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THE FORCE DENSITY METHOD FOR FORM FINDING AND COMPUTATION OF GENERAL NETWORKS

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A new method for network analysis, the "force density method", is presented. This concept is based upon the force-length ratios or force densities which are defined for each branch of the net structure. It is shown that the force densities are very suitable parameters for the description of the equilibrium state of any general network: the associated node coordinates of the structure are obtained by solving *one* single system of *linear* equations. A Gaussian transformation of the topological branch-node matrix yields the equation matrix.

A method based on force densities renders a simple linear "analytical form finding" possible. If the free choice of the force densities is restricted by further conditions, the linear method is extended to a nonlinear one. A damped least squares approach is then introduced. In this version the method may be applied to the detailed computation of networks.

Moreover, an interesting cross-connection between geometrical minimum way nets and prestressed unloaded structures is derived using the force densities.

1. Introduction

The need for a new method in the field of network computation becomes obvious if we consider the following practical and also theoretical problem: Given a pin-joint network consisting of cables or bars, we wish to determine all possible shapes of equilibrium which the network can attain if we suspend it in a given frame or if we stretch it between given fixed points. Initially we do not constrain the possible network forms by any geometrical restrictions. We only impose one important requirement: The shape shall be an equilibrium state. This means that the sum of all forces is zero at each node. We further suppose that the branches (cables or bars) are connected at the nodes as joints. With respect to the branches we introduce no further assumptions; they may be elastic or inflexible. If we consider all possible net shapes, we obviously include all special shapes, especially all orthogonal or all equidistant nets or shapes with positive or negative Gaussian curvature.

1.1. Connection with experimental methods and practical applications

The variety of possible shapes applicable to the design of large spanned structures demands new ideas in the area of analytical computation of static structures. Up to now one evaluated the forces and deformations belonging to a shape that was known in advance. This knowledge of the shape of net structures however is not generally available except in very simple cases. Therefore the search for a suitable form plays an important role, because the shape influences to a great extent the deformations and the forces in the unloaded and loaded state.

In the field of experimental form finding we have mainly the work of the Institut für Leichte Flächentragwerke (directed by F. Otto) at the University of Stuttgart. A survey and discussion of these problems is provided by E. Bubner [1]. These methods have been utilized in the construction of cable net structures (e.g. the Olympic stadium in Munich).

Further experimental work has been done by B. Oleiko [2] on the solidification of fluid minimal surfaces and by L. Gruendig and J. Hennicke [3] on a combined experimental and analytical method for the form finding of grid shells.

The experimental methods help in visualizing the problem, but they have two severe disadvantages: the number of variants is very limited and the models must be measured. A *simple* analytical method for the shape search ("building models in the computer") admits a large number of shape variants which are already in digital form suitable for further computation or for a display output with an interactive system. Fig. 1 shows central-perspective views of a small prestressed net which was used as a demonstration example in this report. A small experimental program for an IBM 2250 display allows one to observe the structure from any desired viewpoint.

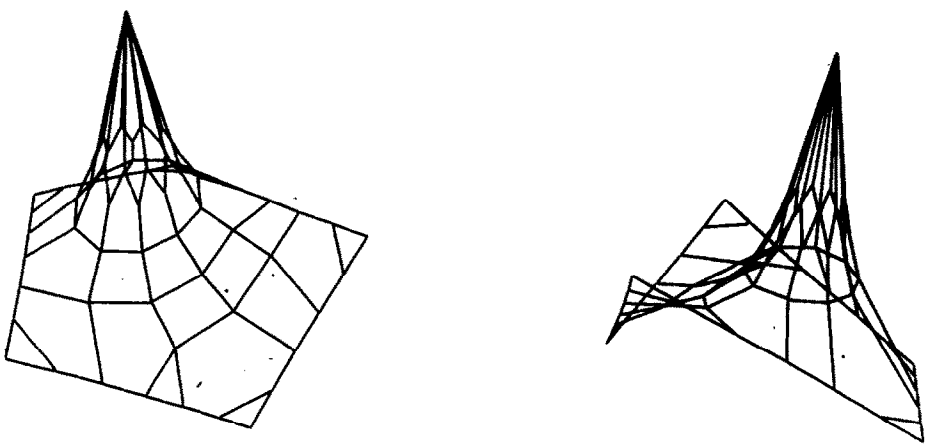


Fig. 1. Views of a prestressed net (IBM 2250 Display).

1.2. Existing methods for analytical form finding

To overcome the difficulties of obtaining models and their measurements, E. Haug [4] and similarly T. Angelopoulos [5] proposed starting the nonlinear computation with an initially plane net. By altering the positions of the fixed points in many incremental steps, the net is stretched from the plane to the desired spatial situation. This approach is based upon known and established methods for nonlinear network computation (see E. Haug [6], J.H. Argyris and D. Scharpf [7], and K. Linkwitz and H.-J. Schek [8]). Due to the nonlinearity the number of iterations can be very high (10 to 20 or more) for each shape variant.

An exception in the nonlinearity is reported by A. Siev and J. Eidelman [9, 10]: They have shown that the heights of a prestressed unloaded net which has orthogonal quadratic meshes in the horizontal projection and which is anchored in a frame can be obtained by one system of

linear equations if the horizontal projections of the cable forces are given. However, because of the supposed properties, these resulting shapes are very restricted.

1.3. Characteristics of the new method

The main advantage of the new method lies in the fact that any state of equilibrium of a general net structure can be obtained by the solution of *one* system of *linear* equations. This system is constructed using the force-length ratios or "force densities" in the branches as network description parameters ("degrees of freedom"). In other words we prescribe one single quantity for each branch – namely the force density – and obtain as a unique result the appropriate state of equilibrium by solving one system of linear equations. This equilibrium structure has the prescribed force-length ratio in each branch. Because of this simple rule a lot of very different shapes can be computed and displayed in a short time. This is shown in sections 2 and 5. Thus the suggestion of C.H. Thornton and C. Birnstiel [11] that for the computation of general initial shapes only trial and error methods are possible can no longer be accepted.

Of course no other conditions are fulfilled in the structure, e.g. rectangular or equidistant meshes. Some examples show that such further conditions are not necessary for a first impression or a global view of the shape.

Clearly, in a detailed analysis it will be necessary to compute net structures under additional geometrical conditions or force constraints. For this purpose we could either use the above mentioned conventional methods [6, 7, 8] using the linearly found equilibrium state as initial values or – and this is the second main advantage of the force density method – we could extend our linear approach to the nonlinear force density method shown in sections 3 and 5. The nonlinear computations are also started with the above linearly found shape. But – in contrast to the methods [6, 7, 8] – the number of nonlinear equations here is identical with the number of additional conditions and is independent of the number of net nodes. Therefore this approach may be simpler and less expensive than the conventional solutions.

Another more theoretical advantage of the force densities as net description parameters lies in the fact that we can reveal an interesting cross-connection: geometrical minimum way nets can easily be interpreted as prestressed nets and vice versa.

1.4. Comment on the notation

A vector is interpreted as a one-column-matrix and written in lower case, and a general matrix is written as a capital letter. The same symbols are used for the components but they have an additional index i, j, or k. The m-dimensional vector $\boldsymbol{a}$ – called the m-vector $\boldsymbol{a}$ – has therefore a_j as j-th component. Further, we often need the diagonal matrix $\boldsymbol{A}$ belonging to any vector $\boldsymbol{a}$: $\boldsymbol{A}$ is simply defined to have $\boldsymbol{a}$ as diagonal, for example

$$\boldsymbol{a} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

2. Force density method for form finding

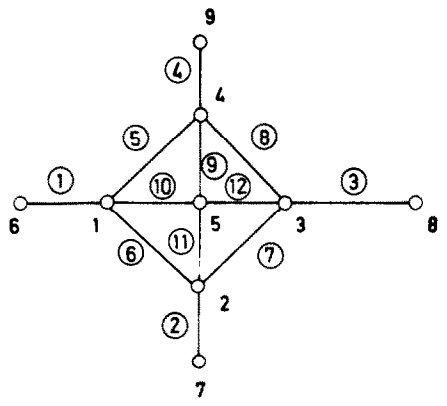
The shape of a net structure depends strongly on the given rule or list determining which nodes have to be connected by branches. For this topological description the usual branch-node matrix (J.H. Argyris [12] and S.J. Fenves and F.H. Branin [13]) is applied. In the force density method this matrix $\boldsymbol{C}$ has a special importance: The Gaussian transformation $\boldsymbol{C}^t\boldsymbol{Q}\boldsymbol{C}$ with the diagonal matrix $\boldsymbol{Q}$ containing the force densities is identical with the equation matrix of the linear system for the node coordinates.

2.1. Branch-node matrix

We start with the graph of a network and number all nodes from 1 to n_s, all branches from 1 to m in any order. For a later application it is advisable that the nodes which are declared afterwards to be fixed points (with number n_f) are taken at the end of the sequence. All other nodes are "free" or "variable" points (with number n). So we have $n_s = n + n_f$. With the aid of the graph we have for each branch j the two matched node numbers $i(j)$ and $k(j)$. The usual branch-node matrix $\boldsymbol{C}_s$ is defined by

$$c(j,i) = \begin{cases} +1 & \text{for } i(j) = 1 \\ -1 & \text{for } k(j) = 1 \\ 0 & \text{in the other cases .} \end{cases}$$

It has therefore m rows and n_s columns. Referring to the classification into free and fixed nodes we define the matrices $\boldsymbol{C}$ and $\boldsymbol{C}_f$ (see fig. 2).



	1	2	3	4	5	6	7	8	9
1	1	.	.	.	.	−1	.	.	.
2	.	1	.	.	.	.	−1	.	.
3	.	.	1	.	.	.	.	−1	.
4	.	.	.	1	.	.	.	.	−1
5	1	.	.	−1	.	.	.	.	.
6	1	−1	.	.	.	.	.	.	.
7	.	1	−1	.	.	.	.	.	.
8	.	.	1	−1	.	.	.	.	.
9	.	.	.	1	−1	.	.	.	.
10	1	.	.	.	−1	.	.	.	.
11	.	1	.	.	−1	.	.	.	.
12	.	.	1	.	−1	.	.	.	.
			$\boldsymbol{C}$					$\boldsymbol{C}_f$	

Fig. 2. Graph and branch-node matrix $\boldsymbol{C}_s = [\boldsymbol{C} \;\; \boldsymbol{C}_f]$.

2.2. Force density method

We interpret the nodes as points P_i with coordinates (x_i, y_i, z_i), $i = 1, \ldots, n$, and the fixed nodes as given fixed points with coordinates (x_{fi}, y_{fi}, z_{fi}), $i = 1, \ldots, n_f$. The coordinates of all points constitute the n-vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ and the n_f-vectors $\boldsymbol{x}_f, \boldsymbol{y}_f, \boldsymbol{z}_f$. The lengths l_j of the branches (physically

either cables or bars) and the branch forces s_j form the m-vectors $\boldsymbol{l}$ and $\boldsymbol{s}$. The load vectors are $\boldsymbol{p}_x, \boldsymbol{p}_y, \boldsymbol{p}_z$. So the i-th node has the load components p_{xi}, p_{yi}, p_{zi} in the x-, y- and z-direction.

The coordinate differences $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ of the connected points can be obtained using the branch-node matrix $\boldsymbol{C}$ and $\boldsymbol{C}_f$

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{C}\boldsymbol{x} + \boldsymbol{C}_f\boldsymbol{x}_f , \\ \boldsymbol{v} &= \boldsymbol{C}\boldsymbol{y} + \boldsymbol{C}_f\boldsymbol{y}_f , \\ \boldsymbol{w} &= \boldsymbol{C}\boldsymbol{z} + \boldsymbol{C}_f\boldsymbol{z}_f . \end{aligned} \tag{1}$$

With the diagonal matrices $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}$ and $\boldsymbol{L}$ belonging to $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ and $\boldsymbol{l}$, it can easily be shown that the sum of the forces in each node is zero and therefore the network is in the state of equilibrium if the following equilibrium equations are implemented

$$\begin{aligned} \boldsymbol{C}^t\boldsymbol{U}\boldsymbol{L}^{-1}\boldsymbol{s} &= \boldsymbol{p}_x , \\ \boldsymbol{C}^t\boldsymbol{V}\boldsymbol{L}^{-1}\boldsymbol{s} &= \boldsymbol{p}_y , \\ \boldsymbol{C}^t\boldsymbol{W}\boldsymbol{L}^{-1}\boldsymbol{s} &= \boldsymbol{p}_z . \end{aligned} \tag{2}$$

Here we have used the obvious representation for the Jacobian matrices

$$\frac{\partial \boldsymbol{l}}{\partial \boldsymbol{x}} = \boldsymbol{C}^t\boldsymbol{U}\boldsymbol{L}^{-1}, \qquad \frac{\partial \boldsymbol{l}}{\partial \boldsymbol{y}} = \boldsymbol{C}^t\boldsymbol{V}\boldsymbol{L}^{-1}, \qquad \frac{\partial \boldsymbol{l}}{\partial \boldsymbol{z}} = \boldsymbol{C}^t\boldsymbol{W}\boldsymbol{L}^{-1} . \tag{3}$$

We search for equations equivalent to (2) and set

$$\boldsymbol{q} = \boldsymbol{L}^{-1}\boldsymbol{s} . \tag{4}$$

The components q_j of the m-vector $\boldsymbol{q}$ are obviously the force-length ratios or force densities of the branches. With (4) we have now as equations of equilibrium

$$\begin{aligned} \boldsymbol{C}^t\boldsymbol{U}\boldsymbol{q} &= \boldsymbol{p}_x , \\ \boldsymbol{C}^t\boldsymbol{V}\boldsymbol{q} &= \boldsymbol{p}_y , \\ \boldsymbol{C}^t\boldsymbol{W}\boldsymbol{q} &= \boldsymbol{p}_z . \end{aligned} \tag{5}$$

With the identities

$$\boldsymbol{U}\boldsymbol{q} = \boldsymbol{Q}\boldsymbol{u} , \qquad \boldsymbol{V}\boldsymbol{q} = \boldsymbol{Q}\boldsymbol{v} , \qquad \boldsymbol{W}\boldsymbol{q} = \boldsymbol{Q}\boldsymbol{w} , \tag{6}$$

($\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}$, and $\boldsymbol{Q}$ are the diagonal matrices belonging to $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$, and $\boldsymbol{q}$) and using u, v, w in (1) we get instead of (5)

$$C^tQCx + C^tQC_fx_f = p_x ,$$
$$C^tQCy + C^tQC_fy_f = p_y ,$$
$$C^tQCz + C^tQC_fz_f = p_z . \tag{7}$$

For simplicity we set $D = C^tQC$ and $D_f = C^tQC_f$, so we have the equations of equilibrium in the form

$$Dx = p_x - D_fx_f ,$$
$$Dy = p_y - D_fy_f ,$$
$$Dz = p_z - D_fz_f . \tag{8}$$

which are very well suited for the form finding of networks. We arrive at the following important conclusions:

1. Equations (8) are *linear* for the determination of the free node coordinates. The (n, n) equation matrix D is the generalized Gaussian transformation of C. The elements q_j of the diagonal matrix Q are the force densities. The matrix D is positive definite if we have a prestressed network ($q_j > 0$) without isolated points.
2. With a given load and a given position of fixed points we get for each set of prescribed force densities exactly one equilibrium state with the shape

$$x = D^{-1}(p_x - D_fx_f) ,$$
$$y = D^{-1}(p_y - D_fy_f) ,$$
$$z = D^{-1}(p_z - D_fz_f) , \tag{9}$$

and the forces

$$s = Lq \tag{10}$$

with L evaluated from x, y, z by eq. (9).

3. Therefore, with the given interconnection (fixed branch-node matrix), given load, and given fixed points we have as many equilibrium shapes as vectors q. The variety of vectors q and the variety of equilibrium shapes are identical. Thus the force densities are suitable for net description parameters.

The above formulas can be used directly for practical applications, and we summarize them in fig. 3. Fig. 4 shows the simple structure of the matrix D belonging to the graph of fig. 2.

3. Force density method for the computation of networks under additional constraints

Up to now we have the totality of equilibrium states. In the following the simple linear form finding method will be extended when there are further conditions restricting the free choice of the force densities. The additional conditions are generally nonlinear. Thus the extended force

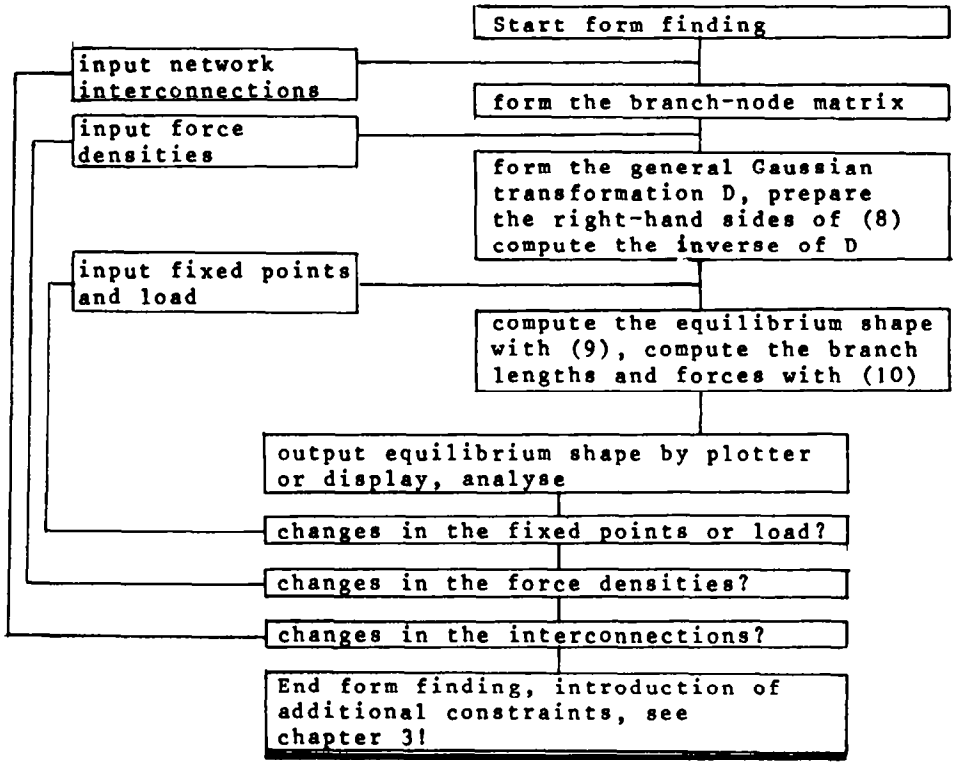


Fig. 3. Diagram of the form finding.

$$D = \begin{bmatrix} q_1+q_5+q_6+q_{10} & -q_6 & \cdot & -q_5 & -q_{10} \\ -q_6 & q_2+q_6+q_7+q_{11} & -q_7 & \cdot & -q_{11} \\ \cdot & -q_7 & q_3+q_7+q_8+q_{12} & -q_8 & -q_{12} \\ -q_5 & \cdot & -q_8 & q_4+q_5+q_8+q_9 & -q_9 \\ -q_{10} & -q_{11} & -q_{12} & -q_9 & q_9+q_{10}+q_{11}+q_{12} \end{bmatrix}$$

Fig. 4. Structure of the matrix D.

density method will also be nonlinear. But since we only consider additional constraints when we perform initial form finding, we have this shape to start the iterations to solve the resulting nonlinear problem.

3.1. General consideration

We suppose that we have r additional conditions of the general form

$$g_1(x, y, z, q) = 0 ,$$
$$g_2(x, y, z, q) = 0 ,$$
$$\vdots$$
$$g_r(x, y, z, q) = 0 , \tag{11a}$$

or, with the r-vector $g = (g_j)$,

$$g(x, y, z, q) = 0 , \tag{11b}$$

where we have assumed that the additional conditions can be easily formulated as functions of the coordinates and force densities. But since the coordinates in (9) are represented as functions of the force densities, the additional conditions are dependent on the force densities alone:

$$g^*(q) = g(x(q), y(q), z(q), q) = 0 . \tag{12}$$

These additional equations – generally nonlinear – will not be fulfilled at $q^{(0)}$, where $q^{(0)}$ are the force densities belonging to the shape which was found during the initial linear form finding.

We seek therefore $q^{(1)} = q^{(0)} + \Delta q$ with $g^*(q^{(1)}) = 0$. Because of the nonlinearity we construct the following iterative method:

Instead of $g^*(q^{(1)}) = 0$ we ask for Δq fulfilling the linearized conditions

$$g^*(q^{(0)}) + \frac{\partial g^*(q^{(0)})}{\partial q} \Delta q = 0 . \tag{13}$$

For simplicity we set

$$G^t = \frac{\partial g^*(q^{(0)})}{\partial q} , \tag{14}$$

$$r = -g^*(q^{(0)}) , \tag{15}$$

whereby we have the linear conditions for Δq in the form

$$G^t \Delta q = r . \tag{16}$$

Since the number r of the conditions is less than the number of branches in many practical applications ($r < m$), the equations (16) have $m - r$ linearly independent solutions. From this totality of solutions we choose a single solution by investigating an additional principle.

An obvious and numerically simple one is the least squares principle

$$\Delta q^t \Delta q \to \min . \tag{17}$$

Quite similar is the following (P is a diagonal weighting matrix)

$$\Delta q^t \Delta q + a^t P a \rightarrow \min \tag{18}$$

if we start with the "damped" conditions instead of (16):

$$G^t \Delta q = r + a \, . \tag{19}$$

Practical experience has shown that the best damped strategy is based upon the "modified damped" conditions

$$G^t \Delta q = A r \, , \tag{20}$$

where A is the diagonal matrix belonging to the vector of damping factors $a = (a_j)$. The associated minimum principle is

$$\Delta q^t \Delta q + (e - a)^t P (e - a) \rightarrow \min \tag{21}$$

with $e = \{1, 1, ..., 1\}$. This approach is especially useful if there are large changes in the force densities or in the shape.

In any case we have a minimum problem with additional equations. The Lagrange-factors k will be obtained by the solution of the (r, r) system

$$Tk = r \tag{22}$$

with

$$T = \begin{cases} G^t G & \text{in case (16, 17)} \, , \\ G^t G + P^{-1} & \text{in case (18, 19)} \, , \\ G^t G + P^{-1} R^2 & \text{in case (20, 21)} \, . \end{cases} \tag{23}$$

One recognizes that the Gaussian transformation of G is simply modified by the addition of the diagonal matrices P^{-1} or $P^{-1}R^2$ in both damped approaches. With

$$k = T^{-1} r \tag{24}$$

the solution for Δq is

$$\Delta q = G k \, . \tag{25}$$

For the next iteration we set

$$q^{(0)} := q^{(0)} + \Delta q$$

and start again with (13) until $g^*(q^{(0)})$ is zero within a given tolerance.

3.2. Representation of the Jacobian matrix G^t

For practical computation one needs a simple representation of the Jacobian matrix $\boldsymbol{G}^t$. In many applications the conditions $\boldsymbol{g}^*$ are not explicitly given but are rather in the form (11) as functions of $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ and $\boldsymbol{q}$. The important matrix $\boldsymbol{G}^t$ can then be obtained by the "chain rule"

$$\boldsymbol{G}^t = \frac{\partial \boldsymbol{g}^*}{\partial \boldsymbol{q}} = \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}} \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{q}} + \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{y}} \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{q}} + \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{z}} \frac{\partial \boldsymbol{z}}{\partial \boldsymbol{q}} + \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}} . \tag{26}$$

The real problem is a representation of the Jacobian matrices of the coordinates referring to the force densities, namely $\partial \boldsymbol{x}/\partial \boldsymbol{q}$, $\partial \boldsymbol{y}/\partial \boldsymbol{q}$ and $\partial \boldsymbol{z}/\partial \boldsymbol{q}$. The first idea to use eqs. (9) for this purpose would not be reasonable because we have difficulties in deriving the matrix $(\boldsymbol{C}^t\boldsymbol{Q}\boldsymbol{C})^{-1}$ referring to $\boldsymbol{q}$. This bottleneck may be bypassed by using implicit derivatives: We look at the equations of equilibrium (5) for the x-coordinates

$$\boldsymbol{C}^t\boldsymbol{U}\boldsymbol{q} = \boldsymbol{P}_x .$$

Any changes $\mathrm{d}\boldsymbol{q}$ and $\mathrm{d}\boldsymbol{x}$ should leave untouched the state of equilibrium. Therefore

$$\mathrm{d}(\boldsymbol{C}^t\boldsymbol{U}\boldsymbol{q}) = \frac{\partial(\boldsymbol{C}^t\boldsymbol{U}\boldsymbol{q})}{\partial \boldsymbol{q}} \mathrm{d}\boldsymbol{q} + \frac{\partial(\boldsymbol{C}^t\boldsymbol{U}\boldsymbol{q})}{\partial \boldsymbol{x}} \mathrm{d}\boldsymbol{x} = \boldsymbol{0} .$$

Since

$$\frac{\partial(\boldsymbol{C}^t\boldsymbol{U}\boldsymbol{q})}{\partial \boldsymbol{q}} = \boldsymbol{C}^t\boldsymbol{U} ,$$

$$\frac{\partial(\boldsymbol{C}^t\boldsymbol{U}\boldsymbol{q})}{\partial \boldsymbol{x}} = \frac{\partial(\boldsymbol{C}^t\boldsymbol{Q}\boldsymbol{u})}{\partial \boldsymbol{x}} = \boldsymbol{C}^t\boldsymbol{Q} \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} = \boldsymbol{C}^t\boldsymbol{Q}\boldsymbol{C} ,$$

the desired matrix is

$$\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{q}} = -(\boldsymbol{C}^t\boldsymbol{Q}\boldsymbol{C})^{-1}\boldsymbol{C}^t\boldsymbol{U} = -\boldsymbol{D}^{-1}\boldsymbol{C}^t\boldsymbol{U} . \tag{27}$$

Analogously the matrices $\partial \boldsymbol{y}/\partial \boldsymbol{q}$ and $\partial \boldsymbol{z}/\partial \boldsymbol{q}$ are obtained. The result is

$$\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{q}} = -\boldsymbol{D}^{-1}\boldsymbol{C}^t\boldsymbol{U} ,$$

$$\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{q}} = -\boldsymbol{D}^{-1}\boldsymbol{C}^t\boldsymbol{V} ,$$

$$\frac{\partial \boldsymbol{z}}{\partial \boldsymbol{q}} = -\boldsymbol{D}^{-1}\boldsymbol{C}^t\boldsymbol{W} . \tag{28}$$

Now we have all preparations for some examples of additional conditions which may be of practical interest.

3.3. Node distance conditions

We suppose that the lengths of the first r branches (bars or cables) have prescribed values l_{vj}. Conditions of this kind may arise, e.g. if one uses prefabricated bars with extremely high stiffness. The additional conditions in this case are

$$\begin{aligned} g_1(x, y, z) &= l_1(x, y, z) - l_{v1} = 0 , \\ g_2(x, y, z) &= l_2(x, y, z) - l_{v2} = 0 , \\ &\vdots \\ g_r(x, y, z) &= l_r(x, y, z) - l_{vr} = 0 . \end{aligned} \tag{29}$$

To simplify the notation we introduce the vector $\bar{l}$ to contain the first r components of the m-vector l, and similarly we write $\bar{l}_v$, $\bar{L}$, $\bar{U}$, $\bar{C}$ etc. So we have

$$g = \bar{l} - \bar{l}_v = 0 . \tag{29}$$

For an application of (26) we need (using (29) and (3))

$$\begin{aligned} \frac{\partial g}{\partial x} &= \frac{\partial \bar{l}}{\partial x} = \bar{C}^t \bar{U} \bar{L}^{-1} , \\ \frac{\partial g}{\partial y} &= \frac{\partial \bar{l}}{\partial y} = \bar{C}^t \bar{V} \bar{L}^{-1} , \\ \frac{\partial g}{\partial z} &= \frac{\partial \bar{l}}{\partial z} = \bar{C}^t \bar{W} \bar{L}^{-1} , \end{aligned} \tag{30}$$

With (28) we have finally the Jacobian matrix G_d belonging to distance conditions

$$G_d^t = -\bar{L}^{-1}(\bar{U}\bar{C}D^{-1}C^tU + \bar{V}\bar{C}D^{-1}C^tV + \bar{W}\bar{C}D^{-1}C^tW) . \tag{31}$$

One can see that the matrix $CD^{-1}C^t$ is simply scaled by the diagonals $\bar{U}$ and U or $\bar{V}$ and V etc.

3.4. Force conditions

It may be desirable or necessary to influence the forces in the branches by prescribing a given force-value for some selected branches. For that we assume the values s_{vj} for the first r forces s_j (without loss of generality) and therefore the conditions

$$g(x, y, z, q) = \bar{L}\bar{q} - \bar{s}_v = \bar{Q}\bar{l} - \bar{s}_v = 0 . \tag{32}$$

Again we need

$$\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}} = \bar{\boldsymbol{Q}} \frac{\partial \bar{\boldsymbol{l}}}{\partial \boldsymbol{x}},$$

$$\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{y}} = \bar{\boldsymbol{Q}} \frac{\partial \bar{\boldsymbol{l}}}{\partial \boldsymbol{y}},$$

$$\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{z}} = \bar{\boldsymbol{Q}} \frac{\partial \bar{\boldsymbol{l}}}{\partial \boldsymbol{z}},$$

$$\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}} = \bar{\boldsymbol{L}}. \tag{33}$$

With (33), (30) and (28) we obtain the Jacobian $\boldsymbol{G}_f^t$ for the force conditions

$$\boldsymbol{G}_f^t = \bar{\boldsymbol{L}} - \bar{\boldsymbol{Q}}\bar{\boldsymbol{L}}^{-1}(\bar{\boldsymbol{U}}\bar{\boldsymbol{C}}\boldsymbol{D}^{-1}\boldsymbol{C}^t\boldsymbol{U} + \bar{\boldsymbol{V}}\bar{\boldsymbol{C}}\boldsymbol{D}^{-1}\boldsymbol{C}^t\boldsymbol{V} + \bar{\boldsymbol{W}}\bar{\boldsymbol{C}}\boldsymbol{D}^{-1}\boldsymbol{C}^t\boldsymbol{W}). \tag{34}$$

3.5. Length conditions

Whereas the distance conditions refer to the lengths of the cables or bars in the strained state (which are therefore identical with the spatial distances between the nodes), we look now at the branches in the unstrained state. The difference between the strained and the unstrained length may be ignored if the branches have a high stiffness.

Now we prescribe the values l_{uvj} for the *unstrained* lengths l_{uj} in the first r branches. For the derivation of the associated conditions we need a relation between the branch-force and the strained and unstrained length. This function may be generally nonlinear.

Especially if we assume the linear Hooke's law (h_j is the stiffness of the j-th branch)

$$l_{uj} = \frac{h_j}{h_j + s_j} l_j,$$

the Jacobian matrix $\boldsymbol{G}_u^t$ for unstrained length conditions can easily be derived. The simple result is

$$\boldsymbol{G}_u^t = -\bar{\boldsymbol{L}}_u^2\bar{\boldsymbol{H}}^{-1} - \bar{\boldsymbol{L}}_u^2\bar{\boldsymbol{L}}^3(\bar{\boldsymbol{U}}\bar{\boldsymbol{C}}\boldsymbol{D}^{-1}\boldsymbol{C}^t\boldsymbol{U} + \bar{\boldsymbol{V}}\bar{\boldsymbol{C}}\boldsymbol{D}^{-1}\boldsymbol{C}^t\boldsymbol{V} + \bar{\boldsymbol{W}}\bar{\boldsymbol{C}}\boldsymbol{D}^{-1}\boldsymbol{C}^t\boldsymbol{W}). \tag{35}$$

It is obvious from (31) and (35) that $\boldsymbol{G}_u^t \to \boldsymbol{G}_d^t$ if $\bar{b} \to \infty$ and $\bar{l}_u \to \bar{l}$.

3.6. Mixed conditions

In the general case we should take into account that more than one type of additional condition may occur. For example, a prestressed net usually is an equidistant net within the region bordered by main cables and should have uniform forces.

In such cases we divide the totality of the branches into different classes referring to the type of the additional condition and obtain the whole matrix $\boldsymbol{G}$ simply by writing the different class matrices sequentially

$$G = [G_d | G_f | G_u | \ldots] .$$

In this way one can take advantage of the very similar structure of the matrices G_d, G_f and G_u.

4. Geometrical minimum properties of nets

With a few short remarks it is possible to reveal an interesting cross-connection between prestressed nets and geometrical minimum nets. We forget statical nets and forces for the moment and ask for a spatial distribution of points such that the squared sum of way lengths between connected points is minimal. The connection by the branch-node matrix $C_s = [C | C_f]$ and at least 4 fixed points in general position are given.

With the aid of the coordinate differences u, v, w we have the squared sum of way lengths l and the minimum condition

$$l^t l = u^t u + v^t v + w^t w \to \min , \tag{36}$$

where

$$\begin{aligned} u &= Cx + C_f x_f , \\ v &= Cy + C_f y_f , \\ w &= Cz + C_f z_f . \end{aligned} \tag{3}$$

The necessary conditions for minimum are obtained by taking the derivative of $l^t l$ with respect to x, y and z.

$$\begin{aligned} \frac{\partial (l^t l)}{\partial x} &= \frac{\partial (u^t u)}{\partial x} = 2u^t \frac{\partial u}{\partial x} , \\ \frac{\partial (l^t l)}{\partial y} &= \frac{\partial (v^t v)}{\partial y} = 2v^t \frac{\partial v}{\partial y} , \\ \frac{\partial (l^t l)}{\partial z} &= \frac{\partial (w^t w)}{\partial z} = 2w^t \frac{\partial w}{\partial z} . \end{aligned} \tag{37}$$

With (3) we recognize

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial w}{\partial z} = C . \tag{38}$$

With eqs. (3), (37) and (38) together we obtain the linear equations for the coordinates x, y, z in the desired minimal spatial distribution

$$\begin{aligned} C^t C x &= - C^t C_f x_f , \\ C^t C y &= - C^t C_f y_f , \\ C^t C z &= - C^t C_f z_f . \end{aligned} \tag{39}$$

We compare eq. (39) with (7) and recognize that the result is identical if we have the force densities = 1 in (7) for all branches ($\boldsymbol{Q}$ will now be the unit matrix) and if the load is zero. It is surprising that we obtain a prestressed unloaded equilibrium net by such a simple and only geometrical minimum condition!

More generally, we obtain *each* unloaded equilibrium net if we minimize the *weighted* squared sum of way lengths: With the diagonal weighting matrix $\boldsymbol{P}$ we introduce instead of (36)

$$\boldsymbol{l}^t\boldsymbol{P}\boldsymbol{l} = \boldsymbol{u}^t\boldsymbol{P}\boldsymbol{u} + \boldsymbol{v}^t\boldsymbol{P}\boldsymbol{v} + \boldsymbol{w}^t\boldsymbol{P}\boldsymbol{w} \to \min . \tag{40}$$

Now the necessary conditions for minimum are

$$\boldsymbol{u}^t\boldsymbol{P}\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} = \boldsymbol{v}^t\boldsymbol{P}\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{y}} = \boldsymbol{w}^t\boldsymbol{P}\frac{\partial \boldsymbol{w}}{\partial \boldsymbol{z}} = \boldsymbol{0} .$$

With (3) and (38) we get

$$\begin{aligned}(\boldsymbol{C}^t\boldsymbol{P}\boldsymbol{C})\boldsymbol{x} &= -(\boldsymbol{C}^t\boldsymbol{P}\boldsymbol{C}_f)\boldsymbol{x}_f ,\\ (\boldsymbol{C}^t\boldsymbol{P}\boldsymbol{C})\boldsymbol{y} &= -(\boldsymbol{C}^t\boldsymbol{P}\boldsymbol{C}_f)\boldsymbol{y}_f ,\\ (\boldsymbol{C}^t\boldsymbol{P}\boldsymbol{C})\boldsymbol{z} &= -(\boldsymbol{C}^t\boldsymbol{P}\boldsymbol{C}_f)\boldsymbol{z}_f .\end{aligned} \tag{41}$$

With (7) we again recognize that eqs. (41) are identical with the general linear equations for unloaded equilibrium structures since $\boldsymbol{P}$ and $\boldsymbol{Q}$ are both diagonal matrices. Therefore we can interpret the force densities as weighting factors of the geometrical minimum problem. Summarizing, we have the following

THEOREM 1. Each equilibrium state of an unloaded network structure with force densities q_j is identical with the net, whose sum of squared way lengths weighted by q_j is minimal.

The following theorem is a consequence of the general theorem 1. We do not minimize the *squared* sum but rather the *sum* of way lengths alone. Nets with a minimal sum of lengths are usually called "minimum way nets".

THEOREM 2. A minimum way net is prestressed net with a constant force throughout all branches (cables).

This fact was already pointed out by E. Haug [6]. Here is a simple mathematical proof:
The general minimum condition (40) will become the special minimum way condition

$$\sum_{j=1}^{m} l_j \to \min .$$

if we set for the weighting factors – interpreted as force densities –

$$p_j = q_j = \text{const}/l_j .$$

That means constant forces over all branches (s_j = const). Since $l_j > 0$, we have also $q_j > 0$ and therefore a prestressed network. This completes the proof.

5. Examples of applications

As shown in the diagram of fig. 3 we begin with the graph of the network (Fig. 5) giving the interconnections and therefore the branch-node matrix.

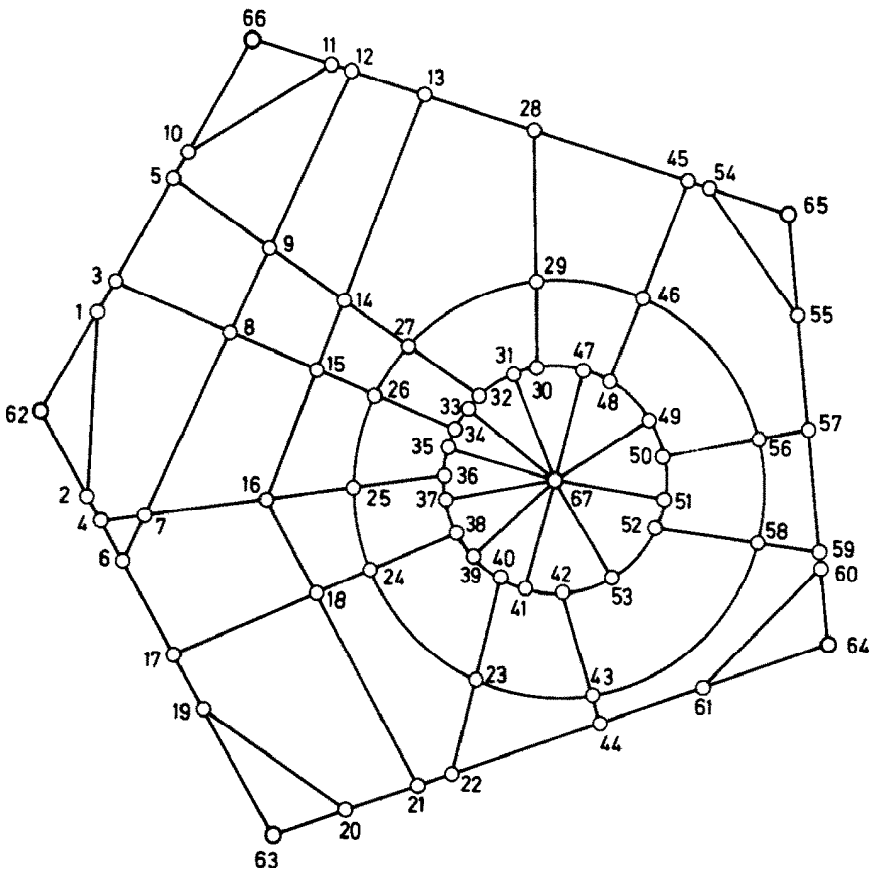


Fig. 5. Graph of the first example.

For the border branches (index pairs (62,2), (2,4), ..., (1,62) we prescribe the force densities 100. Thereby the borders will be nearly straight lines. For the radial branches to the top (point 67) we assign a value of 0.5 for the force densities and for all others the value 1. The force density 0.5 instead of 1 will lengthen the radial branches in relation to the others.

The nodes with the numbers 62, 63, ..., 67 will be interpreted as fixed points and we input their coordinates. Further, a small vertical load in each node is supposed. This describes the full input for a computer program. The computed equilibrium shape is shown in fig. 6.1. By changing the x- and y-coordinates of the fixed point 67 we get the shapes of figs. 6.2 and 6.3, where one

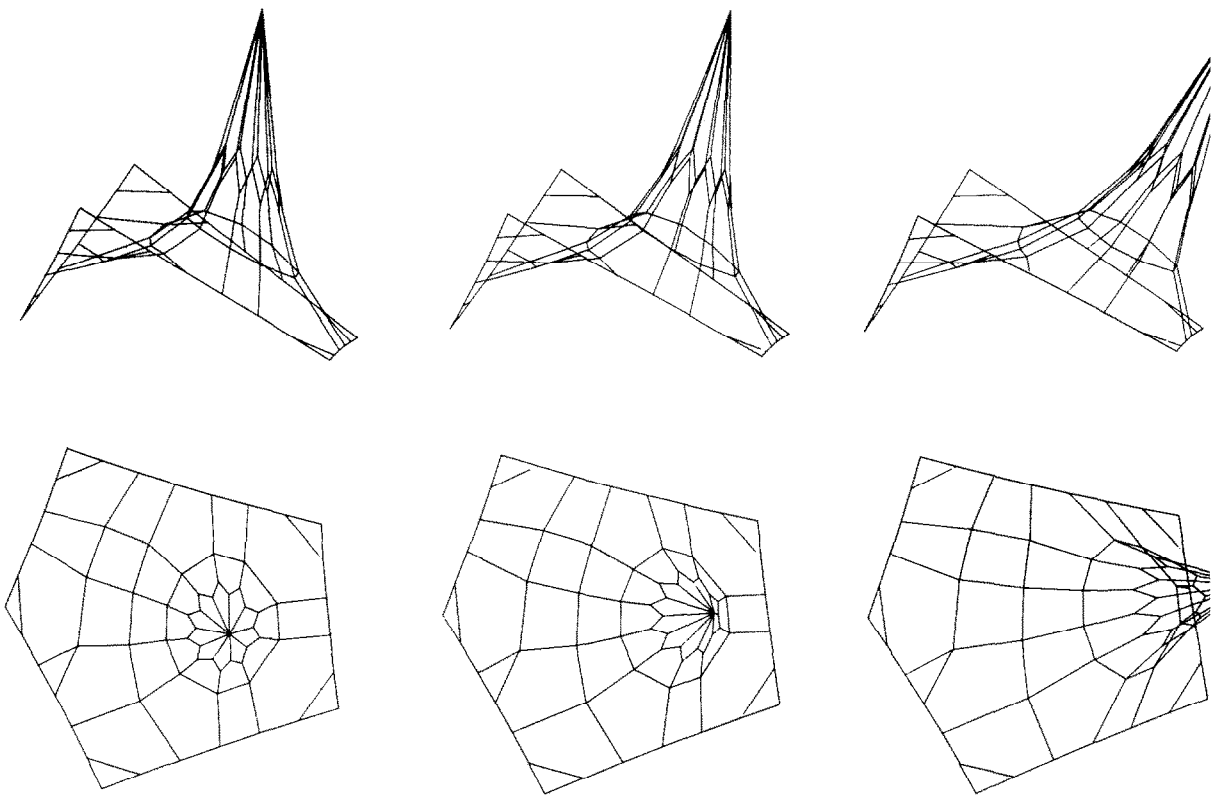


Fig. 6. Different fixed points.

matrix multiplication is necessary for each of these two shapes.

To get an impression of how global force density changes do influence the network form, we input the values 1.0, 0.7, and 0.3 respectively for the force densities in the cables at the top instead of the initial value 0.5. The corresponding shapes are figs. 7.1, 7.2 and 7.3, where the changes in the radial branches can be observed clearly.

The same effect will be obtained if we input the values 10, 5, and 1 instead of 100 for the border branches (main cables): the borders lengthen and the main cable diminish (see figs. 8.1, 8.2 and 8.3). Fig. 8.3 has the force density 1 over all branches. It is therefore at the same time a net whose distribution of nodes has the minimal sum of squared way lengths.

Coming to the next sequence, we give an example for additional constraints. We require that all branches which are not connected with the border have lengths with prescribed uniform values. For example, all radial branches to the top should have the length of 6 meters. Taking the shape of fig. 6.1 as initial shape for the iterations that are now necessary, the lengths differ from 4 to 7 m. Looking at the intermediate shape 9.1 we can see that the requested constarints cannot be fulfilled by a prestressed network because of the prescribed lengths in the ring branches. We shorten them and get the shape of fig. 9.2 after 5 automatic iterations. The shape 9.3 is another result of prescribing uniforn lengths.

The following example was influenced by a small model which was manufactured at the Institut für Leichte Flächetragwerke (see [3]) for the form finding of grid shells. It consists of 112 identical

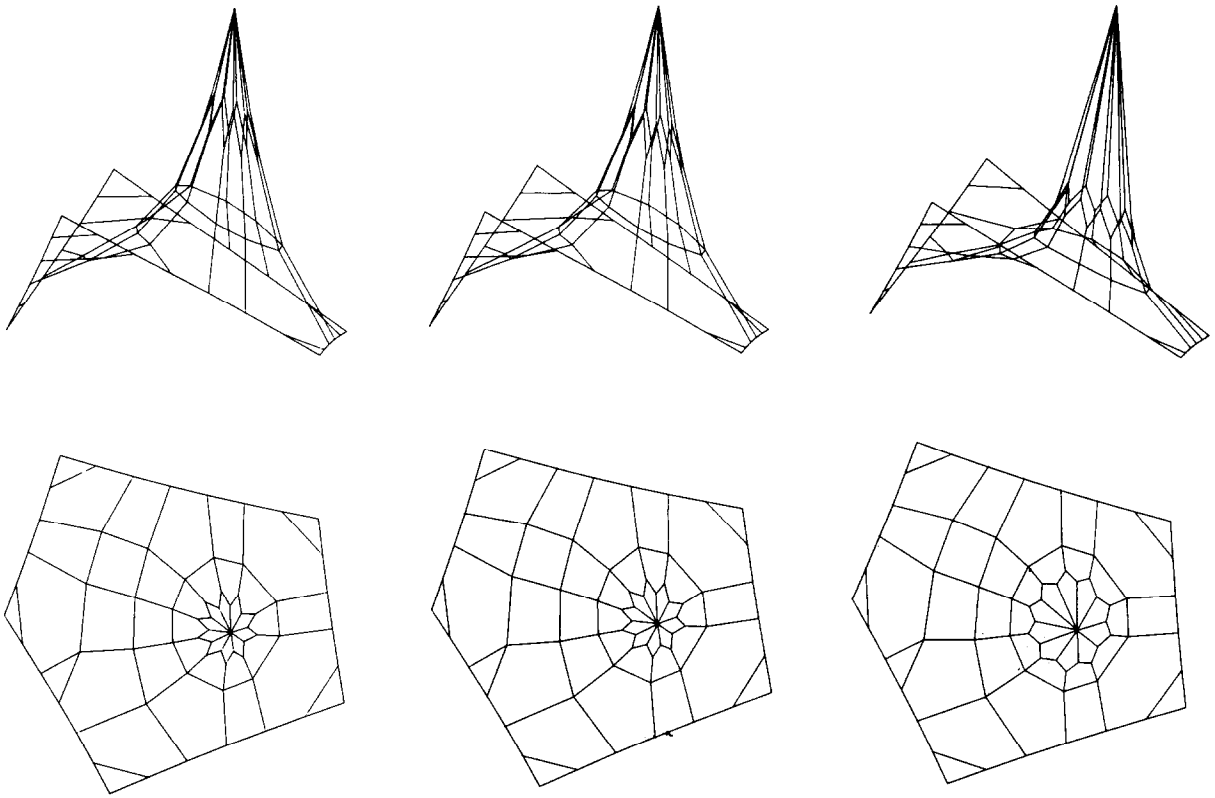


Fig. 7. Global changes in the force densities and lengthening of the radial branches.

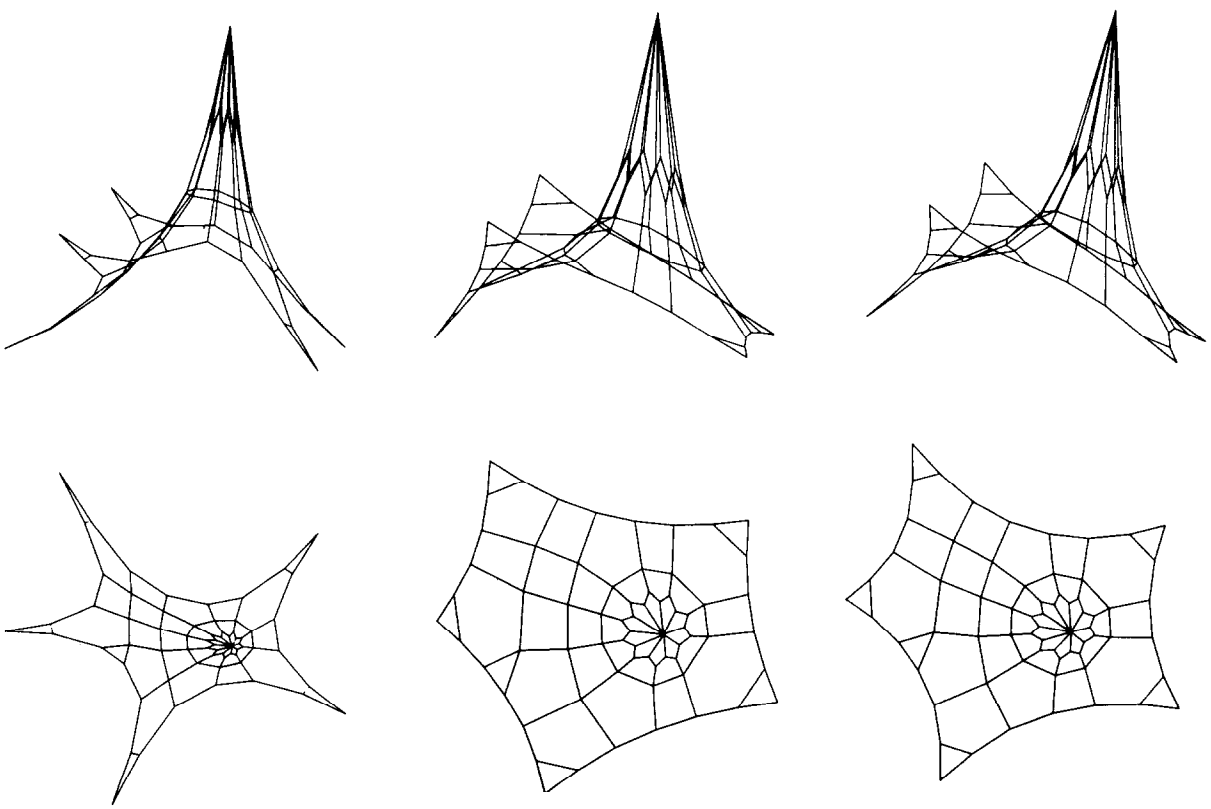


Fig. 8. Global changes in the force densities of the main cables.

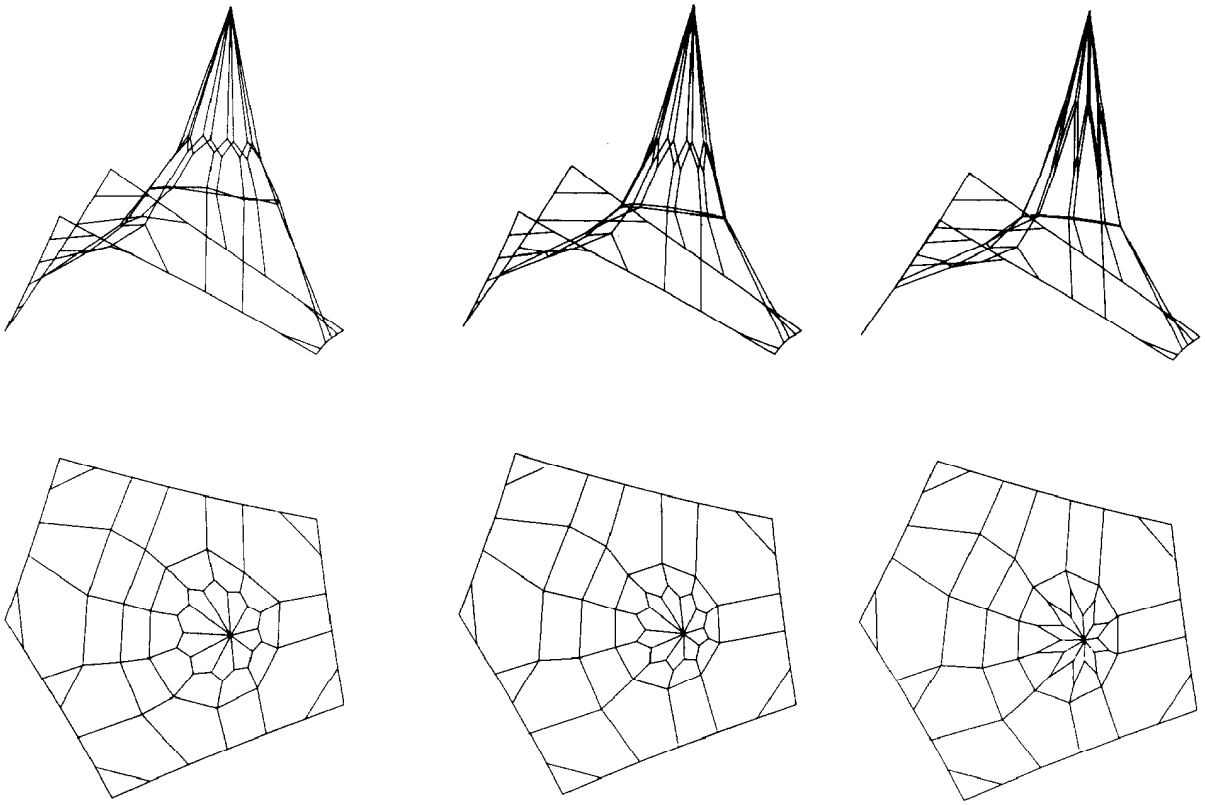


Fig. 9. Additional distance conditions.

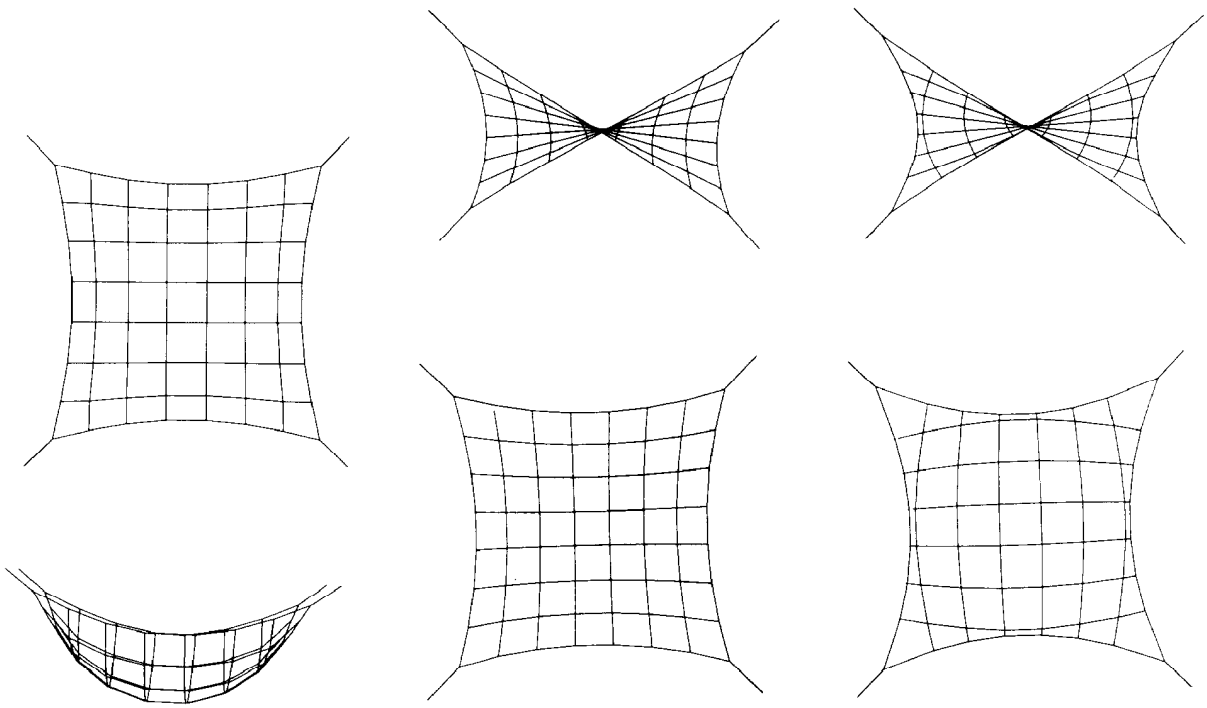


Fig. 10. Hanging and prestressed net with the same structure.

metallic bars which are linked at the nodes by little metallic rings. The resulting network hangs as a "chain in two dimensions" because of its weight. The coordinates of the node points were measured, thus providing input for the computation of the equilibrium shape with the aid of a modified conventional program [3].

Independent of this somewhat time-consuming procedure, the author has computed the same network with the aid of the force density method. Only the coordinates of the anchor points, the bar length, and the bar weight were needed. The result is shown in fig. 10.1. Naturally it is identical with the above result.

The next shapes show that by changing the coordinates of the fixed points, the hanging net can be bent to a prestressed net. In fig. 10.2 the inner branches have the force density 1 and the border has the value 10. Beginning with this shape, we want an equidistant net in the inner region and a constant force over all branches connected with the border. The result is the shape shown in fig. 10.3.

For such computations (figs. 9 and 10) it has been very useful to take advantage of the *damped* least squares approach as mentioned above. The main reasons for this are: The damped iterations may converge without useless oscillations and may be faster. Moreover, in the case of conditions which cannot be implemented (see fig. 7.1) a damped sequence may stop at a shape which fulfills the conditions as close as physically possible.

Thus the linear force densities method and the extension for additional constraints – together with damping principles – may be a useful approach for the analytical form finding and computation of general networks.

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References

[1] E. Bubner, Zum Problem der Formfindung vorgespannter Seilnetzflächen, Dissertation, Universität Stuttgart, 1972.
[2] B. Oleiko, Dissertation, Universität Stuttgart, in preparation.
[3] Institut für Leichte Flächentragwerke, Gitterschalen, IL 10 (Universität Stuttgart) in preparation.
[4] E. Haug, Formermittlung von Netzen, Bautechnik 9 (1971) 293.
[5] J.H. Argyris and T. Angelopoulos, Ein Verfahren für die Formfindung von Netzkonstruktionen, IVBH-Vorberichte 9 (1972) 383.
[6] E. Haug, Berechnung von Seilfachwerken, Institut für Leichte Flächentragwerke, Bericht 1/70 (Universität Stuttgart, 1970).
[7] J.H. Argyris and D.W. Scharpf, Berechnung vorgespannter Netzwerke, Bayr. Akad. der Wiss. Sonderdruck 4 der Sitzungsber. (1970);
J.H. Argyris and D.W. Scharpf, Large deflection analysis of prestressed networks, Journal of the Structural Division, Proc. ASCE (1972) 633–654.
[8] K. Linkwitz and H.-J. Schek, Einige Bemerkungen zur Berechnung vorgespannter Seilnetzkonstruktionen, Ing.-Arch. 40 (1971) 145.

[9] A. Siev and J. Eidelman, Stress analysis of prestressed suspended roofs, Proc. of the ASCE, Journ. of the Struct. Div. (August 1964) 103.
[10] A. Siev, Prestressed suspended roofs bounded by main cables, IVBH-Berichte (1967) 171.
[11] C.H. Thornton and C. Birnstiel, Three-dimensional suspension structures, Proc. ASCE, Journ. of the Struct. Div. (1967) 247.
[12] J.H. Argyris, Recent advances in matrix methods of structural analysis, Progress in Aeronautical Sciences (Pergamon Press, 1964).
[13] S.J. Fenves and F.H. Branin, Network topological formulation of structural analysis, Proc. of the ASCE, Journ. of the Struct. Division (August 1963) 483.
[14] G. Eras and H. Elze, Zur Berechnung und statisch vorteilhaften Formgebung von Seilnetzwerken, Proc. IASS on Hanging Roofs (1963).
[15] Institut für Leichte Flächentragwerke, Experimentelle Ermittlung von Minimalnetzen, IL 1, Karl Krämer, Universität Stuttgart (1969).
[16] K. Linkwitz and H.-J. Schek, Über eine neue Methode zur Berechnung vorgespannter Seilnetze und ihre praktische Anwendung auf die Olympiadächer München, IVBH-Vorberichte, 9 (1972) 393.
[17] K. Linkwitz, New methods for the determination of cutting pattern of prestressed cable nets and their application to the Olympic roofs München, IASS Tokyo (1971).
[18] A. Siev and J. Eidelman, Shapes and suspended roofs, Proc. IASS on Hanging Roofs (1963).