$$\int \sqrt{x^2+1}\,dx$$

**Substitution ($u = a\tan\theta$ with $a = 1$, $u = x$):** $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$, $\sqrt{x^2+1} = \sqrt{\tan^2\theta+1} = \sec\theta$.

$$\int \sqrt{x^2+1}\,dx = \int \sec\theta \cdot \sec^2\theta\,d\theta = \int \sec^3\theta\,d\theta$$

**Let $I = \int \sec^3\theta\,d\theta$. Integration by parts:** $I = \displaystyle\int \sec\theta \cdot \sec^2\theta\,d\theta$, take $f = \sec\theta$, $g' = \sec^2\theta \Rightarrow f' = \sec\theta\tan\theta$, $g = \tan\theta$.

$$I = \sec\theta\tan\theta - \int (\sec\theta\tan\theta)\tan\theta\,d\theta$$

**Identity:** $\tan^2\theta = \sec^2\theta - 1$.

$$I = \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)\,d\theta = \sec\theta\tan\theta - \int \sec^3\theta\,d\theta + \int \sec\theta\,d\theta$$

$$2I = \sec\theta\tan\theta + \int \sec\theta\,d\theta$$

**Formula:** $\displaystyle\int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$.

$$I = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

**Back-substitution:** $\sec\theta = \sqrt{1+x^2}$, $\tan\theta = x$.

$$\boxed{\int \sqrt{x^2+1}\,dx = \frac{1}{2}\,x\sqrt{x^2+1} + \frac{1}{2}\,\ln\left|x + \sqrt{x^2+1}\right| + C}$$