Derivation of Tensegrity Geometry

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If $\left[\vec{F}_{PO}\right]$ is the compressive vector force exerted by the strut QPO upon point O, and
$\left[\vec{F}_{OA}\right]$ is the tensive vector force exerted by the string OA upon point O, and
$\left[\vec{F}_{OB}\right]$ is the tensive vector force exerted by the string OB upon point O, and
$\left[\vec{F}_{OC}\right]$ is the tensive vector force exerted by the string OC upon point O, and
$\left[\vec{F}_{OD}\right]$ is the tensive vector force exerted by the string OD upon point O
then the geometry of the tensegrity structure is completely characterized by the force equilibrium equ

$$\left[\vec{F}_{PO}\right] = \left[\vec{F}_{OA}\right] + \left[\vec{F}_{OB}\right] + \left[\vec{F}_{OC}\right] + \left[\vec{F}_{OD}\right]$$

which states that point O is not moving because all applied forces are in static equilibrium (no net fo
detail, separating magnitude from direction of force) as follows:

$$m_{PO}\left[\vec{i}_{PO}\right] = m_{OA}\left[\vec{i}_{OA}\right] + m_{OB}\left[\vec{i}_{OB}\right] + m_{OC}\left[\vec{i}_{OC}\right] + m_{OD}\left[\vec{i}_{OD}\right]$$

where m_{pq} is the magnitude and $\left[\vec{i}_{pq}\right]$ is the unit vector in the direction from p to q of each force.

Now, by symmetry strings OA and OD apply the same force to point O, and similarly for strings OB

$$m_{PO}\left[\vec{i}_{PO}\right] = m_{ss}\left[\vec{i}_{OA}\right] + m_{ls}\left[\vec{i}_{OB}\right] + m_{ls}\left[\vec{i}_{OC}\right] + m_{ss}\left[\vec{i}_{OD}\right]$$

where the force-magnitude subscripts $_{ss}$ and $_{ls}$ refer to the "short" and "long" strings in the diagram, res

At this point, each of the force directions (the unit vectors) can be represented as the direction of
string) along which the force acts. That is to say, if we represent each such component as a vector ($\vec{V}_{pq}$)
other, q, oriented in the direction of action of the force along the component, then we can take the direc
of this vector. This is written as the vector, $\vec{V}_{pq}$, divided by its length, l_{pq}, so:

$$\left[\vec{i}_{pq}\right] = \left[\frac{\vec{V}_{pq}}{l_{pq}}\right]$$

Substituting this expression for the force direction in the previous equation:

$$m_{pq}\left[\vec{i}_{pq}\right] = m_{pq}\left[\frac{\vec{V}_{pq}}{l_{pq}}\right] = \frac{m_{pq}}{l_{pq}}\left[\vec{V}_{pq}\right]$$

so, therefore the equilibrium equation can be rewritten:

$$\frac{m_{PO}}{l_{PO}}\left[\vec{V}_{PO}\right] = \frac{m_{ss}}{l_{ss}}\left[\vec{V}_{OA}\right] + \frac{m_{ls}}{l_{ls}}\left[\vec{V}_{OB}\right] + \frac{m_{ls}}{l_{ls}}\left[\vec{V}_{OC}\right] + \frac{m_{ss}}{l_{ss}}\left[\vec{V}_{OD}\right]$$

or, combining terms with identical coefficients:

$$\frac{m_{PO}}{l_{PO}}\left[\vec{V}_{PO}\right] = \frac{m_{ss}}{l_{ss}}\left(\left[\vec{V}_{OA}\right] + \left[\vec{V}_{OD}\right]\right) + \frac{m_{ls}}{l_{ls}}\left(\left[\vec{V}_{OB}\right] + \left[\vec{V}_{OC}\right]\right)$$

Now, if we consider only the components of force which act at point O perpendicular to the axis (QPO
force contributed by the strut will be zero, so this part of the force equilibrium equation is:

$$\left[\vec{0}\right] = \frac{m_{ss}}{l_{ss}}\left(\left[\vec{R}_{OA}\right] + \left[\vec{R}_{OD}\right]\right) + \frac{m_{ls}}{l_{ls}}\left(\left[\vec{R}_{OB}\right] + \left[\vec{R}_{OC}\right]\right)$$

where each of the vectors R is the *component* of the corresponding vector V in the previous equation, tha
strut.

Then, once again, we may consider separately the forces acting at point O

1. parallel to the axis, SP, of rotation of the strut through the transformation angle parameter, ϕ, anc

2. perpendicular to the strut axis and in the plane of the transformation rotation through angle ϕ

The two resulting scalar equations can be written in terms of ϕ — the angular transformation parameter (point P) from the center of spherical symmetry, S, of the entire structure. The equations can then be s formula for d in terms of ϕ.

The two equations are:

$$0 = \frac{m_{ss}}{l_{ss}}\left(z_{OA} + z_{OD}\right) + \frac{m_{ls}}{l_{ls}}\left(z_{OB} + z_{OC}\right)$$

and:

$$0 = \frac{m_{ss}}{l_{ss}}\left(x_{OA} + x_{OD}\right) + \frac{m_{ls}}{l_{ls}}\left(x_{OB} + x_{OC}\right)$$

With only slight rearrangement, they become:

$$-\frac{m_{ss}l_{ls}}{l_{ss}m_{ls}} = \frac{\left(z_{OB} + z_{OC}\right)}{\left(z_{OA} + z_{OD}\right)}$$

and

$$-\frac{m_{ss}l_{ls}}{l_{ss}m_{ls}} = \frac{\left(x_{OB} + x_{OC}\right)}{\left(x_{OA} + x_{OD}\right)}$$

Combining the two, in preparation for solution:

$$\frac{\left(z_{OB} + z_{OC}\right)}{\left(z_{OA} + z_{OD}\right)} = \frac{\left(x_{OB} + x_{OC}\right)}{\left(x_{OA} + x_{OD}\right)}$$

which is the same as:

$$\left(z_{OB} + z_{OC}\right)\left(x_{OA} + x_{OD}\right) = \left(x_{OB} + x_{OC}\right)\left(z_{OA} + z_{OD}\right)$$

Note: Everything to this point has been done without reference to any coordinate system, and without refe tensegrity (as to 3–symmetry, 4–symmetry, etc.). We now introduce a coordinate system in order to write each of the points that appear in the force equilibrium equation above, in terms of the parameters, d and the coordinates, we will solve the equation above for d in terms of ϕ, and will then be in a position to prod angles of the tensegrity structure as well as coordinates for all points.

If the the configuration above is taken to be from a six-strut (tetrahedral) tensegrity structure, and consic
cartesian coordinate system:

1. whose origin, S, is at the center of gravity of the entire spherical tensegrity, and
2. whose Z axis passes through point P of strut OPQ, where point P is distance d from point S,
3. whose X, Y and Z units are each equal to half the length of the strut (i.e. $\overline{PO}$), and
4. such that the axis of the strut lies at a 45 degree angle from the XZ and YZ planes, with po both X and Y are positive.

then the coordinates of each of the labelled points in the figure can be written by inspection at three geometry is known at these points:

$$\phi = 0 \Longrightarrow d = \tfrac{\sqrt{2}}{2}$$

$$\begin{array}{ll}
\textit{point} & \textit{coordinates} \\
O: & \left[\ \tfrac{\sqrt{2}}{2}\ ,\ \tfrac{\sqrt{2}}{2}\ ,\ \tfrac{\sqrt{2}}{2}\ \right] \\
P: & \left[\ 0\ ,\ 0\ ,\ \tfrac{\sqrt{2}}{2}\ \right] \\
Q: & \left[-\tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}\right] \\
A: & \left[\ \tfrac{\sqrt{2}}{2}\ ,\ \tfrac{\sqrt{2}}{2}\ ,\ \tfrac{\sqrt{2}}{2}\ \right] \\
B: & \left[-\tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}\right] \\
C: & \left[-\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}\right] \\
D: & \left[\ \tfrac{\sqrt{2}}{2}\ ,\ \tfrac{\sqrt{2}}{2}\ ,\ \tfrac{\sqrt{2}}{2}\ \right] \\
S: & \left[\ 0\ ,\ 0\ ,\ 0\ \right]
\end{array}$$

$$\phi = \tfrac{\pi}{4} \Longrightarrow d = \tfrac{1}{2}$$

$$\begin{array}{ll}
\textit{point} & \textit{coordinates} \\
O: & \left[\ 0\ ,\ 1\ ,\ \tfrac{1}{2}\ \right] \\
P: & \left[\ 0\ ,\ 0\ ,\ \tfrac{1}{2}\ \right] \\
Q: & \left[\ 0\ , -1, \tfrac{1}{2}\ \right] \\
A: & \left[\ \tfrac{1}{2}\ ,\ 0\ ,\ 1\ \right] \\
B: & \left[-\tfrac{1}{2}, 0\ ,\ 1\ \right] \\
C: & \left[-1, \tfrac{1}{2}\ ,\ 0\ \right] \\
D: & \left[\ 1\ ,\ \tfrac{1}{2}\ ,\ 0\ \right] \\
S: & \left[\ 0\ ,\ 0\ ,\ 0\ \right]
\end{array}$$

$$\phi = \tfrac{\pi}{2} \Longrightarrow$$

$$\begin{array}{ll}
\textit{point} & \textit{coor} \\
O: & \left[-\tfrac{\sqrt{2}}{2},\right. \\
P: & \left[\ 0\ ,\right. \\
Q: & \left[\ \tfrac{\sqrt{2}}{2}\ ,-\right. \\
A: & \left[\ \tfrac{\sqrt{2}}{2}\ ,-\right. \\
B: & \left[-\tfrac{\sqrt{2}}{2},\right. \\
C: & \left[-\tfrac{\sqrt{2}}{2},\right. \\
D: & \left[\ \tfrac{\sqrt{2}}{2}\ ,\right. \\
S: & \left[\ 0\ ,\right.
\end{array}$$

From this position, strut OPQ can be rotated about the Z axis through the angle ϕ (where ϕ is betw distance d from the origin to get coordinate formulas for the general-case position of the strut. To do this, first three rows above, but converted to *homogeneous coordinates*, by adding a fourth coordinate whose Then we post-multiply this coordinate matrix by the bilinear transformation matrix for rotation about th

$$\begin{bmatrix} \frac{\sqrt{2}}{2} , & \frac{\sqrt{2}}{2} , & \frac{\sqrt{2}}{2} , 1 \\ 0 , & 0 , & \frac{\sqrt{2}}{2} , 1 \\ -\frac{\sqrt{2}}{2} , & -\frac{\sqrt{2}}{2} , & \frac{\sqrt{2}}{2} , 1 \end{bmatrix} \begin{bmatrix} \cos\phi & , \sin\phi , & 0 & , 0 \\ -\sin\phi & , \cos\phi , & 0 & , 0 \\ 0 & , \ 0 \ , & 1 & , 0 \\ 0 & , \ 0 \ , & -(\frac{\sqrt{2}}{2} - d) & , 1 \end{bmatrix} = \left[\begin{array}{ll} \frac{\sqrt{2}}{2}(\cos\phi - \sin\phi) , & \frac{\sqrt{2}}{2}(\sin\phi \\ 0 , & \\ \frac{\sqrt{2}}{2}(\sin\phi - \cos\phi) , & -\frac{\sqrt{2}}{2}(\sin \end{array} \right.$$

Top to back (for CD) — rotate strut OPQ -90 w/o Z, then -90 w/o X:

$$\begin{bmatrix} 0\,, -1\,, 0\,, 0 \\ 1\,,\ \ 0\ \ , 0\,, 0 \\ 0\,,\ \ 0\ \ , 1\,, 0 \\ 0\,,\ \ 0\ \ , 0\,, 1 \end{bmatrix} \begin{bmatrix} 1\,, 0\,,\ \ 0\ \ , 0 \\ 0\,, 0\,, -1\,, 0 \\ 0\,, 1\,,\ \ 0\ \ , 0 \\ 0\,, 0\,,\ \ 0\ \ , 1 \end{bmatrix} = \begin{bmatrix} 0\,, 0\,, 1\,, 0 \\ 1\,, 0\,, 0\,, 0 \\ 0\,, 1\,, 0\,, 0 \\ 0\,, 0\,, 0\,, 1 \end{bmatrix}$$

For D and C ($O \to D$ and $Q \to C$):

$$\begin{bmatrix} \frac{\sqrt{2}}{2}(\cos\phi - \sin\phi)\,, & \frac{\sqrt{2}}{2}(\sin\phi + \cos\phi) & , d\,, 1 \\ \frac{\sqrt{2}}{2}(\sin\phi - \cos\phi)\,, & -\frac{\sqrt{2}}{2}(\sin\phi + \cos\phi) & , d\,, 1 \end{bmatrix} \begin{bmatrix} 0\,, 0\,, 1\,, 0 \\ 1\,, 0\,, 0\,, 0 \\ 0\,, 1\,, 0\,, 0 \\ 0\,, 0\,, 0\,, 1 \end{bmatrix} = \Bigg[\begin{matrix} \frac{\sqrt{2}}{2}(\sin\phi + \cos\phi) & , d\,, \\ -\frac{\sqrt{2}}{2}(\sin\phi + \cos\phi) & , d\,, \end{matrix}$$

Top to left (for B) — rotate strut OPQ -90 w/o Z, then -90 w/o Y:

$$\begin{bmatrix} 0\,, -1\,, 0\,, 0 \\ 1\,,\ \ 0\ \ , 0\,, 0 \\ 0\,,\ \ 0\ \ , 1\,, 0 \\ 0\,,\ \ 0\ \ , 0\,, 1 \end{bmatrix} \begin{bmatrix} 0\ \ , 0\,, 1\,, 0 \\ 0\ \ , 1\,, 0\,, 0 \\ -1\,, 0\,, 0\,, 0 \\ 0\ \ , 0\,, 0\,, 1 \end{bmatrix} = \begin{bmatrix} 0\ \ , -1\,, 0\,, 0 \\ 0\ \ ,\ \ 0\ \ , 1\,, 0 \\ -1\,,\ \ 0\ \ , 0\,, 0 \\ 0\ \ ,\ \ 0\ \ , 0\,, 1 \end{bmatrix}$$

For B ($O \to B$):

$$O \left[\ \frac{\sqrt{2}}{2}(\cos\phi - \sin\phi)\,, \frac{\sqrt{2}}{2}(\sin\phi + \cos\phi)\,, d\,, 1\ \right] \begin{bmatrix} 0\ \ , -1\,, 0\,, 0 \\ 0\ \ ,\ \ 0\ \ , 1\,, 0 \\ -1\,,\ \ 0\ \ , 0\,, 0 \\ 0\ \ ,\ \ 0\ \ , 0\,, 1 \end{bmatrix} = \Big[\ -d\,, \frac{\sqrt{2}}{2}(\sin\phi - \cos\phi)\,,$$

Top to right (for A) — rotate strut OPQ 90 w/o Z, then 90 w/o Y:

$$\begin{bmatrix} 0 & ,1 & ,0 & ,0 \\ -1 & ,0 & ,0 & ,0 \\ 0 & ,0 & ,1 & ,0 \\ 0 & ,0 & ,0 & ,1 \end{bmatrix} \begin{bmatrix} 0 & ,0 & ,-1 & ,0 \\ 0 & ,1 & ,\ 0 & ,0 \\ 1 & ,0 & ,\ 0 & ,0 \\ 0 & ,0 & ,\ 0 & ,1 \end{bmatrix} = \begin{bmatrix} 0 & ,1 & ,0 & ,0 \\ 0 & ,0 & ,1 & ,0 \\ 1 & ,0 & ,0 & ,0 \\ 0 & ,0 & ,0 & ,1 \end{bmatrix}$$

For A ($O \rightarrow A$):

$$\left[\ \tfrac{\sqrt{2}}{2}(\cos\phi - \sin\phi)\ ,\ \tfrac{\sqrt{2}}{2}(\sin\phi + \cos\phi)\ ,\ d\ ,\ 1\ \right] \begin{bmatrix} 0 & ,1 & ,0 & ,0 \\ 0 & ,0 & ,1 & ,0 \\ 1 & ,0 & ,0 & ,0 \\ 0 & ,0 & ,0 & ,1 \end{bmatrix} = \left[\ d\ ,\ \tfrac{\sqrt{2}}{2}(\cos\phi - \sin\phi)\ ,\ \tfrac{\sqrt{2}}{2}\right.$$

So, in the general case:

$$\begin{array}{cl} \textit{point} & \qquad\qquad\qquad\qquad \textit{coordinates} \\ O: & \left[\ \tfrac{\sqrt{2}}{2}(\cos\phi - \sin\phi)\ ,\ \tfrac{\sqrt{2}}{2}(\sin\phi + \cos\phi)\ ,\ d\ \right] \\ P: & \left[\ 0\ ,\ 0\ ,\ d\ \right] \\ Q: & \left[\ \tfrac{\sqrt{2}}{2}(\sin\phi - \cos\phi)\ ,-\tfrac{\sqrt{2}}{2}(\sin\phi + \cos\phi),\ d\ \right] \\ A: & \left[\ d\ ,\ \tfrac{\sqrt{2}}{2}(\cos\phi - \sin\phi)\ ,\tfrac{\sqrt{2}}{2}(\sin\phi + \cos\phi)\right] \\ B: & \left[\ -d\ ,\ \tfrac{\sqrt{2}}{2}(\sin\phi - \cos\phi)\ ,\tfrac{\sqrt{2}}{2}(\sin\phi + \cos\phi)\right] \\ C: & \left[-\tfrac{\sqrt{2}}{2}(\sin\phi + \cos\phi),\ d\ ,\tfrac{\sqrt{2}}{2}(\sin\phi - \cos\phi)\right] \\ D: & \left[\ \tfrac{\sqrt{2}}{2}(\sin\phi + \cos\phi)\ ,\ d\ ,\tfrac{\sqrt{2}}{2}(\cos\phi - \sin\phi)\right] \\ S: & \left[\ 0\ ,\ 0\ ,\ 0\ \right] \end{array}$$

Next, rotate the coordinate system $-\pi/4$, then ϕ around the Z axis, by use of the following transform

$$\begin{bmatrix} \frac{\sqrt{2}}{2} , \frac{\sqrt{2}}{2} , 0 , 0 \\ -\frac{\sqrt{2}}{2} , \frac{\sqrt{2}}{2} , 0 , 0 \\ 0 , 0 , 1 , 0 \\ 0 , 0 , 0 , 1 \end{bmatrix} \begin{bmatrix} \cos\phi , -\sin\phi , 0 , 0 \\ \sin\phi , \cos\phi , 0 , 0 \\ 0 , 0 , 1 , 0 \\ 0 , 0 , 0 , 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2}(\sin\phi+\cos\phi) , \frac{\sqrt{2}}{2}(\cos\phi - \sin \\ \frac{\sqrt{2}}{2}(\sin\phi-\cos\phi) , \frac{\sqrt{2}}{2}(\sin\phi + \cos \\ 0 , 0 \\ 0 , 0 \end{bmatrix}$$

Applying the transformation to the points:

$$\begin{bmatrix} \frac{\sqrt{2}}{2}(\cos\phi-\sin\phi) , \frac{\sqrt{2}}{2}(\sin\phi+\cos\phi) , d , 1 \\ 0 , 0 , d , 1 \\ \frac{\sqrt{2}}{2}(\sin\phi-\cos\phi) , -\frac{\sqrt{2}}{2}(\sin\phi+\cos\phi) , d , 1 \\ d , \frac{\sqrt{2}}{2}(\cos\phi-\sin\phi) , \frac{\sqrt{2}}{2}(\sin\phi+\cos\phi) , 1 \\ -d , \frac{\sqrt{2}}{2}(\sin\phi-\cos\phi) , \frac{\sqrt{2}}{2}(\sin\phi+\cos\phi) , 1 \\ -\frac{\sqrt{2}}{2}(\sin\phi+\cos\phi) , d , \frac{\sqrt{2}}{2}(\sin\phi-\cos\phi) , 1 \\ \frac{\sqrt{2}}{2}(\sin\phi+\cos\phi) , d , \frac{\sqrt{2}}{2}(\cos\phi-\sin\phi) , 1 \\ 0 , 0 , 0 , 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2}(\sin\phi+\cos\phi) , \frac{\sqrt{2}}{2}(\\ \frac{\sqrt{2}}{2}(\sin\phi-\cos\phi) , \frac{\sqrt{2}}{2}(\\ 0 , \\ 0 , \end{bmatrix}$$

The first point (O) is:

$$\left[\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}(\cos\phi+\sin\phi)(\cos\phi-\sin\phi) - \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}(\cos\phi+\sin\phi)(\cos\phi-\sin\phi) , \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}(\cos\phi-\sin\phi)^2 + \right.$$

Or:

$$\left[0 , \frac{1}{2}\left(\cos^2\phi - 2\cos\phi\sin\phi + \sin^2\phi\right) + \frac{1}{2}\left(\cos^2\phi + 2\cos\phi\sin\phi + \sin^2\phi\right) , d \right.$$

Or:

$$\left[0 , \frac{1}{2}\left(\cos^2\phi - 2\cos\phi\sin\phi + \sin^2\phi + \cos^2\phi + 2\cos\phi\sin\phi + \sin^2\phi\right) , d , \right.$$

Or:

$$\left[\; 0 , 1 , d , 1 \; \right]$$

The second point (A) is:

$$\Big[\ \tfrac{\sqrt{2}}{2}d\left(\sin\phi+\cos\phi\right)-\tfrac{\sqrt{2}}{2}\tfrac{\sqrt{2}}{2}\left(\cos\phi-\sin\phi\right)^2,\ -\tfrac{\sqrt{2}}{2}d\left(\sin\phi-\cos\phi\right)\tfrac{\sqrt{2}}{2}\tfrac{\sqrt{2}}{2}\left(\cos\phi+\sin\phi\right)\left(\cos\phi-\sin\right.$$

Or:

$$\Big[\ \tfrac{\sqrt{2}}{2}d\left(\sin\phi+\cos\phi\right)-\tfrac{1}{2}\left(\cos\phi-\sin\phi\right)^2,\ \tfrac{1}{2}\left(\cos^2\phi-\sin^2\phi\right)-\tfrac{\sqrt{2}}{2}d\left(\sin\phi-\cos\phi\right),\ \tfrac{\sqrt{2}}{2}\left(\mathrm{c}\right.$$

The third point (B) is:

$$\Big[\ -\tfrac{\sqrt{2}}{2}d\left(\sin\phi+\cos\phi\right)+\tfrac{\sqrt{2}}{2}\tfrac{\sqrt{2}}{2}\left(\cos\phi-\sin\phi\right)^2,\ \tfrac{\sqrt{2}}{2}d\left(\sin\phi-\cos\phi\right)+\tfrac{\sqrt{2}}{2}\tfrac{\sqrt{2}}{2}\left(\sin\phi+\cos\phi\right)\left(\sin\phi-\mathrm{c}\right.$$

Or:

$$\Big[\ -\tfrac{\sqrt{2}}{2}d\left(\sin\phi+\cos\phi\right)+\tfrac{1}{2}\left(\cos\phi-\sin\phi\right)^2,\ \tfrac{\sqrt{2}}{2}d\left(\sin\phi-\cos\phi\right)+\tfrac{1}{2}\left(\sin^2\phi-\cos^2\phi\right),\ \tfrac{\sqrt{2}}{2}\left(\right.$$

The fourth point (C) is:

$$\Big[\ -\tfrac{\sqrt{2}}{2}\tfrac{\sqrt{2}}{2}\left(\sin\phi+\cos\phi\right)^2-\tfrac{\sqrt{2}}{2}d\left(\cos\phi-\sin\phi\right),\ \tfrac{\sqrt{2}}{2}\tfrac{\sqrt{2}}{2}\left(\sin\phi+\cos\phi\right)\left(\sin\phi-\cos\phi\right)+\tfrac{\sqrt{2}}{2}d\left(\sin\phi+\mathrm{c}\right.$$

Or:

$$\Big[\ -\tfrac{1}{2}\left(\sin\phi+\cos\phi\right)^2-\tfrac{\sqrt{2}}{2}d\left(\cos\phi-\sin\phi\right),\ \tfrac{1}{2}\left(\sin^2\phi-\cos^2\phi\right)+\tfrac{\sqrt{2}}{2}d\left(\sin\phi+\cos\phi\right),\ \tfrac{\sqrt{2}}{2}\left(\right.$$

The fifth point (D) is:

$$\Big[\ \tfrac{\sqrt{2}}{2}\tfrac{\sqrt{2}}{2}\left(\sin\phi+\cos\phi\right)^2-\tfrac{\sqrt{2}}{2}d\left(\cos\phi-\sin\phi\right),\ -\tfrac{\sqrt{2}}{2}\tfrac{\sqrt{2}}{2}\left(\sin\phi+\cos\phi\right)\left(\sin\phi-\cos\phi\right)+\tfrac{\sqrt{2}}{2}d\left(\sin\phi+\mathrm{c}\right.$$

Or:

$$\Big[\ \tfrac{1}{2}\left(\sin\phi+\cos\phi\right)^2-\tfrac{\sqrt{2}}{2}d\left(\cos\phi-\sin\phi\right),\ -\tfrac{1}{2}\left(\sin^2\phi-\cos^2\phi\right)+\tfrac{\sqrt{2}}{2}d\left(\sin\phi+\cos\phi\right),\ \tfrac{\sqrt{2}}{2}\left(\right.$$

So, gathering all the points together:

$$\begin{array}{rl}
Q: & \Big[\ 0,\quad 1,\ \ldots \\
A: & \Big[\ \tfrac{\sqrt{2}}{2}d\left(\sin\phi+\cos\phi\right)-\tfrac{1}{2}\left(\cos\phi-\sin\phi\right)^2,\ -\tfrac{\sqrt{2}}{2}d\left(\sin\phi-\cos\phi\right)+\tfrac{1}{2}\left(\cos^2\phi-\sin^2\phi\right),\ \ldots \\
B: & \Big[-\tfrac{\sqrt{2}}{2}d\left(\sin\phi+\cos\phi\right)+\tfrac{1}{2}\left(\cos\phi-\sin\phi\right)^2,\ \tfrac{\sqrt{2}}{2}d\left(\sin\phi-\cos\phi\right)+\tfrac{1}{2}\left(\sin^2\phi-\cos^2\phi\right),\ \ldots \\
C: & \Big[-\tfrac{1}{2}\left(\sin\phi+\cos\phi\right)^2-\tfrac{\sqrt{2}}{2}d\left(\cos\phi-\sin\phi\right),\ \tfrac{1}{2}\left(\sin^2\phi-\cos^2\phi\right)+\tfrac{\sqrt{2}}{2}d\left(\sin\phi+\cos\phi\right),\ \ldots \\
D: & \Big[\ \tfrac{1}{2}\left(\sin\phi+\cos\phi\right)^2-\tfrac{\sqrt{2}}{2}d\left(\cos\phi-\sin\phi\right),\ -\tfrac{1}{2}\left(\sin^2\phi-\cos^2\phi\right)+\tfrac{\sqrt{2}}{2}d\left(\sin\phi+\cos\phi\right),\ \ldots
\end{array}$$

We can subtract 1 from the y-coordinate and d from the z-coordinate to center our coordinate system

$$\begin{array}{ll}
Q: & \left[\quad 0 \quad , \quad 0 \quad ,\right. \\
A: & \left[\ \frac{\sqrt{2}}{2}d\left(\sin\phi+\cos\phi\right)-\frac{1}{2}\left(\cos\phi-\sin\phi\right)^2 \ , -\frac{\sqrt{2}}{2}d\left(\sin\phi-\cos\phi\right)+\frac{1}{2}\left(\cos^2\phi-\sin^2\phi\right)-1\ ,\right. \\
B: & \left[-\frac{\sqrt{2}}{2}d\left(\sin\phi+\cos\phi\right)+\frac{1}{2}\left(\cos\phi-\sin\phi\right)^2 , \ \frac{\sqrt{2}}{2}d\left(\sin\phi-\cos\phi\right)+\frac{1}{2}\left(\sin^2\phi-\cos^2\phi\right)-1 \ ,\right. \\
C: & \left[-\frac{1}{2}\left(\sin\phi+\cos\phi\right)^2-\frac{\sqrt{2}}{2}d\left(\cos\phi-\sin\phi\right) , \ \frac{1}{2}\left(\sin^2\phi-\cos^2\phi\right)+\frac{\sqrt{2}}{2}d\left(\sin\phi+\cos\phi\right)-1 \ ,\right. \\
D: & \left[\ \frac{1}{2}\left(\sin\phi+\cos\phi\right)^2-\frac{\sqrt{2}}{2}d\left(\cos\phi-\sin\phi\right) \ , -\frac{1}{2}\left(\sin^2\phi-\cos^2\phi\right)+\frac{\sqrt{2}}{2}d\left(\sin\phi+\cos\phi\right)-1\ ,\right.
\end{array}$$

Recall the equilibrium equation for our class of symmetrical tensegrities:

$$\left(z_{OB}+z_{OC}\right)\left(x_{OA}+x_{OD}\right)=\left(x_{OB}+x_{OC}\right)\left(z_{OA}+z_{OD}\right)$$

By simple sustituting of coordinates we get:

$$(z_B+z_C)(x_A+x_D)=(x_B+x_C)(z_A+z_D)$$

where z_p and x_p are the z and x components of point p in our coordinate system centered at O.

Looking at each factor in turn:

$$\begin{aligned}
z_B+z_C &= \frac{\sqrt{2}}{2}\left(\cos\phi+\sin\phi\right)-d+\frac{\sqrt{2}}{2}\left(\sin\phi-\cos\phi\right)-d=\sqrt{2}\sin\phi-2d \\
x_A+x_D &= \frac{\sqrt{2}}{2}d\left(\sin\phi+\cos\phi\right)-\frac{1}{2}\left(\cos\phi-\sin\phi\right)^2+\frac{1}{2}\left(\sin\phi+\cos\phi\right)^2-\frac{\sqrt{2}}{2}d\,(\mathrm{co} \\
&= 2\cos\phi\sin\phi+\sqrt{2}d\sin\phi \\
x_B+x_C &= -\frac{\sqrt{2}}{2}d\left(\sin\phi+\cos\phi\right)+\frac{1}{2}\left(\cos\phi-\sin\phi\right)^2-\frac{1}{2}\left(\sin\phi+\cos\phi\right)^2-\frac{\sqrt{2}}{2}d\,(\\
&= -2\cos\phi\sin\phi-\sqrt{2}d\cos\phi \\
z_A+z_D &= \frac{\sqrt{2}}{2}\left(\cos\phi+\sin\phi\right)-d+\frac{\sqrt{2}}{2}\left(\cos\phi-\sin\phi\right)-d=\sqrt{2}\cos\phi-2d
\end{aligned}$$