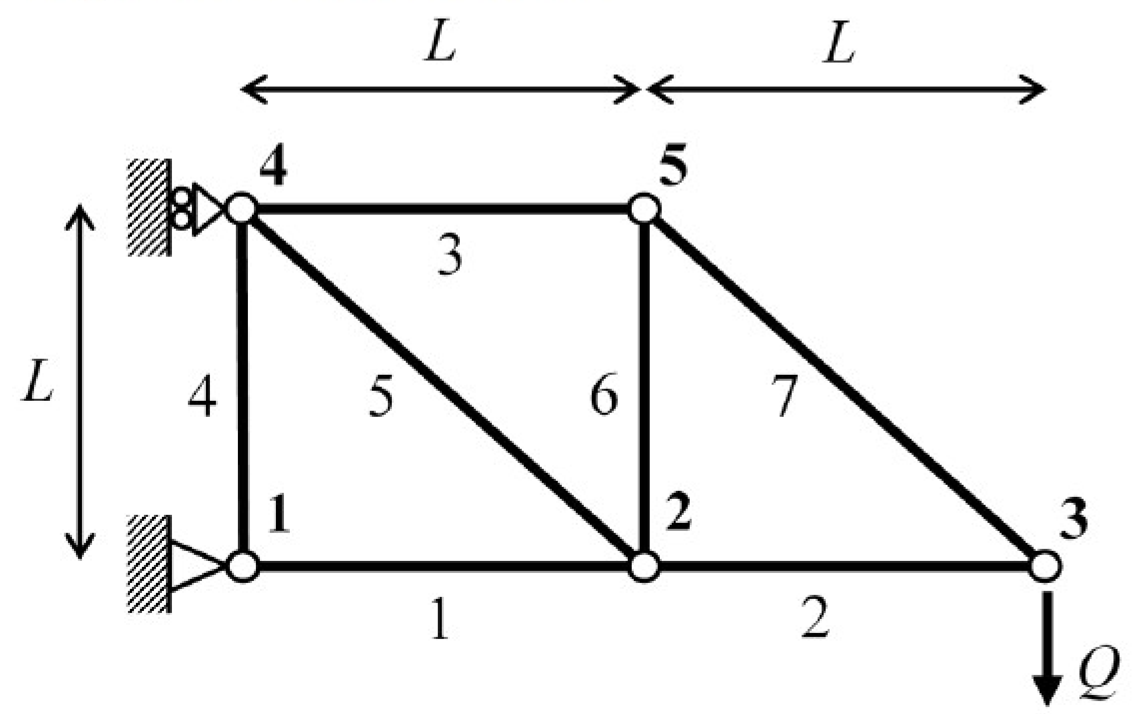
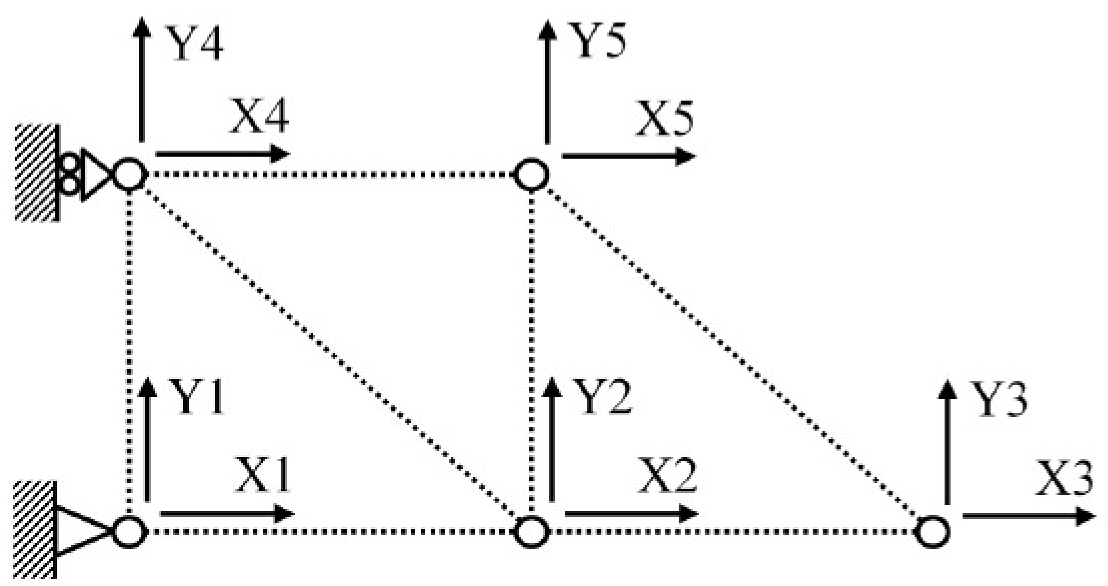
Lecture 1
Method of Joints and Truss Model

Method of Joints

Degrees of Freedom





Degrees of Freedom

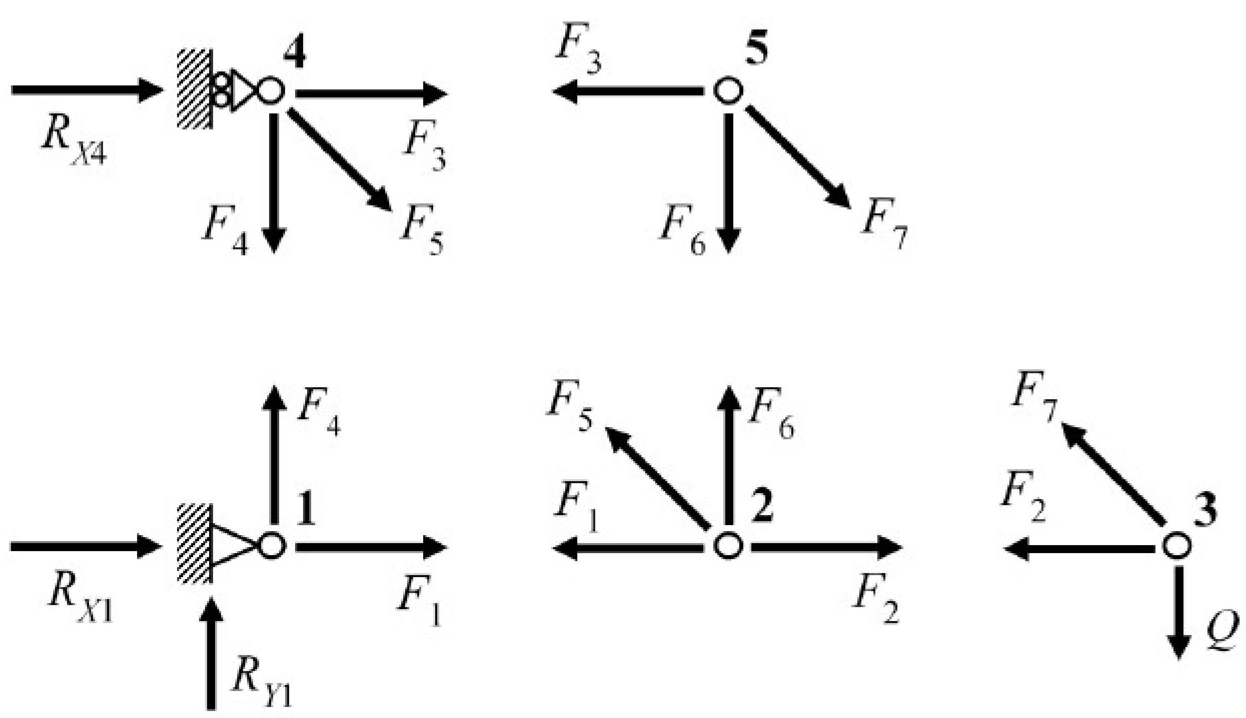
This truss has 5 nodes, 7 elements and 3 fixities provided by 2 supports.

A degree of freedom (DOF) is the combination of a node and a translational or rotational axis. In the 2D model of a truss, each node has 2 DOFs corresponding to 2 translational axes. A natural pair of axes are the vertical and horizontal ones.

In the above example, X1, Y1 and X4 are fixed DOFs, and the other 7 DOFs are mobile DOFs. Mobile DOFs introduce displacements, and fixed DOFs introduce reactions.

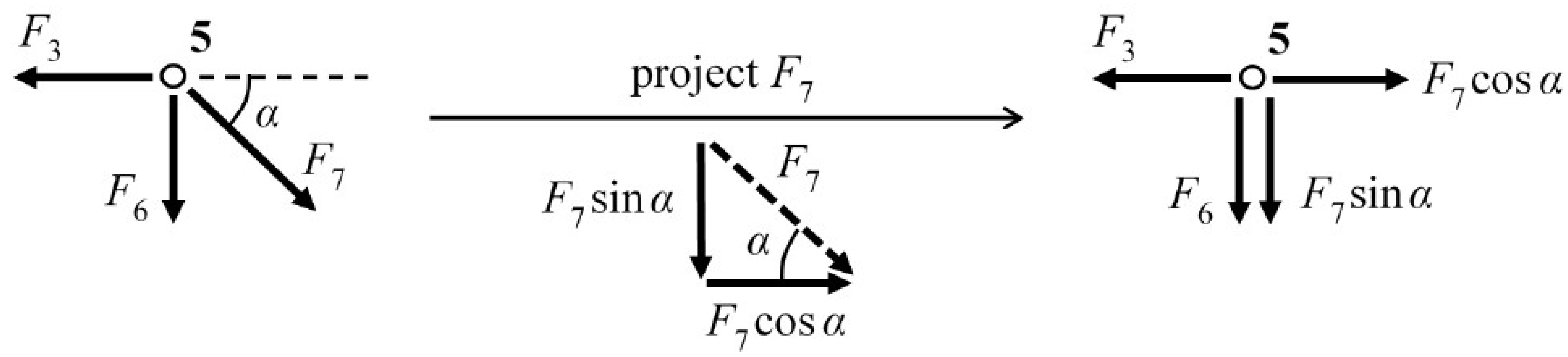
Equilibrium Equations

When the load Q is applied, the supports develop reactions to keep the structure in equilibrium and the elements develop axial forces to transfer the load to the supports. The loads, element forces and reactions acting on the nodes are as follows:



Note: the above diagram defines the positive direction of each force. If the value of a force is positive, that force acts in the direction of the vector shown in the diagram (e.g. downward if $Q > 0$). The above diagram follows a convention where the force in an element is positive when the element is in tension (e.g. when $F_7 > 0$, element 7 pulls nodes 3 and 5 toward each other).

The diagonal forces can be projected onto the directions of the DOFs. Consider for example force F_7 acting on node 5:



Equilibrium of the structure requires equilibrium of every node in both directions, i.e. equilibrium of every DOF. We define the factor β to simplify the force projections:

$$\beta = \sin\alpha = \cos\alpha = \frac{1}{\sqrt{2}} \qquad \text{since} \qquad \alpha = \frac{\pi}{4}$$

The full set of equilibrium equations is then:

Node	DOF	Equilibrium equation
1	X1	$F_1 + R_{X1} = 0$
	Y1	$F_4 + R_{Y1} = 0$
2	X2	$-F_1 + F_2 - \beta F_5 = 0$
	Y2	$\beta F_5 + F_6 = 0$
3	X3	$-F_2 - \beta F_7 = 0$
	Y3	$\beta F_7 - Q = 0$
4	X4	$F_3 + \beta F_5 + R_{X4} = 0$
	Y4	$-F_4 - \beta F_5 = 0$
5	X5	$-F_3 + \beta F_7 = 0$
	Y5	$-F_6 - \beta F_7 = 0$

There are 10 equations involving 10 variables. The number of equations and variables could have been determined directly from the topology of the truss, using the following quantities:

N number of nodes (= 5 in the current example)
E number of elements (= 7)
F number of fixities (= 3)

Each node has 2 DOFs, and each DOF provides an equilibrium equation.
→ The number of equations is $2N$ (= 10).

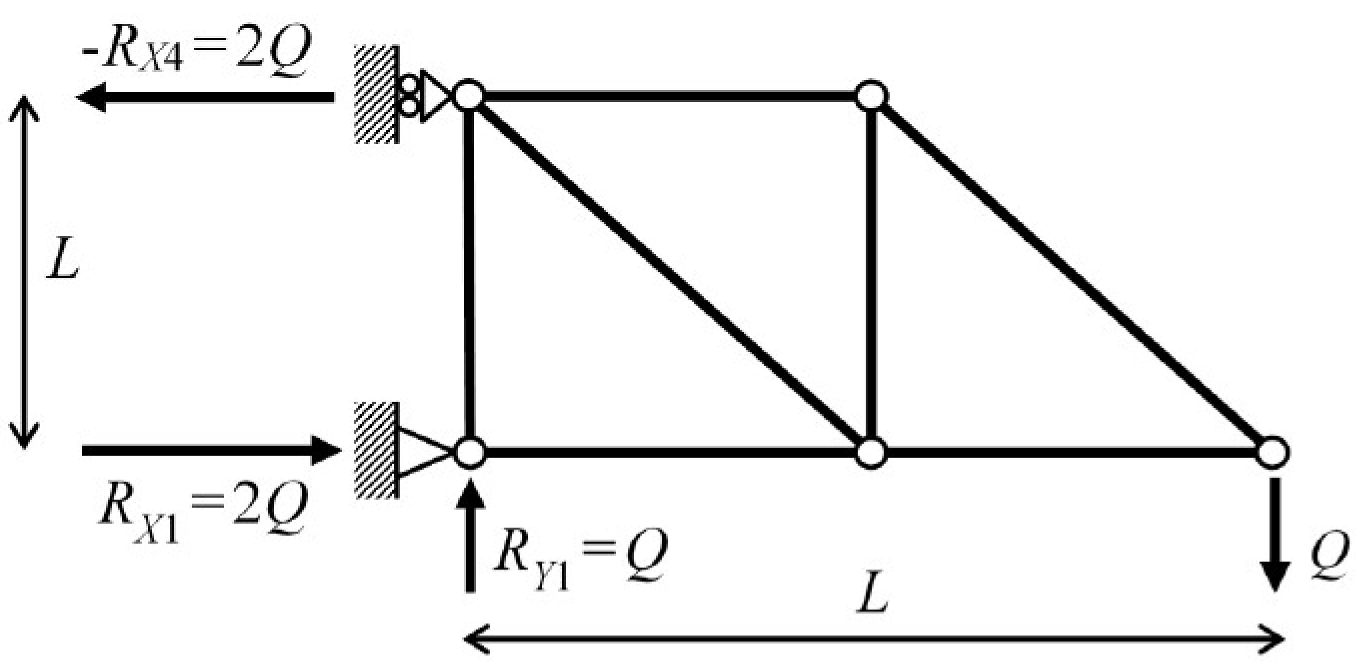
The variables are the forces in the elements and the reactions at the fixities.
→ The number of variables is $E + F$ (= 10).

Solution

The system of equations is sparse, with each equation involving only a few variables. It can be solved by considering the DOFs in an order such that a single force is still unknown at the DOF currently considered. In trusses, the first DOF to consider if often at a free extremity or a support. We can then jump from node to node, considering both DOFs of each node successively. The following sequence solves the current example:

DOF	Equation		Solution
Y3	$\beta F_7 - Q = 0$	$\rightarrow$	$F_7 = \dfrac{Q}{\beta}$
X3	$-F_2 - \beta F_7 = 0$	$\rightarrow$	$F_2 = -\beta F_7 = -Q$
X5	$-F_3 + \beta F_7 = 0$	$\rightarrow$	$F_3 = \beta F_7 = Q$
Y5	$-F_6 - \beta F_7 = 0$	$\rightarrow$	$F_6 = -\beta F_7 = -Q$
Y2	$\beta F_5 + F_6 = 0$	$\rightarrow$	$F_5 = -\dfrac{F_6}{\beta} = \dfrac{Q}{\beta}$
X2	$-F_1 + F_2 - \beta F_5 = 0$	$\rightarrow$	$F_1 = F_2 - \beta F_5 = -2Q$
Y4	$-F_4 - \beta F_5 = 0$	$\rightarrow$	$F_4 = -\beta F_5 = -Q$
X4	$F_3 + \beta F_5 + R_{X4} = 0$	$\rightarrow$	$R_{X4} = -F_3 - \beta F_5 = -2Q$
X1	$F_1 + R_{X1} = 0$	$\rightarrow$	$R_{X1} = -F_1 = 2Q$
Y1	$F_4 + R_{Y1} = 0$	$\rightarrow$	$R_{Y1} = -F_4 = Q$

We can verify that the truss as a whole is in equilibrium under the loads and reactions:



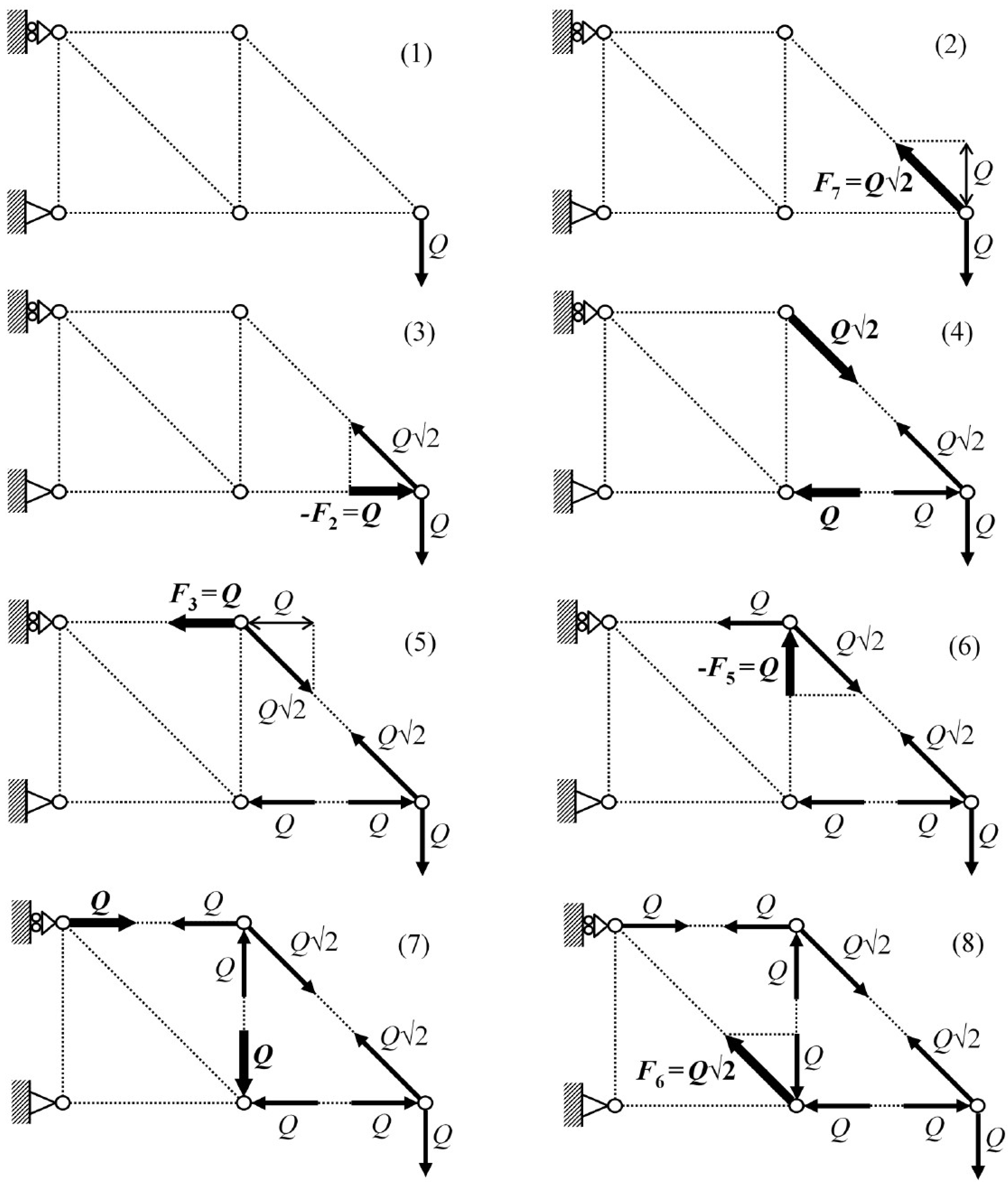
Sum of horizontal forces: $2Q - 2Q = 0$

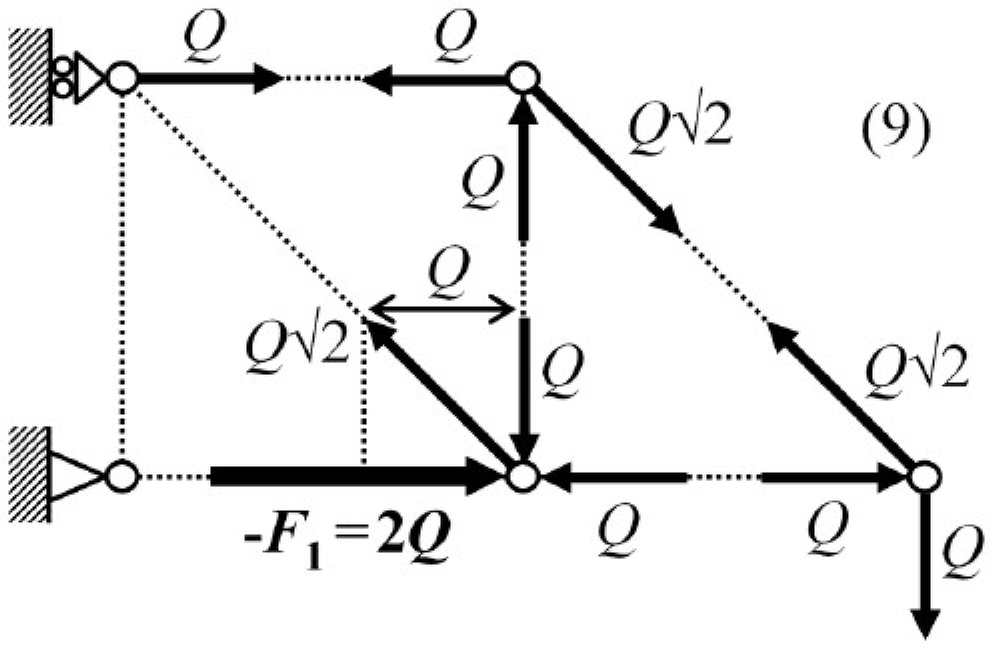
Sum of vertical forces: $Q - Q = 0$

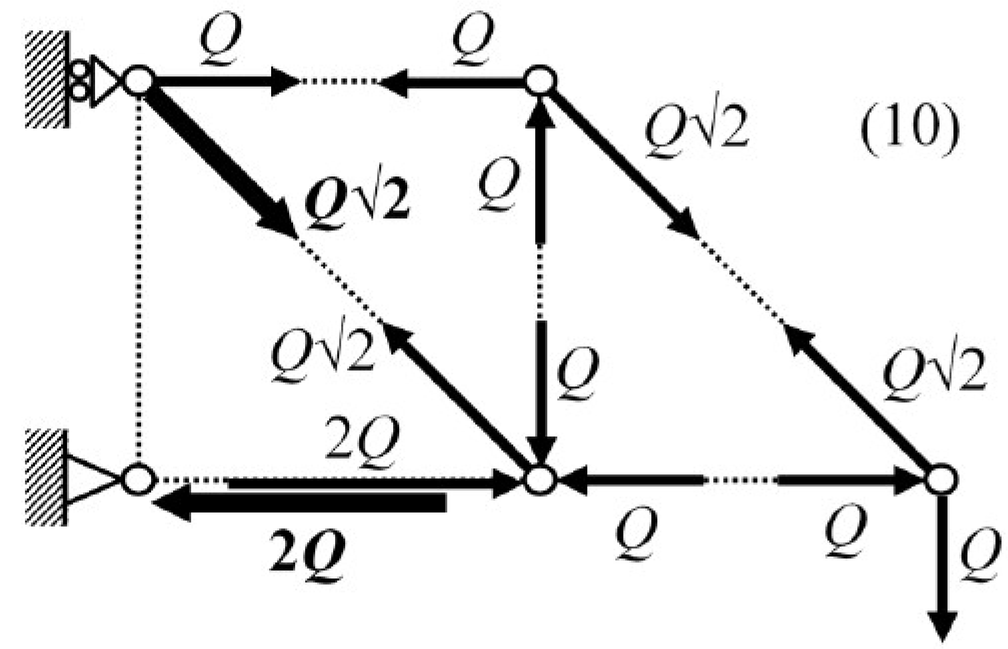
Sum of moments about node 1: $(2Q)(L) - (Q)(2L) = 0$

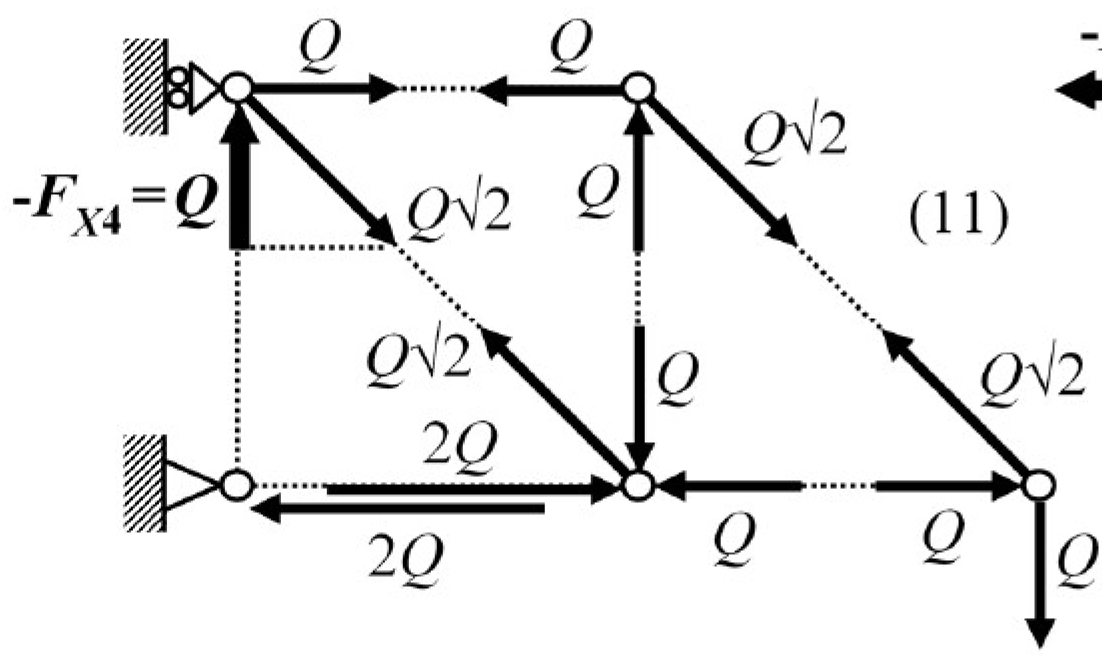
$\rightarrow$ Equilibrium is satisfied.

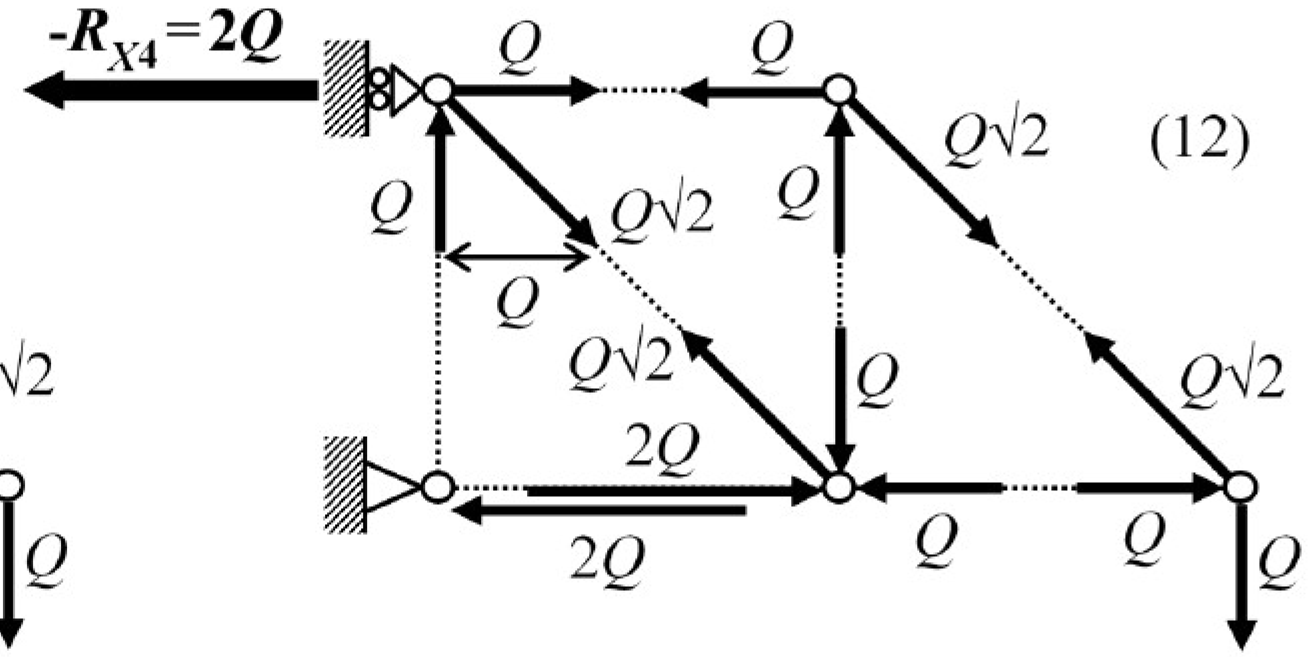
Note: in the above diagram we chose to change the orientation of the vector representing R_{X4}, so it now represents $-R_{X4}$. Because we obtained a negative value for R_{X4}, the force $-R_{X4}$ is positive and the new vector is therefore in the actual direction of the force. Switching both the direction of a vector and the sign of the variable associated with the vector keeps the notation consistent. Following the same solution sequence as above, the element forces and support reactions could have been determined graphically:

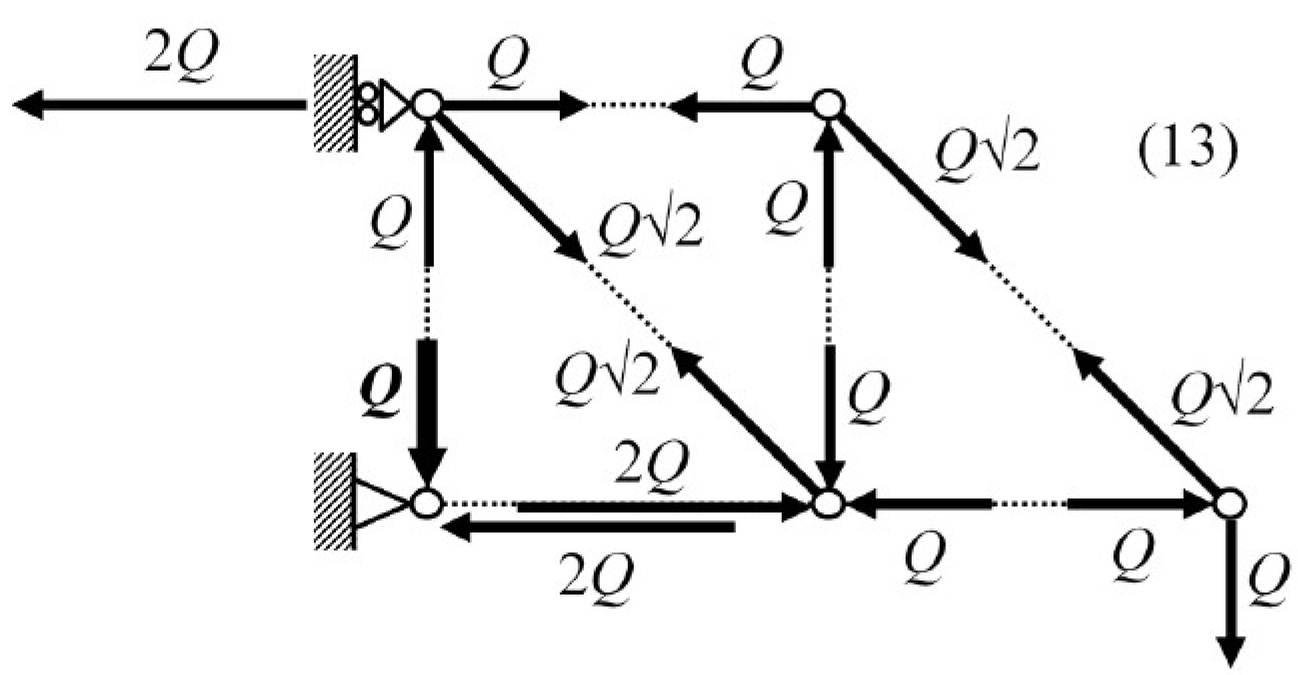


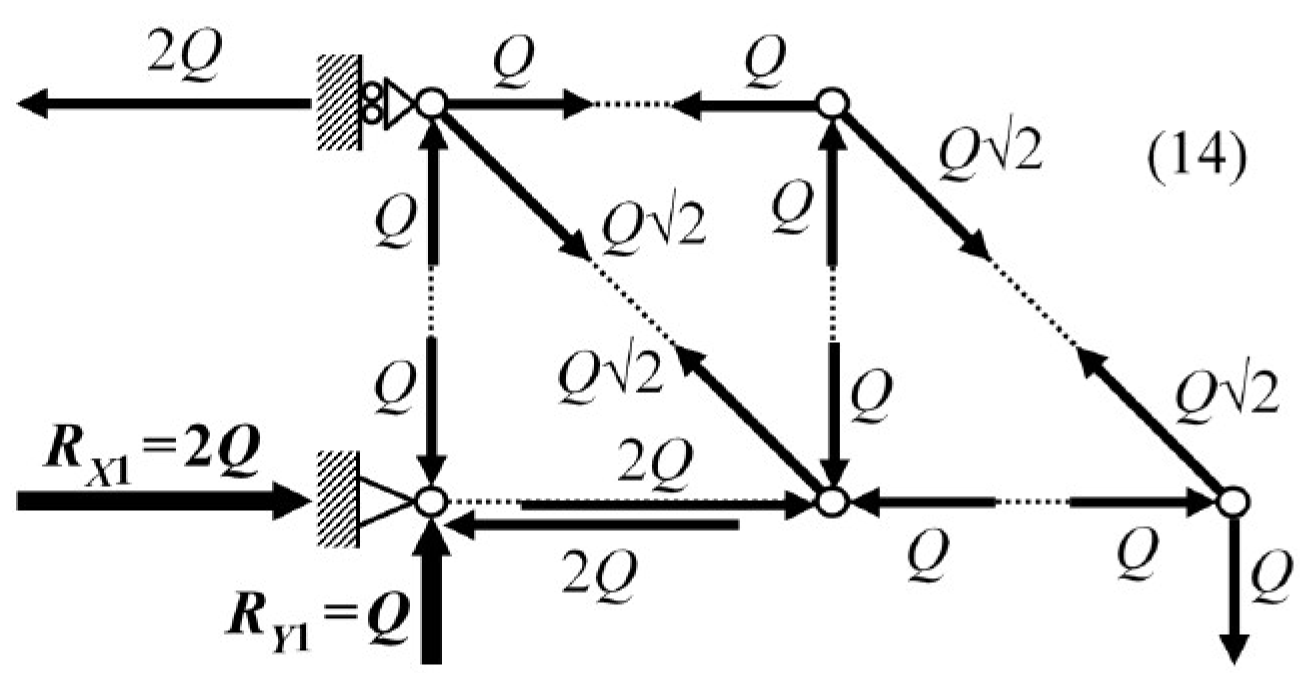

(9)
Q
Q
Q√2
Q
Q
Q√2
Q
Q√2
-F1 = 2Q
Q
Q
Q


(10)
Q
Q
Q√2
Q
Q√2
Q√2
Q
Q√2
2Q
2Q
Q
Q
Q


(11)
Q
Q
Q√2
-FX4 = Q
Q√2
Q
Q√2
Q
Q√2
2Q
2Q
Q
Q
Q


(12)
-RX4 = 2Q
Q
Q
Q√2
Q
Q√2
Q
Q
Q√2
Q
Q√2
2Q
2Q
Q
Q
Q


(13)
2Q
Q
Q
Q√2
Q
Q√2
Q
Q
Q√2
Q
Q√2
2Q
2Q
Q
Q
Q


(14)
2Q
Q
Q
Q√2
Q
Q√2
Q
Q
Q√2
Q
Q√2
RX1 = 2Q
2Q
2Q
RY1 = Q
Q
Q
Q

Matrix Formulation

Vectors and matrices can be used to express a system of linear equations. In matrix form, the equilibrium equations of the current truss example are as follows:

$$\begin{pmatrix} 1 & & & & & & & 1 & & \\ & & & 1 & & & & & 1 & \\ -1 & 1 & & & -\beta & & & & & \\ & & & & \beta & 1 & & & & \\ & -1 & & & & & -\beta & & & \\ & & & & & & \beta & & & \\ & & 1 & & \beta & & & & & 1 \\ & & & -1 & -\beta & & & & & \\ & & -1 & & & & \beta & & & \\ & & & & & -1 & -\beta & & & \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ R_{X1} \\ R_{Y1} \\ R_{X4} \end{pmatrix} + \begin{pmatrix} \\ \\ \\ \\ \\ -Q \\ \\ \\ \\ \\ \end{pmatrix} = \overline{0}$$

<u>Note</u>: the blank terms in the matrices and vectors represent zeros, and the $\overline{0}$ vector is full of zeros. The number of terms in the $\overline{0}$ vector matches the number of rows in the system.

In compact form:

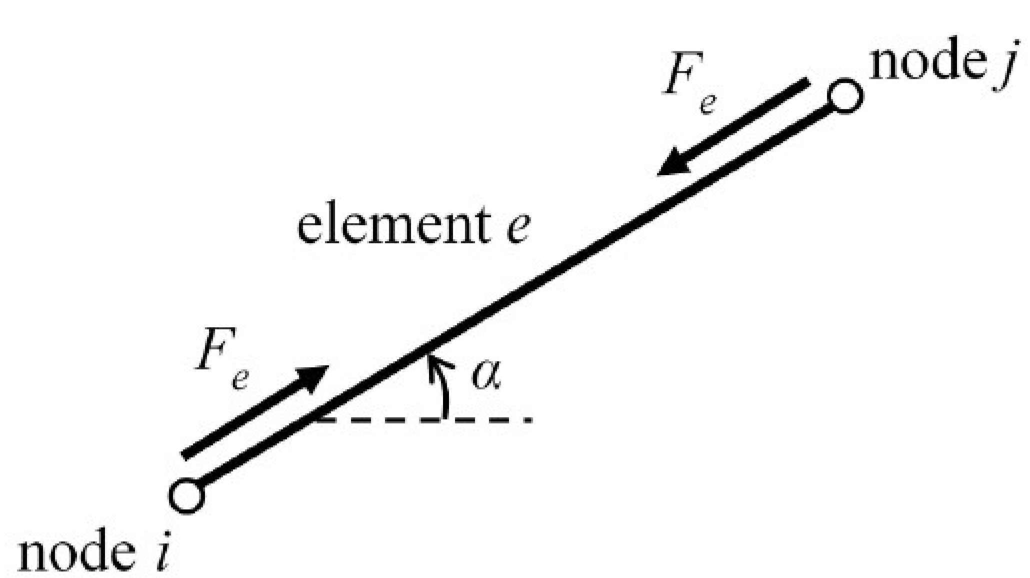
$$\underline{A}\overline{F} + \overline{Q} = \overline{0}$$

$\underline{A}$ is the force projection matrix, $\overline{F}$ the force vector and $\overline{Q}$ the load vector. $\overline{F}$ contains the element forces and support reactions, both of which are to be determined. If $\underline{A}$ can be inversed:

$$\overline{F} = -\underline{A}^{-1}\overline{Q}$$

Matrix $\underline{A}$ has a row for each DOF and a column for each element force and support reaction. With N, E and F the number of nodes, elements and fixities, $\underline{A}$ has $2N$ rows and $E + F$ columns. A necessary (but not sufficient) condition to inverse $\underline{A}$ is that it is square, and therefore the truss topology must satisfy $2N = E + F$.

Each truss element contributes to the force projection matrix $\underline{A}$ as follows:



column e

$\downarrow$

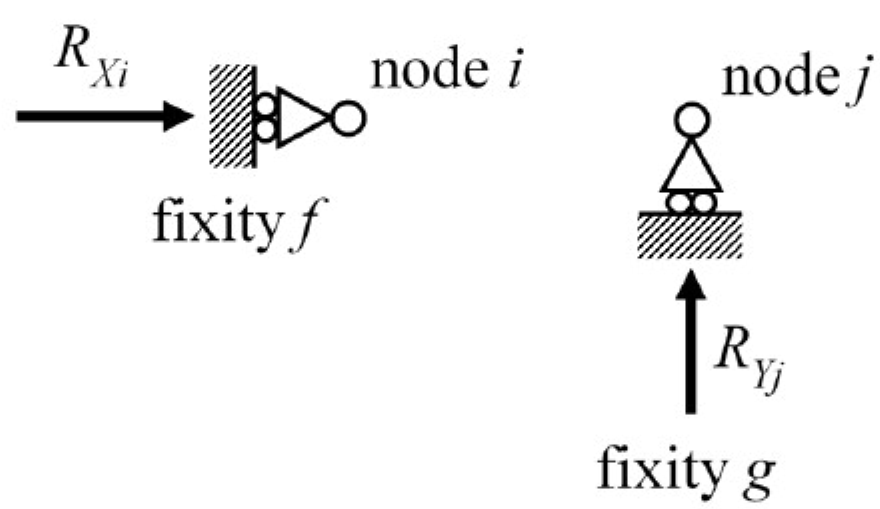
$$
\underline{A} = \begin{array}{c}\left(\begin{array}{c} \cos\alpha \\ \sin\alpha \\ \\ -\cos\alpha \\ -\sin\alpha \end{array}\right)\end{array}
\begin{array}{l} \leftarrow \ \text{row } 2i-1 \\ \leftarrow \ \text{row } 2i \\ \\ \leftarrow \ \text{row } 2j-1 \\ \leftarrow \ \text{row } 2j \end{array}
$$

Note: sines and cosines should be calculated directly, without computing the angle first. Suppose the coordinates of nodes i and j are (x_i, y_i) and (x_j, y_j):

$$
L = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2} \qquad \cos\alpha = \frac{x_j - x_i}{L} \qquad \sin\alpha = \frac{y_j - y_i}{L}
$$

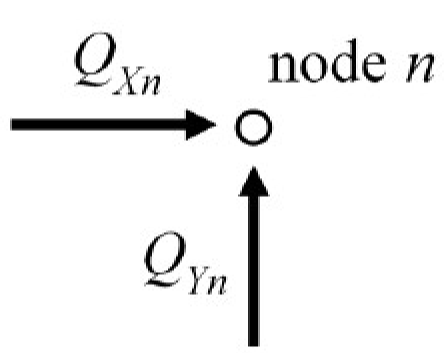
Computing an angle using an inverse trigonometric function (arcsin, arccos, arctan) is risky because any sine, cosine or tangent value is associated with a pair of different angles, only one of which is returned by the function.

Each fixity contributes to the force projection matrix $\underline{A}$ as follows (E is the number of elements, and fixities are numbered from 1 to F):



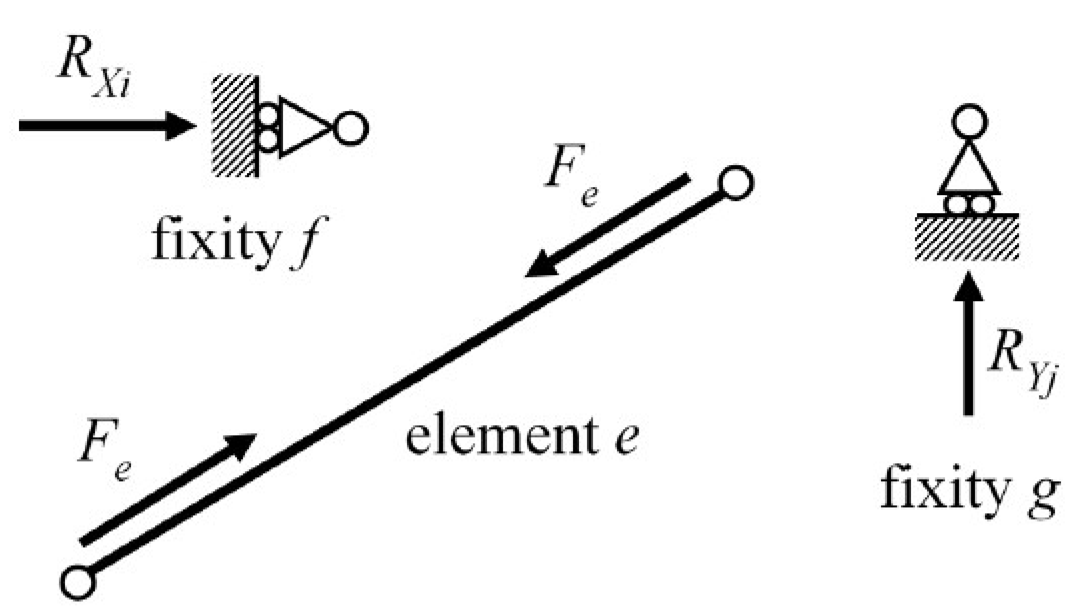
$$
\begin{array}{cl}
\begin{array}{cc} \text{column } (E+f) & \text{column } (E+g) \\ \downarrow & \downarrow \end{array} & \\
\underline{A} = \left(\begin{array}{cc} 1 & \\ & \\ & 1 \end{array}\right) & \begin{array}{l} \leftarrow \ \text{row } 2i-1 \\ \\ \leftarrow \ \text{row } 2j \end{array}
\end{array}
$$

Each load contributes to the load vector $\overline{Q}$ as follows:



$$
\overline{Q} = \left(\begin{array}{c} Q_{Xn} \\ Q_{Yn} \end{array}\right) \begin{array}{l} \leftarrow \ \text{row } 2n-1 \\ \leftarrow \ \text{row } 2n \end{array}
$$

Once the force vector $\overline{F}$ has been computed, the element forces and support reactions are retrieved as follows:



$$\overline{F} = \left(\begin{array}{c} F_e \\ \\ R_{Xi} \\ \\ R_{Yj} \end{array}\right) \begin{array}{l} \leftarrow \text{row} \quad e \\ \\ \leftarrow \text{row} \quad E+f \\ \\ \leftarrow \text{row} \quad E+g \end{array}$$

Truss Stability and Determinacy

Characterization

The following characterizations apply to any structure, i.e. not only 2D trusses:

A structure is in equilibrium when every DOF is in equilibrium under the loads, inner actions (e.g. element forces in a truss) and support reactions.

A structure is stable if equilibrium can be achieved with a load applied to any DOF. It is unstable otherwise. Unstable equilibrium can be achieved by applying a particular combination of loads to an unstable structure.

A stable structure is determinate if the inner actions and support reactions can be determined by expressing equilibrium only. It is indeterminate otherwise.

Other common terminology: determinate = statically determinate = isostatic
indeterminate = statically indeterminate = hyperstatic

Topological Analysis

An equilibrium equation can be formulated for each DOF, and each node has 2 DOFs in a 2D truss. With N the number of nodes, we can formulate $2N$ equilibrium equations. The variables in these equations are the element forces and support reactions. With E and F the number of elements and fixities, the equilibrium equations involve $E + F$ variables.

If $2N > E + F$, the number of variables is insufficient to satisfy every equilibrium equation under some load combinations, and the truss is unstable. The truss is not necessarily stable when $2N \leq E + F$, and stability must be established by other means than a topological analysis (see examples below).

If the truss is stable, there is always at least one solution for the element forces and support reactions under any load combination. If $2N = E + F$, the number of equations and variables match and there can only be one solution for the element forces and support reactions. The truss is determinate, and the force projection matrix can be inversed. If $2N < E + F$, the number of equations is insufficient and there is an infinite number of solutions for the element forces and support reactions. Additional equations involving the deformation of the structure must be introduced to identify the correct solution, and the truss is indeterminate.