Derivation for Double-Layer Tensegrity Structures

I hypothesize that a double-layer (and, thus, trusswork) tensegrity sphere can be constructed by using two of the spheres I've usually studied – of different diameters – nested concentrically.

My goal in the current project is to derive useful characterizations of each of the important aspects of the form I identify as based on the tetrahedron. A later goal will be to implement animated computer visualizations of this class of structures and, subsequently, to build physical examples of these structures in the real world, for testing of their features and methods of fabricating them.

Eventually, doing the same exercise for the octahedral and icosahedral forms will be valuable. The final goal is to devise how to build structures similarly that derive from multi-frequency subdivision of the icosahedral form, so that practical real-world structures can be manufactured from exhaustively complete specifications, and that such structures can be designed and their real-world performance thoroughly analyzed in simulation.

The two simpler tensegrities, from which the current exercise will compose a result, must be of opposite rotation directions, and the struts from each must be moved to positions that connect the inner with the outer and then each strut will need two additional cables which, also, will connect the inner to the outer layers.

The rotation directions I speak about above can be understood via a crucial observation about the spherical tensegrity spheres I've usually studied (they can be six-strut, twelve-strut, and thirty-strut, corresponding to tetrahedral, octahedral, or icosahedral symmetries, respectively, to be spherically omni-symmetrical). This observation is that the function of the struts (or axial-compression members) can be thought of as stretching as far as possible one of the two diagonals of the quadrilateral corresponding to the strut in the cable net of the tensegrity structure.

If the cable network is kept taught and the structure is moved from a position where those quadrilaterals have one diagonal maximized, to a position where the opposite diagonal is maximized, the centers of the large and small triangles can be held fixed and the triangles can be seen to rotate in opposite directions about each of their centers, as they move closer or farther from the center of the sphere. The geometry of the whole constrains the quadrilateral shape to adjust no farther than a fixed maximum length for each quadrilateral diagonal. When the net is in one of the two possible tensegrity positions, the cables will be held taught as well so, in that position, no additional constraint is necessary but the presence of the correct-length strut, to maintain structural integrity.

The current exercise, focusing on the tetrahedral symmetry (the simplest, to start), will be comprised of twelve struts and seventy-two cables in the final structure but, due to symmetry, will have far fewer distinct lengths. There will be four large and four small triangles of cable in each of the two spherical layers (4*2*3*2=48 cables, of four different lengths), two different lengths of inter-layer cables for each of four large triangles (2*4*3=24 cables of two lengths), and three identical strut lengths for each of four large triangles (3*4=12 struts of a single length). Thus there will be twelve identical cables of each of six lengths, and twelve struts of a single length.

I will begin with the inner differing by 20% in diameter from the outer six-strut dual-bridle cable net. I will also plan on the shorter cable to be 20% of the length of the longer. Each of the four large triangles of each concentric sphere will have three struts connecting it to the corresponding large triangle of the other sphere, and the additional cables making that component a three-strut tensegrity in its own right. These four three-strut tensegrity structures will be connected to the others to form the final structure, requiring addition of the cables comprising the small triangles of each concentric spherical tensegrity – but not the twelve struts those structures would need if they were to have structural integrity independently. The inner and outer sets of these cables are thought to counter opposite degrees of freedom of twisting of the large triangles of the inner and outer tensegrity nets.

The first step will be to write specifications of a three-strut tensegrity to be used four times in the scenario described above.

It will have cylindrical symmetry, so the inner and outer large triangles will be equilateral, and be twisted with respect to each other by an angle of from 120 degrees to 180 degrees. I'll begin by assuming that 150 degrees will work. The distance from the inner to the outer triangle will be 20% of the radius of the outer size of the eventual structure. Ultimately, it will be useful to have a derivation that allows this twist angle to vary between the limits.

Each strut-end, in addition to the two cables of the triangle to which it is connected, will have two more cables connecting it to the strut ends of the other two struts in the opposite triangle. This will result in twelve cables in all connecting the three struts.

I will choose the distance from the center of the outer large triangle to a vertex to be two units, and assign parameters to any remaining undetermined distances, so as to allow derivation of all distances and geometry but the ones intentionally left to be assigned freely. This assignment allows the eventual results to be parameterized by assigning this two-unit scale factor any real-world value.

These assumptions will allow writing immediately the cartesian coordinates of the six points where cable end-points connect to strut end-points, given a choice of convenient positioning of the structure in the coordinate system.

I will choose the outer large triangle to have vertices in the x-y plane of a 3D cartesian coordinate system, where one vertex is at $[0, -2, 0]$ and its center is at the origin.

This implies that the other two vertices will be at $[\sqrt{3}, 1, 0]$ and $[-\sqrt{3}, 1, 0]$.

Next, we will identify coordinates for the inner large triangle. It would be satisfying to parameterize its size (and, thus its distance from the outer large triangle and from the center of the eventual double-layer structure) as a fraction, but that seems complicated and not crucial for the current derivation.

I'll choose 1 as the z coordinate for all three vertices. For the purpose of deriving formulas for the current three-strut tensegrity, this will be adequate. When we wish to compose four of these structures to result in the desired double-layer structure this dimension, taken as a fraction of the distance of the large triangle from the center of the complete structure, should be the same as the fraction that the smaller triangle is of the larger triangle. Assuming the 150-degree rotation described above, the vertex of this second (inner) triangle corresponding to the first vertex specified above for the outer triangle will then have an x coordinate of $-.8$ and a y-coordinate of $\sqrt{3}$. The next corresponding vertex will be at $[2, 0]$ and the third will be at $[-.8, -\sqrt{3}]$. The final results, in tabular form are:

(first strut)	(second strut)	(third strut)
$[\ 0,\ -2,\ 0\ ]$	$[\ \sqrt{3},\ 1,\ 0\ ]$	$[\ -\sqrt{3},\ 1,\ 0\ ]$
$[\ -.8,\ \sqrt{3},\ 1\ ]$	$[\ 2,\ 0,\ 1\ ]$	$[\ -.8,\ -\sqrt{3},\ 1\ ]$

The columns in the table show end-points of the three struts, and the rows show the vertices of the outer triangle above those for the inner.

Next, it will be convenient to derive formulas for the lengths of the struts and the cables. After that, formulas for angles between struts and cables. Symmetry implies there will be a single length for all three struts, four possibly distinct cable lengths and eight possibly distinct angles. We won't need to characterize all angles between cables emanating from the same vertex until such time as design of the fixtures that facilitate connection of cables to struts is done.

The six 3D points shown in the table are sufficient for generating a computer model of the completed three-strut tensegrity for purposes of visualization.

A final step for this component model is to derive formulas for the force equilibrium of the tension and compression members of the structure. It will be sufficient to make this derivation for each end of a single strut. The rest are determined by symmetry. A key goal, here, is to verify that the configuration specified above achieves this force equilibrium. If we specify an arbitrary compression force borne by a strut, we hope to deduce the consequent tension forces on all cables, under equilibrium conditions.

To do that, we can observe that there are four cables attached to each end of the typical strut. If we choose four pairs of these eight cables appropriately, by symmetry, each pair should be bearing the same tension force, calculated by distributing the compression force from the strut to the cables via the angles the cables make with respect to the strut. In addition, the components that are perpendicular to the strut, of the tension forces induced into the four cables connected to each strut end, should add up to the zero vector.

— John Kirk