**CHAPTER SIX**

# Force density method

## Design of a timber shell

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**LEARNING OBJECTIVES**

- Derive the static equilibrium equations of a single node, with four bar forces and a load applied to it.
- Explain the consequences of introducing force densities into those equations.
- Apply matrix algebra and branch-node matrices to generalize this single-node formulation to arbitrary networks.
- Use the force density method to generate the shape for a shell based on such a network.

The use of 'force densities' presents an approach for the rapid generation of feasible shapes for prestressed and (inverted) hanging structures. This method allows, especially in the early stages of a new project, the instant exploration of large numbers of alternative, feasible solutions.

This chapter explains the basic premise and application of the Force Density Method (FDM), also known as the '(Stuttgart) direct approach'. It has been applied to the design of many built structures, particularly to tensioned roofs, but also to the timber shell roofs of the 1974 Mannheim Multihalle (see page 58) and the 1987 Solemar Therme in Bad Dürrheim, discussed in further detail in Chapter 12.

### The brief

To expand its sports offerings, the municipality of Stuttgart is developing a new sports complex, which includes a swimming pool and an ice rink (Fig. 6.1). Each has to be covered independently to maintain a hot and a cold climate. Both structures are adjacent in order to exchange heat between their heating and cooling installations. For convenient access, both facilities share a central entrance. The Olympic-size swimming pool (25m × 50m) will be naturally ventilated. For this reason, a high-point roof is envisioned,

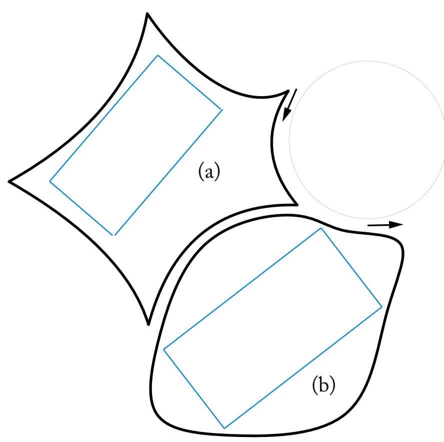


**Figure 6.1** Outline for sports complex with (a) an Olympic-size swimming pool (25m × 50m) and (b) a standard-size hockey rink (30m × 60m)

creating a stack effect. The controlled climate for the standard-size hockey rink (30m × 60m) is maintained by a domed enclosure. The client wishes to use locally sourced timber for the main structure, so a timber gridshell is proposed as the structural system.

## 6.1 Equilibrium shapes

As part of the conceptual design of structures, especially domes, shells and membrane structures, generating an adequate structural shape is crucial to the load-bearing behaviour and aesthetic expression of the design. Their shapes cannot be freely chosen and conceived directly, due to the intrinsic interaction between form and forces. For such a problem one needs form finding. Typical structural systems that require form finding include:

- soap films within a given boundary;
- prestressed, or hanging fabric membranes;
- prestressed, or hanging cable nets;
- structures generated by pressure (e.g. air, water).

Membranes or cable nets can be used for the design of shell structures such as thin gridshells, but are only partially valid as the constituting elements are not necessarily free from bending.

For these types of structures, the force density method has proven an invaluable approach to generate equilibrium solutions, and thus feasible shapes for potential designs. Solutions are generated from simple linear systems of equations.

Another advantage of using force densities is that they do not require any information about the material for the later realization of the design. As we are dealing with non-materialized equilibrium shapes, no limitations with respect to material laws exist. The materialization follows in a second step. When introducing material, we may choose (independently for each bar in the net) the material, without changing the shape created with force densities.

## 6.2 A thought experiment

Looking at any prestressed, lightweight surface structure, we observe that the continuous surface is doubly curved at each point. In other words, when considering its discretization as a pin-jointed net, at each interior intersection point, two bars are curved downwards, or 'standing', and two curved upwards, or 'hanging' (Fig. 6.2). This opposite curvature is called 'anticlastic' or 'negative' curvature.

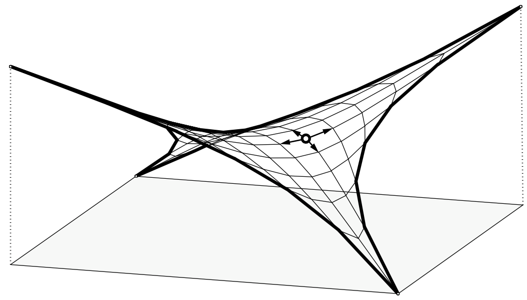

**Figure 6.2** A prestressed surface, when discretized shows at each interior point, two 'hanging' bars curved upwards, and two 'standing' downwards

This characteristic doubly curved shape immediately becomes understandable in the following thought experiment. We put up four elastic rubber bands in a box. They will sag under their self-weight and assume the shape of catenaries (Fig. 6.3a). To stabilize these four hanging rubber bands, we place four new ones perpendicularly over them, connecting them at their intersections and attaching them to the bottom of the box (Fig. 6.3b). We now stabilize the net further by applying tension to the bands, by prestressing them. We pull the hanging bands by their ends, located at the sides of the box. Their initial lengths decrease, and as they are lifted up, they pull the net upwards (Fig. 6.3c). We continue by alternately pulling both sets of four elastic bands, as the geometry of the net changes less and less (Fig. 6.3d). Meanwhile, the net itself becomes increasingly stiff, and anticlastic curvature results at every point. Precisely this principle is applied everywhere in a prestressed, structural net.

### 6.2.1 A single node in equilibrium

The thought experiment in Section 6.2 can be further simplified. The stationary, prestressed net is characterized by the fact that at every node equilibrium must exist between the four cable-forces, induced by initial prestress, and any load acting on that node.

Let us consider such a single node $P_0$ in equilibrium (Fig. 6.4). The node $P_0$ is connected to fixed points

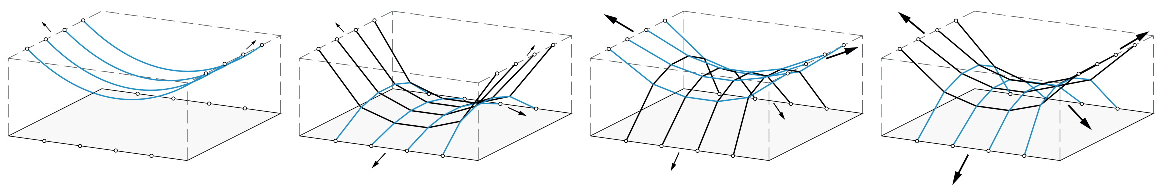


**Figure 6.3** A thought experiment resulting from (a) four hanging, (b) four standing rubber bands and alternately (c, d) tensioning them

$P_1$,$P_2$,$P_3$,$P_4$ in three-dimensional space in an 'anticlastic' configuration, meaning that of the opposite pairs of points, $P_1$,$P_3$ must be 'high' and $P_2$,$P_4$ must be 'low' points, or vice versa.

The four elastic bars $a$,$b$,$c$,$d$ between these points, are connected as pin-joints. In their slack state the four bars are too short to be connected at $P_0$. Consequently, tension forces $F_a$,$F_b$,$F_c$,$F_d$ are generated when they are connected at $P_0$. A fifth force, representing self-weight, is applied as an external load $P_z$ at node $P_0$.

### 6.2.2 Translation to equations

Now that we have modelled the prestressed net as a spatial, four-bar, pin-jointed network, the question is how to find a state of equilibrium and the resulting geometry. To this end, we first formulate three basic relationships:

1. Every individual bar is increased in length due to the tension force acting in it. The difference between the non-stretched and elastically stretched length of the bar results from material behaviour.
2. In the prestressed state of the net, every length of an elastically extended bar has to be equal to the distance of the nodes to which it is connected. This describes the compatibility between the elongation of the bars and the geometry of the net in the final, prestressed state.
3. The tension forces and self-weight applied to the unsupported node must be in equilibrium.

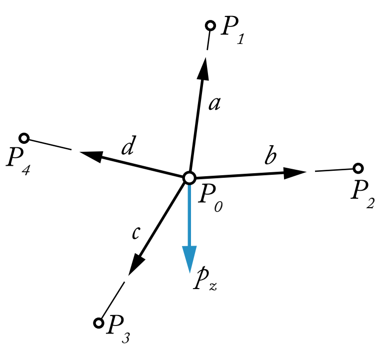


**Figure 6.4** A single node with four forces and a load

If we translate these three basic facts to mathematical formulae, we obtain the following relationships, for a single node and its four neighbours.

First, Hooke's law of elasticity applies without loss of generality to the changes in length. The tension force $F_i$, with $i = a,b,c,d$, in each of the four bars is

$$F_i = \left[\frac{EA}{l_0} \cdot e\right]_i. \tag{6.1}$$

where $EA$ is the (axial) stiffness of the material, $l_0$ is the non-stretched length, and $e$ the elastic elongation of the bar.

Second, the length $l_i$ of every elastically elongated bar must be exactly equal to the spatial distance between the nodes at its ends

$$l_i = \sqrt{(x_k - x_0)^2 + (y_k - y_0)^2 + (z_k - z_0)^2}, \tag{6.2}$$

where $x$,$y$,$z$ are the coordinates of the nodes, and $k = 1,2,3,4$.

The elastic elongation $e$ is the difference of stretched and non-stretched lengths $l$ and $l_0$, so by substituting $l$ with equation (6.2), the elongation of each of the four bars

$$e_i = l_i - l_{0,i}. \tag{6.3}$$

Third and last, there must be equilibrium in every node. This must also hold in each of the three dimensions, $x$, $y$ and $z$. We decompose the force $F_i$ in each bar into three components, which for example in $x$-direction, gives

$$F_{i,x} = F_i \cdot \cos\alpha_i, \tag{6.4}$$

where $\cos\alpha_i$ is the direction cosine, and $\alpha_i$ is the angle between bar $i$ and the $x$-direction. The equilibrium of the four forces in point $P_0$ in the $x$-direction, with a load $p_x$ acting on the node, is

$$F_a \cdot \cos\alpha_a + \ldots + F_d \cdot \cos\alpha_d + p_x = 0. \tag{6.5}$$

Writing the direction cosine between points $P_k$ and $P_0$ with reference to the three coordinate axes results in the expression

$$\text{direction cos} = \frac{\text{coordinate difference}}{\text{distance in space}}$$

and substituted as an equation in (6.5) gives

$$\frac{x_1 - x_0}{l_a} \cdot F_a + \ldots + \frac{x_4 - x_0}{l_d} \cdot F_d + p_x = 0. \tag{6.6}$$

In structural mechanics, the following relationships apply:

- Static equilibrium requires the formulation of *equilibrium equations*, which relate external loads and internal forces. In our case, they relate load $p$ to forces $F_i$ in equation (6.6).
- Material behaviour is determined by *constitutive equations* relating the internal forces to deformations (more generally, strain) such as elongations and/or curvatures. In this case, Hooke's law of elasticity (without loss of generality) describes the material relationship (6.1) between forces $F_i$ and elongations $e_i$.
- Geometry is governed by the *compatibility* or *kinematic equations*, relating the deformations to translations.

In order to find a solvable system of equations from (6.1–6.6), we carry out a number of substitutions.

First, by substituting equation (6.3) into (6.1), we can write

$$F_i = \left[\frac{EA}{l_0}(l - l_0)\right]_i. \tag{6.7}$$

We insert these expressions for the forces into the equilibrium equations (6.6), and get

$$\frac{x_1 - x_0}{l_a} \cdot \frac{EA_a}{l_{0,a}}(l_a - l_{0,a}) + \ldots$$
$$+ \frac{x_4 - x_0}{l_d} \cdot \frac{EA_d}{l_{0,d}}(l_d - l_{0,d}) + p_x = 0. \tag{6.8}$$

If the coordinates of the fixed points $x_k, y_k, z_k$ and the unstressed lengths of the elastic bars $l_{0,a}, \ldots, l_{0,d}$ are given, we are now able to determine the unknown coordinates $x_0, y_0, z_0$ of point $P_0$ and thus its position in three-dimensional space. We have to solve the system of the preceding equations for the unknown coordinates $x_0, y_0, z_0$. However, this is by no means trivial. The system to be solved is nonlinear as the unknown coordinates $x_0, y_0, z_0$ are also contained in the lengths $l_a, \ldots, l_d$, in equation (6.2).

Thus, we have to linearize the system, observing that the system is nonlinear with respect to geometry and material.

### 6.2.3 Force densities

To deal with the nonlinearity of the problem, we introduce force densities, also known as 'tension coefficients'. These are defined as

$$\text{force density} = \frac{\text{force in a bar}}{\text{stressed length of the bar}}.$$

Quantities of this type can already be recognized in our previous equations, (6.6) and (6.8). To find the spatial coordinates of $P_0$, we take the following approach to overcome the nonlinearity problem. First, we rewrite equation (6.6) to

$$(x_1 - x_0) \cdot \frac{F_a}{l_a} + \ldots + (x_4 - x_0) \cdot \frac{F_d}{l_d} + p_x = 0. \tag{6.9}$$

The quotients $F/l$ are declared as new variables $q$, called force densities, defined as

$$q_i := \frac{F_i}{l_i} \tag{6.10}$$

and equation (6.9) thus becomes

$$(x_1 - x_0) \cdot q_a + \ldots + (x_4 - x_0) \cdot q_d + p_x = 0. \tag{6.11}$$

Equation (6.11) is reordered in such a way, that the terms with the unknowns (i.e. the coordinates of $P_0$) are on the left-hand side and the terms with the constant factors (i.e. the given values of the coordinates of the fixed points and the force densities) are on the right-hand side of the equation. The resulting system of equations

$$-(q_a + q_b + q_c + q_d) \cdot x_0 =$$
$$-p_x - (x_1 \cdot q_a + x_2 \cdot q_b + x_3 \cdot q_c + x_4 \cdot q_d) \quad (6.12)$$

has the solution, now given in three dimensions,

$$x_0 = \frac{p_x + x_1 \cdot q_a + x_2 \cdot q_b + x_3 \cdot q_c + x_4 \cdot q_d}{q_a + q_b + q_c + q_d},$$
$$y_0 = \frac{p_y + y_1 \cdot q_a + y_2 \cdot q_b + y_3 \cdot q_c + y_4 \cdot q_d}{q_a + q_b + q_c + q_d}, \quad (6.13)$$
$$z_0 = \frac{p_z + z_1 \cdot q_a + z_2 \cdot q_b + z_3 \cdot q_c + z_4 \cdot q_d}{q_a + q_b + q_c + q_d}.$$

For each chosen set of four force densities we get a unique solution of the unknown point $P_0(x_0,y_0,z_0)$ from the linear system of equations (6.13). These unique solutions are equivalent and identical with the solutions of the nonlinear equations (6.6) and (6.8). Notice the equivalence of the equations (6.6) and (6.12), where the former is the nonlinear description, and the latter is the linear description, of the very same equilibrium solution.

## 6.3 Matrix formulations

So far, we have found a solution if we only have one unknown three-dimensional point in space. Practically, the solution for the single node is by no means sufficient. We are dealing with nets with arbitrary topology and numbers of given fixed and unknown free points. To find solutions for arbitrarily large nets, we have to extend our mathematical tools, and introduce matrix formulations combined with graph theory. In the following sections we discuss some conventions in our notation, then introduce two specific concepts: the branch-node matrix and the Jacobian. Using these, we rewrite the single-node problem in matrix form, before generalizing to arbitrary networks.

## 6.4 Notation

A vector is interpreted as a one-column matrix and written in bold lower case, and a general matrix is written as a bold capital letter. The same symbols are used for the components but they have an additional index $i, j$ or $k$. The $m$-dimensional vector $\mathbf{a}$ – called the $m$-vector $\mathbf{a}$ – has therefore $a_j$ as $j$-th component. The transpose of a vector is a one-row matrix,

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = [1 \;\; 2 \;\; 3]^T, \mathbf{a}^T = [1 \;\; 2 \;\; 3].$$

Further, we often need the diagonal matrix $\mathbf{A}$ belonging to any vector $\mathbf{a}$ : $\mathbf{A}$ is simply defined to have $\mathbf{a}$ as diagonal, for example

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{A} = \mathrm{diag}(\mathbf{a}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

### 6.4.1 Branch-node matrix

Before proceeding to generalize our equations for a single node to those for an arbitrary network, we discuss some fundamental concepts of graph theory that can be used to describe net-like entities and are therefore useful for such a general formulation. In graph theory, a net-like entity consists of an aggregation of $n$ nodes (also called points) and an aggregation of $m$ branches (also called edges). Each branch connects two nodes.

The topological relationships between nodes and branches can be described in graph theory by a branch-node matrix $\mathbf{C}$ (or incidence matrix $\mathbf{C}^T$), consisting of the elements +1, –1 or 0 in each row, so

$$C_{ij} = \begin{cases} +1 & \text{if branch } j \text{ ends in node } i, \\ -1 & \text{if branch } j \text{ begins in node } i, \\ 0 & \text{otherwise.} \end{cases}$$

A few remarks characterizing the branch-node matrix $\mathbf{C}$:

- $\mathbf{C}$ does not contain ‘geometry’, only topology, that is, there are no metric relationships;
- there is precisely one element +1 and one element –1 in each row;
- the matrix is not necessarily regular with respect to

its columns, that is, it can have a different number of elements in each column;

- in the case of a contiguous net, **C** has the rank $m - 1$.

Now we are able to treat our 'single-node problem' using the corresponding branch-node matrix. The branch-node matrix **C** of the point $P_0$ with its neighbours $P_1,\ldots,P_4$ is

$$\mathbf{C} = \begin{array}{c} \begin{array}{ccccc} P_0 & P_1 & P_2 & P_3 & P_4 \end{array} \\ \begin{bmatrix} +1 & -1 & 0 & 0 & 0 \\ +1 & 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & -1 & 0 \\ +1 & 0 & 0 & 0 & -1 \end{bmatrix} \end{array} \begin{array}{l} a \\ b. \\ c \\ d \end{array} \qquad (6.14)$$

The usefulness of **C** can be demonstrated by noting that the coordinate differences,

$$\mathbf{u} = [x_1 - x_0 \quad x_2 - x_0 \quad x_3 - x_0 \quad x_4 - x_0]^{\mathrm{T}}, \qquad (6.15)$$

are obtained through the multiplication of the coordinate vectors

$$\mathbf{x} = [x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4] \qquad (6.16)$$

with the branch-node matrix **C**, such that

$$\mathbf{u} = \mathbf{C}\mathbf{x}. \qquad (6.17)$$

We subdivide the branch-node matrix **C** into the part $\mathbf{C}_\mathrm{N}$ containing the new, unknown points, and the part $\mathbf{C}_\mathrm{F}$ containing the fixed points, so

$$\mathbf{C} = [\mathbf{C}_\mathrm{N} \quad \mathbf{C}_\mathrm{F}] \qquad (6.18)$$

and in a similar manner we subdivide the vector of coordinates **x** into new, unknown points and fixed points, so

$$\mathbf{x} = [\mathbf{x}_\mathrm{N} \quad \mathbf{x}_\mathrm{F}] \qquad (6.19)$$

Substituting equations (6.18) and (6.19) into (6.16), and similarly for the $y$- and $z$-direction, we get

$$\begin{aligned} \mathbf{u} &= \mathbf{C}_\mathrm{N}\mathbf{x}_\mathrm{N} + \mathbf{C}_\mathrm{F}\mathbf{x}_\mathrm{F}, \\ \mathbf{v} &= \mathbf{C}_\mathrm{N}\mathbf{y}_\mathrm{N} + \mathbf{C}_\mathrm{F}\mathbf{y}_\mathrm{F}, \\ \mathbf{w} &= \mathbf{C}_\mathrm{N}\mathbf{z}_\mathrm{N} + \mathbf{C}_\mathrm{F}\mathbf{z}_\mathrm{F}, \end{aligned} \qquad (6.20)$$

and using their diagonal matrices U, V, and W the corresponding bar lengths

$$\mathbf{L} = (\mathbf{U}^2 + \mathbf{V}^2 + \mathbf{W}^2)^{\frac{1}{2}}. \qquad (6.21)$$

Declaring force densities and bar lengths as vectors **q** and **l**, or as diagonal matrices **Q** and **L**, we have everything at our disposal that allows us to solve the 'single-node problem' automatically.

### 6.4.2 Jacobian

To write the equations (6.6) and (6.8) in matrix notation, we also determine the gradient $\nabla$ in Euclidean space, or the Jacobian $\partial\mathbf{f}(x_0)/\partial x_0$, of the function

$$\mathbf{f}(x_0) = \begin{bmatrix} f_a(x_0) \\ f_b(x_0) \\ f_c(x_0) \\ f_d(x_0) \end{bmatrix} = \begin{bmatrix} l_a \\ l_b \\ l_c \\ l_d \end{bmatrix} = \mathbf{l}. \qquad (6.22)$$

As a result, we get for the transposed Jacobian

$$\left(\frac{\partial\mathbf{f}(x_0)}{\partial x_0}\right)^{\mathrm{T}} = \begin{bmatrix} \dfrac{\partial l_a}{\partial x_0} & \dfrac{\partial l_b}{\partial x_0} & \dfrac{\partial l_c}{\partial x_0} & \dfrac{\partial l_d}{\partial x_0} \end{bmatrix} = \begin{bmatrix} \dfrac{-(x_1 - x_0)}{l_a} \\ \dfrac{-(x_2 - x_0)}{l_b} \\ \dfrac{-(x_3 - x_0)}{l_c} \\ \dfrac{-(x_4 - x_0)}{l_d} \end{bmatrix}^{\mathrm{T}}. \qquad (6.23)$$

Using the branch-node matrix, we can write the Jacobian

$$\left(\frac{\partial\mathbf{f}(\mathbf{x})}{\partial\mathbf{x}}\right)^{\mathrm{T}} = \mathbf{C}_\mathrm{N}^{\mathrm{T}}\mathbf{U}\mathbf{L}^{-1}. \qquad (6.24)$$

The Jacobian corresponds exactly to the direction cosines of equation (6.6).

### 6.4.3 Solution in matrix form

Introducing the vector of forces **f** and the vector of load components **p**, equation (6.6) is then equivalent to

$$-\left(\frac{\partial \mathbf{f}(x_0)}{\partial x_0}\right)^{\mathrm{T}}\begin{bmatrix}F_a\\F_b\\F_c\\F_d\end{bmatrix}+p_x=0. \tag{6.25}$$

This equation can be written as

$$\mathbf{C}_{\mathrm{N}}^{\mathrm{T}}\mathbf{U}\mathbf{L}^{-1}\mathbf{f}+\mathbf{p}=\mathbf{0}. \tag{6.26}$$

With the definition for the force densities known already from equation (6.10),

$$\mathbf{q}=\mathbf{L}^{-1}\mathbf{f}, \tag{6.27}$$

we receive, by substitution, the system of equations

$$\mathbf{C}_{\mathrm{N}}^{\mathrm{T}}\mathbf{U}\mathbf{q}+\mathbf{p}=\mathbf{0}. \tag{6.28}$$

We want to find the linear system of equations for the determination of the solution. Given that $\mathbf{Uq}=\mathbf{Qu}=\mathbf{QCx}$, we rewrite equation (6.28) to equations of the form

$$\mathbf{C}_{\mathrm{N}}^{\mathrm{T}}\mathbf{QCx}+\mathbf{p}=\mathbf{0} \tag{6.29}$$

or

$$\mathbf{C}_{\mathrm{N}}^{\mathrm{T}}\mathbf{QC}_{\mathrm{N}}\mathbf{x}_{\mathrm{N}}+\mathbf{C}_{\mathrm{N}}^{\mathrm{T}}\mathbf{QC}_{\mathrm{F}}\mathbf{x}_{\mathrm{F}}+\mathbf{p}=\mathbf{0}. \tag{6.30}$$

We observe the independence of the equations for the respective coordinate components. For simplicity, we set $\mathbf{D}_{\mathrm{N}}=\mathbf{C}_{\mathrm{N}}^{\mathrm{T}}\mathbf{QC}_{\mathrm{N}}$ and $\mathbf{D}_{\mathrm{F}}=\mathbf{C}_{\mathrm{N}}^{\mathrm{T}}\mathbf{QC}_{\mathrm{F}}$, and obtain the system of equations of equilibrium in the form

$$\mathbf{D}_{\mathrm{N}}\mathbf{x}_{\mathrm{N}}=\mathbf{p}-\mathbf{D}_{\mathrm{F}}\mathbf{x}_{\mathrm{F}}, \tag{6.31}$$

which is a system of linear equations of the standard form $\mathrm{A}\mathbf{x}=\mathbf{b}$. This equation linearly defines the free node coordinates $\mathbf{x}_{\mathrm{N}}$. This system can be solved efficiently by using, for example, Cholesky decomposition (see Section 13.5.1). With a given load and a given position of fixed points, we get for each set of prescribed force densities exactly one equilibrium state with the shape, now given in three dimensions,

$$\begin{aligned}\mathbf{x}_{\mathrm{N}}&=\mathbf{D}_{\mathrm{N}}^{-1}(\mathbf{p}_x-\mathbf{D}_{\mathrm{F}}\mathbf{x}_{\mathrm{F}}),\\ \mathbf{y}_{\mathrm{N}}&=\mathbf{D}_{\mathrm{N}}^{-1}(\mathbf{p}_y-\mathbf{D}_{\mathrm{F}}\mathbf{y}_{\mathrm{F}}),\\ \mathbf{z}_{\mathrm{N}}&=\mathbf{D}_{\mathrm{N}}^{-1}(\mathbf{p}_z-\mathbf{D}_{\mathrm{F}}\mathbf{z}_{\mathrm{F}}),\end{aligned} \tag{6.32}$$

and the branch forces

$$\mathbf{f}=\mathbf{Lq}. \tag{6.33}$$

### 6.4.4 Generalization

We can generalize the solution in equation (6.32) for arbitrarily large nets of $m$ branches and $n=n_{\mathrm{N}}+n_{\mathrm{F}}$ nodes, consisting of $n_{\mathrm{N}}$ unknown, new nodes, and $n_{\mathrm{F}}$ fixed nodes. Previously, we had only $m=4$ branches and $n=n_{\mathrm{N}}+n_{\mathrm{F}}=1+4$ nodes. So long as the $m\times n$ branch-node matrix $\mathbf{C}$ correctly describes the topology of our network, and consistent indexes are used for all the matrices and vectors describing properties of branches and nodes, equation (6.32) holds for any problem. We apply the same equation to another topology, belonging to that of the net drawn in Figure 6.5.

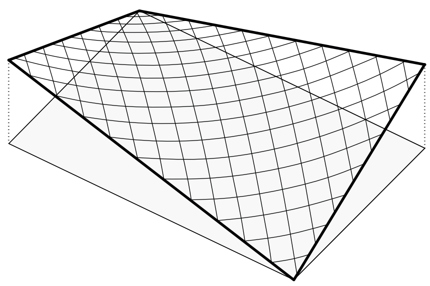

**Figure 6.5** An arbitrary net with a large number of nodes and branches

The example, without external loads, for force densities varying in the edges and in the interior of the net, and the boundary conditions shown in Figure 6.5, leads to the solutions in Figure 6.6. Here the ratio of force densities in the edge to the interior branches is varied from 5:1, 2:1, 1:1 to 1:2, suggesting possible shapes for anticlastic surfaces. When vertical loads are introduced to Figure 6.7a in the range $\mathbf{p}_z=0.1,0.2,0.5$, and the boundary edges are either free or fixed, it yields different, synclastic shapes shown in Figure 6.7b–d.

## 6.5 Materialization

A net has been determined with 'pure' force densities, without any information about the material used for its realization. Subsequently, any materialization of each and every individual bar is possible.

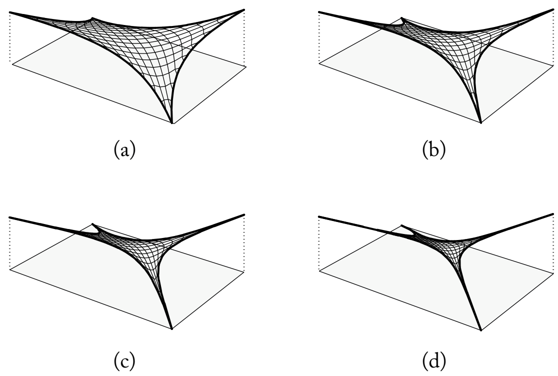


**Figure 6.6** Figures of equilibrium for varying proportions of edge to interior force densities, (a) 5:1, (b) 2:1, (c) 1:1 and (d) 1:2

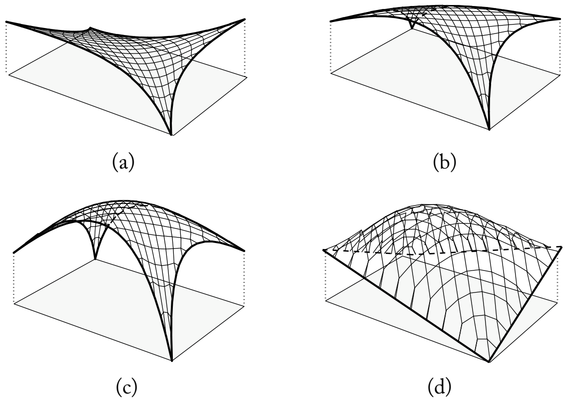


**Figure 6.7** Figures of equilibrium for varying loads (a) $\mathbf{p}_z = 0$, (b) $\mathbf{p}_z = 0.1$, (c) $\mathbf{p}_z = 0.2$ and (d) $\mathbf{p}_z = 0.2$ with fixed, straight edges

We know the forces from equation (6.33), where the force densities $\mathbf{q}$ are given, and the lengths are calculated for each bar from equation (6.3) using the appropriate coordinates.

Then, selecting a diagonal matrix of axial stiffnesses $\mathbf{EA}$, we can calculate the corresponding vectors of elastic elongations $\mathbf{e}$ and initial lengths $\mathbf{l}_0$. The principle here is to select the initial lengths $\mathbf{l}_0$ in such a manner that the elongations $\mathbf{e} = \mathbf{l} - \mathbf{l}_0$, that is, the elongation which is necessary to generate the solution. According to Hooke's law of elasticity, rewritten and generalized from equation (6.1),

$$\begin{aligned} \mathbf{e} &= \mathbf{l} - \mathbf{l}_0 = \mathbf{L}_0(\mathbf{EA})^{-1}\mathbf{f} \\ &= \mathbf{L}_0(\mathbf{EA})^{-1}\mathbf{Lq}, \end{aligned} \tag{6.34}$$

$$\mathbf{q} = \mathbf{eEAL}_0^{-1}\mathbf{L}^{-1}. \tag{6.35}$$

Continuing to rewrite equation (6.34)

$$\begin{aligned} \mathbf{l} &= \mathbf{l}_0 + \mathbf{L}_0(\mathbf{EA})^{-1}\mathbf{f} \\ &= (\mathbf{I} + (\mathbf{EA})^{-1}\mathbf{F})\mathbf{l}_0, \end{aligned} \tag{6.36}$$

$$\begin{aligned} \mathbf{l}_0 &= (\mathbf{I} + (\mathbf{EA})^{-1}\mathbf{F})^{-1}\mathbf{l} \\ &= (\mathbf{I} + (\mathbf{EA})^{-1}\mathbf{QL})^{-1}\mathbf{l}, \end{aligned} \tag{6.37}$$

where $\mathbf{I}$ is an identity matrix of size $m$.

As long as each force $F$ is larger than zero, the denominator of the fraction is always >1 and therefore we have tension in the bar if $l_0 < 1$ and, vice versa, compression. Furthermore, the initial length $l_0$ of the bar, or cable segment, necessary to realize a given equilibrium shape, is only dependent on the chosen, individual axial stiffness $EA$.

## 6.6 Procedure

The steps in the force density method (Fig. 6.8) are simply constructing the right-hand side of equation (6.32), which consists of the boundary conditions, or fixed points $\mathbf{x}_F$, the topology described by $\mathbf{C}$, the force densities $\mathbf{q}$ and external loads $\mathbf{p}$. If the resulting solution, described by the coordinates $\mathbf{x} = [\mathbf{x}_N\ \mathbf{x}_F]$, is unsatisfactory to the designer, each of these four quantities can be changed to generate a new, unique solution.

The loads could also be calculated from the surface area or bar lengths surrounding each node, to approximate self-weight of the structure. For discrete structures, this is done using equation (10.28), while for continuous surfaces, approaches such as those adopted in Chapters 7, 13 and 14 can be used.

The effect of particular sets of force densities $\mathbf{q}$ on the resulting equilibrium shapes may be difficult to anticipate. Their value can also be determined indirectly: either by the user controlling the horizontal thrust components as in thrust network analysis (Chapter 7), or by adding constraints to form a least-squares problem (Chapters 12 and 13).

## 6.7 Design development

The roof of the ice rink is designed as a synclastic surface structure by applying vertical loads $\mathbf{p}$. Starting from a quadrilateral topology, Figure 6.9 shows some design possibilities by varying the loads **p** and the force densities **q**. Because both these parameters are given by the designer, and because their relation (6.28) is linear, one can obtain the same geometry, by scaling both parameters equally. For example, the solution in Figure 6.9a with **q** = 2 and **p** = 0.5 also results by scaling the loads and force densities by a factor 2, so with **q** = 4 and **p** = 1. In other words, once the geometry is found, one can calculate the real load afterwards and simply scale the force densities accordingly to obtain actual forces in the structure.

Figure 6.10 shows a few variations for the swimming pool roof, obtained by changing the boundary conditions (the height and size of the opening in the middle), and by adding point loads to the nodes in Figure 6.10c. The initial topology is radial, with its origin in the centre of the high point. Without providing a load, the resulting forms are anticlastic. The design team settles on the geometries shown in Figures 6.9 and 6.10b for the initial design of the sports complex, shown in Figure 6.11. This design can be materialized and then tested for other load combinations. However, the project may have additional architectural or structural constraints, ultimately expressed in the form of specific positions of nodes, target lengths of branches or values of forces. These constraints lead to a nonlinear FDM, as explained further in Chapter 12.

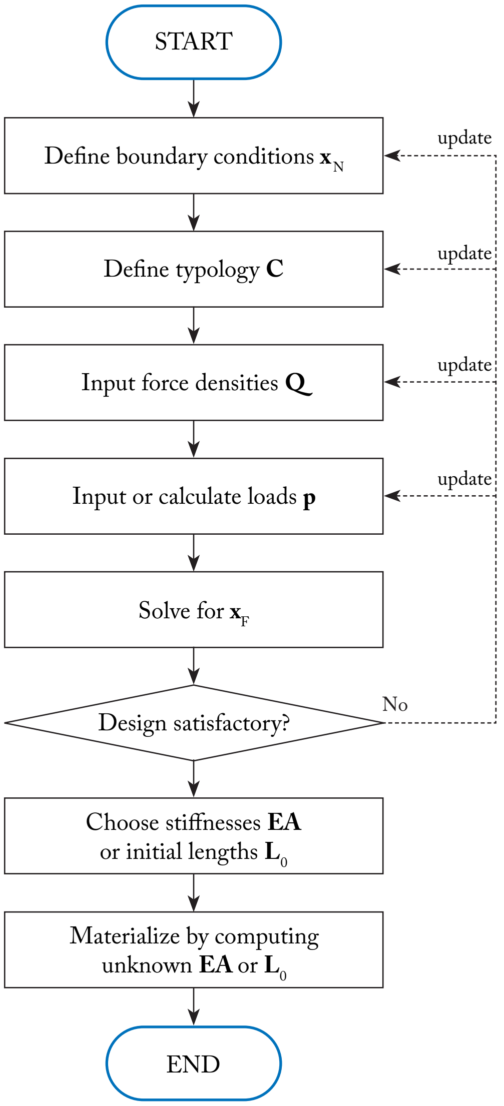


**Figure 6.8** Flowchart for FDM

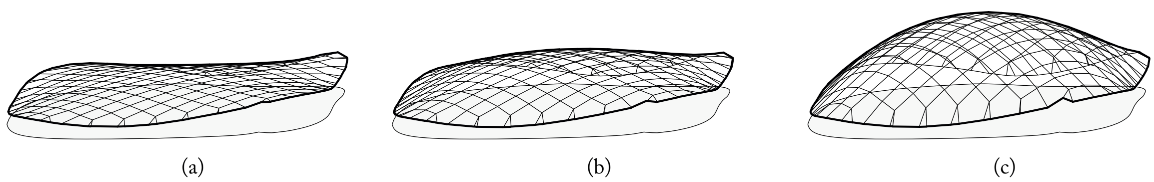


**Figure 6.9** Variations for the ice rink roof with (a) **q** = 2, **p** = 0.5, (b) **q** = 2, **p** = 1, and (c) **q** = 1, **p** = 1

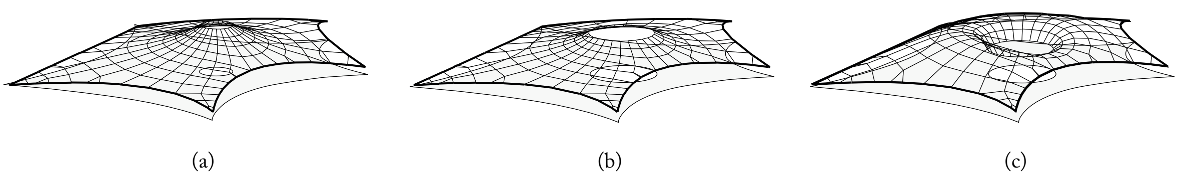


**Figure 6.10** Variations for the swimming pool roof with **q** = 1 for (a) high point $h$ = 12m, small opening, (b) $h$ = 10m, large opening and (c) $h$ = 6m, large opening and **p** = 1

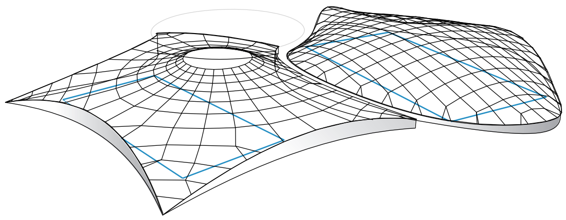

**Figure 6.11** Preliminary design for the Stuttgart sports complex

## 6.8 Conclusion

The force density method is able to generate solutions of discrete networks, through linear systems of equations, that are in an exact state of equilibrium, without needing iterations or some kind of convergence criterion. Applied to the design brief, it enabled us to quickly generate solutions for a given topology, by varying the force densities and external loads. The method, originally developed for cable nets, is, to this day, very common in the design practice of tensioned membrane roofs. By introducing loads, it also allows the form finding of synclastic structures, highly suitable for efficient shell structures. Because the method is entirely independent of material properties, two interesting opportunities arise. First, resulting designs can be materialized arbitrarily, giving the initial lengths of the network in undeformed state, without affecting the final shape. Second, one can simply multiply the loads to any realistic value, and then calculate the internal force distribution, again without changing the geometry.

## Key concepts and terms

The **force density** is the ratio of force over (stressed) length in a bar or cable segment. It is also known as the 'tension coefficient'.

A **branch-node matrix** is a matrix that shows the (topological) relationship between *m* branches and *n* nodes. The matrix has m rows and n columns. The entry in a certain row and column is 1 or -1 if the corresponding branch and node are related, and 0 if they are not. The sign depends on the direction of the branch. The transpose of the branch-node matrix is known as the incidence matrix.

The **Jacobian** is the gradient for functions in Euclidean space. It is the matrix of all first-order partial derivatives of a vector- or scalar-valued function with respect to another vector. In other words, the variation in space of any quantity can be represented by a slope. The gradient represents the steepness and direction of that slope.

## Further reading

- 'Einige bemerkungen zur Berechnung von vorgespannten Seilnetzkonstruktionen or Some remarks on the calculation of prestressed cable-net structures', Linkwitz and Schek (1971). This German journal publication laid all the groundwork for what later would be called force densities and the force density method.
- 'The force density method for form finding and computation of general networks', Schek (1974). This seminal paper by Hans-Jörg Schek concisely describes the force density method and explains how constraints can be introduced.
- 'Formfinding by the "direct approach" and pertinent strategies for the conceptual design of prestressed and hanging structures', Linkwitz (1999). This journal paper explains the force density method and also linearizes the nonlinear, materialized equations for static analysis. This shows, indirectly, how the force density method relates to static analysis.

## Exercises

- Four points, $P_1(0,0,0)$, $P_2(5,0,3)$, $P_3(0,7,3)$ and $P_4(7,5,0)$ are connected to a central node $P_0$ through links *a*, *b*, *c* and *d*. Each link has a force of 1kN. When a gravity load $p = 5$kN is applied, determine the position of node $P_0$. Calculate the sum of forces in node $P_0$. What do you observe?
- Now, determine the position of node $P_0$ once more, except by imposing force densities $q = 1$ in the four bars, instead of forces. What is the sum of forces in node $P_0$? What happens if the force density $q = 2$ in bars *b* and *d*?
- Compose a branch-node matrix for the standard example grid in Figure 6.12 and assemble the vectors of force densities **q** and coordinates **x,y,z**.

- Change the boundary conditions (vertical position of anchor points $\mathbf{z}_F$), force densities $\mathbf{q}$ and external loads $\mathbf{p}_z$, to shape a shell structure.

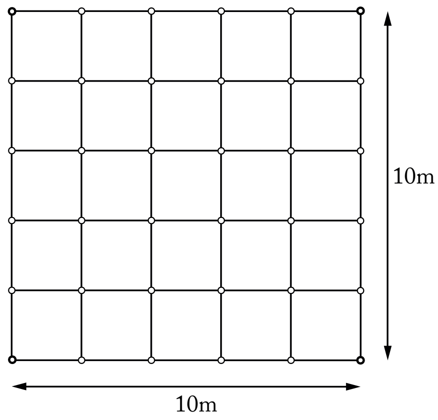


**Figure 6.12** Standard grid