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Proof by picture: A selection of nice picture proofs

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Just like any other cultural group, mathematicians like to tell stories. We tell heroic stories about famous mathematicians, to inspire or reinforce our cultural values, and we encase our results in narratives to explain how they are interesting and how they relate to other results. We also tell stories to convince others that our results are valid, and preferably also to explain why they are true. These stories are what you know as "proofs".

You might think (perhaps with some trepidation) of proofs as formal, written sequences of logical steps. However, proofs can come in a variety of formats, some of which are more active or intuitive than a formal written proof. In this article, we show how proofs can be in the form of pictures, and we show a selection of these picture proofs that we find particularly elegant and instructive.

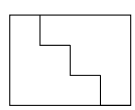
A classic way to add integers

Let us begin with one of our favourite picture proofs, of the following identity (which is not quite as interesting as its various proofs).

Theorem 1.

$$1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$$

Proof



Each of the two regions demarked in the rectangle above consists of $1 + 2 + \cdots + n$ squares (here, $n = 3$). The rectangle has n rows of squares and $n + 1$ columns, so the rectangle consists of $n(n+1)$ squares in total. Each of the two regions is half of this number, just as the theorem states. □

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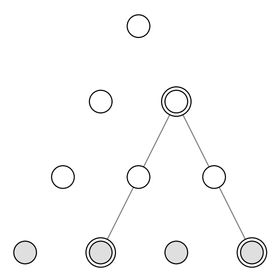
A different way to add integers

Looking at the term $\frac{1}{2}n(n+1)$ gives us a clue for another picture proof of Theorem 1. This term, also written as the *binomial coefficient* $\binom{n+1}{2}$, is the number of ways of choosing two things out of an unordered collection of $n+1$ things. For instance, if we have $n+1=10$ people and want to count the ways in which any two of them can shake hands, then we could first choose any one of the people (10 choices) and then choose a second person from the remaining nine people (9 choices). This will give $10 \times 9 = 90$ choices, but we have counted the handshakes twice. For instance, choosing Ada and then Ben, and later choosing Ben and then Ada. The number of handshakes is therefore $\frac{1}{2} \times 90 = 45$, or $\binom{n+1}{2} = \frac{1}{2}n(n+1)$. The following picture proof expresses the sum $1+2+\cdots+n$ as the white circles in the first n rows, where $n=3$ here.



The bottom row has $n+1$ grey circles, and any choice of two of these corresponds to exactly one of the white circles, and vice versa; this is illustrated by the three encircled circles. So, instead of counting white circles, we can count the ways of choosing any 2 of the grey circles, and this number is, as we have seen, $\binom{n+1}{2} = \frac{1}{2}n(n+1)$.[4]

How to add squares, or how to play with LEGO

We have seen how to count the sum $1+2+\cdots+n$ by drawing pictures. What about the sum of squares? It turns out that this is also not so hard:

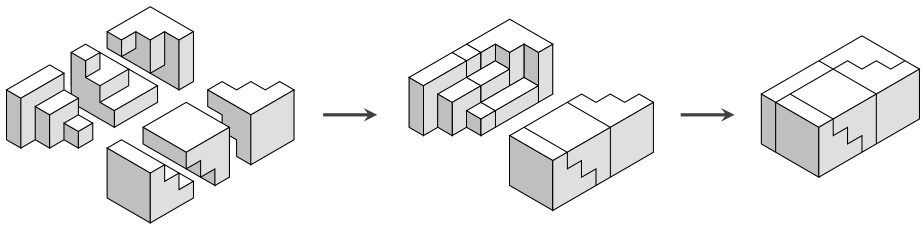
Theorem 2.

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

By looking at the term $\frac{1}{6}n(n+1)(2n+1)$ and by thinking about our simpler sum, we might be inspired to form a rectangle-type object with side lengths n, $n+1$, and $2n+1$ out of 6 triangle-type objects, which, given that we are now working in three dimensions, would probably be pyramid-type objects. This does in fact give us our proof:[5]

[4]This elegant proof was first presented, along with many other proofs, in [L.C. Larson, *A discrete look at* $1+2+\cdots+n$, College Math. Journal **16** (1985), 369–382].

[5]Thomas discovered this proof when he was still a student, during a particularly boring lecture. It had already been discovered, however, presumably many times, the first known instance being by Yang Hui in 1261 AD (see [M.-K. Sui, *Pyramid, pile, and sum of squares*, Historia Math. **8** (1981), 61–66]).



It could be a little hard to visualise how the middle of the rectangle fits together, so we made a YouTube video in which we built the pyramids in LEGO and put them together. You can find the video here:

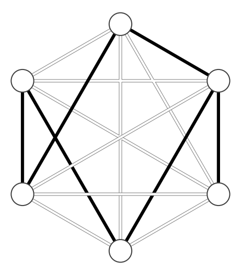
https://www.youtube.com/watch?v=p8dL7kHMsII

(Alternatively, you can google "LEGO proof YouTube".)

Love/hate-triangles among six people

So far, our proofs have used pictures to prove algebraic identities. We will now use pictures to prove a mathematical result that could itself be seen as a picture.

Imagine putting six people in a room together and letting them get to know each other. One person might like or dislike some of the others, or not really care about them, say. Suppose, though, that each pair of these people either forms a mutual like or mutual dislike to each other. We can draw this situation by drawing a *graph* in which each person is represented by a circle, and any two people are connected by a line that is either white (—) if they like each other or black (—) if they don't:

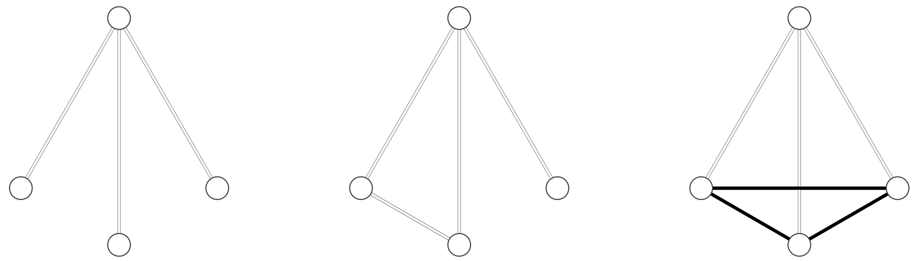


Theorem 3. *Among any six people as described above, you can always find three people who pairwise like each other or three people who pairwise dislike each other.*

In pictures, the theorem states that there will always be either a white triangle or a black triangle. In the specific example above, there are no black triangles but there are white triangles. This is a simple case of *Ramsey Theory* which is a beautiful but difficult mathematical area.

Proof

Look at one particular person and consider the 5 lines that connect that person. Of those lines, at least 3 must be white or at least 3 must be black. Let us suppose that at least 3 are white, perhaps as in Case 1 below:



Case 1 Subcase 1 Subcase 2

If, among the 3 people that are thus connected to our first person, any of two of these like each other (i.e. are connected by a white line), then we have a white triangle, for instance as in Subcase 1. However, if no pair of these three people like each other, then we get a black triangle as in Subcase 2. If the first person had been connected by at least 3 black lines, then just repeat the above Case 1 considerations with black and white swapped. □

Figures in elementary geometry

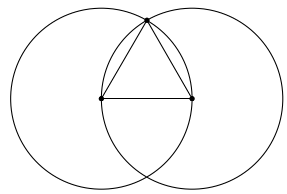
We have now seen pictures used as proofs or as parts of proofs, as well as pictures being the objects of interest in the theorems themselves. Moving on to classical Euclidean geometry, we will now see how pictures can be important for describing the processes involved in devising and proving theorems.

Stating the obvious: Diagrams and geometry

Euclid's work *The Elements* (c. 295 BC) is one of the most influential mathematical texts. Its first book treats elementary planar geometry by deriving theorems from statements known as "axioms" or "postulates" that Euclid saw as self-evidently true. These included statements such as "when two quantities are each equal to a third quantity, they are also equal to each other", and the postulates that any two points can be joined by a line and that a circle can be drawn around any centre with any radius.

Euclid's theorems are rigidly presented by a statement of the theorem, a geometric construction, and a proof showing that the theorem follows from the construction. The constructions can be seen as recipes for how to draw diagrams, and so we see that picture proofs are inherent in Euclid's work. Some of the oldest still existing sources — papyri from c. 300 AD — display diagrams.

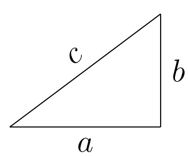
The very first theorem in Euclid's work is encapsuled in the following diagram and proves the existence of an equilateral triangle on a given line segment. The construction is to draw the two circles using the segment as their radius and to identify their intersection. The proof then uses the axioms to prove that the three sides are equal. The diagram serves both to illustrate the construction and to support the proof. That is also true for the rest of Euclid's work which includes much more complicated diagrams.



You may have immediately understood Euclid's proof from the drawing, and "read" the story that we told above. Yet, you may also have been misled by the apparently convincing argument. For instance, how could you be sure that the two circles did in fact intersect? None of Euclid's axioms claim that the circles do intersect, and it was not until the end of the nineteenth century, more than two thousand years after Euclid, that mathematics developed beyond Euclidean intuitions enough to require that the existence of such intersections should be stated explicitly, adding this statement to Euclid's age-old set of assumptions.

Pythagoras without words

You have undoubtedly heard of, and used, Pythagoras' Theorem.



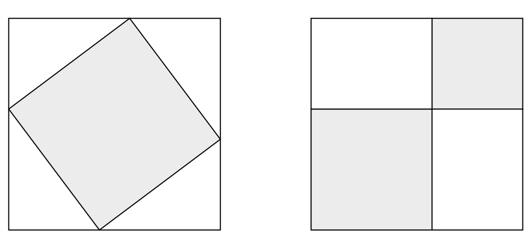
It lies at the heart of algebraic geometry, relating the three side lengths of a right-angled triangle in an elegant way:

Pythagoras' Theorem

$$a^2 + b^2 = c^2$$

There are several picture proofs of this theorem but we will present just one, this time without any accompanying words.

Proof



$\square$

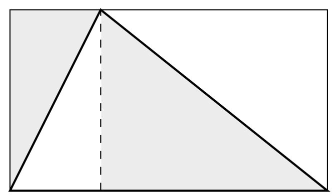
When you see the picture and convince yourself that it proves Pythagoras' theorem, you are, again, telling a story that could also be useful (and amended as necessary) if you were to describe the proof to a friend.

Areas of triangles: Right for the wrong reasons

Just like written formal proofs, picture proofs can contain flaws or holes that are not always immediately obvious. Let us for instance prove the following simple fact:

Theorem 4. *The area of a triangle is half of its base width times its height.*

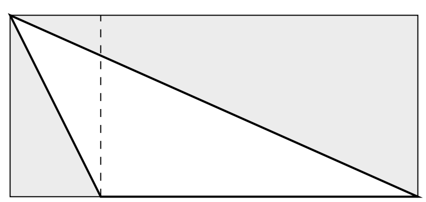
Proof



The big triangle is formed from two smaller triangles, one white and white grey, each of which has an different-coloured twin outside the big triangle. The area inside the big triangle is therefore the same as the area outside of it. The four triangles form the full rectangle, so the big triangle's area is half that of the rectangle which is the base width times the height. □

This is a simple proof of a simple theorem, and you might even have understood it at a glance, without even reading the explaining text. However, the proof is wrong! Or rather, the proof does not work for triangles having an angle greater than $90°$.

By using the picture below and by modifying our arguments from above, can you fix the proof by showing that the theorem also holds for triangles with an angle greater than $90°$? (*Hint*: First identify the rectangle that has twice the triangle's area.)



Explaining ideas

Ideally, a proof should not only convince us that some mathematical theorem is true; it should also explain the theorem so that we understand it, not just believe it. A picture proof is particularly useful if it can provide us with this insight at a quick glance. So far, our proofs have been nice and elegant but they might not all have provided you with clear insight. Let us therefore look at a picture proof that clearly explains as well as convinces.

To infinity and beyond

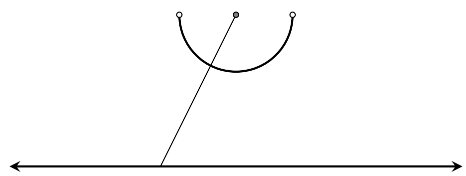
Two finite sets have the same size if we can match each element of the one set with an element of the other set, and vice versa. For instance, the sets $\{1, 2, 4\}$ and $\{a, b, c\}$ have the same size (3) since we can match up their elements, for instance in the obvious way:

$$1 — a \qquad 2 — b \qquad 4 — c$$

If our sets are infinite, then we could use the same matching-up criterium to determine whether the sets are, in this sense, of the same size.

Theorem 5. *The real interval* $(0, 1)$ *can be matched up with the whole set of real numbers* $\mathbb{R}$*.*

Proof

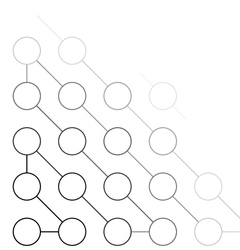


Here, the interval $(0, 1)$ is drawn as a semicircle, and $\mathbb{R}$ as an infinite line. Drawing a line starting from the semicircle's center, down through the $(0, 1)$ interval, and onto $\mathbb{R}$ gives a direct matching up of the two sets. $\square$

This idea of matching up can be further used to prove even more interesting (and perhaps even counter-intuitive) results, such as that there are "equally many" rational and natural numbers using a proof famously known as the "diagonal argument":

Theorem 6. *The natural numbers* $\mathbb{N} = \{1, 2, \ldots\}$ *can be matched up with all pairs* $\mathbb{N}^2 = \{(1, 1), (1, 2), (2, 1), \ldots\}$ *of natural numbers.*

Proof



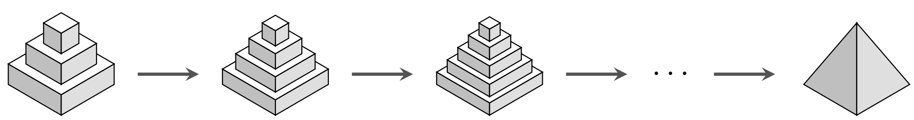
Draw a line through the pairs in $\mathbb{N}^2$, presented as the infinite array of circles above. This matches up the numbers $\mathbb{N}$ and the pairs $\mathbb{N}^2$: 1 is matched up with $(1, 1)$; 2 is matched up with $(2, 1)$; 3 is matched up with $(1, 2)$; 4 is matched up with $(1, 3)$; and so on. $\square$

But are these pictures really proofs?

We have presented a variety of our favourite picture proofs and hope that you have enjoyed them too.[6] But, you might ask, are these *really* proofs? Can they stand alone without accompanying text? Are they too imprecise — or are they sometimes too specific, for instance to represent induction arguments or the infinite? Do we really need pictures when words and symbols might suffice? And do these pictures allow us to find new results, or are they just pretty distractions? These and other questions are sometimes asked by some mathematicians, and the ensuing discussions for and against picture proofs can quickly lead to very deep and interesting questions about the nature of mathematics, the nature of truth, the nature of the world, and the nature of our own understanding.

We will not even scratch the surface of these fascinating questions here in this article but leave it up to you, the reader, to search for ways of answering them yourself. In that process, you may perhaps consider subquestions such as: Were you convinced by each of these proofs? Would you prefer a formal written proof to any of these picture proofs? Or, if you liked them and were convinced by them, can you find more nice picture proofs of your own?

As a final challenge, and to show how these pictures can lead to new results, we challenge you to prove that a pyramid with width, length, and height 1 has volume $\frac{1}{3}$, by using Theorem 2 to fill in the details of the following picture proof:[7]



[6]You can find many more nice picture proofs, for instance in the books *Proofs without Words* and *Proofs without Words II* by Roger B. Nelsen.

[7]This proof was first published (in written form) in 1086 AD by Shen Kuo, who wrote about *zao wei zhi shu*, or "the art of piling up very small things". We here see an early instance of infinite limits and integration. (See [M.-K. Sui, *Pyramid, pile, and sum of squares*, Historia Math. **8** (1981), 61–66].)