Derivation of Tensegrity Geometry

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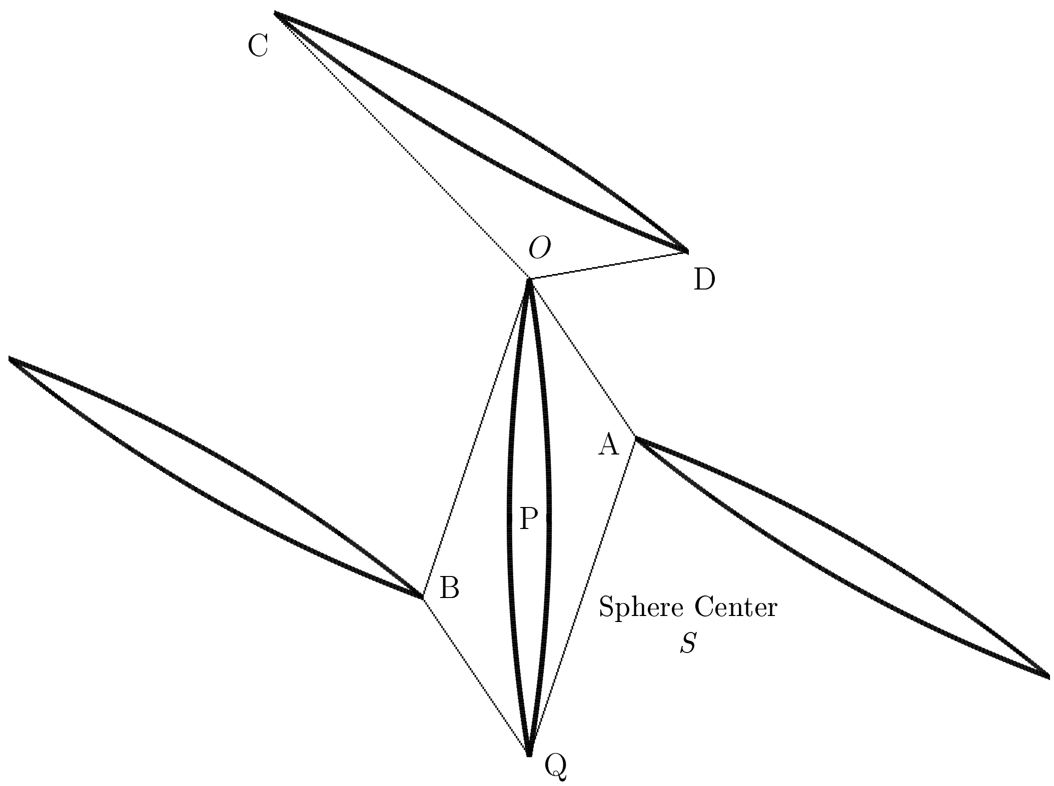

If $\left[\vec{F}_{PO}\right]$ is the compressive vector force exerted by the strut OPQ upon point O, and
$\left[\vec{F}_{OA}\right]$ is the tensive vector force exerted by the string OA upon point O, and
$\left[\vec{F}_{OB}\right]$ is the tensive vector force exerted by the string OB upon point O, and
$\left[\vec{F}_{OC}\right]$ is the tensive vector force exerted by the string OC upon point O, and
$\left[\vec{F}_{OD}\right]$ is the tensive vector force exerted by the string OD upon point O

then the geometry of the tensegrity structure is completely characterized by the force equilibrium equation for point O:

$$\left[\vec{F}_{PO}\right] = \left[\vec{F}_{OA}\right] + \left[\vec{F}_{OB}\right] + \left[\vec{F}_{OC}\right] + \left[\vec{F}_{OD}\right]$$

which states that point O is not moving because all applied forces are in static equilibrium (no net force). This can be restated (in more detail, separating magnitude from direction of force) as follows:

$$m_{PO}\left[\vec{i}_{PO}\right] = m_{OA}\left[\vec{i}_{OA}\right] + m_{OB}\left[\vec{i}_{OB}\right] + m_{OC}\left[\vec{i}_{OC}\right] + m_{OD}\left[\vec{i}_{OD}\right]$$

where m_{pq} is the magnitude and $\left[\vec{i}_{pq}\right]$ is the unit vector in the direction from p to q of each force.

Now, by symmetry strings OA and OD apply the same force to point O, and similarly for strings OB and OC, so:

$$m_{PO}\left[\vec{i}_{PO}\right] = m_{ss}\left[\vec{i}_{OA}\right] + m_{ls}\left[\vec{i}_{OB}\right] + m_{ls}\left[\vec{i}_{OC}\right] + m_{ss}\left[\vec{i}_{OD}\right]$$

where the force-magnitude subscripts $_{ss}$ and $_{ls}$ refer to the "short" and "long" strings in the diagram, respectively.

At this point, each of the force directions (the unit vectors) can be represented as the direction of the structural component (strut or string) along which the force acts. That is to say, if we represent each such component as a vector $(\vec{V}_{pq})$ from one of its endpoints, p, to the other, q, oriented in the direction of action of the force along the component, then we can take the direction of the force to be the direction of this vector. This is written as the vector, $\vec{V}_{pq}$, divided by its length, l_{pq}, so:

$$\left[\vec{i}_{pq}\right] = \left[\frac{\vec{V}_{pq}}{l_{pq}}\right]$$

Substituting this expression for the force direction in the previous equation:

$$m_{pq}\left[\vec{i}_{pq}\right] = m_{pq}\left[\frac{\vec{V}_{pq}}{l_{pq}}\right] = \frac{m_{pq}}{l_{pq}}\left[\vec{V}_{pq}\right]$$

so, therefore the equilibrium equation can be rewritten:

$$\frac{m_{PO}}{l_{PO}}\left[\vec{V}_{PO}\right] = \frac{m_{ss}}{l_{ss}}\left[\vec{V}_{OA}\right] + \frac{m_{ls}}{l_{ls}}\left[\vec{V}_{OB}\right] + \frac{m_{ls}}{l_{ls}}\left[\vec{V}_{OC}\right] + \frac{m_{ss}}{l_{ss}}\left[\vec{V}_{OD}\right]$$

or, combining terms with identical coefficients:

$$\frac{m_{PO}}{l_{PO}}\left[\vec{V}_{PO}\right] = \frac{m_{ss}}{l_{ss}}\left(\left[\vec{V}_{OA}\right] + \left[\vec{V}_{OD}\right]\right) + \frac{m_{ls}}{l_{ls}}\left(\left[\vec{V}_{OB}\right] + \left[\vec{V}_{OC}\right]\right)$$

Now, if we consider only the components of force which act at point O perpendicular to the axis (OPQ) of the strut, then the compressive force contributed by the strut will be zero, so this part of the force equilibrium equation is:

$$\left[\vec{0}\right] = \frac{m_{ss}}{l_{ss}}\left(\left[\vec{R}_{OA}\right] + \left[\vec{R}_{OD}\right]\right) + \frac{m_{ls}}{l_{ls}}\left(\left[\vec{R}_{OB}\right] + \left[\vec{R}_{OC}\right]\right)$$

where each of the vectors R is the *component* of the corresponding vector V in the previous equation, that lies in the plane orthogonal to the strut.

Then, once again, we may consider separately the forces acting at point O

1. parallel to the axis, SP, of rotation of the strut through the transformation angle parameter, ϕ, and
2. perpendicular to the strut axis and in the plane of the transformation rotation through angle ϕ

The two resulting scalar equations can be written in terms of ϕ — the angular transformation parameter, and d — the distance of the strut (point P) from the center of spherical symmetry, S, of the entire structure. The equations can then be solved simultaneously to derive in a formula for d in terms of ϕ.

The two equations are:

$$0 = \frac{m_{ss}}{l_{ss}} \left(z_{OA} + z_{OD} \right) + \frac{m_{ls}}{l_{ls}} \left(z_{OB} + z_{OC} \right)$$

and:

$$0 = \frac{m_{ss}}{l_{ss}} \left(x_{OA} + x_{OD} \right) + \frac{m_{ls}}{l_{ls}} \left(x_{OB} + x_{OC} \right)$$

With only slight rearrangement, they become:

$$-\frac{m_{ss} l_{ls}}{l_{ss} m_{ls}} = \frac{\left(z_{OB} + z_{OC} \right)}{\left(z_{OA} + z_{OD} \right)}$$

and

$$-\frac{m_{ss} l_{ls}}{l_{ss} m_{ls}} = \frac{\left(x_{OB} + x_{OC} \right)}{\left(x_{OA} + x_{OD} \right)}$$

Combining the two, in preparation for solution:

$$\frac{\left(z_{OB} + z_{OC} \right)}{\left(z_{OA} + z_{OD} \right)} = \frac{\left(x_{OB} + x_{OC} \right)}{\left(x_{OA} + x_{OD} \right)}$$

which is the same as:

$$\left(z_{OB} + z_{OC} \right) \left(x_{OA} + x_{OD} \right) = \left(x_{OB} + x_{OC} \right) \left(z_{OA} + z_{OD} \right)$$

Note: Everything to this point has been done without reference to any coordinate system, and without reference to any particular spherical tensegrity (as to 3–symmetry, 4–symmetry, etc.). We now introduce a coordinate system in order to write formulas for the coordinates of each of the points that appear in the force equilibrium equation above, in terms of the parameters, d and ϕ. When we have formulas for the coordinates, we will solve the equation above for d in terms of ϕ, and will then be in a position to produce formulas for all lengths and angles of the tensegrity structure as well as coordinates for all points.