CHAPTER 2
METHODS OF INTEGRATION

2.1 integrals involving powers of sine and cosine

$$\int \sin^m v \cos^n v dx$$

Case1: When m is a positive odd integer and n is any number.

$$\int \sin^m v \cos^n v dx = \int \left(\sin^{m-1} v \cos^n v\right) \sin v$$

Use the trigonometric identity $\boxed{\sin^2 v = 1 - \cos^2 v}$

Then it can be reduced to $\int (\text{sum of powers of } \cos v) \sin v dx$

and apply the Power Formula.

Case 2: When m is any number and n is a positive odd integer.

$$\int \sin^m v \cos^n v dx = \int \left(\sin^{m} v \cos^{n-1} v\right) \cos v$$

Use the trigonometric identity $\boxed{\cos^2 v = 1 - \sin^2 v}$

Then it can be reduced to $\int (\text{sum of powers of } \sin v) \cos v dx$

and apply the Power Formula.

<u>Note:</u> When m & n are both positive odd integers, use either Case 1 (if m <n) or Case 2.

Case 3: When m and n are both even integers (either both positive or one positive and one zero)

$$\int \sin^m v \cos^n v dx = \int \left(\sin^2 v\right)^{\frac{m}{2}} \left(\cos^2 v\right)^{\frac{n}{2}} dx$$

Use one or both of the following trigonometric identities

$$\boxed{\sin^2 v = \frac{1 - \cos 2v}{2}} \quad , \quad \boxed{\cos^2 v = \frac{1 + \cos 2v}{2}}$$

To reduce the given integral to an integrable form.

Activity / Exercise No. 8
I. INTEGRATION INVOLVING POWERS OF SINE & COSINE

I. Evaluate the following:

1. $\int \sin^3 x \cos^2 x \, dx$
2. $\int \frac{\sin^3 x}{\cos x} dx$
3. $\int e^x \sin^3(e^x)\cos^5(e^x)\, dx$
4. $\int e^{-2x} \sin^5(e^{-2x})\cos^4(e^{-2x})\, dx$
5. $\int \frac{\cot^3 x}{\csc x} dx$
6. $\int \frac{\cot^5 x}{\csc^3 x} dx$
7. $\int \sqrt[3]{\sin^7 x} \cos^5 x \, dx$
8. $\int e^{-x} \sin^2(e^{-x})\cos^3(e^{-x})\, dx$
9. $\int \sin^2 2x \cos^2 2x \, dx$
10. $\int \sin^4 3x \cos^2 3x \, dx$
11. $\int \sin^4 x \, dx$
12. $\int \cos^4 x \, dx$
13. $\int \left(\sqrt{\sin x} + \cos x\right)^2 dx$
14. $\int \sin^7 x \cos^3 x \, dx$

INTEGRALS INVOLVING POWERS OF TANGENT (COTANGENT) AND SECANT (COSECANT)

A. $\int \tan^m v \sec^n v dx$

Case1: When m is any number and n is a positive even integer greater than 2.

$$\int \tan^m v \sec^n v dx = \int \left(\tan^m v \sec^{n-2} v\right)\sec^2 v dx$$

Use the trigonometric identity $\boxed{\sec^2 v = 1 + \tan^2 v}$

Then it can be reduced to $\int (\text{sum of powers of } \tan v)\sec^2 v dx$ and apply the Power Formula.

Case 2: When m is a positive odd integer and n is any number.

$$\int \tan^m v \sec^n v dx = \int \left(\tan^{m-1} v \sec^{n-1} v\right)\sec v \tan v dx$$

Use the trigonometric identity $\boxed{\tan^2 v = \sec^2 v - 1}$

Then it can be reduced to $\int (\text{sum of powers of } \sec v)\sec v \tan v dx$ and apply the Power Formula.

Case 3: When m is a positive integer and n is zero.

$$\int \tan^m v \, dx = \int \left(\tan^{m-2} v\right)\tan^2 v dx$$

Use the trigonometric identity $\boxed{\tan^2 v = \sec^2 v - 1}$

To reduce the given integral to an integrable form.

B. $\int \cot^m v \csc^n v dx$

The technique involved is similar to that of evaluating the integral of $\int \tan^m v \sec^n v dx$.

Hence, the identity $\boxed{\csc^2 v = 1 + \cot^2 v}$ or $\boxed{\cot^2 v = \csc^2 v - 1}$ is used to reduce the original expression into an integrable form.

Activity / Exercise No. 9
INTEGRALS INVOLVING POWERS OF TANGENT (COTANGENT) AND SECANT (COSECANT)

I. Evaluate the following:

1. $\int \sec^4 3x \tan^3 3x \, dx$
2. $\int \tan^4 \frac{x}{2} \sec^4 \frac{x}{2} dx$
3. $\int \left(\frac{\sec x}{\tan x}\right)^4 dx$
4. $\int \frac{\csc^4 x}{\cot^2 x} dx$
5. $\int \csc^6 6x \, dx$
6. $\int \sqrt{\cot 3x} \csc^4 3x dx$
7. $\int \tan x \sqrt{\sec x} \, dx$
8. $\int \frac{1}{x} \tan^3(\ln x) \sec^6(\ln x) \, dx$
9. $\int \tan^3 2x \sec^5 2x \, dx$
10. $\int \frac{\tan^3 x}{\sec x} dx$
11. $\int \cot^5 4x \, dx$
12. $\int \tan^7 7x \, dx$
13. $\int (\tan^4 3x - \sec^4 3x) \, dx$
14. $\int \frac{dx}{\tan^4 6x}$

INTEGRATION BY PARTS

$$\int u dv = uv - \int v du$$

Activity / Exercise No. 10

I. Evaluate the following:

1. $\int x (4x+1)^{-5} dx$
2. $\int x(2x-1)^7 \, dx$
3. $\int x \sin x \, dx$
4. $\int x \cos x \, dx$
5. $\int \ln x \, dx$
6. $\int x \ln x \, dx$
7. $\int x^3 \sqrt{x^2+4} \, dx$
8. $\int x^5 (x^3-1)^{\frac{3}{2}} dx$
9. $\int e^x \cos x dx$
10. $\int e^{-x} \sin x dx$
11. $\int \sec^3 x \, dx$
12. $\int \csc^3 x \, dx$

INTEGRATION BY ALGEBRAIC SUBSTITUTION

Case1: Integrands containing $\boxed{\sqrt[n]{ax+b}}$

To eliminate the radical $\sqrt[n]{ax+b}$, let $z=\sqrt[n]{ax+b} \quad \Rightarrow \quad z^n = ax+b$

$$\text{and} \quad nz^{n-1}dz = adx \quad \Rightarrow \quad dx = \frac{nz^{n-1}dz}{a}$$

Case 2: Integrands Containing Multiple Radicals

If an integrand contains several radicals, it can be simplified by the substitution

$$\boxed{x = z^n}$$

where n is the least common multiple (LCM) of the indices of the radicals or the least common denominator (LCD) of the denominator of the fractional exponents.

Case 3: Reciprocal Substitution

If the integrand contains an expression of the form $\boxed{\dfrac{P(x)}{x\sqrt{x^2+a^2}}}$, can be solved usually by substituting $\boxed{x=\dfrac{1}{z}}$.

I. Activity / Exercise No. 11

I. Evaluate the following:

1. $\int x\sqrt[3]{3x-1}\,dx$
2. $\int (3x-4)\sqrt{3x+2}$
3. $\int \dfrac{dy}{\sqrt{y+2}-1}$
4. $\int \dfrac{xdx}{(2x+3)^{\frac{4}{3}}}$
5. $\int \dfrac{dx}{(x+2)^{\frac{3}{4}}-(x+2)^{\frac{1}{2}}}$
6. $\int \dfrac{dx}{(2x-1)^{\frac{1}{2}}-(2x-1)^{\frac{3}{4}}}$
7. $\int \sqrt{1+\sqrt{x}}\,dx$
8. $\int \dfrac{dx}{\sqrt{x}\left(1+\sqrt{x}\right)}$
9. $\int \dfrac{\sqrt[3]{x}}{1+\sqrt[3]{x^2}}dx$
10. $\int \dfrac{dx}{\sqrt[3]{x}+\sqrt[4]{x}}$
11. $\int \dfrac{\left(x-x^3\right)^{1/3}dx}{x^4}$
12. $\int \dfrac{\left(1-x^2\right)^{1/2}dx}{x^4}$
13. $\int \dfrac{dx}{x\sqrt{x^2+2x-1}}$
14. $\int \dfrac{dx}{x\sqrt{4x-x^2}}$

INTEGRATION BY TRIGONOMETRIC SUBSTITUTION

Case1: Integrands containing $\boxed{a^2 - u^2}$, use the substitution $\boxed{u = a\sin\theta}$.

Case 2: Integrands containing $\boxed{u^2 + a^2}$, use the substitution $\boxed{u = a\tan\theta}$.

Case 3: Integrands containing $\boxed{u^2 - a^2}$, use the substitution $\boxed{u = a\sec\theta}$.

Activity / Exercise No. 12

I. Prove the following:

1. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\frac{u}{a} + c$

2. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\frac{u}{a} + c$

3. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{u}{a} + c$

4. $\displaystyle\int \sqrt{a^2 - u^2}\,du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\arcsin\frac{u}{a} + c$

II. Evaluate the following:

1. $\displaystyle\int \frac{dx}{(9 - x^2)^{\frac{3}{2}}}$

2. $\displaystyle\int \frac{x^2dx}{\sqrt{4 - x^2}}$

3. $\displaystyle\int \frac{\sqrt{9 - 4x^2}dx}{x^2}$

4. $\displaystyle\int \frac{\sqrt{9 - x^2}dx}{x^2}$

5. $\displaystyle\int \frac{x^2dx}{\sqrt{x^2 + 4}}$

6. $\displaystyle\int \frac{dx}{(x^2 + 4)^2}$

7. $\displaystyle\int \frac{dx}{x^4\sqrt{x^2 + 16}}$

8. $\displaystyle\int \frac{dx}{x\sqrt{9x^2 + 4}}$

9. $\displaystyle\int \frac{(x^2 - 16)^{3/2}}{x^3}dx$

10. $\displaystyle\int \frac{x^3}{\sqrt{16x^2 - 25}}dx$

11. $\displaystyle\int \frac{\sqrt{5x^2 - 9}}{x}dx$

12. $\displaystyle\int \frac{\sqrt{4x^2 - 25}}{8x^3}dx$

INTEGRATION BY PARTIAL FRACTIONS

Let $\frac{N(x)}{D(x)}$ where $D(x) \neq 0$ be given rational fraction. If the degree of the numerator $N(x)$ is less than the degree of the denominator $D(x)$, then $\frac{N(x)}{D(x)}$ is called **proper rational fraction** or **partial fractions**.

Case 1: Distinct Linear Factors

Every distinct linear factor of the form $ax+b$ in the denominator $D(x)$, there corresponds a partial fraction of the form $\boxed{\frac{A}{ax+b}}$ where A is constant to be determined.

Case 2: Repeated Linear Factors

Every repeated linear factor of the form $(ax+b)^n$ in the denominator $D(x)$, there corresponds the sum of n partial fractions of the form $\boxed{\frac{A}{ax+b}+\frac{B}{(ax+b)^2}+\cdots+\frac{Z}{(ax+b)^n}}$ where A, B, …, Z are constants to be determined.

Case 3: Distinct Quadratic Factors

Every irreducible distinct quadratic factor ax^2+bx+c in the denominator $D(x)$, there corresponds a partial fraction of the form $\boxed{\frac{A(2ax+b)+B}{ax^2+bx+c}}$ where A and B are constants to be determined and $2ax+b=\frac{d}{dx}(ax^2+bx+c)$.

Case 4: Repeated Quadratic Factors

Every irreducible repeated quadratic factor $(ax^2+bx+c)^n$ in the denominator $D(x)$, there corresponds the sum of n partial fractions of the form $\boxed{\frac{A(2ax+b)+B}{ax^2+bx+c}+\frac{C(2ax+b)+D}{(ax^2+bx+c)^2}+\ldots+\frac{Y(2ax+b)+Z}{(ax^2+bx+c)^n}}$ where A, B, …, Z are constants to be determined and $2ax+b=\frac{d}{dx}(ax^2+bx+c)$.

Activity / Exercise No. 13

I. Evaluate the following:

1. $\int \frac{x+2}{x^2-1}dx$

2. $\int \frac{3x-4}{x^2-2x}dx$

3. $\int \frac{x^2+2}{x^3-4x^2+x+6}dx$

4. $\int \frac{x^2-2x-5}{12x^3-28x^2-7x+5}dx$

Hint: one of the factor is 2x + 1

5. $\int \frac{(5y-4)dy}{(y^3+4y^2)}$

6. $\int \frac{x^2}{x^3-x^2-x+1}dx$

7. $\int \frac{x^5}{x^2-4x+4}dx$

8. $\int \frac{2x^4+2x^3-5x^2+3x-1}{x^4-x^3}dx$

9. $\int \frac{6x^2+3x-2}{x^3+2x^2+2x+1}dx$

10. $\int \frac{2x+1}{(3x-1)(x^2+2x+2)}dx$

11. $\int \frac{x^2+6x+4}{x^4+5x^2+4}dx$

12. $\int \frac{4x+9}{x^2-4x+20}dx$

13. $\int \frac{x^5+2x^3-3x}{(x^2+1)^3}dx$

14. $\int \frac{x^3+4x^2-4x-1}{(x^2+1)^2}dx$

15. $\int \frac{3x^4+x^3+20x^2+3x+31}{(x+1)(x^2+4)^2}dx$

16. $\int \frac{3x^4-10x^3+7x^2-3x-48}{(x+5)(x^2-x+3)^2}$