text{, then } du=\frac{x^2-1}{x^2}dx \\ &= \int u^{3/2}du \\ &= \frac{1}{\frac{3}{2}+1}u^{\frac{3}{2}+1}+C \\ &= \frac{1}{\frac{5}{2}}u^{\frac{5}{2}}+C \\ &\quad \text{but } u=\frac{x^2+1}{x} \\ &= \frac{2}{5}\left(\frac{x^2+1}{x}\right)^{\frac{5}{2}}+C\end{aligned}$$

Lesson 4.3: Integration Leading to Logarithm

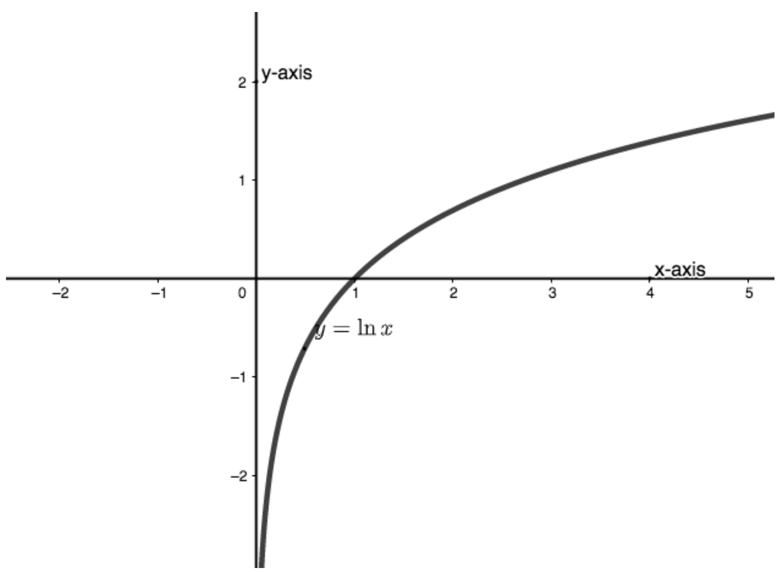
Learning Outcomes

At the end of this lesson, you should be able to:

1. Apply the concept of u-substitution in power rule when $n = -1$; and
2. Evaluate integrals leading to logarithm.

In this section, we will extend the concept of the power rule $\displaystyle\int x^n dx = \frac{1}{n+1}x^{n+1} + C$ whenever $n = -1$.

Recall:
The **natural logarithmic function** is defined for all $x > 0$



Properties of the Natural Logarithmic Function

The following are immediate from the definition.

1. $\ln x > 0$ if $x > 1$.
2. $\ln x < 0$ if $0 < x < 1$.
3. $\ln x = 0$ if and only if $x = 1$.

Theorem 46

If a, $b > 0$, then the following hold.

1. $\ln(ab) = \ln a + \ln b$.
2. $\ln\left(\dfrac{a}{b}\right) = \ln a - \ln b$.
3. $\ln(a^r) = r \cdot \ln a$, for all rational number r.

Recall: From Differential Calculus,

Theorem 47

Let u be differentiable function in x.

1. $\dfrac{d}{dx}(\log_a u) = \dfrac{1}{u \ln a}\dfrac{du}{dx}$
2. $\dfrac{d}{dx}(\ln u) = \dfrac{1}{u}\dfrac{du}{dx}$

Now, we will introduce the integration leading to natural logarithm.

Integration Leading to Natural Logarithm

In $\displaystyle\int x^n dx$ if $n = -1$, then $\dfrac{1}{n+1}$ will be undefined. Since the derivative of natural logarithm of x is $\dfrac{dx}{x}$; that is $\dfrac{d}{dx}(\ln x) = \dfrac{dx}{x}$, integrating both sides gives as $\displaystyle\int \frac{dx}{x} = \ln x + C$. In general, we have this theorem,

Theorem 48

If u is differentiable function then

$$\int \frac{du}{u} = ln|u| + C$$

where $u \neq 0$.

Example 145. Evaluate the following integrals.

1. $\displaystyle\int \frac{dx}{x \ln x}$

   Solution: Let $u = \ln x$, then $du = \dfrac{dx}{x}$.

$$\begin{aligned}
\int \frac{dx}{x\ln x} &= \int \frac{du}{u} \\
&= \ln|u| + C \\
&\quad \text{but } u = \ln x \\
&= \ln|\ln x| + C
\end{aligned}$$

2. $\displaystyle\int \frac{dx}{2x+5}$

Solution: Let $u = 2x+5$, then $du = 2dx$ and $\dfrac{1}{2}du = dx$.

$$\begin{aligned}
\int \frac{dx}{2x+5} &= \int \frac{\frac{1}{2}du}{u} \\
&= \frac{1}{2}\int \frac{du}{u} \\
&= \frac{1}{2}\ln|u| + C \\
&\quad \text{but } u = 2x+5 \\
&= \frac{1}{2}\ln|2x+5| + C
\end{aligned}$$

3. $\displaystyle\int \left(\frac{x^2-3x+2}{x+1}\right)dx$

Solution:

$$\begin{aligned}
\int \left(\frac{x^2-3x+2}{x+1}\right)dx &= \int \left(x - 2 + \frac{4}{x+1}\right)dx \\
&= \int xdx - \int 2dx + \int \frac{4dx}{x+1} \\
&= \int xdx - 2\int dx + 4\int \frac{dx}{x+1} \\
&\quad \text{let } u = x+1, \text{ then } du = dx \\
&= \int xdx - 2\int dx + 4\int \frac{du}{u} \\
&= \frac{1}{2}x^2 - 2x + 4\ln|u| + C \\
&\quad \text{but } u = x+1 \\
&= \frac{1}{2}x^2 - 2x + 4\ln|x+1| + C
\end{aligned}$$

4. $\displaystyle\int \frac{\sec x \tan x}{1+\sec x}dx$

Solution: Let $u = 1+\sec x$, then $du = \sec x \tan x dx$.

$$\begin{aligned}
\int \frac{\sec x \tan x}{1+\sec x}dx &= \int \frac{du}{u} \\
&= \ln|u| + C \\
&\quad \text{but } u = 1+\sec x \\
&= \ln|1+\sec x| + C
\end{aligned}$$

5. $\displaystyle\int \frac{5}{x(1+\ln x^2)}dx$

Solution: Let $u = 1 + 2\ln x$, then $du = 2\dfrac{dx}{x}$ and $\dfrac{1}{2}du = \dfrac{dx}{x}$.

$$\begin{aligned}
\int \frac{5}{x(1+\ln x^2)}dx &= \int \frac{5}{x(1+2\ln x)}dx \\
&= 5\int \left(\frac{1}{1+2\ln x}\right)\left(\frac{dx}{x}\right) \\
&= 5\int \frac{1}{u}\left(\frac{1}{2}du\right) \\
&= \frac{5}{2}\int \frac{du}{u} \\
&= \frac{5}{2}\ln|u| + C \\
&\quad \text{but } u = 1+2\ln x \\
&= \frac{5}{2}\ln|1+2\ln x| + C
\end{aligned}$$

6. $\displaystyle\int \frac{dx}{e^x+1}$

Solution: Multiply both numerator and denominator by e^{-x}.

$$\begin{aligned}
\int \frac{dx}{e^x+1} &= \int \frac{dx}{e^x+1}\cdot\frac{e^{-x}}{e^{-x}} \\
&= \int \frac{e^{-x}dx}{1+e^{-x}}
\end{aligned}$$

let $u = 1+e^{-x}$, then $du = -e^{-x}dx$ and $-du = e^{-x}dx$

$$\begin{aligned}
\int \frac{dx}{e^x+1} &= \int \frac{-du}{u} \\
&= -\int \frac{du}{u} \\
&= -\ln|u| + C \\
&\quad \text{but } u = 1+e^{-x} \\
&= -\ln|1+e^{-x}| + C
\end{aligned}$$

Lesson 4.4: Integrals of Trigonometric Functions

Learning Outcomes

At the end of this lesson, you should be able to:

1. Find the relationship between the derivative and integral of trigonometric functions;
2. Enumerate and apply the integral of trigonometric functions; and
3. Evaluate the integrals of trigonometric functions.

In this section, we will discuss the integrals of some functions involving trigonometric functions.

Now, we will introduce the integrals of trigonometric functions.

Theorem 49: Integrals of Trigonometric Functions

If u is differentiable function then,

1. $\displaystyle\int \sin u du = -\cos u + C$
2. $\displaystyle\int \cos u du = \sin u + C$
3. $\displaystyle\int \sec^2 u du = \tan u + C$
4. $\displaystyle\int \csc^2 u du = -\cot u + C$
5. $\displaystyle\int \sec u \tan u du = \sec u + C$
6. $\displaystyle\int \csc u \cot u du = -\csc u + C$
7. $\displaystyle\int \tan u du = \ln|\sec u| + C$
8. $\displaystyle\int \cot u du = \ln|\sin u| + C$
9. $\displaystyle\int \sec u du = \ln|\sec u + \tan u| + C$
10. $\displaystyle\int \csc u du = \ln|\csc u - \cot u| + C$

The first six integrals of trigonometric functions can be easily proven from the concept of derivatives of trigonometric functions. However, the last four formulas were derived using integrals leading to logarithm.

Proof. Let us prove formula 9.
To derive the formula, we multiply the numerator and denominator of the integrand by $\sec u + \tan u$, thus we have

$$\begin{aligned} \int \sec u du &= \int \sec u \cdot \frac{\sec u + \tan u}{\sec u + \tan u} du \\ &= \int \frac{\sec^2 u + \sec u \tan u}{\sec u + \tan u} du \end{aligned}$$

Let $v = \sec u + \tan u$, then $dv = (\sec u \tan u + \sec^2 u)du$. Therefore we have,

$$\begin{aligned}
\int \sec u du &= \int \frac{dv}{v} \\
&= \ln|v| + C \\
&\quad \text{but } v = \sec u + \tan u \\
&= \ln|\sec u + \tan u| + C
\end{aligned}$$

Thus, $\int \sec u du = \ln|u + \tan u| + C$. □

Example 146. Evaluate the following integrals.

1. $\int \sec(5x+1)\tan(5x+1)dx$

   Solution: Let $u = 5x + 1$, then $du = 5dx$ and $\frac{1}{5}du = dx$.

$$\begin{aligned}
\int \sec(5x+1)\tan(5x+1)dx &= \int \sec u \tan u \frac{1}{5}du \\
&= \frac{1}{5}\int \sec u \tan u du \\
&= \frac{1}{5}\sec u + C \\
&\quad \text{but } u = 5x + 1 \\
&= \frac{1}{5}\sec(5x+1) + C
\end{aligned}$$

2. $\int \frac{dx}{\sin(2x)\tan(2x)}$

   Solution: Let $u = 2x$, then $du = 2dx$ and $\frac{1}{2}du = dx$.

$$\begin{aligned}
\int \frac{dx}{\sin(2x)\tan(2x)} &= \int \frac{\frac{1}{2}du}{\sin u \tan u} \\
&= \frac{1}{2}\int \csc u \cot u du \\
&= \frac{1}{2}(-\csc u) + C \\
&\quad \text{but } u = 2x \\
&= -\frac{1}{2}\csc(2x) + C
\end{aligned}$$

3. $\displaystyle\int \frac{\sin x + \cos(2x)}{\sin^2 x} dx$

Solution:

$$\begin{aligned}
\int \frac{\sin x + \cos(2x)}{\sin^2 x} dx &= \int \frac{\sin x + 1 - 2\sin^2 x}{\sin^2 x} dx \\
&= \int \left(\frac{\sin x}{\sin^2 x} + \frac{1}{\sin^2 x} - \frac{2\sin^2 x}{\sin^2 x} \right) dx \\
&= \int \Big(\csc x + \csc^2 x - 2 \Big)\, dx \\
&= \int \csc x dx + \int \csc^2 x dx - 2 \int dx \\
&= \ln|\csc x - \cot x| - \cot x - 2x + C
\end{aligned}$$

4. $\displaystyle\int (\cos^4 x - \sin^4 x) dx$

Solution:

$$\begin{aligned}
\int (\cos^4 x - \sin^4 x) dx &= \int (\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x) dx \\
&\quad \text{but } \cos^2 x + \sin^2 x = 1 \text{ and } \cos^2 x - \sin^2 x = \cos(2x) \\
&= \int \cos(2x) dx \\
&\quad \text{let } u = 2x \text{, then } du = 2dx \text{ and } \frac{1}{2} du = dx
\end{aligned}$$

$$\begin{aligned}
\int (\cos^4 x - \sin^4 x) dx &= \int \cos u \left(\frac{1}{2} du \right) \\
&= \frac{1}{2} \int \cos u du \\
&= \frac{1}{2} \sin u + C \\
&\quad \text{but } u = 2x \\
&= \frac{1}{2} \sin(2x) + C
\end{aligned}$$

Lesson 4.5: Integrals Leading to Inverse Trigonometric Functions

Learning Outcomes

At the end of this lesson, you should be able to:

1. Find the relationship between the derivative of inverse trigonometric functions and the integrals leading to inverse trigonometric functions; and
2. Evaluate integrals leading to inverse trigonometric functions.

In this section, we will discuss the integrals leading to inverse trigonometric functions. As of this lesson there is no integrals for inverse trigonometric functions.

Recall: In differential calculus,

Theorem 50: Derivative of Inverse Trigonometric Functions

If u is differentiable function, then

1. $\frac{d}{dx}(\arcsin u) = \frac{1}{\sqrt{1-u^2}} \cdot \frac{d}{dx}(u)$
2. $\frac{d}{dx}(\arccos u) = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{d}{dx}(u)$
3. $\frac{d}{dx}(\arctan u) = \frac{1}{1+u^2} \cdot \frac{d}{dx}(u)$
4. $\frac{d}{dx}(\text{arccot} u) = \frac{-1}{1+u^2} \cdot \frac{d}{dx}(u)$
5. $\frac{d}{dx}(\text{arcsec} u) = \frac{1}{u\sqrt{u^2-1}} \cdot \frac{d}{dx}(u)$
6. $\frac{d}{dx}(\text{arccsc} u) = \frac{-1}{u\sqrt{u^2-1}} \cdot \frac{d}{dx}(u)$

Now, we will introduce the integrals leading to inverse trigonometric functions.

Theorem 51: Integrals Leading to Inverse Trigonometric Functions

If u is differentiable function and a be any constant, then

1. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C$
2. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$
3. $\displaystyle\int \frac{du}{|u|\sqrt{u^2-a^2}} = \frac{1}{a}\text{arcsec}\left(\frac{u}{a}\right) + C$

Proof. Let us prove number 1.
Recall that $\frac{d}{dx}(\arcsin u) = \frac{1}{\sqrt{1-u^2}} \cdot \frac{d}{dx}(u)$. If we replace u by $\frac{u}{a}$, we have

$$\begin{aligned}
\frac{d}{dx}\left(\arcsin\left(\frac{u}{a}\right)\right) &= \frac{1}{\sqrt{1-\left(\frac{u}{a}\right)^2}}\cdot\frac{d}{dx}\left(\frac{u}{a}\right)\\
&= \frac{1}{\sqrt{1-\frac{u^2}{a^2}}}\cdot\frac{1}{a}\cdot\frac{du}{dx}\\
&= \frac{1}{a\sqrt{\frac{a^2-u^2}{a^2}}}\cdot\frac{du}{dx}\\
&= \frac{1}{a\frac{\sqrt{a^2-u^2}}{\sqrt{a^2}}}\cdot\frac{du}{dx}\\
&= \frac{1}{a\left(\frac{1}{a}\right)\sqrt{a^2-u^2}}\cdot\frac{du}{dx}\\
&= \frac{1}{\sqrt{a^2-u^2}}\cdot\frac{du}{dx}
\end{aligned}$$

Thus, $\frac{1}{\sqrt{a^2-u^2}}$ is an antiderivative of $\arcsin\left(\frac{u}{a}\right)$.
Therefore,

$$\int\frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right)+C$$

□

Example 147. Evaluate the following integrals.

1. $\displaystyle\int\frac{dx}{\sqrt{9-x^2}}$

   Solution: Let $u = x$, then $du = dx$ and $a = 3$.

$$\begin{aligned}
\int\frac{dx}{\sqrt{9-x^2}} &= \int\frac{du}{\sqrt{a^2-u^2}}\\
&= \arcsin\left(\frac{u}{a}\right)+C\\
&\quad \text{but } u = x \text{ and } a = 3\\
&= \arcsin\left(\frac{x}{3}\right)+C
\end{aligned}$$

2. $\displaystyle\int \frac{e^{3x}dx}{9+4e^{6x}}$

Solution: Let $u = 2e^{3x}$, then $du = 6e^{3x}dx$ and $\dfrac{1}{6}du = e^{3x}dx$, $a = 3$.

$$\begin{aligned}
\int \frac{e^{3x}dx}{9+4e^{6x}} &= \int \frac{\frac{1}{6}du}{a^2+u^2} \\
&= \frac{1}{6}\int \frac{du}{a^2+u^2} \\
&= \frac{1}{6}\cdot\frac{1}{a}\arctan\left(\frac{u}{a}\right)+C \\
&\quad \text{but } u = 2e^{3x} \text{ and } a = 3 \\
&= \frac{1}{18}\arctan\left(\frac{2e^{3x}}{3}\right)+C
\end{aligned}$$

3. $\displaystyle\int \frac{dx}{\sqrt{e^{4x}-16}}$

Solution: Let $u = e^{2x}$ and $a = 4$, then $du = 2e^{2x}dx$ and $\dfrac{1}{2}du = e^{2x}dx$

$$\begin{aligned}
\int \frac{dx}{\sqrt{e^{4x}-16}} &= \int \frac{e^{2x}dx}{e^{2x}\sqrt{e^{4x}-16}} \\
&= \int \frac{\frac{1}{2}du}{u\sqrt{u^2-a^2}} \\
&= \frac{1}{2}\int \frac{du}{u\sqrt{u^2-a^2}} \\
&= \frac{1}{2}\cdot\frac{1}{a}\text{arcsec}\left(\frac{u}{a}\right)+C \\
&\quad \text{but } u = e^{2x} \text{ and } a = 4 \\
&= \frac{1}{8}\text{arcsec}\left(\frac{e^{2x}}{4}\right)+C
\end{aligned}$$

Lesson 4.6: Integrals of Exponential Functions

Learning Outcomes

At the end of this lesson, you should be able to:

1. Find the relationship between the derivative and integral of exponential functions; and
2. Evaluate the integrals of exponential functions.

In this lesson, we will discuss the integrals of exponential functions.

Recall: From differential calculus,

Theorem 52: Derivative of Exponential Functions

If u is differentiable function and a be any constant, then

1. $\dfrac{d}{dx}(a^u) = a^u \ln a \cdot \dfrac{d}{dx}(u)$
2. $\dfrac{d}{dx}(e^u) = e^u \cdot \dfrac{d}{dx}(u)$

Now, we will introduce the integrals of exponential functions.

Theorem 53: Integrals of Exponential Functions

If u is differential function, a is constant and $e = 2.71\ldots$ then,

1. $\displaystyle\int a^u du = \frac{a^u}{\ln a} + C$
2. $\displaystyle\int e^u du = e^u + C$

Example 148. Evaluate the following integrals.

1. $\displaystyle\int e^{\cos x} \sin x dx$

   Solution: Let $u = \cos x$, then $du = -\sin x dx$ and $-du = \sin x dx$.

$$\begin{aligned}
\int e^{\cos x} \sin x dx &= \int e^u(-du) \\
&= -\int e^u du \\
&= -e^u + C \\
&\quad \text{but } u = \cos x \\
&= -e^{\cos x} + C
\end{aligned}$$

2. $\displaystyle\int \frac{e^{2x}dx}{e^x+4}$

Solution:

$$\begin{aligned}
\int \frac{e^{2x}dx}{e^x+4} &= \int \left(e^x - \frac{4e^x}{e^x+4}\right)dx \\
&= \int e^x dx - 4\int \frac{e^x dx}{e^x+4} \\
&\quad \text{let } u = e^x+4, \text{ then } du = e^x dx \\
&= \int du - 4\int \frac{du}{u} \\
&= u - 4\ln|u| + C \\
&\quad \text{but } u = e^x + 4 \\
&= e^x + 4 - 4\ln|e^x+4| + C
\end{aligned}$$

3. $\displaystyle\int \frac{4^{\csc(2x)}dx}{\sin(2x)\tan(2x)}$

Solution:

$$\begin{aligned}
\int \frac{4^{\csc(2x)}dx}{\sin(2x)\tan(2x)} &= \int 4^{\csc(2x)}\csc(2x)\cot(2x)dx \\
&\quad \text{let } u = \csc(2x) \text{ and } a = 4, \text{ then } du = -2\csc(2x)\cot(2x)dx \\
&\quad \text{and } -\frac{1}{2}du = \csc(2x)\cot(2x)dx \\
&= \int a^u\left(-\frac{1}{2}du\right) \\
&= -\frac{1}{2}\int a^u du \\
&= \left(-\frac{1}{2}\right)\left(\frac{a^u}{\ln a}\right) + C \\
&\quad \text{but } u = \csc(2x) \text{ and } a = 4 \\
&= -\frac{4^{\csc(2x)}}{2\ln 4} + C
\end{aligned}$$

Lesson 4.7: Integrals of Hyperbolic Functions

Learning Outcomes

At the end of this lesson, you should be able to:

1. Define the different hyperbolic functions;
2. Find the relationship between the derivative and integral of hyperbolic functions; and
3. Evaluate the integrals of hyperbolic functions.

In this section, we will discuss the integrals of hyperbolic functions.

Definition 35: Hyperbolic Functions

1. **Hyperbolic sine:** $\sinh x = \dfrac{e^x - e^{-x}}{2}$
2. **Hyperbolic cosine:** $\cosh x = \dfrac{e^x + e^{-x}}{2}$
3. **Hyperbolic tangent:** $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
4. **Hyperbolic cotangent:** $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$
5. **Hyperbolic secant:** $\text{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$
6. **Hyperbolic cosecant:** $\text{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$

Here are some identities for Hyperbolic Functions that can help in simplifying the some functions. The following identities are direct implication of the definitions of Hyperbolic Functions.

Theorem 54: Identities of Hyperbolic Functions

1. $\cosh x + \sinh x = e^x$
2. $\cosh x - \sinh x = e^{-x}$
3. $\cosh^2 x - \sinh^2 x = 1$
4. $1 - \tanh^2 x = \text{sech}^2 x$
5. $\coth^2 x - 1 = \text{csch}^2 x$
6. $\cosh(-x) = \cosh x$
7. $\sinh(-x) = -\sinh x$

8. $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$
9. $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$
10. $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$
11. $\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$
12. $\sinh 2x = 2\sinh x \cosh x$
13. $\cosh 2x = \cosh^2 x + \sinh^2 x$
14. $\cosh 2x = 2\sinh^2 x + 1$
15. $\cosh 2x = 2\cosh^2 x - 1$

From differential calculus,

Theorem 55: Derivative of Hyperbolic Functions

If u is differentiable function in x, then

1. $\dfrac{d}{dx}[\sinh u] = \cosh u \cdot \dfrac{d}{dx}(u)$
2. $\dfrac{d}{dx}[\cosh u] = \sinh u \cdot \dfrac{d}{dx}(u)$
3. $\dfrac{d}{dx}[\tanh u] = \text{sech}^2 u \cdot \dfrac{d}{dx}(u)$
4. $\dfrac{d}{dx}[\coth u] = -\text{csch}^2 u \cdot \dfrac{d}{dx}(u)$
5. $\dfrac{d}{dx}[\text{sech} u] = -\text{sech} u \tanh u \cdot \dfrac{d}{dx}(u)$
6. $\dfrac{d}{dx}[\text{csch} u] = -\text{csch} u \coth u \cdot \dfrac{d}{dx}(u)$

Now, we will introduce the integrals of hyperbolic functions.

Theorem 56: Integrals of Hyperbolic Functions

If u is differentiable function, then

1. $\displaystyle\int \cosh u du = \sinh u + C$
2. $\displaystyle\int \sinh u du = \cosh u + C$
3. $\displaystyle\int \text{sech}^2 u du = \tanh u + C$
4. $\displaystyle\int \text{csch}^2 u du = -\coth u + C$
5. $\displaystyle\int \text{sech} u \tanh u du = -\text{sech} u + C$
6. $\displaystyle\int \text{csch} u \coth u du = -\text{csch} u + C$
7. $\displaystyle\int \tanh u du = \ln|\cosh u| + C$
8. $\displaystyle\int \coth u du = \ln|\sinh u| + C$
9. $\displaystyle\int \text{csch} u du = \ln|\text{csch} u - \coth u| + C$

Example 149. Evaluate the following integrals.

1. $\displaystyle\int \cosh(2x-3)dx$

   Solution: Let $u = 2x-3$, then $du = 2dx$ and $\frac{1}{2}du = dx$.

$$\begin{aligned}\int \cosh(2x-3)dx &= \int \cosh u \cdot \frac{1}{2}du \\ &= \frac{1}{2}\int \cosh u du \\ &= \frac{1}{2}\sinh u + C \\ &\quad \text{but } u = 2x-3 \\ &= \frac{1}{2}\sinh(2x-3) + C\end{aligned}$$

2. $\displaystyle\int \sqrt{\tanh x}\,\text{sech}^2 x dx$

   Solution: Let $u = \tanh x$, then $du = \text{sech}^2 x dx$.

$$\begin{aligned}\int \sqrt{\tanh x}\,\text{sech}^2 x dx &= \int \sqrt{u} du \\ &= \frac{1}{\frac{1}{2}+1}u^{\frac{1}{2}+1} + C \\ &= \frac{1}{\frac{3}{2}}u^{\frac{3}{2}} + C \\ &\quad \text{but } u = \tanh x \\ &= \frac{2}{3}(\tanh x)^{\frac{3}{2}} + C\end{aligned}$$

3. $\displaystyle\int e^{\sec x}\sec x\tan x\,\text{sech}(e^{\sec x})\tanh(e^{\sec x})dx$

   Solution: Let $u = e^{\sec x}$, then $du = e^{\sec x}\sec x\tan x dx$.

$$\begin{aligned}\int e^{\sec x}\sec x\tan x\,\text{sech}(e^{\sec x})\tanh(e^{\sec x})dx &= \int \text{sech} u \tanh u du \\ &= -\text{sech} u + C \\ &\quad \text{but } u = e^{\sec x} \\ &= -\text{sech}(e^{\sec x}) + C\end{aligned}$$

Lesson 4.8: Integrals Yielding Inverse Hyperbolic Functions

Learning Outcomes

At the end of this lesson, you should be able to:

1. Define the different inverse hyperbolic functions; and
2. Evaluate the integrals yielding inverse hyperbolic functions.

Since the hyperbolic functions are constructed using exponential functions, we expect that their inverses can be written in terms of logarithms. In fact, the following hold.

Theorem 57: Inverse Hyperbolic Functions

1. $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$
2. $\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$
3. $\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$
4. $\coth^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{x+1}{x-1}\right)$
5. $\operatorname{sech}^{-1} x = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right)$
6. $\operatorname{csch}^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{1+x^2}}{|x|}\right)$

Recall that the derivatives of inverse hyperbolic functions states

Theorem 58: Derivatives of Inverse Hyperbolic Functions

1. $\dfrac{d}{dx}\left(\sinh^{-1} u\right) = \dfrac{1}{\sqrt{u^2+1}} \cdot \dfrac{du}{dx}$
2. $\dfrac{d}{dx}\left(\cosh^{-1} u\right) = \dfrac{1}{\sqrt{u^2-1}} \cdot \dfrac{du}{dx}$
3. $\dfrac{d}{dx}(\tanh^{-1} u) = \dfrac{1}{1-u^2} \cdot \dfrac{du}{dx}, \ |u| < 1$
4. $\dfrac{d}{dx}(\coth^{-1} u) = \dfrac{1}{1-u^2} \cdot \dfrac{du}{dx}, \ |u| > 1$
5. $\dfrac{d}{dx}(\operatorname{sech}^{-1} u) = -\dfrac{1}{u\sqrt{1-u^2}} \cdot \dfrac{du}{dx}$
6. $\dfrac{d}{dx}(\operatorname{csch}^{-1} u) = -\dfrac{1}{|u|\sqrt{u^2+1}} \cdot \dfrac{du}{dx}$

Now, we will provide the integrals yielding the inverse hyperbolic functions.

Theorem 59: Integrals Yielding Inverse Hyperbolic Functions

Let $a > 0$.

1. $\displaystyle\int \frac{1}{\sqrt{u^2+a^2}}du = \sinh^{-1}\left(\frac{u}{a}\right) + C = \ln\left(u+\sqrt{u^2+a^2}\right) + C$

2. $\displaystyle\int \frac{1}{\sqrt{u^2-a^2}}du = \cosh^{-1}\left(\frac{u}{a}\right) + C = \ln\left(u+\sqrt{u^2-a^2}\right) + C$

3. $\displaystyle\int \frac{1}{a^2-u^2}du = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C, & \text{if } |u| < a \\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C, & \text{if } |u| > a \end{cases} = \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + C$

Example 150. Evaluate the following integrals.

1. $\displaystyle\int \frac{3}{\sqrt{4x^2+1}}dx$

   Solution: Let $u = 2x$. Then $du = 2dx$ and $\frac{1}{2}du = dx$. Thus,

$$\begin{aligned}\int \frac{3}{\sqrt{4x^2+1}}dx &= 3\int \frac{\frac{1}{2}du}{\sqrt{u^2+1}} \\ &= \frac{3}{2}\int \frac{du}{u^2}+1^2 \\ &= \frac{3}{2}\sinh^{-1}u + C \\ & \quad \text{but } u = 2x \\ &= \frac{3}{2}\sinh^{-1}(2x) + C.\end{aligned}$$

2. $\displaystyle\int \frac{5x^2}{x^6-49}dx$

   Solution: Let $u = x^3$. Then $du = 3x^2dx$ and $\frac{1}{3}du = x^2dx$

$$\begin{aligned}\int \frac{5x^2}{x^6-49}dx &= 5\int \frac{\frac{1}{3}du}{u^2-7^2} \\ &= \frac{5}{3}\int \frac{du}{u^2-7^2} \\ &= -\frac{5}{3}\int \frac{du}{7^2-u^2} \\ &= -\frac{5}{3}\cdot\frac{1}{14}\ln\left|\frac{7+u}{7-u}\right| + C \\ & \quad \text{but } u = x^3 \\ &= -\frac{5}{42}\ln\left|\frac{7+x^3}{7-x^3}\right| + C\end{aligned}$$

4.9 Unit Test 4

Antiderivatives and Indefinite Integrals

Instruction: Write all your official answers and solutions on sheets of yellow pad paper, using only either black or blue pens.

I. **Each question is a multiple-choice question with four answer choices. Read each question and answer choice carefully and choose the ONE best answer.**

1. Evaluate $\int (x^2 - 4x - 15)dx$.

   a. $x^3 - 4x^2 - 15x + C$
   b. $\dfrac{x^3}{3} - \dfrac{2x^2}{3} - 15x + C$
   c. $3x^3 - 4x^2 - 15x + C$
   d. $\dfrac{x^3}{3} - 2x^2 - 15x + C$

2. What is $\int 9x^2(9 + 3x^3)^7 dx$?

   a. $(9 + 3x^3)^8 + C$
   b. $\dfrac{(9 + 3x^3)^8}{8} + C$
   c. $x^2(9 + 3x^3)^8 + C$
   d. $\dfrac{x^2(9 + 3x^3)^8}{8} + C$

3. Determine the general antiderivative of $f(x) = \dfrac{2x^3 - 54}{x - 3}$.

   a. $\dfrac{2x^3}{3} - 3x^2 + 18x + C$
   b. $\dfrac{x^3}{3} + 3x^2 + 18x + C$
   c. $\dfrac{2x^3}{3} + 3x^2 + 18x + C$
   d. $\dfrac{2x^3}{3} + \dfrac{3x^2}{2} + 18x + C$

4. Evaluate $\int 6\cos(3x)dx$.

   a. $2\sin(3x) + C$
   b. $-2\sin(3x) + C$
   c. $3\sin(3x) + C$
   d. $-3\sin(3x) + C$

5. What is $\int 3\sec(2x)dx$?

   a. $\dfrac{3}{2}\tan(2x) + C$
   b. $\dfrac{3}{2}\ln|\sec(2x) + \tan(2x)| + C$
   c. $\dfrac{2}{3}\tan(2x) + C$
   d. $\dfrac{2}{3}\ln|\sec(2x) + \tan(2x)| + C$

6. Determine the general antiderivative of $f(x) = \frac{3}{4}\csc^2\left(\frac{5}{2}x\right)$.

   a. $-\frac{3}{10}\cot\left(\frac{5}{2}x\right) + C$
   b. $\frac{3}{10}\cot\left(\frac{5}{2}x\right) + C$
   c. $-\frac{15}{8}\cot\left(\frac{5}{2}x\right) + C$
   d. $\frac{15}{8}\cot\left(\frac{5}{2}x\right) + C$

7. Evaluate $\int 4\tan\left(\frac{3}{2}x\right)\,dx$.

   a. $-6\ln\left|\sec\left(\frac{3}{2}x\right)\right| + C$
   b. $6\ln\left|\sec\left(\frac{3}{2}x\right)\right| + C$
   c. $-\frac{8}{3}\ln\left|\sec\left(\frac{3}{2}x\right)\right| + C$
   d. $\frac{8}{3}\ln\left|\sec\left(\frac{3}{2}x\right)\right| + C$

8. What is $\int \frac{-16}{\sqrt{4-x^2}}dx$?

   a. $16\arcsin\left(\frac{x}{2}\right) + C$
   b. $16\arcsin\left(\frac{2}{x}\right) + C$
   c. $-16\arcsin\left(\frac{x}{2}\right) + C$
   d. $-16\arcsin\left(\frac{2}{x}\right) + C$

9. Determine the general antiderivative of $f(x) = \frac{4x}{x^4 - 12x^2 + 45}$.

   a. $\frac{2}{3}\arctan\left(\frac{x^2-6}{3}\right) + C$
   b. $\frac{2}{3}\arctan\left(\frac{x^2-6}{2}\right) + C$
   c. $\frac{1}{3}\arctan\left(\frac{x^2-6}{3}\right) + C$
   d. $\frac{1}{3}\arctan\left(\frac{x^2-6}{2}\right) + C$

10. Evaluate $\int \frac{1}{x\sqrt{4x^2-25}}dx$.

    a. $\operatorname{arcsec}\left(\frac{2x}{5}\right) + C$
    b. $\frac{1}{5}\operatorname{arcsec}\left(\frac{2x}{5}\right) + C$
    c. $\frac{2}{5}\operatorname{arcsec}\left(\frac{2x}{5}\right) + C$
    d. $\frac{4}{5}\operatorname{arcsec}\left(\frac{2x}{5}\right) + C$

11. What is $\int 3^{2x}dx$?

    a. $\frac{3^{2x}}{\ln 3} + C$
    b. $\frac{3^{2x+1}}{\ln 3} + C$
    c. $\frac{3^{2x}}{\ln 9} + C$
    d. $\frac{2\cdot 3^{2x}}{\ln 3} + C$

12. Determine the antiderivative of $f(x) = 5^{\cos(3x)}\sin(3x)$.

    a. $\frac{5^{\cos(3x)}}{\ln 5} + C$
    b. $-\frac{5^{\cos(3x)}}{\ln 5} + C$
    c. $\frac{5^{\cos(3x)}}{\ln 125} + C$
    d. $-\frac{5^{\cos(3x)}}{\ln 125} + C$

13. Evaluate $\int e^{x^2+6x+5}(x+3)dx$.

a. $2e^{x^2+6x+5}+C$

b. $\frac{1}{2}e^{x^2+6x+5}+C$

c. $e^{x^2+6x+5}(x+3)^2+C$

d. $\frac{e^{x^2+6x+5}(x+3)^2}{2}+C$

14. What is $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$?

a. $\frac{1}{2}e^{\sqrt{x}}+C$

b. $e^{\sqrt{x}}+C$

c. $2e^{\sqrt{x}}+C$

d. $4e^{\sqrt{x}}+C$

15. Determine the general antiderivative of $f(x)=\frac{x^3}{x^4+10}$.

a. $\ln\left|x^4-10\right|+C$

b. $4\ln\left|x^4-10\right|+C$

c. $\ln|x|+\ln 10+C$

d. $\frac{1}{4}\ln|x|+C$

16. Evaluate $\int \frac{4x-11}{2x^2-11x+15}dx$.

a. $\ln\left|2x^2-11x+15\right|+C$

b. $2\ln\left|2x^2-11x+15\right|+C$

c. $4\ln\left|2x^2-11x+15\right|+C$

d. $8\ln\left|2x^2-11x+15\right|+C$

17. What is $\int(-12\sinh(6x))dx$?

a. $2\cosh(6x)+C$

b. $3\cosh(6x)+C$

c. $-2\cosh(6x)+C$

d. $-3\cosh(6x)+C$

18. Determine the antiderivative of $f(x)=3e^{2x}\text{csch}(e^{2x})$.

a. $\frac{3}{2}\coth(e^{2x})+C$

b. $\frac{3}{2}\ln\left|\text{csch}(e^{2x})-\coth(e^{2x})\right|+C$

c. $-\frac{3}{2}\coth(e^{2x})+C$

d. $-\frac{3}{2}\ln\left|\text{csch}(e^{2x})-\coth(e^{2x})\right|+C$

19. Evaluate $\int \frac{8}{3}\text{sech}^2(4x)dx$.

a. $\frac{2}{3}\tanh(4x)+C$

b. $\tanh(4x)+C$

c. $\frac{8}{3}\tanh(4x)+C$

d. $6\tanh(4x)+C$

20. What is $\int \frac{10\text{sech}\sqrt{x}\tanh\sqrt{x}}{\sqrt{x}}dx$?

a. $5\,\text{sech}\sqrt{x}+C$

b. $-10\,\text{sech}\sqrt{x}+C$

c. $20\,\text{sech}\sqrt{x}+C$

d. $-20\,\text{sech}\sqrt{x}+C$

21. Determine the general antiderivative of $f(x) = \dfrac{4x}{\sqrt{4x^4+9}}$.

   a. $2\cosh^{-1}\left(\dfrac{2x^2}{3}\right) + C$
   b. $\cosh^{-1}\left(\dfrac{2x^2}{3}\right) + C$
   c. $2\sinh^{-1}\left(\dfrac{2x^2}{3}\right) + C$
   d. $\sinh^{-1}\left(\dfrac{2x^2}{3}\right) + C$

22. Evaluate $\displaystyle\int \frac{4}{\sqrt{x^2-8x-9}}dx$.

   a. $\cosh^{-1}\left(\dfrac{x-4}{5}\right) + C$
   b. $4\cosh^{-1}\left(\dfrac{x-4}{5}\right) + C$
   c. $\sinh^{-1}\left(\dfrac{x-4}{5}\right) + C$
   d. $4\sinh^{-1}\left(\dfrac{x-4}{5}\right) + C$

23. What is $\displaystyle\int \frac{6e^{2x}}{3-e^{4x}}dx$?

   a. $\dfrac{\sqrt{3}}{2}\ln\left|\dfrac{e^{2x}+\sqrt{3}}{e^{2x}-\sqrt{3}}\right| + C$
   b. $\dfrac{\sqrt{3}}{2}\ln\left|\dfrac{e^{2x}-\sqrt{3}}{e^{2x}+\sqrt{3}}\right| + C$
   c. $\dfrac{3}{2}\ln\left|\dfrac{e^{2x}+\sqrt{3}}{e^{2x}-\sqrt{3}}\right| + C$
   d. $\dfrac{3}{2}\ln\left|\dfrac{e^{2x}-\sqrt{3}}{e^{2x}+\sqrt{3}}\right| + C$

24. Evaluate $\displaystyle\int \sqrt{x}dx$ and determine its particular antiderivative passing through the point $(9, 19)$.

   a. $y = \dfrac{2}{3}x^{\frac{3}{2}} - 1$
   b. $y = \dfrac{1}{2}x^{-\frac{1}{2}} - 1$
   c. $y = \dfrac{2}{3}x^{\frac{3}{2}} + 1$
   d. $y = \dfrac{1}{2}x^{-\frac{1}{2}} - 1$

25. What is the particular antiderivative of $f(x) = -\dfrac{2}{x^2}$ passing through $(4, -1)$?

   a. $y = -\dfrac{2}{x} - \dfrac{3}{2}$
   b. $y = -\dfrac{2}{x} + \dfrac{3}{2}$
   c. $y = \dfrac{2}{x} - \dfrac{3}{2}$
   d. $y = \dfrac{2}{x} + \dfrac{3}{2}$

26. Evaluate $\displaystyle\int \sec\left(\frac{2x}{5}\right)\tan\left(\frac{2x}{5}\right) dx$ and determine its particular antiderivative whose y-intercept is 2.

   a. $y = -\dfrac{1}{2}\left[5\sec\left(\dfrac{2x}{5}\right) + 1\right]$
   b. $y = -\dfrac{1}{2}\left[5\sec\left(\dfrac{2x}{5}\right) - 1\right]$
   c. $y = \dfrac{1}{2}\left[5\sec\left(\dfrac{2x}{5}\right) + 1\right]$
   d. $y = \dfrac{1}{2}\left[5\sec\left(\dfrac{2x}{5}\right) - 1\right]$

27. What is the particular antiderivative of $f(x) = \dfrac{3}{x^2+9}$ whose x-intercept is 3?

a. $y = \arctan\left(\dfrac{x}{3}\right) - \dfrac{\pi}{4}$

b. $y = \arctan\left(\dfrac{x}{3}\right) - \dfrac{3\pi}{4}$

c. $y = \arctan\left(\dfrac{x}{3}\right) + \dfrac{\pi}{4}$

d. $y = \arctan\dfrac{x}{3} + \dfrac{3\pi}{4}$

28. Evaluate $\int e^{3x}dx$ and determine its particular antiderivative whose x-intercept is $\ln 3$.

a. $y = \dfrac{1}{3}e^{3x} - 9$

b. $y = \dfrac{1}{3}e^{3x} + 9$

c. $y = \dfrac{1}{3}e^{3x} - 3$

d. $y = \dfrac{1}{3}e^{3x} + 3$

29. What is the particular antiderivative of $f(x) = \dfrac{6x}{x^2-1}$ whose y-intercept is -4?

a. $y = 4\ln\left|x^2-1\right| - 3$

b. $y = 4\ln\left|x^2-1\right| + 3$

c. $y = 3\ln\left|x^2-1\right| - 4$

d. $y = 3\ln\left|x^2-1\right| + 4$

30. Evaluate $\int \dfrac{7dx}{1-x^2}$ determine its particular antiderivative passing through the point of origin.

a. $y = \dfrac{7}{2}\ln\left|\dfrac{1+x}{1-x}\right|$

b. $y = \dfrac{7}{2}\ln\left|\dfrac{1+x}{1-x}\right| + \dfrac{1}{2}$

c. $y = \dfrac{7}{2}\ln\left|\dfrac{1+x}{1-x}\right| + \dfrac{7}{2}$

d. $y = \dfrac{7}{2}\ln\left|\dfrac{1+x}{1-x}\right| + 7$

Unit 5: Definite Integrals and Its Applications

This unit focuses o finding a solution to the problem of finding the area of a plane region bounded by a curve lying above the x-axis with equation $y = f(x)$, the x-axis and the vertical lines $x = a$ and $x = b$. We will then define a definite integral from the ensuing ideas. We will also discuss the two fundamental theorems of calculus.

Lesson 5.1: Area of the Plane Region using Rectangular Method

Learning Outcomes

At the end of this lesson, you should be able to:

1. Find the area of the plane region bounded by a curve lying above or below the x-axis and by the curve lying to the right or to the left of y-axis using rectangular method; and
2. Define and evaluate definite integrals.

Definition 36: Sigma Notation

Let n be a positive integer and F be a function such that $\{1, 2, \ldots, n\}$ is in the domain of F. We define:

$$\sum_{i=1}^{n} F(i) := F(1) + F(2) + \cdots + F(n)$$

The left hand side is read "the summation of F of i, where i is evaluated from 1 to n.

Theorem 60

Let n be a positive integer, c be a real number, and F and G be functions defined on the set $\{1, 2, \ldots, n\}$.

1. $\displaystyle\sum_{i=1}^{n} c = cn$
2. $\displaystyle\sum_{i=1}^{n} cF(i) = c\sum_{i=1}^{n} F(i)$
3. $\displaystyle\sum_{i=1}^{n}[F(i)+G(i)] = \sum_{i=1}^{n} F(i) + \sum_{i=1}^{n} G(i)$
4. $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
5. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
6. $\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

Example 151. Express $\displaystyle\sum_{n=1}^{k}(n+1)^2$ in closed form.

Solution:

$$\begin{aligned}
\sum_{n=1}^{k}(n+1)^2 &= \sum_{n=1}^{k}(n^2+2n+1)\\
&= \sum_{n=1}^{k} n^2 + 2\sum_{n=1}^{k} n + \sum_{n=1}^{k} 1\\
&= \frac{k(k+1)(2k+1)}{6} + \frac{2k(k+1)}{2} + k\\
&= \frac{k(2k^2+3k+1)}{6} + k(k+1) + k\\
&= \frac{2k^3+3k^2+k}{6} + k^2 + k + k\\
&= \frac{2k^3+3k^2+k+6k^2+12k}{6}\\
&= \frac{2k^3+9k^2+13k}{6}
\end{aligned}$$

Example 152. Evaluate the following:

1. $\displaystyle\sum_{i=1}^{5}(3i-7)$

   Solution:

$$\begin{aligned}
\sum_{i=1}^{5}(3i+7) &= \sum_{i=1}^{5} 3i + \sum_{i=1}^{5} 7\\
&= 3\sum_{i=1}^{5} i + \sum_{i=1}^{5} 7\\
&= 3\cdot\frac{5\cdot 6}{2} + 7(5)\\
&= 80
\end{aligned}$$

2. $\sum_{i=1}^{7}(2i-3)^2$

Solution:

$$\begin{aligned}
\sum_{i=1}^{7}(2i-3)^2 &= \sum_{i=1}^{7}(4i^2-12i+9) \\
&= \sum_{i=1}^{7}4i^2 - \sum_{i=1}^{7}12i + \sum_{i=1}^{7}9 \\
&= 4\sum_{i=1}^{7}i^2 - 12\sum_{i=1}^{7}i + \sum_{i=1}^{7}9 \\
&= 4\cdot\frac{7\cdot 8\cdot 15}{6} - 12\cdot\frac{7\cdot 8}{2} + 9\cdot 7 \\
&= 287
\end{aligned}$$

The Area of a Plane Region

Procedure 1: Area of a Plane Region

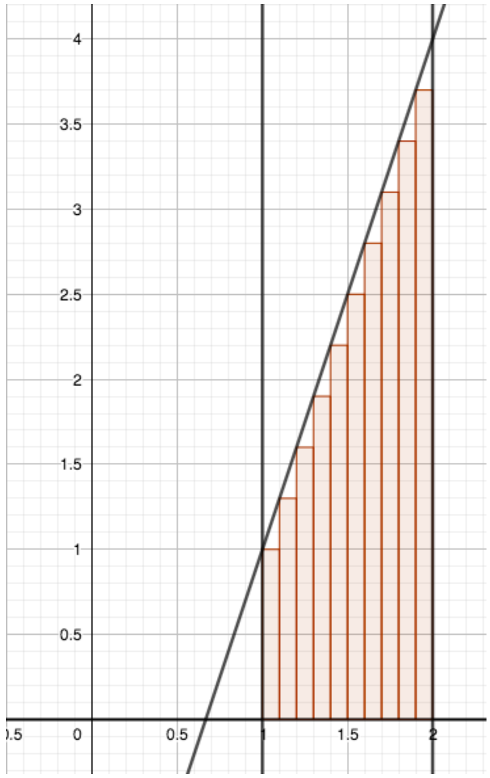
1. First, we divide the interval $[a, b]$ into n subintervals of equal length $\Delta x = \dfrac{b-a}{n}$.
2. Let c_i be the right endpoint of the i^{th} subinterval, $i = 1, 2, 3, \ldots, n$.
3. Cover the region with n circumscribed rectangles of width Δx and height $f(c_i)$.
4. Let A_i be the area of the i^{th} rectangle. Thus, $A_i = f(c_i)\cdot\Delta x$.
5. The area of the region can be approximated by taking the sum of the areas of the n circumscribed rectangles. Thus,

$$A_R \approx \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} f(c_i)\Delta x.$$

6. Simplify the right hand side using summation formulas.
7. Evaluate the limit as the value of n is increasing.

Example 153. Find the area of the region bounded by the curves.

1. Region bounded by the curves $y = 3x - 2$, $x = 1$, $x = 2$ and x-axis.



Solution:
First, divide the interval $[1, 2]$ into n subintervals of equal length Δx.

$$\Delta x = \frac{2-1}{n} = \frac{1}{n}.$$

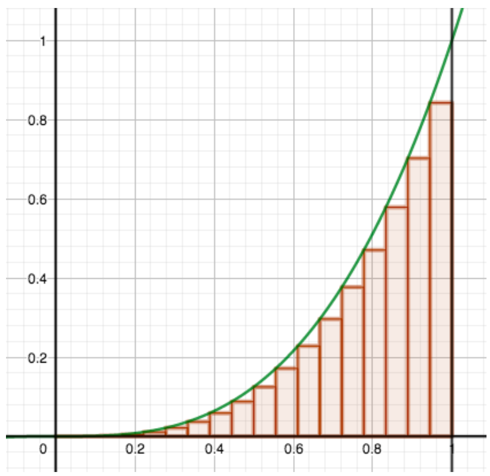
Let $A_i = f(c_i) \cdot \Delta x$ be the area of the i^{th} rectangle where $f(c_i)$ is the height and Δx is the width.

A_i	c_i	$f(c_i)$	Δx	Area
A_1	$1+\frac{1}{n}$	$3\left(1+\frac{1}{n}\right)-2$	$\frac{1}{n}$	$\frac{n+3}{n^2}$
A_2	$1+\frac{2}{n}$	$3\left(1+\frac{2}{n}\right)-2$	$\frac{1}{n}$	$\frac{n+6}{n^2}$
A_3	$1+\frac{3}{n}$	$3\left(1+\frac{3}{n}\right)-2$	$\frac{1}{n}$	$\frac{n+9}{n^2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_n	$1+\frac{n}{n}$	$3\left(1+\frac{n}{n}\right)-2$	$\frac{1}{n}$	$\frac{4n}{n^2}$

Thus,

$$\begin{aligned} A_R = \lim_{n\to+\infty}\sum_{i=1}^{n} A_i &= \lim_{n\to+\infty}\sum_{i=1}^{n}\frac{n+3i}{n^2} \\ &= \lim_{n\to+\infty}\frac{5n+3}{2n} \\ &= \frac{5}{2} \end{aligned}$$

2. Region bounded by $y = x^3$, $x = 0$, $x = 1$ and x-axis.



Solution:

First, divide the interval [0, 1] into n subintervals of equal length Δx.

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}$$

Let $A_i = f(c_i)\cdot\Delta x$ be the area of the i^{th} rectangle where $f(c_i)$ is the height and Δx is the width.

A_i	c_i	$f(c_i)$	Δx	Area
A_1	$\frac{1}{n}$	$\left(\frac{1}{n}\right)^3$	$\frac{1}{n}$	$\frac{(1)^3}{n^4}$
A_2	$\frac{2}{n}$	$\left(\frac{2}{n}\right)^3$	$\frac{1}{n}$	$\frac{(2)^3}{n^4}$
A_3	$\frac{3}{n}$	$\left(\frac{3}{n}\right)^3$	$\frac{1}{n}$	$\frac{(3)^3}{n^4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_n	$\frac{n}{n}$	$\left(\frac{n}{n}\right)^3$	$\frac{1}{n}$	$\frac{(n)^3}{n^4}$

Thus,

$$\begin{aligned} A_R = \lim_{n\to+\infty}\sum_{i=1}^{n} A_i &= \lim_{n\to+\infty}\sum_{i=1}^{n}\frac{i^3}{n^4} \\ &= \lim_{n\to+\infty}\frac{n^2+2n+1}{4n^2} \\ &= \frac{1}{4} \end{aligned}$$

5.1.1 The Definite Integral

Definition 37: Definite Integral

Let f be defined on $[a,\, b]$. the **definite integral** of f from a to b is

$$\int_a^b f(x)dx = \lim_{||\Delta||\to 0}\sum_{i=1}^{n} f(x_i^*)\cdot \Delta x_i,$$

if the limit exists and does not depend on the choice of numbers x_i^* in the subintervals. If these are true, the function is said to be integrable on $[a,\, b]$.

Theorem 61

If a function is continuous on $[a,\, b]$, then it is integrable on $[a,\, b]$.

Remark 12: Properties of the Definite Integral

Let f and g be integrable on $[a,\, b]$, and c, p, q and r be real numbers.

1. $\displaystyle\int_a^b f(x)dx = -\int_b^a f(x)dx$
2. $\displaystyle\int_a^a f(x)dx = 0$
3. $\displaystyle\int_a^b cdx = c(b-a)$
4. $\displaystyle\int_a^b cf(x)dx = c\int_a^b f(x)dx$
5. $\displaystyle\int_a^b [f(x)\pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
6. If $[a,\, b]$ contains p, q and r, then $\displaystyle\int_p^q f(x)dx = \int_p^r f(x)dx + \int_r^q f(x)dx$ regardless of the order of p, q and r.

Example 154. Do as indicated.

1. If $\int_{-2}^{5} f(x)dx = 7$ and $\int_{-2}^{5} g(x)dx = 13$, find $\int_{-2}^{5} [5f(x) - 3g(x)]dx$.
   Solution:

$$\begin{aligned}
\int_{-2}^{5} [5f(x) - 3g(x)]dx &= \int_{-2}^{5} 5f(x)dx - \int_{-2}^{5} 3g(x)dx \\
&= 5\int_{-2}^{5} f(x)dx - 3\int_{-2}^{5} g(x)dx \\
&= (5)(7) - (3)(13) \\
&= 35 - 39 = -4
\end{aligned}$$

2. Find $\int_{1}^{3} [f(x) + g(x)]^2 \, dx$ if $\int_{1}^{3} [f(x)]^2 dx = 5$, $\int_{3}^{1} [g(x)]^2 dx = 7$ and $\int_{1}^{3} f(x)g(x)dx = -2$.
   Solution:

$$\begin{aligned}
\int_{1}^{3} [f(x) + g(x)]^2 \, dx &= \int_{1}^{3} \left[(f(x))^2 + 2f(x)g(x) + (g(x))^2\right] dx \\
&= \int_{1}^{3} [f(x)]^2 dx + 2\int_{1}^{3} f(x)g(x)dx + \int_{1}^{3} [g(x)]^2 dx \\
&= \int_{1}^{3} [f(x)]^2 dx + 2\int_{1}^{3} f(x)g(x)dx - \int_{3}^{1} [g(x)]^2 dx \\
&= 5 + 2(-2) - 7 \\
&= -6
\end{aligned}$$

Lesson 5.2: Fundamental Theorem of Calculus

Learning Outcomes

At the end of this lesson, you should be able to:

1. State and discuss the first and second fundamental theorem of calculus;
2. Evaluate the definite integrals of even and odd functions; and
3. Evaluate integrals using wallis' formula.

Theorem 62: The First Fundamental Theorem of Calculus

Let f be a function continuous on $[a, b]$ and let x be any number in $[a, b]$. If F is the function defined by $F(x) = \int_a^x f(t)dt$, then

$$F'(x) = f(x)$$

Example 155. Find the derivative of each of the following functions.

1. $F(x) = \int_1^x t^2 dt$
   Solution: Let $f(t) = t^2$. Applying the First Fundamental Theorem of Calculus, we obtain $F'(x) = x^2$.

2. $F(x) = \int_{\frac{\pi}{2}}^x \sin(t^3 + 1)dt$
   Solution: Let $f(t) = \sin(t^3 + 1)$. Applying the First Fundamental Theorem of Calculus, we obtain $F'(x) = \sin(x^3 + 1)$.

3. $F(x) = \int_x^1 \cos(3t)dt$
   Solution: To apply the First Fundamental Theorem of Calculus, we need to switch the upper and lower limits of integration,

   $$F(x) = -\int_1^x \cos(3t)dt = \int_1^x [-\cos(3t)]\,dt.$$

   Now, let $f(t) = -\cos(3t)$. Applying the First Fundamental Theorem of Calculus, we obtain $F'(x) = -\cos(3x$.

Remark 13

Suppose $F(x) = \int_a^x f(t)dt$, where f is continuous on $[a, b]$ and let $g(x) \in [a, b]$. If we let $H(x) = \int_a^x f(t)dt$, then $F(x) = H(g(x))$. Using the Chain Rule, we get $F'(x) = H'(g(x)) \cdot g'(x)$. By the First Fundamental Theorem of Calculus, $H'(x) = f(x)$. So we have

$$F'(x) = f(g(x)) \cdot g'(x).$$

Example 156. Find the derivative of each of the following functions.

1. $F(x) = \int_{-2}^{x^2} t^3 dt$

   Solution: Let $f(t) = t^3$ and $g(x) = x^2$, then $g'(x) = 2x$. Applying **Remark 5.2.1**, we obtain

$$\begin{aligned} F'(x) &= f(g(x)) \cdot g'(x) \\ &= f(x^2) \cdot 2x \\ &= (x^2)^3 \cdot 2x \\ &= 2x^7 \end{aligned}$$

2. $F(x) = \int_1^{\sqrt{x}} \frac{\sin(t^2)}{t^2} dt$

   Solution: Let $f(t) = \frac{\sin(t^2)}{t^2}$ and $g(x) = \sqrt{x}$, then $g'(x) = \frac{1}{2\sqrt{x}}$. Applying **Remark 5.2.1**, we obtain

$$\begin{aligned} F'(x) &= f(g(x)) \cdot g'(x) \\ &= f(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} \\ &= \frac{\sin(\sqrt{x})^2}{(\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} \\ &= \frac{\sin x}{x} \cdot \frac{1}{2\sqrt{x}} \\ &= \frac{\sin x}{2x\sqrt{x}} \end{aligned}$$

3. $F(x) = \displaystyle\int_2^{\sin x} \frac{dt}{t}$

   Solution: Let $f(t) = \dfrac{1}{t}$ and $g(x) = \sin x$, then $g'(x) = \cos x$. Applying **Remark 5.2.1**, we obtain

$$\begin{aligned} F'(x) &= f(g(x)) \cdot g'(x) \\ &= f(\sin x) \cdot \cos x \\ &= \frac{1}{\sin x} \cdot \cos x \\ &= \frac{\cos x}{\sin x} \\ &= \cot x \end{aligned}$$

Theorem 63: The Second Fundamental Theorem of Calculus

Let f be a function continuous on $[a, b]$. If F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)dx = F(x)\Bigg|_{x=a}^{x=b} = F(b) - F(a)$$

Example 157. Evaluate the following definite integrals.

1. $\displaystyle\int_1^2 (3x^2 - 7x + 5)dx$

   Solution:

$$\begin{aligned} \int_1^2 (3x^2 - 7x + 5)dx &= \left(x^3 - \frac{7}{2}x + 5x\right)\Bigg|_{x=1}^{x=2} \\ &= \left[(2)^3 - \frac{7}{2}(2) + 5(2)\right] - \left[(1)^3 - \frac{7}{2}(1) + 5(1)\right] \\ &= (8 - 7 + 10) - \left(1 - \frac{7}{2} + 5\right) \\ &= 11 - 1 + \frac{7}{2} - 5 \\ &= \frac{17}{2} \end{aligned}$$

2. $\displaystyle\int_{-2}^2 (7x^5 + 4x^3 - 2x)dx$

   Solution:

$$\begin{aligned} \int_{-2}^2 (7x^5 + 4x^3 - 2x)dx &= \left(\frac{7}{6}x^6 + x^4 - x^2\right)\Bigg|_{x=-2}^{x=2} \\ &= \left[\frac{7}{6}(2)^6 + (2)^4 - (2)^2\right] - \left[\frac{7}{6}(-2)^6 + (-2)^4 - (-2)^2\right] \\ &= \left(\frac{224}{3} + 16 - 4\right) - \left(\frac{224}{3} + 16 - 4\right) \\ &= 0 \end{aligned}$$

Remark 14

By the Second Fundamental Theorem of Calculus and the Substitution Rule,

$$\int_a^b f(g(x)) \cdot g'(x)dx = F(g(x))\Bigg|_{x=a}^{x=b} = F(g(b)) - F(g(b)).$$

If we let $u = g(x)$, we have

$$\int_a^b f(g(x)) \cdot g'(x)dx = F(u)\Bigg|_{u=g(a)}^{u=g(b)}.$$

Therefore,

$$\int_a^b f(g(x)) \cdot g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Example 158. Evaluate the following definite integrals using **Remark 5.2.2**.

1. $\displaystyle\int_{-2}^{3} \frac{dx}{(5x-4)^3}$

   Solution: Let $u = 5x - 4$, then $du = 5dx$ and $\frac{1}{5}du = dx$

Upper Limit:	**Lower Limit:**
$x = 3$	$x = -2$
$u = 5x - 4 = 5(3) - 4 = 11$	$u = 5x - 4 = 5(-2) - 4 = -14$

$$\begin{aligned}
\int_{-2}^{3} \frac{dx}{(5x-4)^3} &= \frac{1}{5}\int_{-14}^{11} u^{-3}du \\
&= \left(\frac{1}{5} \cdot \frac{1}{-2}u^{-2}\right)\Bigg|_{x=-14}^{x=11} \\
&= \left[-\frac{1}{10}(11)^{-2}\right] - \left[-\frac{1}{10}(-14)^{-2}\right] \\
&= \left(-\frac{1}{10}\right)\left(\frac{1}{121}\right) - \left(-\frac{1}{10}\right)\left(\frac{1}{196}\right) \\
&= \left(-\frac{1}{1210}\right) - \left(-\frac{1}{1960}\right) \\
&= -\frac{15}{47432}
\end{aligned}$$

2. $\displaystyle\int_{-3}^{2} |x|dx$

Solution: Recall that $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$ We split the interval at $x = 0$ and we obtain,

$$\begin{aligned} \int_{-3}^{2} |x|dx &= \int_{-3}^{0} |x|dx + \int_{0}^{2} |x|dx \\ &= \int_{-3}^{0} (-x)dx + \int_{0}^{2} xdx \\ &= \left(-\frac{x^2}{2}\right)\Bigg|_{x=-3}^{x=0} + \left(\frac{x^2}{2}\right)\Bigg|_{x=0}^{x=2} \\ &= \left(0 + \frac{9}{2}\right) + (2 - 0) \\ &= \frac{13}{2} \end{aligned}$$

Remark 15

1. The Fundamental Theorems of Calculus establish a close connection between antiderivatives and definite integrals. For this reason, $\int f(x)dx$ is also referred to as an indefinite integral, and the process of antidifferentiation as integration. However, note that in Advanced Calculus, the integral is defined independently of the antiderivative and does not always coincide with the antiderivative of a function.

2. To use the Second Fundamental Theorem of Calculus, the function f must be continuous on $[a, b]$. For instance, $\displaystyle\int_{-1}^{1} \frac{1}{x^2}dx \neq \left(-\frac{1}{x}\right)\Bigg|_{x=-1}^{x=1} = \frac{1}{2}$. In fact, the definite integral of $f(x) = \dfrac{1}{x^2}$ on $[-1, 1]$ cannot even be defined since the interval is not contained in the domain of f.

5.2.1 Definite Integrals of Even and Odd Functions

Definition 38: Even and Odd Functions

If a function f is defined on $[-a, a]$ and $f(-x) = f(x)$ for all $x \in [-a, a]$, then f is called an **even function**.
If a function f is defined on $[-a, a]$ and $f(-x) = -f(x)$ for all $x \in [-a, a]$, then f is called an **odd function**.

Example 159. .

1. The function $f(x) = 4x^6 - 7x^4 + 13x^2 - 11$ is an even function since

$$\begin{aligned} f(-x) &= 4(-x)^6 - 7(-x)^4 + 13(-x)^2 - 11 \\ &= 4x^6 - 7x^4 + 13x^2 - 11 \\ &= f(x) \end{aligned}$$

2. The function $f(x) = 3x^5 - 7x^3 + 4x$ is an odd function since

$$\begin{aligned} f(-x) &= 3(-x)^5 - 7(-x)^3 + 4(-x) \\ &= 3(-x^5) - 7(-x^3) - 4x \\ &= -3x^5 + 7x^3 - 4x \\ &= -(3x^5 - 7x^3 + 4x) \\ &= -f(x) \end{aligned}$$

3. The function $f(x) = \dfrac{x^2+1}{x^2}$ is an even function since

$$\begin{aligned} f(-x) &= \frac{(-x)^2+1}{(-x)^2} \\ &= \frac{x^2+1}{x^2} \\ &= f(x) \end{aligned}$$

4. The function $f(x) = 2x^3 - 5$ is neither even nor odd function since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$

$$\begin{aligned} f(-x) &= 2(-x)^3 - 5 \\ &= 2(-x^3) - 5 \\ &= -2x^3 - 5 \\ &= -(2x^3 + 5) \end{aligned}$$

Theorem 64

If f is defined on $[-a, a]$ and an even function, then

$$\int_{-a}^{a} f(x)dx = 2\int_{-a}^{a} f(x)dx.$$

If f is defined on $[-a, a]$ and an odd function, then

$$\int_{-a}^{a} f(x)dx = 0$$

Example 160. Evaluate the following definite integral.

1. $\displaystyle\int_{-2}^{2} (4x^6 - 7x^4 + 13x^2 - 11)dx$

   Solution: Since $f(x) = 4x^6 - 7x^4 + 13x^2 - 11$ is an even function, then

$$\begin{aligned}
\int_{-2}^{2} (4x^6 - 7x^4 + 13x^2 - 11)dx &= 2\int_{0}^{2} (4x^6 - 7x^4 + 13x^2 - 11)dx \\
&= 2\left(4 \cdot \frac{1}{7}x^7 - 7 \cdot \frac{1}{5}x^5 + 13 \cdot \frac{1}{3}x^3 - 11x\right)\Bigg|_{x=0}^{x=2} \\
&= 2\left[\frac{4}{7}(2)^7 - \frac{7}{5}(2)^5 + \frac{13}{3}(2)^3 - 11(2)\right] - \\
&\quad\; 2\left[\frac{4}{7}(0)^7 - \frac{7}{5}(0)^5 + \frac{13}{3}(0)^3 - 11(0)\right] \\
&= 2\left[\frac{4}{7}(128) - \frac{7}{5}(32) + \frac{13}{3}(8) - 22\right] - 2(0) \\
&= 2\left[\frac{512}{7} - \frac{224}{5} + \frac{104}{3} - 22\right] \\
&= 2\left(\frac{4306}{105}\right) \\
&= \frac{8612}{105}
\end{aligned}$$

2. $\displaystyle\int_{-3}^{3} (3x^5 - 7x^3 + 4x)dx$

   Solution: Since $f(x) = 3x^5 - 7x^3 + 4x$ is an odd function, then

$$\int_{-3}^{3} (3x^5 - 7x^3 + 4x)dx = 0$$

5.2.2 Wallis' Formula

Some integrals involving powers of sine and cosine may be evaluated easily by using the **Wallis' Formula**. This formula was named in honor of John Wallis who is an English mathematician.

Theorem 65

Let n and m be a positive integers.

1. $\displaystyle\int_0^{\frac{\pi}{2}} \sin^n \theta d\theta = \frac{(n-1)(n-3)(n-5)\cdots(1\ or\ 2)}{(n)(n-2)(n-4)\cdots(1\ or\ 2)} \cdot \alpha$

   where $\alpha = 1$ if n is odd and $\alpha = \dfrac{\pi}{2}$ if n is even.

2. $\displaystyle\int_0^{\frac{\pi}{2}} \cos^n \theta d\theta = \frac{(n-1)(n-3)(n-5)\cdots(1\ or\ 2)}{(n)(n-2)(n-4)\cdots(1\ or\ 2)} \cdot \alpha$

   where $\alpha = 1$ if n is odd and $\alpha = \dfrac{\pi}{2}$ if n is even.

3. $\displaystyle\int_0^{\frac{\pi}{2}} \sin^n \theta \cos^m \theta d\theta = \frac{(n-1)(n-3)\cdots(1\ or\ 2)(m-1)(m-3)\cdots(1\,or\,2)}{(n+m)(n+m-2)(n+m-4)\cdots(1\,or\,2)} \cdot \alpha$

   where $\alpha = \dfrac{\pi}{2}$ if n and m are both even and $\alpha = 1$ otherwise.

Example 161. Use Wallis' Formula to evaluate each of the following definite integrals.

1. $\displaystyle\int_0^{\frac{\pi}{2}} \sin^5 x \cos^7 x dx$

   Solution:

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^5 x \cos^7 x dx &= \frac{(4)(2)(6)(4)(2)}{(12)(10)(8)(6)(4)(2)} \cdot 1 \\ &= \frac{1}{120}\end{aligned}$$

2. $\displaystyle\int_0^{\frac{\pi}{2}} \sin^8 x dx$

   Solution:

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^8 x dx &= \frac{(7)(5)(3)(1)}{(8)(6)(4)(2)} \cdot \left(\frac{\pi}{2}\right) \\ &= \frac{35\pi}{256}\end{aligned}$$

3. $\displaystyle\int_0^{\frac{\pi}{4}} \cos^4(2x)dx$

Solution: Let $u = 2x$, then $du = 2dx$ and $\frac{1}{2}du = dx$.

Upper Limit:	**Lower Limit:**
$x = \frac{\pi}{4}$ $u = 2x = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$	$x = 0$ $u = 2x = 2(0) = 0$

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \cos^4(2x)dx &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^2 u du \\
&= \left(\frac{1}{2}\right)\cdot\frac{(3)(1)}{(4)(2)}\cdot\left(\frac{\pi}{2}\right) \\
&= \frac{3\pi}{32}
\end{aligned}$$

4. $\displaystyle\int_0^{\frac{\pi}{6}} \sin^4(3x)\cos^7(3x)dx$

Solution: Let $u = 3x$, then $du = 3dx$ and $\frac{1}{3}du = dx$.

Upper Limit:	**Lower Limit:**
$x = \frac{\pi}{6}$ $u = 3x = 3\left(\frac{\pi}{6}\right) = \frac{\pi}{2}$	$x = 0$ $u = 3x = 3(0) = 0$

$$\begin{aligned}
\int_0^{\frac{\pi}{6}} \sin^4(3x)\cos^7(3x)dx &= \frac{1}{3}\int_0^{\frac{\pi}{2}} \sin^4 u\cos^7 u du \\
&= \left(\frac{1}{3}\right)\cdot\frac{(3)(1)(6)(4)(2)}{(11)(9)(7)(5)(3)(1)}\cdot(1) \\
&= \frac{16}{3465}
\end{aligned}$$

5. $\displaystyle\int_0^{3} x^4\sqrt{9-x^2}dx$

Solution: Let $x = 3\sin\theta$, then $dx = 3\cos\theta d\theta$ and $9 - x^2 = 9\cos^2\theta$.

Upper Limit:	**Lower Limit:**
$x = 3$ $\theta = \arcsin\left(\frac{x}{3}\right) = \arcsin\left(\frac{3}{3}\right) = \arcsin(1) = \frac{\pi}{2}$	$x = 0$ $\theta = \arcsin\left(\frac{x}{3}\right) = \arcsin\left(\frac{0}{3}\right) = \arcsin(0) = 0$

$$\begin{aligned}
\int_0^3 x^4\sqrt{9-x^2}dx &= \int_0^{\frac{\pi}{2}} (3\sin\theta)^4 \left(\sqrt{9\cos^2\theta}\right)(3\cos\theta d\theta) \\
&= \int_0^{\frac{\pi}{2}} (81\sin^4\theta)(3\cos\theta)(3\cos\theta d\theta) \\
&= 729\int_0^{\frac{\pi}{2}} \sin^4\theta\cos^2\theta d\theta \\
&= 729\cdot\frac{(3)(1)(1)}{(6)(4)(2)}\cdot\frac{\pi}{2} \\
&= \frac{729\pi}{32}
\end{aligned}$$

Lesson 5.3: Applications of the Definite Integral

Learning Outcomes

At the end of this lesson, you should be able to:

1. Calculate the area of the plane region using definite integrals;
2. Compute the areas of plane region between two curves;
3. Find the arc length of some curves in a specified interval; and
4. Calculate the volume of the solid of revolution using disk method, washer method and cylindrical shell method.

5.3.1 Area of a Plane Region

Learning Outcomes

At the end of this lesson, you should be able to:

1. State and apply the formula for finding the area of a plane region bounded by a curve, the x-axis, and the vertical lines $x = a$ and $x = b$; and
2. Calculate the area of a plane region by integration.

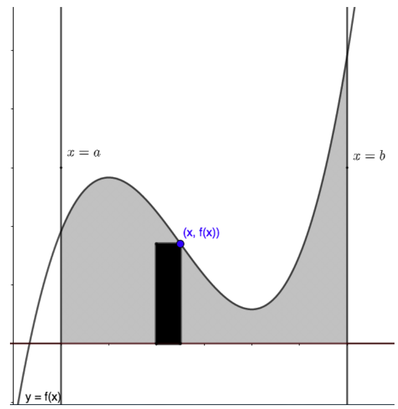
The computation of areas of a plane region has been introduced in lower mathematics subjects. Usually, these plane regions are bounded by straight edges, which are known as polygons. We start by defining the area of a rectangle to be its length times its width. From this, we successively derive the formulas for the area of a parallelogram, a triangle or other polygons. Partitioning these into rectangles and summing up each area of the rectangles derives such polygons.

In integral calculus, our ancient mathematicians develop a simpler and more accurate tool in solving plane areas that are bounded by either arcs or straight lines.

Formula for the Area of a Plane Region

Formula 1: Vertical Rectangle

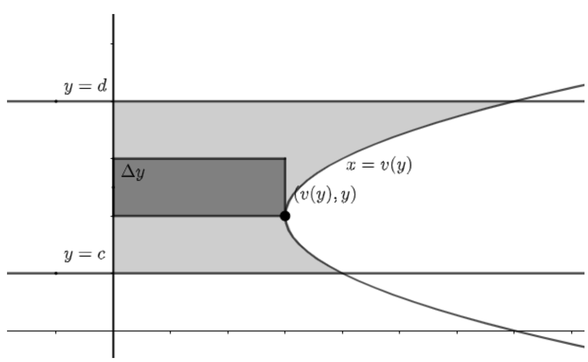
If the graph of $y = f(x)$ lies entirely above the x-axis on the interval $[a, b]$, $\int_a^b f(x)dx$ gives the area of the region bounded by the curves $y = f(x)$, the x-axis and the vertical lines $x = a$ and $x = b$.



If the graph of $y = f(x)$ lies entirely below the x-axis on the interval $[a, b]$, $-\int_a^b f(x)dx$ gives the area of the region bounded by the curves $y = f(x)$, the x-axis and the vertical lines $x = a$ and $x = b$.

Formula 2: Horizontal Rectangle

f the graph $x = v(y)$ lies entirely on the right of the y-axis on the interval $[c, d]$, $\int_c^d v(y)dy$ gives the area of the region bounded by the curves $x = v(y)$, the y-axis and the horizontal lines $y = c$ and $y = d$.



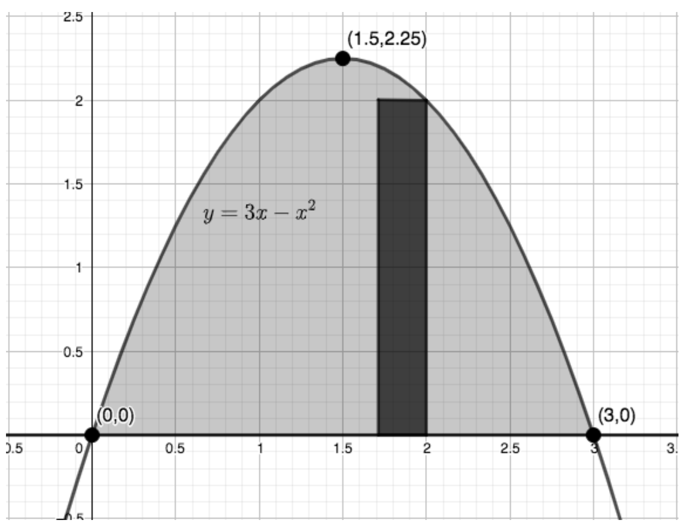
If the graph $x = v(y)$ lies entirely on the left of the y-axis on the interval $[c, d]$, $-\int_c^d v(y)dy$ gives the area f the region bounded by the curves $x = v(y)$, the y-axis and the horizontal lines $y = c$ and $y = d$.

Procedure 2: Suggested Steps in Calculating Areas by Integration

1. Sketch the region.
2. Shade the area to be determined.
3. Draw a representative rectangle (vertical or horizontal rectangle) of area and denote its base and altitude.
4. Set up the area integral.

Example 162. Find the area of the region R bounded by the following curves.

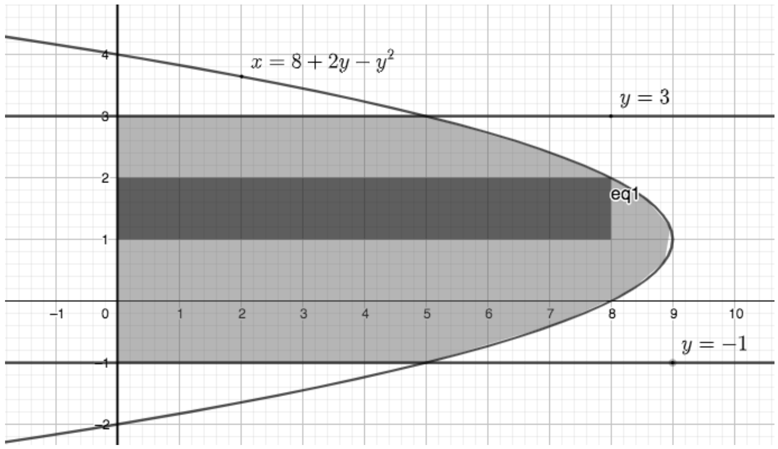
1. $y = 3x - x^2$ and the x-axis



Solution: Using **Vertical Rectangle**,the area of the region R bounded above by $f(x) = 3x - x^2$ and below by x-axis and the lines $x = 0$ and $x = 3$ is

$$\begin{aligned} A_R &= \int_a^b f(x)dx \\ &= \int_0^3 (3x - x^2)dx \\ &= \left(\frac{3}{2}x^2 - \frac{1}{3}x^3\right)\Bigg|_{x=0}^{x=3} \\ &= \left(\frac{27}{2} - 9\right) - (0) \\ &= \frac{45}{2} \text{ square units} \end{aligned}$$

2. $x = 8 + 2y - y^2$, y-axis, $y = -1$ and $y = 3$



Solution: Using **Horizontal Rectangle**, the area of the region R bounded on the right by $x = 8 + 2y - y^2$ and on the left by the y-axis and the lines $y = -1$ and $y = 3$ is

$$\begin{aligned}
A_R &= \int_c^d v(y)dy \\
&= \int_{-1}^{3} (8 + 2y - y^2)dy \\
&= \left(8y + y^2 - \frac{1}{3}y^3\right)\Bigg|_{y=-1}^{y=3} \\
&= (24 + 9 - 9) - \left(-8 + 1 + \frac{1}{3}\right) \\
&= 24 - \left(-\frac{20}{3}\right) \\
&= \frac{92}{3} \text{ square units}
\end{aligned}$$

5.3.2 Area of the Region Bounded by Two Curves

Learning Outcomes

At the end of this lesson, you should be able to:

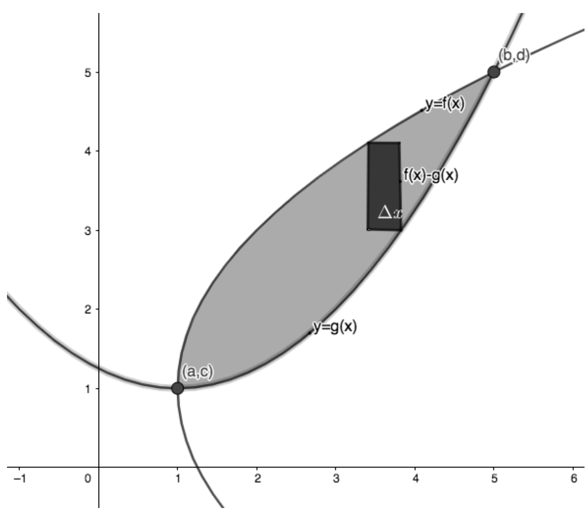
1. State and discuss the formula for computing the area bounded by two curve; and
2. Compute the areas of plane regions between two curves.

Formula for the Area of the Region Bounded by Two Curves

Formula 3: Vertical Rectangle

If f and g are continuous functions on the interval $[a, b]$ and $f(x) \geq g(x))$ for all $x \in [a, b]$, then the area of the region R bounded above by $y = f(x)$, below by $y = g(x)$ and the vertical lines $x = a$ and $x = b$ is

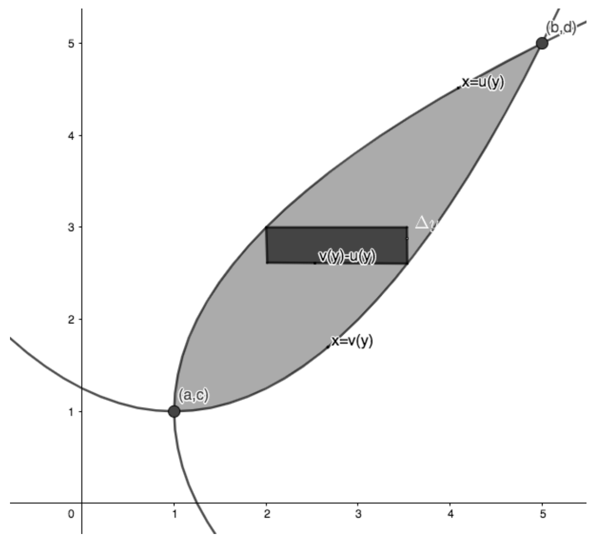
$$A_R = \int_a^b [f(x) - g(x)]dx$$



Formula 4: Horizontal Rectangle

If u and v are continuous functions of y on the interval $[c,\, d]$ and $v(y) \geq u(y)$ for all $y \in [c,\, d]$, then the area of the region R bounded on the left by $x = u(y)$, on the right by $x = v(y)$ and the horizontal lines $y = c$ and $y = d$ is

$$A_R = \int_c^d [v(y) - u(y)]dy$$



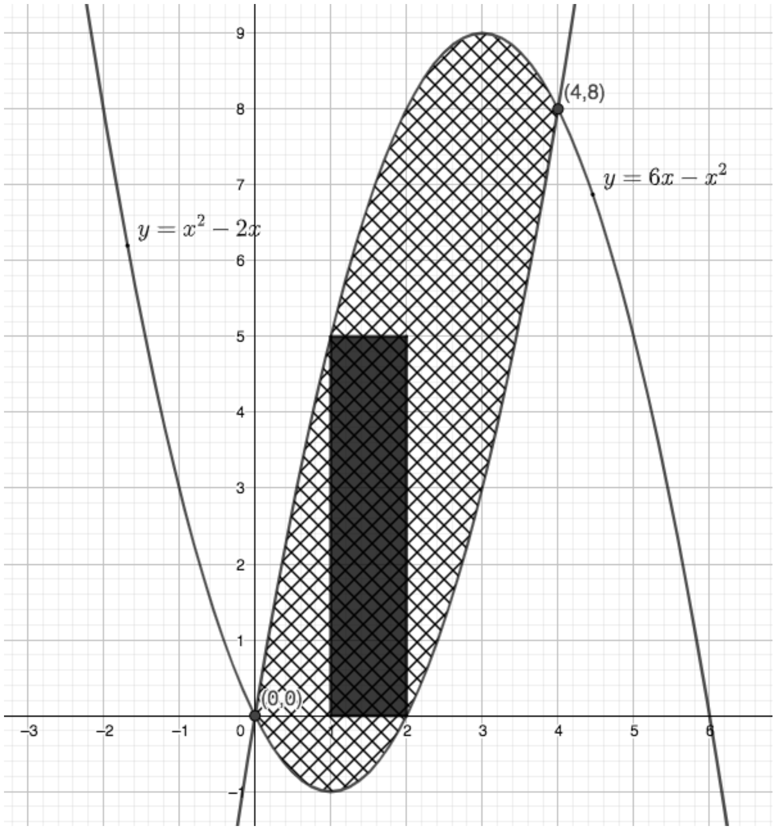
Example 163. Find the area of the following regions.

1. Region bounded by $y = 6x - x^2$ and $y = x^2 - 2x$

   Solution:

   Using Vertical Rectangle:

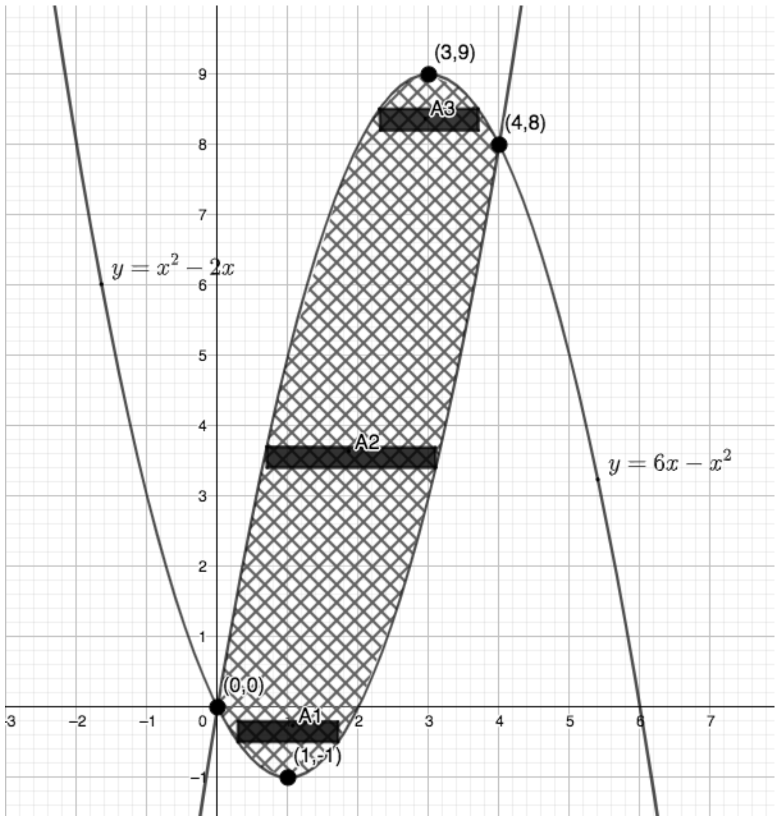
   The area of the region R is bounded above by $y = 6x - x^2$ and below by $y = x^2 - 2x$. The Vertical Rectangle is evaluated from $x = 0$ to $x = 4$.



$$
\begin{aligned}
A_R &= \int_0^4 [(6x - x^2) - (x^2 - 2x)]dx \\
&= \int_0^4 (-2x^2 + 8x)dx \\
&= \left(-\frac{2}{3}x^3 + 4x^2\right)\Bigg|_{x=0}^{x=4} \\
&= \left[-\frac{2}{3}(4)^3 + 4(4)^2\right] - \left[-\frac{2}{3}(0)^3 + 4(0)^2\right] \\
&= -\frac{128}{3} + 64 \\
&= \frac{64}{3} \text{ square units}
\end{aligned}
$$

Using Horizontal Rectangle:

We will divide the region into three parts; first, from $y = -1$ to $y = 0$; second, from $y = 0$ to $y = 8$; and lastly, from $y = 8$ to $y = 9$.



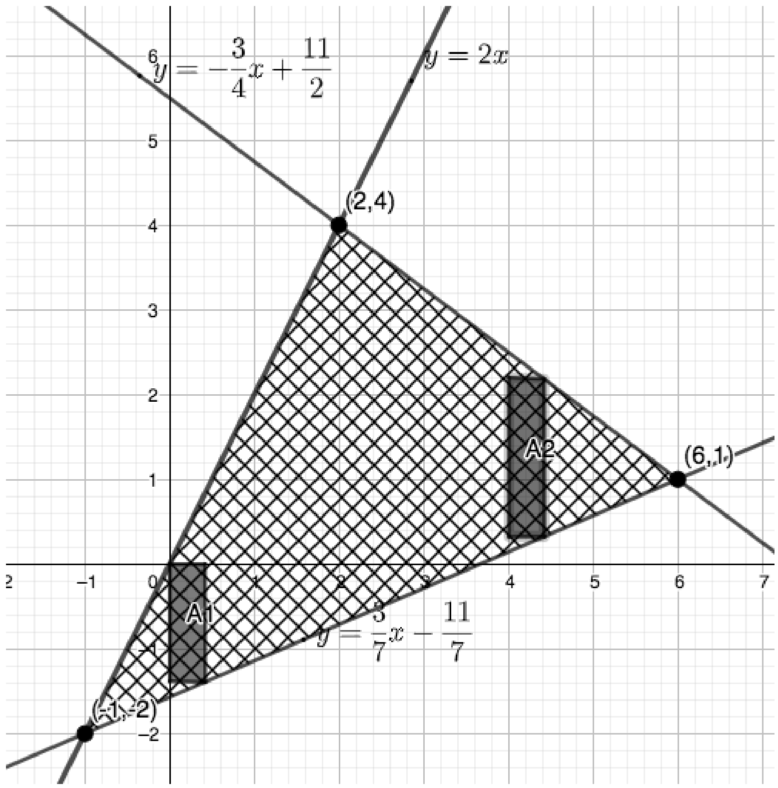
$$
\begin{aligned}
A_R &= A_1 + A_2 + A_3 \\
&= \int_{-1}^{0} [(1+\sqrt{y+1}) - (1-\sqrt{y+1})]dy + \int_{0}^{8} [(1+\sqrt{y+1}) - (3-\sqrt{9-y})]dy + \\
&\quad \int_{8}^{9} [(3+\sqrt{9-y}) - (3-\sqrt{9-y})]dy \\
&= \int_{-1}^{0} 2\sqrt{y+1}dy + \int_{0}^{8} (-2+\sqrt{y+1}+\sqrt{9-y})dy + \int_{8}^{9} 2\sqrt{9-y}dy \\
&= \left[\frac{4}{3}(y+1)^{\frac{3}{2}}\right]\Bigg|_{y=-1}^{y=0} + \left[-2y + \frac{2}{3}(y+1)^{\frac{3}{2}} - \frac{2}{3}(9-y)^{\frac{3}{2}}\right]\Bigg|_{y=0}^{y=8} \\
&\quad \left[-\frac{4}{3}(9-y)^{\frac{3}{2}}\right]\Bigg|_{y=8}^{y=9} \\
&= \left(\frac{4}{3} - 0\right) + \left[\left(-16+18-\frac{2}{3}\right) - \left(0+\frac{2}{3}-18\right)\right] + \left(0+\frac{4}{3}\right) \\
&= \frac{4}{3} + \frac{4}{3} + \frac{52}{3} + \frac{4}{3}
\end{aligned}
$$

2. Triangle whose vertices are the points $(-1, 4)$, $(2, -2)$, and $(5, 1)$.

Solution:

Using Vertical Rectangle:

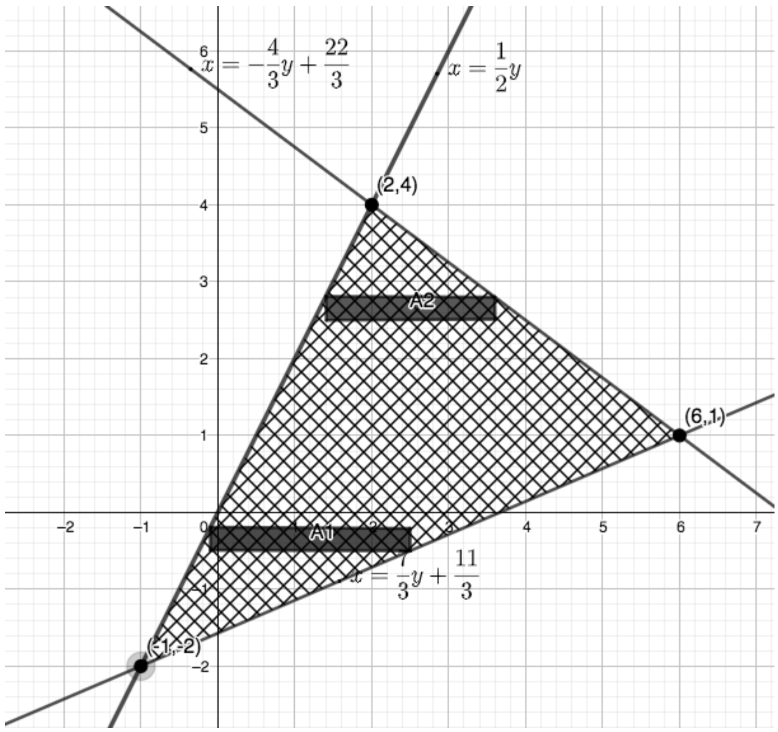
The equation of the line connecting the points $(-1, -2)$ and $(2, 4)$ is $y = 2x$, the equation of the line connecting the points $(-1, -2)$ and $(6, 1)$ is $y = \frac{3}{7}x - \frac{11}{7}$ and the equation of the line connecting the points $(2, 4)$ and $(6, 1)$ is $y = -\frac{3}{4}x + \frac{11}{2}$.



$$
\begin{aligned}
A_R &= A_1 + A_2 \\
&= \int_{-1}^{2} \left[2x - \left(\frac{3}{7}x - \frac{11}{7}\right)\right] dx + \int_{2}^{6} \left[\left(-\frac{3}{4}x + \frac{11}{2}\right) - \left(\frac{3}{7}x - \frac{11}{7}\right)\right] dx \\
&= \int_{-1}^{2} \left(\frac{11}{7}x + \frac{11}{7}\right) dx + \int_{2}^{6} \left(-\frac{33}{28}x + \frac{99}{14}\right) dx \\
&= \left(\frac{11}{14}x^2 + \frac{11}{7}x\right)\Bigg|_{x=-1}^{x=2} + \left(-\frac{33}{56}x^2 + \frac{99}{14}x\right)\Bigg|_{x=2}^{x=6} \\
&= \left[\frac{11}{14}(2)^2 + \frac{11}{7}(2)\right] - \left[\frac{11}{14}(-1)^2 + \frac{11}{7}(-1)\right] + \left[-\frac{33}{56}(6)^2 + \frac{99}{14}(6)\right] - \left[-\frac{33}{56}(2)^2 + \frac{99}{14}(2)\right] \\
&= \frac{22}{7} + \frac{22}{7} - \frac{11}{14} + \frac{11}{7} - \frac{297}{14} + \frac{297}{7} + \frac{33}{14} - \frac{99}{7} \\
&= \frac{33}{2} \text{ square units}
\end{aligned}
$$

Using Horizontal Rectangle:

The equation of the line connecting the points $(-1, -2)$ and $(2, 4)$ is $x = \frac{1}{2}y$, the equation of the line connecting the points $(-1, -2)$ and $(6, 1)$ is $x = \frac{7}{3}y + \frac{11}{3}$ and the equation of the line connecting the points $(2, 4)$ and $(6, 1)$ is $x = -\frac{4}{3}y + \frac{22}{3}$.



$$
\begin{aligned}
A_R &= A_1 + A_2 \\
&= \int_{-2}^{1} \left[\left(\frac{7}{3}y + \frac{11}{3}\right) - \left(\frac{1}{2}y\right)\right] dy + \int_{1}^{4} \left[\left(-\frac{4}{3}y + \frac{22}{3}\right) - \left(\frac{1}{2}y\right)\right] dy \\
&= \int_{-2}^{1} \left(\frac{11}{6}y + \frac{11}{3}\right) dy + \int_{1}^{4} \left(-\frac{11}{6}y + \frac{22}{3}\right) dy \\
&= \left(\frac{11}{12}y^2 + \frac{11}{3}y\right)\Bigg|_{y=-2}^{y=1} + \left(-\frac{11}{12}y^2 + \frac{22}{3}y\right)\Bigg|_{y=1}^{y=4} \\
&= \left[\frac{11}{12}(1)^2 + \frac{11}{3}(1)\right] - \left[\frac{11}{12}(-2)^2 + \frac{11}{3}(-2)\right] + \left[-\frac{11}{12}(4)^2 + \frac{22}{3}(4)\right] - \left[-\frac{11}{12}(1)^2 + \frac{22}{3}(1)\right] \\
&= \frac{11}{12} + \frac{11}{3} - \frac{11}{3} + \frac{22}{3} - \frac{44}{3} + \frac{88}{3} + \frac{11}{12} - \frac{22}{3} \\
&= \frac{33}{2} \text{ square units}
\end{aligned}
$$

5.3.3 Arc Length

Learning Outcomes

At the end of this lesson, you should be able to:

1. Define smooth in the curve of an equation; and
2. Compute the length of an arc of an equation.

Definition 39: Smooth

A curve with equation $y = f(x)$ is said to be **smooth** on $[a, b]$ if f' is continuous on $[a, b]$.

Formula for the Length of an Arc

1. If $y = f(x)$ is a smooth curve on the x-interval $[a, b]$, then the arc length L of this curve from $x = a$ to $x = b$ is
$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$
2. If $x = u(y)$ is a smooth curve on the y-interval $[c, d]$, then the arc length L of this curve from $y = c$ to $y = d$ is
$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Example 164. Find the arc length of the following curves in the specified interval.

1. $y = 2(3x-2)^{\frac{3}{2}}$ from $x = 1$ to $x = 3$.
Solution: First,
$$\frac{dy}{dx} = 3(3x-2)^{\frac{1}{2}} \cdot 3 = 9\sqrt{3x-2},$$
which is continuous on $[1, 3]$. Thus, $y = 2(3x-2)^{\frac{3}{2}}$ is smooth on $[1, 3]$. Therefore,
$$\begin{aligned} L &= \int_1^3 \sqrt{1 + (9\sqrt{3x-2})^2} dx \\ &= \int_1^3 \sqrt{1 + 81(3x-2)} dx \\ &= \int_1^3 \sqrt{243x - 161} dx \\ &= 162(243x-161)^{\frac{3}{2}} \Big|_{x=1}^{x=3} \\ &= 162(568)^{\frac{3}{2}} - 162(82)^{\frac{3}{2}} \end{aligned}$$

2. $3x - 6y + 9 = 0$ between $y = -1$ to $y = 2$.

   Solution: The equation $3x - 6y + 9 = 0$ is the same as $x = 2y - 3$. Also,

$$\frac{dx}{dy} = 2,$$

which is continuous on $[-1, 2]$. Thus, $3x - 6y + 9 = 0$ is smooth on $[-1, 2]$. Therefore,

$$\begin{aligned} L &= \int_{-1}^{2} \sqrt{1 + (2)^2} dy \\ &= \int_{-1}^{2} \sqrt{5} dy \\ &= \sqrt{5} y \Big|_{y=-1}^{y=2} \\ &= \sqrt{5}(2) - \sqrt{5}(-1) \\ &= 3\sqrt{5} \end{aligned}$$

3. $8y = x^4 + 2x^{-2}$ from the point where $x = 1$ and $x = 2$.

   Solution: The equation $8y = x^4 + 2x^{-2}$ is the same as $y = \frac{1}{8}x^4 + \frac{1}{4}x^{-2}$. Also,

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}x^3 - \frac{1}{2}x^{-3} \\ &= \frac{x^3}{2} - \frac{1}{2x^3} \\ &= \frac{x^6 - 1}{2x^3}, \end{aligned}$$

which is continuous on $[1, 2]$. Thus, $8y = x^4 + 2x^{-2}$ is smooth on $[1, 2]$. Therefore,

$$\begin{aligned}
L &= \int_1^2 \sqrt{1+\left(\frac{x^6-1}{2x^3}\right)^2}dx \\
&= \int_1^2 \sqrt{\frac{4x^6+x^{12}-2x^6+1}{4x^6}}dx \\
&= \int_1^2 \sqrt{\frac{x^{12}+2x^6+1}{4x^6}}dx \\
&= \int_1^2 \sqrt{\left(\frac{x^6+1}{2x^3}\right)^2}dx \\
&= \int_1^2 \left(\frac{x^6+1}{2x^3}\right)dx \\
&= \int_1^2 \left(\frac{1}{2}x^3+\frac{1}{2}x^{-3}\right)dx \\
&= \left(\frac{1}{8}x^4-\frac{1}{4}x^{-2}\right)\Bigg|_{x=1}^{x=2} \\
&= \left[\frac{1}{8}(2)^4-\frac{1}{4}(2)^{-2}\right]-\left[\frac{1}{8}(1)^4-\frac{1}{4}(1)^{-2}\right] \\
&= 2-\frac{1}{16}-\frac{1}{8}+\frac{1}{4} \\
&= \frac{33}{16}
\end{aligned}$$

5.3.4 Volumes of Solids of Revolution

Learning Outcomes

At the end of this lesson, you should be able to:

1. Define a solid of revolution;
2. State and illustrate the method of calculating the volume of solids of revolution using: the disk method, washer method and cylindrical shell method; and
3. Calculate volumes of solids of revolution using: the disk method, washer method and cylindrical shell method.

Definition 40: Solid of Revolution

A **solid of revolution** is a solid obtained when a plane region is revolved about a line (in the same plane) called the **axis of revolution**.

In finding the volume of a solid of revolution, the idea is to approximate by very thin rectangles the region to be revolved, and then to revolve each rectangle about the same axis. We shall employ two methods:

1. **Disk or Washer Method**. Use rectangles that are perpendicular to the axis of revolution.
2. **Cylindrical Shell Method**. Use rectangles that are parallel to the axis of revolution.

Formula for the Volume of a Solid of Revolution using Disks or Washers

Formula 5: Vertical Rectangle

Suppose R is the region bounded above by $y = f(x)$, below by $y = g(x)$, and the vertical lines $x = a$ and $x = b$ such that f and g are continuous functions on $[a, b]$. If the line $y = y_0$ does not intersect the interior of R, then the volume of the solid of revolution obtained when R is revolved about the line $y = y_0$ is given by:

a. If only disks are obtained (that is, a boundary of R lies on the axis of revolution), then

$$V = \int_a^b \pi[r(x)]^2 dx$$

where $r(x)$ is the radius of a risk at an arbitrary x in $[a, b]$

b. If washers are obtained (that is, a boundary of R does not lie on the axis of revolution) then

$$V = \int_a^b \pi \left([r_2(x)]^2 - [r_1(x)]^2\right) dx$$

where $r_2(x)$ and $r_1(x)$ are the outer radius and inner radius, respectively, of a washer at an arbitrary x in $[a, b]$.

Formula 6: Horizontal Rectangle

Suppose R is the region bounded on the left by $x = u(y)$, on the right by $x = v(y)$, and the horizontal lines $y = c$ and $y = b$ such that u and v are continuous functions on $[c, d]$. If the line $x = x_0$ does not intersect the interior of R, then the volume of the solid of revolution obtained when R is revolved about the line $x = x_0$ is given by:

a. If only disks are obtained (that is, a boundary of R lies on the axis of revolution), then

$$V = \int_a^b \pi[r(y)]^2 dy$$

where $r(y)$ is the radius of a risk at an arbitrary y in $[c, d]$

b. If washers are obtained (that is, a boundary of R does not lie on the axis of revolution) then

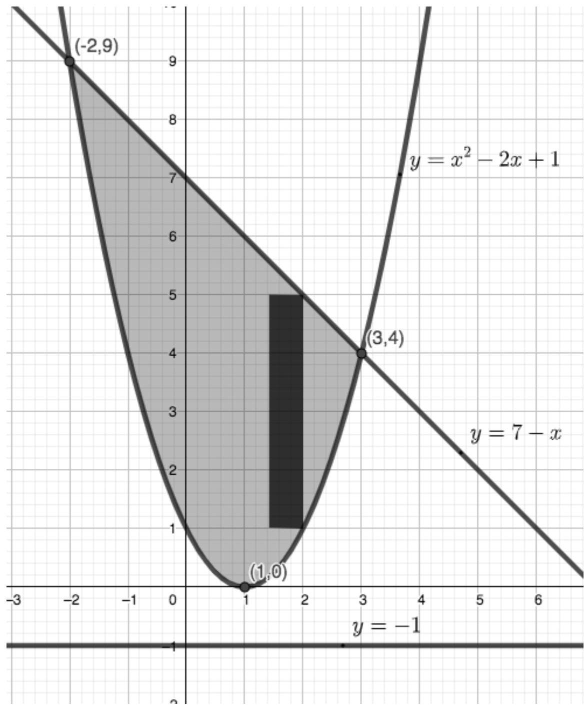
$$V = \int_a^b \pi \left([r_2(y)]^2 - [r_1(y)]^2\right) dy$$

where $r_2(y)$ and $r_1(y)$ are the outer radius and inner radius, respectively, of a washer at an arbitrary y in $[c, d]$.

Example 165. Find the volume of the solid generated when the indicated plane region is revolved about the given axis of revolution.

1. Region bounded by $y = x^2 - 2x + 1$ and $y = 7 - x$

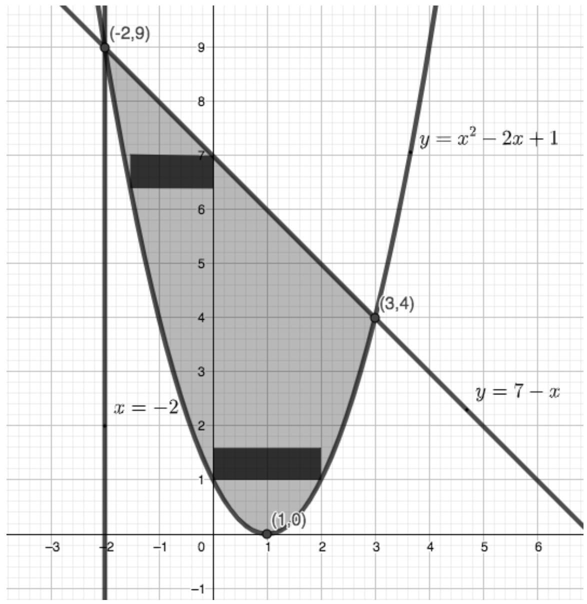
   a. about $y = -1$



Solution: Using the washers we obtain that the value of $r_1(x) = x^2 - 2x + 2$ and $r_2(x) = 8 - x$. Also,

$$\begin{aligned}
V &= \int_{-2}^{3} \pi \left[(8-x)^2 - (x^2 - 2x + 2)^2\right] dx \\
&= \int_{-2}^{3} \pi \left[(64 - 16x + x^2) - (x^4 + 4x^2 + 4 - 4x^3 + 4x^2 - 8x)\right] dx \\
&= \int_{-2}^{3} \pi(-x^4 + 4x^3 - 7x^2 - 8x + 60)dx \\
&= \pi \left[-\frac{1}{5}x^5 + x^4 - \frac{7}{3}x^3 - 4x^2 + 60x\right] \Bigg|_{x=-2}^{x=3} \\
&= \pi \left[-\frac{1}{5}(3)^5 + (3)^4 - \frac{7}{3}(3)^3 - 4(3)^2 + 60(3)\right] - \\
&\quad \pi \left[-\frac{1}{5}(-2)^5 + (-2)^4 - \frac{7}{3}(-2)^3 - 4(-2)^2 + 60(-2)\right] \\
&= \frac{567}{5}\pi - \left(-\frac{1424}{15}\pi\right) \\
&= \frac{625}{3}\pi \text{ cubic units}
\end{aligned}$$

b. about $x = -2$

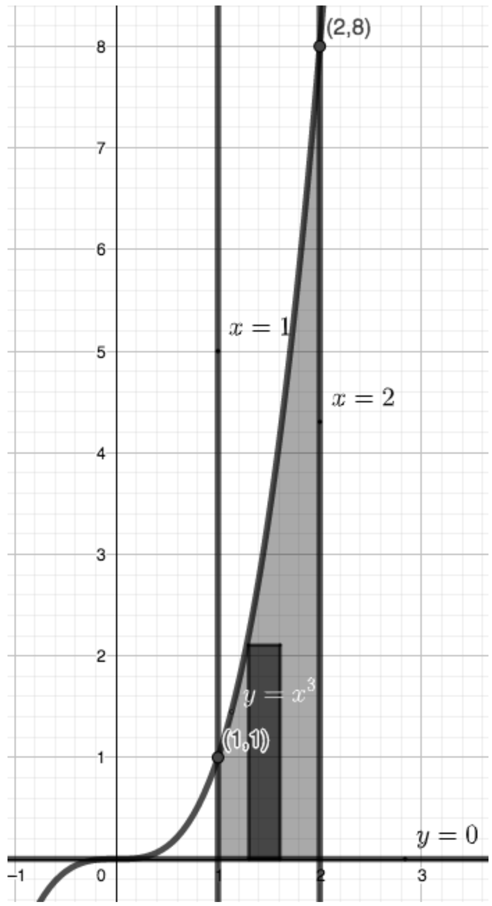


Solution: Using the washers we obtain from $y = 0$ to $y = 4$, $r_1(y) = 3 - \sqrt{y}$ and $r_2(y) = 3 + \sqrt{y}$. From $y = 4$ to $y = 9$, $r_1(y) = 3 - \sqrt{y}$ and $r_2(y) = 9 - y$.

$$\begin{aligned}
V &= V_1 + V_2 \\
&= \int_0^4 \pi \left[(3+\sqrt{y})^2 - (3-\sqrt{y})^2\right] dy + \int_4^9 \pi \left[(9-y)^2 - (3-\sqrt{y})^2\right] dy \\
&= \int_0^4 \pi[(9+6\sqrt{y}+y) - (9-6\sqrt{y}+y)]dy + \int_4^9 \pi[(81-18y+y^2) - (9-6\sqrt{y}+y)]dy \\
&= \int_0^4 12\pi\sqrt{y}dy + \int_4^9 \pi(y^2 - 19y + 6\sqrt{y} + 72)dy \\
&= 8\pi y^{\frac{3}{2}} \Bigg|_{y=0}^{y=4} + \pi\left(\frac{1}{3}y^3 - \frac{19}{2}y^2 + 4y^{\frac{3}{2}} + 72y\right)\Bigg|_{y=4}^{y=9} \\
&= \pi\left[8(4)^{\frac{3}{2}}\right] - \pi\left[8(0)^{\frac{3}{2}}\right] + \pi\left[\frac{1}{3}(9)^3 - \frac{19}{2}(9)^2 + 4(9)^{\frac{3}{2}} + 72(9)\right] - \\
&\quad \pi\left[\frac{1}{3}(4)^3 - \frac{19}{2}(4)^2 + 4(4)^{\frac{3}{2}} + 72(4)\right] \\
&= \pi\left(64 - 0 + \frac{459}{2} - \frac{568}{3}\right) \\
&= \frac{625}{6}\pi \text{ cubic units}
\end{aligned}$$

2. Region bounded by $y = x^3$, $x = 1$, $x = 2$ and $y = 0$

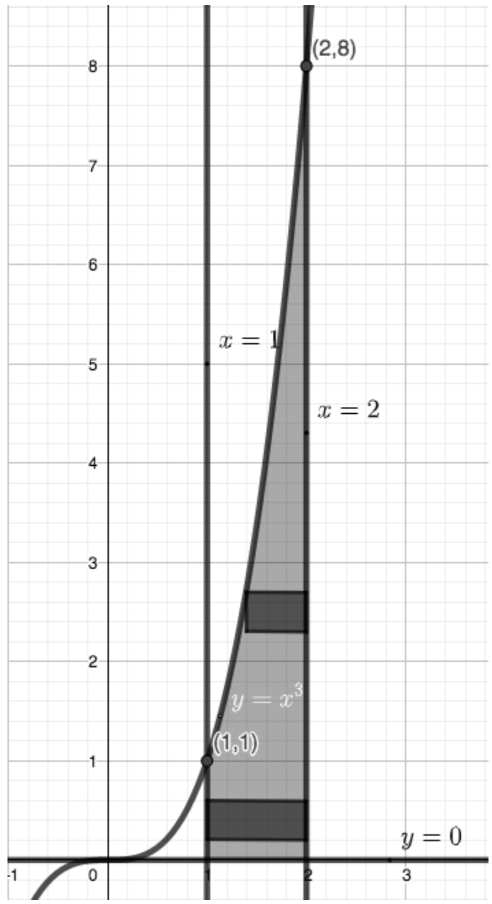
a. about $y = 0$



Solution: Using the disk we obtain that the value of $r(x) = x^3$.

$$\begin{aligned} V &= \int_1^2 \pi \left[r(x)\right]^2 dx \\ &= \int_1^2 \pi \left(x^3\right)^2 dx \\ &= \int_1^2 \pi x^6 dx \\ &= \pi\left(\frac{1}{7}x^7\right)\Bigg|_{x=1}^{x=2} \\ &= \pi\left[\frac{1}{7}(2)^7\right] - \pi\left[\frac{1}{7}(1)^7\right] \\ &= \frac{128}{7}\pi - \frac{1}{7}\pi \\ &= \frac{127}{7}\pi \text{ cubic units} \end{aligned}$$

b. about $x = 2$



Solution: Using the disk method we obtain that the value of $r(x) = 1$ from $y = 0$ to $y = 1$, and $r(x) = 2 - y^{\frac{1}{3}}$ from $y = 1$ to $y = 8$.

$$
\begin{aligned}
V &= \int_0^1 \pi(1)^2 dy + \int_1^8 \pi \left(2 - y^{\frac{1}{3}}\right)^2 dy \\
&= \pi \int_0^1 dy + \pi \int_1^8 \left(4 - 4y^{\frac{1}{3}} + y^{\frac{2}{3}}\right) dy \\
&= \pi y \Bigg|_{y=0}^{y=1} + \pi \left(4y - 3y^{\frac{4}{3}} + \frac{3}{5} y^{\frac{5}{3}}\right) \Bigg|_{y=1}^{y=8} \\
&= \pi(1) - \pi(0) + \pi \left[4(8) - 3(8)^{\frac{4}{3}} + \frac{3}{5}(8)^{\frac{5}{3}}\right] - \pi \left[4(1) - 3(1)^{\frac{4}{3}} + \frac{3}{5}(1)^{\frac{5}{3}}\right] \\
&= \pi + \pi \left(32 - 48 + \frac{96}{5}\right) - \pi \left(4 - 3 + \frac{3}{5}\right) \\
&= \pi + \frac{16}{5}\pi - \frac{8}{5}\pi \\
&= \frac{13}{5}\pi \text{ cubic units}
\end{aligned}
$$

Formula for the Volume of a Solid of Revolution
using Cylindrical Shells

Formula 7: Vertical Rectangle

Suppose R is the region bounded above by $y = f(x)$, below by $y = g(x)$, and the vertical lines $x = a$ and $x = b$ such that f and g are continuous functions on $[a, b]$. If the line $x = x_0$ does not intersect the interior of R, then the volume of the solid of revolution obtained when R is revolved about the line $x = x_0$ is given by

$$V = \int_a^b 2\pi r(x)h(x)dx,$$

where $r(x)$ and $h(x)$ are the radius and height, respectively, of a cylindrical shell at an arbitrary x in $[a, b]$.

Formula 8: Horizontal Rectangle

Suppose R is the region bounded on the left by $x = u(y)$, on the right by $x = v(y)$, and the horizontal lines $y = c$ and $y = d$ such that u and v are continuous functions on $[c, d]$. If the line $y = y_0$ does not intersect the interior of R, then the volume of the solid of revolution obtained when R is revolved about the line $y = y_0$ is given by

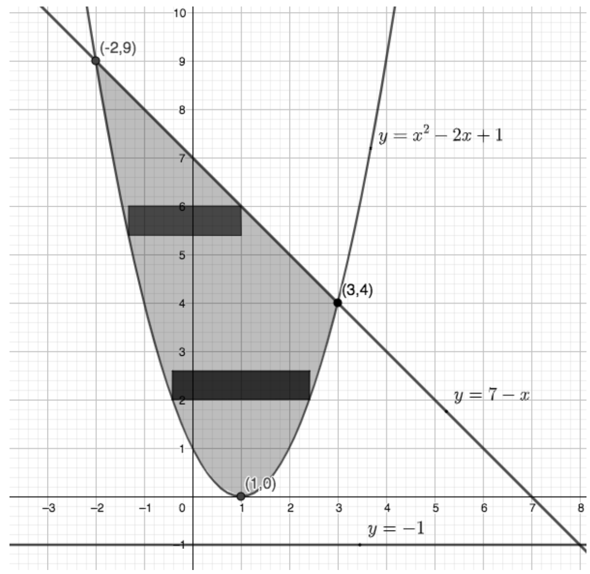
$$V = \int_a^b 2\pi r(y)h(y)dy,$$

where $r(y)$ and $h(y)$ are the radius and height, respectively, of a cylindrical shell at an arbitrary y in $[c, d]$.

Example 166. Find the volume of the solid generated when the indicated plane region is revolved about the given axis of revolution.

1. Region bounded by $y = x^2 - 2x + 1$ and $y = 7 - x$

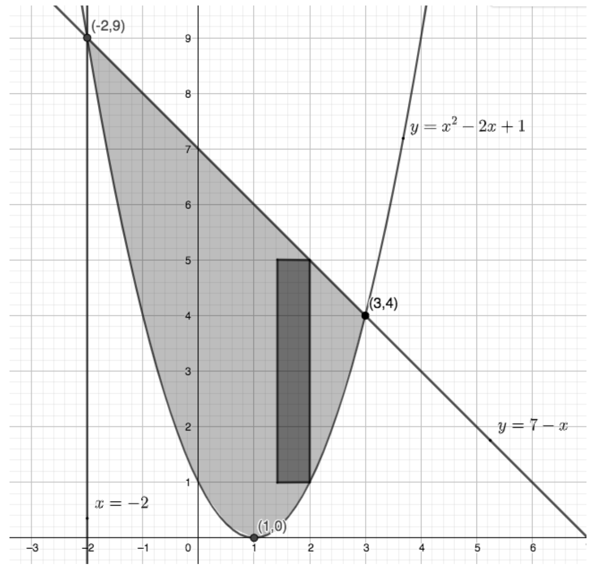
   a. about $y = -1$



Solution: Using the cylindrical shell we obtain that the value of $r(y) = y+1$ and $h(y) = 2y^{\frac{1}{2}}$ from $y = 0$ to $y = 4$, and $r(y) = y+1$ and $h(y) = 6 - y + y^{\frac{1}{2}}$ from $y = 4$ to $y = 9$. Also,

$$
\begin{aligned}
V &= \int_0^4 2\pi\,(y+1)\left(2y^{\frac{1}{2}}\right)dy + \int_4^9 2\pi\,(y+1)\left(6 - y + y^{\frac{1}{2}}\right)dy \\
&= \int_0^4 2\pi\left(2y^{\frac{3}{2}} + 2y^{\frac{1}{2}}\right)dy + \int_4^9 2\pi\left(5y - y^2 + y^{\frac{3}{2}} + 6 + y^{\frac{1}{2}}\right)dy \\
&= 2\pi\left(\frac{4}{5}y^{\frac{5}{2}} + \frac{4}{3}y^{\frac{3}{2}}\right)\Bigg|_{y=0}^{y=4} + 2\pi\left(\frac{5}{2}y^2 - \frac{1}{3}y^3 + \frac{2}{5}y^{\frac{5}{2}} + 6y + \frac{2}{3}y^{\frac{3}{2}}\right)\Bigg|_{y=4}^{y=9} \\
&= 2\pi\left[\frac{4}{5}(4)^{\frac{5}{2}} + \frac{4}{3}(4)^{\frac{3}{2}}\right] - 2\pi\left[\frac{4}{5}(0)^{\frac{5}{2}} + \frac{4}{3}(0)^{\frac{3}{2}}\right] + \\
&\quad 2\pi\left[\frac{5}{2}(9)^2 - \frac{1}{3}(9)^3 + \frac{2}{5}(9)^{\frac{5}{2}} + 6(9) + \frac{2}{3}(9)^{\frac{3}{2}}\right] - \\
&\quad 2\pi\left[\frac{5}{2}(4)^2 - \frac{1}{3}(4)^3 + \frac{2}{5}(4)^{\frac{5}{2}} + 6(4) + \frac{2}{3}(4)^{\frac{3}{2}}\right] \\
&= 2\pi\left(\frac{544}{15}\right) + 2\pi\left(\frac{1287}{10}\right) - 2\pi\left(\frac{304}{5}\right) \\
&= 2\pi\left(\frac{625}{6}\right) \\
&= \frac{625}{3}\pi \text{ cubic units}
\end{aligned}
$$

b. about $x = -2$

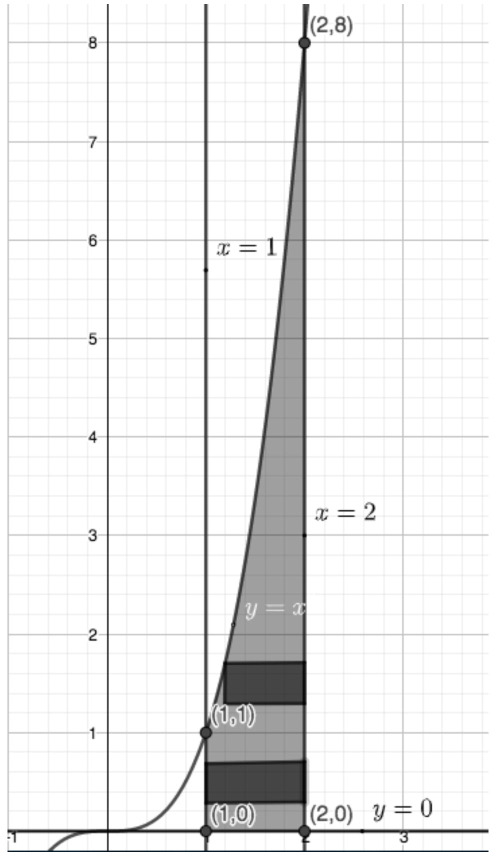


Solution: Using the cylindrical shell we obtain that the value of $r(x) = x + 2$ and $h(x) = -x^2 + x + 6$. Also,

$$
\begin{aligned}
V &= \int_{-2}^{3} 2\pi\,(x+2)\left(-x^2+x+6\right)dx \\
&= \int_{-2}^{3} 2\pi\left(-x^3-x^2+8x+12\right)dx \\
&= 2\pi\left(-\frac{1}{4}x^4-\frac{1}{3}x^3+4x^2+12x\right)\Bigg|_{x=-2}^{x=3} \\
&= 2\pi\left[-\frac{1}{4}(3)^4-\frac{1}{3}(3)^3+4(3)^2+12(3)\right]-2\pi\left[-\frac{1}{4}(-2)^4-\frac{1}{3}(-2)^3+4(-2)^2+12(-2)\right] \\
&= 2\pi\left(\frac{171}{4}\right)-2\pi\left(-\frac{28}{3}\right) \\
&= 2\pi\left(\frac{625}{12}\right) \\
&= \frac{625}{6}\pi \text{ cubic units}
\end{aligned}
$$

2. Region bounded by $y = x^3$, $x = 1$, $x = 2$ and $y = 0$

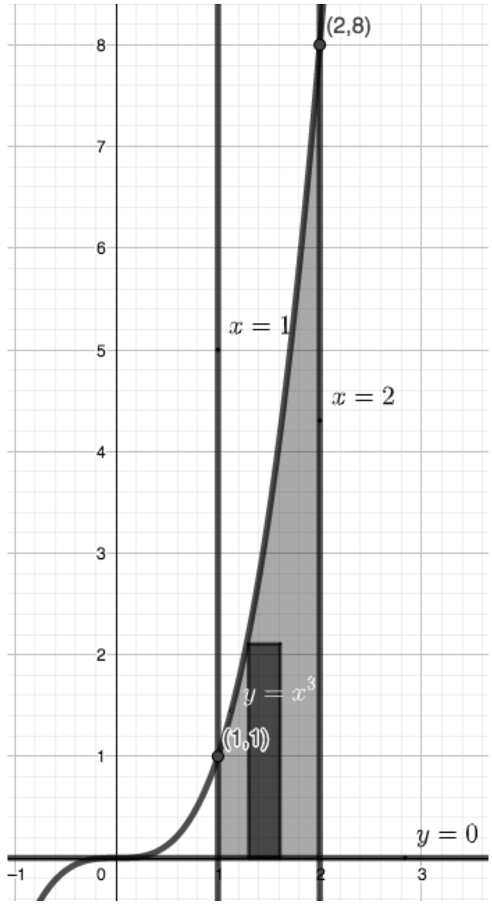
   (a) about $y = 0$



Solution: Using the cylindrical shell we obtain that the value of $r(y) = y$ and $h(y) = 1$ from $y = 0$ to $y = 1$, and $r(y) = y$ and $h(y) = 2 - y^{\frac{1}{3}}$ from $y = 1$ to $y = 8$. Also,

$$\begin{aligned}
V &= \int_0^1 2\pi\,(y)\,(1)\,dy + \int_1^8 2\pi\,(y)\left(2 - y^{\frac{1}{3}}\right)dy \\
&= \int_0^1 2\pi y dy + \int_1^8 2\pi\left(2y - y^{\frac{4}{3}}\right)dy \\
&= 2\pi\left(\frac{1}{2}y^2\right)\Bigg|_{y=0}^{y=1} + 2\pi\left(y^2 - \frac{3}{7}y^{\frac{7}{3}}\right)\Bigg|_{y=1}^{y=8} \\
&= 2\pi\left[\frac{1}{2}(1)^2\right] - 2\pi\left[\frac{1}{2}(0)^2\right] + 2\pi\left[(8)^2 - \frac{3}{7}(8)^{\frac{7}{3}}\right] - 2\pi\left[(1)^2 - \frac{3}{7}(1)^{\frac{7}{3}}\right] \\
&= 2\pi\left(\frac{1}{2}\right) - 2\pi(0) + 2\pi\left(\frac{64}{7}\right) - 2\pi\left(\frac{4}{7}\right) \\
&= 2\pi\left(\frac{127}{14}\right) \\
&= \frac{127}{7}\pi \text{ cubic units}
\end{aligned}$$

(b) about $x = 2$



Solution: Using the cylindrical shell we obtain that the value of $r(x) = 2 - x$ and $h(x) = x^3$ from $x = 1$ to $x = 2$. Also,

$$\begin{aligned}
V &= \int_1^2 2\pi(2-x)(x^3)dx \\
&= \int_1^2 2\pi(2x^3 - x^4)dx \\
&= 2\pi\left(\frac{1}{2}x^4 - \frac{1}{5}x^5\right)\Bigg|_{x=1}^{x=2} \\
&= 2\pi\left[\frac{1}{2}(2)^4 - \frac{1}{5}(2)^5\right] - 2\pi\left[\frac{1}{2}(1)^4 - \frac{1}{5}(1)^5\right] \\
&= 2\pi\left(\frac{8}{5}\right) - 2\pi\left(\frac{3}{10}\right) \\
&= 2\pi\left(\frac{13}{10}\right) \\
&= \frac{13}{5}\pi \text{ cubic units}
\end{aligned}$$

5.4 Unit Test 5

Definite Integrals and Its Application

Instruction: Write all your official answers and solutions on sheets of yellow pad paper, using only either black or blue pens.

I. **Each question is a multiple-choice question with four answer choices. Read each question and answer choice carefully and choose the ONE best answer.**

1. Express $\sum_{n=1}^{k}(n+3)^2$ in closed form.

   a. $\dfrac{2k^3+21k^2+73k}{6}$

   b. $\dfrac{2k^3+6k^2+58k}{6}$

   c. $\dfrac{2k^3+6k^2+73k}{6}$

   d. $\dfrac{2k^3+21k^2+58k}{6}$

2. Express the function of n, $\lim\limits_{n\to+\infty}\sum_{k=1}^{n}\dfrac{5k}{n^2}$ in closed form and then find the limit.

   a. $\dfrac{5(n+1)(2n+1)}{2n^2}$; 5

   b. $\dfrac{5(n+1)(2n+1)}{n^2}$; 10

   c. $\dfrac{5(n+1)}{2n}$; $\dfrac{5}{2}$

   d. $\dfrac{5(n+1)}{n}$; 5

3. If $\int_2^3 [f(x)]^2 dx = 6$, $\int_3^2 [g(x)]^2 dx = -7$ and $\int_2^3 4f(x)g(x)dx = 16$, find $\int_2^3 [f(x)+g(x)]^2 dx$.

   a. 7

   b. 17

   c. 15

   d. 21

4. Let $F(x) = \int_2^x \sqrt{3t^2+1}dt$. Find $F''(2)$

   a. 0

   b. $\sqrt{13}$

   c. $6\sqrt{13}$

   d. $\dfrac{6\sqrt{13}}{13}$

5. Let $F(x) = \int_{\sin x}^{\cos x} t^2 dt$. Find $F'(x)$

   a. $\sin x\cos x(\sin x+\cos x)$

   b. $-\sin x\cos x(\sin x+\cos x)$

   c. $-\sin x\cos x$

   d. $\sin x+\cos x$

6. Evaluate $\int_{-1}^{2} f(x)dx$, given that

$$f(x) = \begin{cases} x^3, & \text{if } x \leq 1 \\ 5x+7, & \text{if } x > 1 \end{cases}$$

a. $\frac{29}{2}$

b. $\frac{15}{4}$

c. $\frac{57}{2}$

d. $\frac{129}{4}$

7. Evaluate $\int_{-2}^{3} |x+1|dx$.

a. $\frac{15}{2}$

b. $\frac{7}{2}$

c. $\frac{17}{2}$

d. $\frac{3}{2}$

8. Evaluate $\int_{-1}^{1} \sin\left(\frac{x^2+1}{x^2}\right)dx$.

a. $2\int_{0}^{1} \sin\left(\frac{x^2+1}{x^2}\right)dx$

b. 0

c. $\int_{0}^{1} \sin\left(\frac{x^2+1}{x^2}\right)dx$

d. 2

9. Evaluate $\int_{0}^{\frac{\pi}{4}} \sin^5(2x)\cos^7(2x)dx$.

a. $\frac{1}{120}$

b. $\frac{1}{240}$

c. $\frac{\pi}{480}$

d. $\frac{\pi}{240}$

10. Evaluate $\int_{0}^{2} x^3\sqrt{4-x^2}dx$.

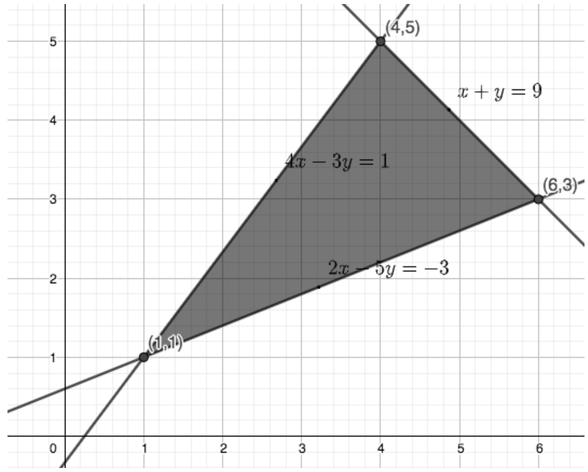
a. $\frac{1}{15}$

b. $\frac{64}{15}$

c. $\frac{2}{15}$

d. $\frac{32}{15}$

11. Find the area of the triangle whose vertices are the points (1, 1), (4, 5) and (6, 3).



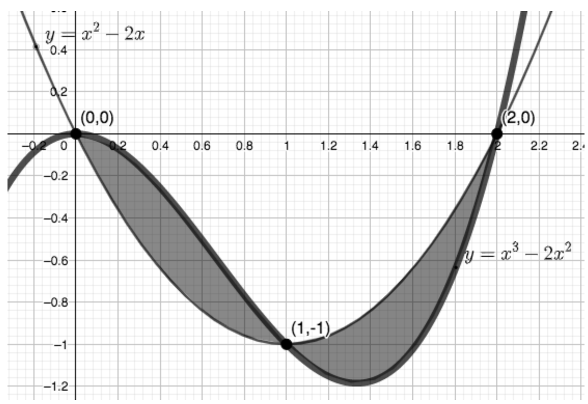
a. 2

b. 4

c. 5

d. 7

12. Find the area of the region bounded by the curves $y = x^3 - 2x^2$ and $y = x^2 - 2x$.



a. $\frac{1}{4}$

b. $\frac{1}{2}$

c. $\frac{1}{8}$

d. 1

13. Find the exact arc length of the curve over the stated interval $y = x^{\frac{2}{3}}$ from $x = 1$ to $x = 8$.

a. $\frac{80\sqrt{10} - 13\sqrt{13}}{27}$

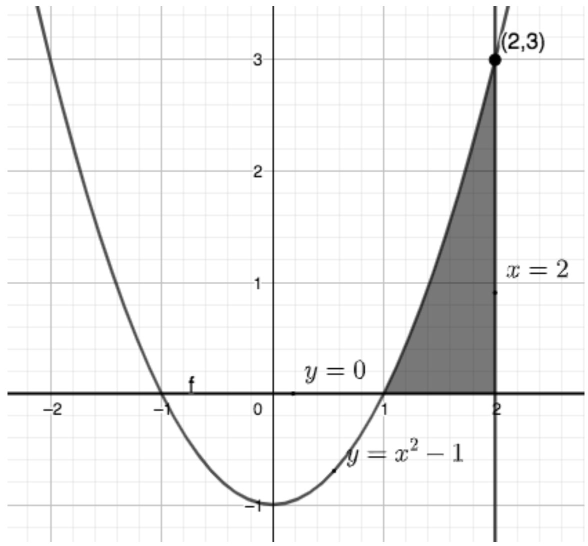
b. $\frac{80\sqrt{10} + 13\sqrt{13}}{27}$

c. $\frac{80\sqrt{13} - 13\sqrt{10}}{27}$

d. $\frac{80\sqrt{13} + 13\sqrt{10}}{27}$

For numbers 14-17:

Let R be the region bounded by $y = x^2 - 1$, $x = 2$ and $y = 0$.



14. Find the area of R.

a. $\dfrac{7}{3}$ square units
c. $\dfrac{4}{3}$ square units
b. $\dfrac{2}{3}$ square units
d. $\dfrac{1}{3}$ square units

15. Find the volume of the solid generated when R is revolved about x-axis using the Washer Method.

a. $\dfrac{9\pi}{2}$ cubic units
c. $\dfrac{4\pi}{3}$ cubic units
b. $\dfrac{5\pi}{2}$ cubic units
d. $\dfrac{5\pi}{3}$ cubic units

16. Find the volume of the solid generated when R is revolved about $x = 2$ using the Disk Method.

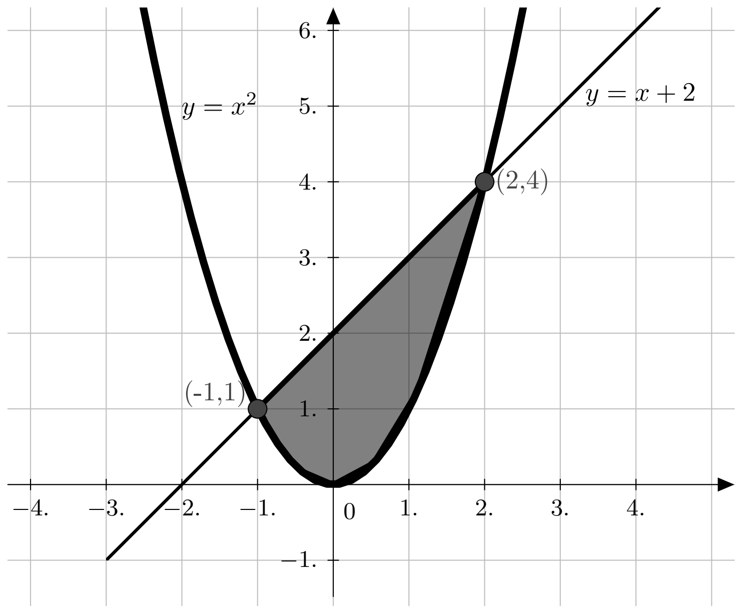
a. $\dfrac{11\pi}{6}$ cubic units
c. $\dfrac{8\pi}{3}$ cubic units
b. $\dfrac{5\pi}{6}$ cubic units
d. $\dfrac{7\pi}{3}$ cubic units

17. Set up the definite integrals required to find the perimeter of R.

a. $\displaystyle\int_1^2 \sqrt{1+x^2}dx + 4$
c. $\displaystyle\int_1^2 \sqrt{1+4x^2}dx + 4$
b. $\displaystyle\int_1^2 \sqrt{1+9x^4}dx + 4$
d. $\displaystyle\int_1^2 \sqrt{1+x^4}dx + 4$

For numbers 18 - 25:

Let M be the region bounded by the parabola $y = x^2$ and the line $y = x + 2$.



18. Set up the integral to find the area of M, using vertical rectangular strips.

a. $\int_1^4 \left[x^2 - (x+2)\right] dx$

c. $\int_{-1}^2 \left[(x+2) - x^2\right] dx$

b. $\int_{-1}^2 \left[x^2 - (x+2)\right] dx$

d. $\int_1^4 \left[(x+2) - x^2\right] dx$

19. Set up the integral to find the area of M, using horizontal rectangular strips.

a. $\int_0^1 [\sqrt{y} - (y-2)]\, dy + \int_1^4 2\sqrt{y} dy$

b. $\int_0^1 2\sqrt{y} dy + \int_1^4 [\sqrt{y} - (y-2)]\, dy$

c. $\int_0^1 2\sqrt{y} dy + \int_1^4 [(y-2) - \sqrt{y}]\, dy$

d. $\int_0^1 [(y-2) - \sqrt{y}]\, dy + \int_1^4 2\sqrt{y} dy$

20. Find the area of M.

a. $\dfrac{9}{2}$ sq. units b. 3 sq. units c. 4 sq. units d. $\dfrac{7}{2}$ sq. units

21. Which of the following represents the integral to find the volume of the solid generated by revolving M about the x-axis by using washer method?

a. $\pi \int_{-1}^2 \left[\left(x^2\right)^2 - (x+2)^2\right] dx$

c. $2\pi \int_{-1}^2 \left[(x+2)^2 - \left(x^2\right)^2\right] dx$

b. $2\pi \int_{-1}^2 \left[\left(x^2\right)^2 - (x+2)^2\right] dx$

d. $\pi \int_{-1}^2 \left[(x+2)^2 - \left(x^2\right)^2\right] dx$

22. What is the volume obtained in number 21?

a. $\dfrac{63\pi}{5}$ cu. units
c. $\dfrac{64\pi}{3}$ cu. units
b. $\dfrac{71\pi}{3}$ cu. units
d. $\dfrac{72\pi}{5}$ cu. units

23. Which of the following represents the integral to find the volume of the solid generated by revolving M about the line $x = -2$ by using washer method?

a. $\pi \displaystyle\int_0^1 4\sqrt{y}dy + \pi \int_1^4 (y + 2\sqrt{y} + 4 - y^2)dy$
b. $\pi \displaystyle\int_0^1 ydy + \pi \int_1^4 (-y^2 + 5y - 4)dy$
c. $\pi \displaystyle\int_0^1 8\sqrt{y}dy + \pi \int_1^4 (y + 4\sqrt{y} + 4 - y^2)dy$
d. $\pi \displaystyle\int_0^1 4ydy + \pi \int_1^4 (y^2 - 5y + 4)dy$

24. Which of the following represents the integral to find the volume of the solid generated by revolving M about the line $x = -2$ by using cylindrical shell?

a. $2\pi \displaystyle\int_{-1}^2 (2 - x)\left[(x + 2) - x^2\right] dx$
c. $2\pi \displaystyle\int_{-1}^2 (2 + x)\left[(x + 2) - x^2\right] dx$
b. $2\pi \displaystyle\int_{-1}^2 (x - 2)\left[(x + 2) - x^2\right] dx$
d. $2\pi \displaystyle\int_{-1}^2 x\left[(x + 2) - x^2\right] dx$

25. What is the volume obtained in number 23?

a. $\dfrac{45\pi}{4}$ cu. units
b. $\dfrac{45\pi}{2}$ cu. units
c. $\dfrac{39\pi}{4}$ cu. units
d. $\dfrac{9\pi}{2}$ cu. units

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Appendix A

Polytechnic University of the Philippines
College of Science
Department of Mathematics and Statistics

Midterm Examination in Mathematical Analysis 1

NAME: ____________________________ DATE: ____________
COURSE, YEAR AND SECTION: ______________ SCORE: ____________

Answer Sheet

	A	B	C	D		A	B	C	D		A	B	C	D
1.	○	○	○	○	11.	○	○	○	○	21.	○	○	○	○
2.	○	○	○	○	12.	○	○	○	○	22.	○	○	○	○
3.	○	○	○	○	13.	○	○	○	○	23.	○	○	○	○
4.	○	○	○	○	14.	○	○	○	○	24.	○	○	○	○
5.	○	○	○	○	15.	○	○	○	○	25.	○	○	○	○
6.	○	○	○	○	16.	○	○	○	○	26.	○	○	○	○
7.	○	○	○	○	17.	○	○	○	○	27.	○	○	○	○
8.	○	○	○	○	18.	○	○	○	○	28.	○	○	○	○
9.	○	○	○	○	19.	○	○	○	○	29.	○	○	○	○
10.	○	○	○	○	20.	○	○	○	○	30.	○	○	○	○

Instructions: On the above answer sheet provided, shade the bubble that corresponds to your answer. Write your solutions on separate yellow/bond papers and attached to this test paper for submission. Use either black- or blue-ink pen in writing.

1. If f and g are polynomial functions in x, which of the following is **NOT** always true?

   a. $\lim_{x\to a} f(x)g(x) = f(a)g(a)$

   b. $\lim_{x\to a} [f(x) + g(x)] = f(a) + g(a)$

   c. $\lim_{x\to a} [f(x) - g(x)] = f(a) - g(a)$

   d. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{f(a)}{g(a)}$

2. Evaluate $\lim_{x\to 2} \sqrt{\dfrac{2x^2 + 10x - 28}{x^2 - 2x}}$.

   a. 4 b. 2 c. 0 d. 3

3. Let $f(x) = \begin{cases} \dfrac{x^2-9}{x+3}, & \text{if } x \neq -3 \\ k, & \text{if } x = -3 \end{cases}$. Find k so that $f(-3) = \lim_{x \to -3} f(x)$.

a. -3 b. -6 c. 1 d. 0

4. Evaluate the $\lim_{x \to 4^-} \dfrac{3}{x^2 - 9x + 20}$.

a. 0 b. $+\infty$ c. $-\infty$ d. does not exist

5. Evaluate the $\lim_{x \to +\infty} \dfrac{5x^2+7}{3x^2-x}$

a. 7 b. 6 c. $\dfrac{5}{3}$ d. $\dfrac{7}{3}$

6. Evaluate: $\lim_{x \to +\infty} \dfrac{x-2}{\sqrt{2x^2+3x}}$.

a. $\dfrac{\sqrt{2}}{2}$ b. 0 c. $\dfrac{2\sqrt{3}}{3}$ d. -2

7. On which of the following intervals is $f(x) = \dfrac{1}{\sqrt{x-2}}$ continuous?

a. $[2, +\infty)$ b. $(-\infty, +\infty)$ c. $(2, +\infty)$ d. $[1, 2)$

8. Find the equation(s) of each horizontal asymptotes for $f(x) = \dfrac{1-|x|}{x}$.

a. $y = 0,\ y = 1$ b. $y = -1,\ y = 0$ c. $y = 0$ d. $y = 1,\ y = -1$

9. If $3 \leq f(x) \leq (x-3)^2 + 2$ for $x \neq 2$, then $\lim_{x \to 2} f(x)$ is:

a. 1 b. 2 c. 3 d. 0

10. Evaluate $\lim_{x \to 0} \dfrac{\cos^2 x - 1}{2x \sin x}$.

a. -1 b. $-\dfrac{1}{2}$ c. 1 d. $\dfrac{1}{2}$

11. Evaluate $\lim_{x \to 0} \dfrac{e^{4x} - e^{2x}}{3x}$.

a. $\dfrac{2}{3}$ b. 2 c. $\dfrac{1}{3}$ d. 3

12. Find the equation of the tangent line to the graph of $y = f(x)$ at $x = -3$ if $f(-3) = 2$ and $f'(-3) = 5$.

a. $2x - y + 11 = 0$ c. $5x - y + 17 = 0$
b. $5x - y - 13 = 0$ d. $2x - y - 13 = 0$

13. Let $y = x(x^2 - 4)$, find the average rate of change over the interval $x = 3$ and $x = 4$.

a. 33 b. 32 c. -32 d. -33

14. Find the instantaneous rate of change of $f(x) = 3x^2 + 5x - 2$ at $x = 3$.

a. 43 b. 40 c. 32 d. 23

15. The following functions are differentiable at $x = 0$ except.

a. $f(x) = |x|$ b. $g(x) = 3x^2 - 5$ c. $h(x) = \sin x$ d. $y = \tan x$

16. Find $\dfrac{dy}{dx}$ if $y = \dfrac{3x}{2x-1}$.

a. $\dfrac{3}{(2x-1)^2}$ b. $-\dfrac{3}{(2x-1)^2}$ c. $-\dfrac{1}{(2x-1)^2}$ d. $\dfrac{1}{(2x-1)^2}$

17. Let $y = (3x+4x^2)(2x-1)^2$, find the $\dfrac{dy}{dx}$.

a. $64x^3 - 12x^2 - 14x + 3$
b. $64x^3 - 26x^2 - 14x + 3$
c. $64x^3 - 26x^2 - 16x + 3$
d. $64x^3 - 12x^2 - 16x + 3$

18. Find $\dfrac{dy}{dx}$ if $y = \left(6x - x^2\right)^{\frac{1}{2}}$.

a. $\dfrac{\sqrt{6x-x^2}}{6-2x}$
b. $\dfrac{\sqrt{6x-x^2}}{2}$
c. $\dfrac{(3-x)\sqrt{6x-x^2}}{6x-x^2}$
d. $(6-2x)\sqrt{6x-x^2}$

19. Suppose $y = \dfrac{\sin x}{1+\cos x}$, find $\dfrac{dy}{dx}$.

a. $\dfrac{\cos^2 x + \sin^2 x}{(1+\cos x)^2}$
b. $\dfrac{\cos x + \cos^2 x + \sin^2 x}{1+\cos^2 x}$
c. $\dfrac{1}{1+\cos x}$
d. $\dfrac{1}{1+\cos^2 x}$

20. Evaluate $\dfrac{dy}{dx}$ if $y = x^2 \sin x + 3x^4 \tan x$.

a. $2x \cos x + 12x^3 \sec^2 x$
b. $x^2 \cos x + 2x \sin x + 3x^4 \sec^2 x + 12x^3 \tan x$
c. $2x \sin x + 12x^3 \tan x$
d. $x^2 \sin x + 2x \cos x + 3x^4 \tan x + 12x^3 \sec^2 x$

21. Let $g(x) = \dfrac{-x - f(x)}{f(x)}$, find $g'(1)$, if $f(1) = 4$ and $f'(1) = 2$.

a. $-\dfrac{1}{2}$ b. $\dfrac{3}{16}$ c. $-\dfrac{1}{8}$ d. $\dfrac{1}{4}$

22. Find $F'(2)$ if $f(2) = -1$, $f'(2) = 4$, $g(2) = 1$, $g'(2) = -5$ and $F(x) = x[f(x) + g(x)]$.

a. 1 b. -1 c. 2 d. -2

23. Find $\dfrac{dy}{dx}$ if $y = \dfrac{2}{(1-t)^2}$ and $t = \dfrac{1}{3}x^3 + 1$.

a. $-\dfrac{108}{x^7}$ b. $\dfrac{108}{x^4}$ c. $\dfrac{108}{x^7}$ d. $-\dfrac{108}{x^4}$

24. Evaluate $f'(x)$ of $f(x) = 5^{\ln \cos 3x}$.

a. $f'(x) = (\ln 25)(5^{\ln \cos 3x})(\cot 3x)$
b. $f'(x) = -3(\ln 5)(5^{\ln \cos 3x})(\tan 3x)$
c. $f'(x) = 3(\ln 5)(5^{\ln \cos 3x})(\cot 3x)$
d. $f'(x) = (\ln 5)(5^{\ln \cos 3x})(\tan 3x)$

25. Find the derivative of $y = \arccos(4x)$ with respect to x.

a. $-\dfrac{4}{1-4x^2}$ b. $\dfrac{4}{\sqrt{1-4x^2}}$ c. $-\dfrac{4}{\sqrt{(4x)^2-1}}$ d. $-\dfrac{4}{\sqrt{1-(4x)^2}}$

26. Find $\dfrac{dy}{dx}$, if $y = \ln(\sin x)$.

a. $\tan x$ b. $\cot x$ c. $\csc x$ d. $\sin x$

27. Find $\dfrac{dy}{dx}$ in the implicit function $x^2 + xy + y^2 = 4$.

a. $-\dfrac{2x+y}{x+2y}$ b. $-\dfrac{x+2y}{2x+y}$ c. $\dfrac{x+2y}{2x+y}$ d. $\dfrac{2x+y}{x+2y}$

28. Find $\dfrac{dy}{dx}$ in $\sin x \cos y + \cos x \sin y = 0$.

a. -1 b. 1 c. 0 d. $-\dfrac{\cos x \cos y + \sin x \sin y}{\sin x \cos y + \cos x \sin y}$

29. Let $f(x) = e^{kx}$, find $f^{(n)}(x)$.

a. 0 b. e^{kx} c. $k^n e^{kx}$ d. nke^{kx}

30. If $y = x^4 - 3x^3 + \dfrac{1}{x^2} + 8x - 10$, find $y^{(5)}$.

a. $-24x^{-5}$ c. $120x^{-7}$

b. $720x^{-5}$ d. $-720x^{-7}$

Appendix B

Polytechnic University of the Philippines

College of Science

Department of Mathematics and Statistics

Final Examination in Mathematical Analysis 1

NAME: ______________________________ DATE: ____________

COURSE, YEAR AND SECTION: ______________________________ SCORE: ____________

Answer Sheet

	A	B	C	D		A	B	C	D		A	B	C	D
1.	○	○	○	○	11.	○	○	○	○	21.	○	○	○	○
2.	○	○	○	○	12.	○	○	○	○	22.	○	○	○	○
3.	○	○	○	○	13.	○	○	○	○	23.	○	○	○	○
4.	○	○	○	○	14.	○	○	○	○	24.	○	○	○	○
5.	○	○	○	○	15.	○	○	○	○	25.	○	○	○	○
6.	○	○	○	○	16.	○	○	○	○					
7.	○	○	○	○	17.	○	○	○	○					
8.	○	○	○	○	18.	○	○	○	○					
9.	○	○	○	○	19.	○	○	○	○					
10.	○	○	○	○	20.	○	○	○	○					

Instructions: On the above answer sheet provided, shade the bubble that corresponds to your answer. Write your solutions on separate yellow/bond papers and attached to this test paper for submission. Use either black- or blue-ink pen in writing.

1. The point where the curves changes from concave upward to concave downward or vice versa is called:
   a. critical point
   b. minimum point
   c. point of inflection
   d. maximum point

2. In $f(x) = x^2 + \frac{k}{x}$, find k so that f has a critical number at $x = 3$.
   a. 6
   b. 27
   c. 216
   d. 54

3. The point of inflection of the curve $y = (x-1)^3(x-2)$ are:
   a. $(1, 0)$ and $\left(\frac{3}{2}, \frac{-1}{16}\right)$
   b. $(0, 1)$ and $\left(\frac{-1}{16}, \frac{3}{2}\right)$
   c. no point of inflection
   d. $(0, 1)$ only

4. At what interval $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x$ is increasing?
   a. $(-\infty,\ -3) \cup (2,\ \infty)$
   b. $(-\infty,\ 2) \cup (3,\ \infty)$
   c. $(2,\ 3)$
   d. $(-3,\ 2)$

5. Find the local linear approximation of $f(x) = \sqrt{x}$ at $x_0 = 1$.
   a. $y = \dfrac{1}{2\sqrt{x}}$
   b. $y = \dfrac{x+1}{2}$
   c. $y = x + 1$
   d. $y = 2x + 2$

6. Find the relative minimum point of the function $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x - \frac{1}{2}$
   a. $\left(2,\ -\dfrac{47}{6}\right)$
   b. $(3,\ -5)$
   c. $(-3,\ 13)$
   d. $\left(-2,\ \dfrac{65}{6}\right)$

7. Find the relative maximum point of the function $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x - \frac{1}{2}$
   a. $\left(2,\ -\dfrac{47}{6}\right)$
   b. $(3,\ -5)$
   c. $(-3,\ 13)$
   d. $\left(-2,\ \dfrac{65}{6}\right)$

8. Divide the number 120 into two parts such that the product of one part and the square of the other is maximum?
   a. 100 and 20
   b. 90 and 30
   c. 80 and 40
   d. 60 and 60

9. The sum of two positive numbers is 20. If the product of the square of the first and the cube of the second is a maximum, then the numbers are:
   a. 6 and 14
   b. 10 and 10
   c. 4 and 16
   d. 8 and 12

10. Find an interval $[a,\ b]$ on which $f(x) = x^4 + x^3 - x^2 + x - 2$ satisfies the hypotheses of Rolles's Theorem.
    a. $[-2,\ 1]$
    b. $[1,\ 2]$
    c. $[-2,\ -1]$
    d. $[-1,\ 2]$

11. Find values of a, b, c and d so that the function $f(x) = ax^3 + bx^2 + cx + d$ has a relative minimum at $(0,\ 0)$ and relative maximum at $(1,\ 1)$.
    a. $a = 1,\ b = 2,\ c = 3,\ d = 4$
    b. $a = -2,\ b = 3,\ c = 0,\ d = 0$
    c. $a = -1,\ b = 2,\ c = 0,\ d = 0$
    d. $a = 3,\ b = -2,\ c = 1,\ d = -2$

12. At what time the particle at the position $s(t) = t^4 - 4t + 2$ will stopped?
    a. 0
    b. 1
    c. 2
    d. 3

For items 13 and 14:

$$\text{Given: } f(x) = 3x^4 + 4x^3 - 12x^2$$

13. Find the interval on which f is decreasing.
    a. $x < -1$ or $x > \dfrac{1}{3}$
    b. $-1 < x < \dfrac{1}{3}$
    c. $-2 < x < 0$ or $x > 1$
    d. $x < -2$ or $0 < x < 1$

14. Which of the following points is not a critical point in f.

a. $(0,\ 0)$ b. $(1,\ -5)$ c. $(-1,\ -13)$ d. $(-2,\ -32)$

15. A square sheet of cardboard of side k is used to make an open box by cutting squares of equal size from the four corners and folding up the sides. What should be the size of the squares that will cut from the corners to obtain a box with the largest possible volume? a. $\frac{1}{2}k$ b. $2k$

16. Evaluate $\int \frac{\left(x^{\frac{1}{3}} - x^{\frac{1}{4}}\right)}{4x^{\frac{1}{2}}} dx$.

a. $\frac{1}{10}x^{\frac{5}{6}} - \frac{1}{9}x^{\frac{3}{4}} + C$

b. $-\frac{1}{3}x^{\frac{5}{6}} + \frac{3}{10}x^{\frac{3}{4}} + C$

c. $\frac{3}{10}x^{\frac{5}{6}} - \frac{1}{3}x^{\frac{3}{4}} + C$

d. $-\frac{3}{10}x^{\frac{5}{6}} + \frac{1}{3}x^{\frac{3}{4}} + C$

17. Evaluate $\int \frac{\sin(2x)dx}{\sqrt{\sin^2 x - 3}}$.

a. $2\sqrt{\sin^2 x - 3} + C$

b. $\frac{2}{\sqrt{\sin^2 x - 3}} + C$

c. $\frac{1}{2}\sqrt{\sin^2 x - 3} + C$

d. $\sqrt{\sin^2 x - 3} + C$

18. If $du = \frac{z+3}{z-2}dz$, then u is equal to

a. $-2z + 3\ln|z-2| + C$

b. $z + \ln|z-2| + C$

c. $z - \frac{1}{z^2} + 3\ln|z-2| + C$

d. $z + 5\ln|z-2| + C$

19. Evaluate $\int x\sqrt{2x+1}dx$.

a. $\frac{1}{15}(2x+1)^{\frac{3}{2}}(3x+1) + C$

b. $\frac{1}{15}(2x+1)^{\frac{3}{2}}(3x-1) + C$

c. $\frac{1}{15}(2x+1)^{\frac{5}{2}}(3x+1) + C$

d. $\frac{1}{15}(2x+1)^{\frac{5}{2}}(3x-1) + C$

20. Evaluate $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$.

a. $\frac{\pi}{6}$ b. $\frac{\pi}{3}$ c. $\frac{\pi}{2}$ d. $\frac{\pi}{4}$

21. Evaluate $\int_0^2 3x\sqrt{4-x^2}dx$.

a. $-\frac{1}{2}$ b. 8 c. 12 d. 24

22. The $\int_0^1 x(x^2+x)^2 dx$ is equal to

a. $\frac{49}{50}$ b. $\frac{49}{60}$ c. $\frac{49}{70}$ d. $\frac{49}{80}$

23. Evaluate $\int_{1-\pi}^{1+\pi} \sec^2\left(\frac{1}{4}x - \frac{1}{4}\right) dx$.

a. 8 b. 4 c. 16 d. 0

24. Find the area between the curves $y^2 = x$ and $y^2 = 2 - x$.

a. $\frac{4}{3}$ square units b. $\frac{5}{3}$ square units c. $\frac{8}{3}$ square units d. $\frac{10}{3}$ square units

25. Find the volume generated by revolving about the x-axis the area bounded by the curves $y = 2x + 1$, $y = 0$, $x = 1$, and $x = 2$.

a. $\frac{128\pi}{3}$ cubic units b. $\frac{125\pi}{3}$ cubic units c. $\frac{98\pi}{3}$ cubic units d. $\frac{49\pi}{3}$ cubic units