

# CS 229, Winter 2024

## Problem Set #4 Solutions

YOUR NAME HERE (YOUR SUNET HERE)

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**Due Wednesday, March 6 at 11:59 pm on Gradescope.**

**Notes:** (1) These questions require thought, but do not require long answers. Please be as concise as possible.

(2) If you have a question about this homework, we encourage you to post your question on our Ed forum, at <https://edstem.org/us/courses/51342/discussion/>.

(3) If you missed the first lecture or are unfamiliar with the collaboration or honor code policy, please read the policy on the course website before starting work.

(4) For the coding problems, you may not use any libraries except those defined in the provided `environment.yml` file. In particular, ML-specific libraries such as scikit-learn are not permitted.

(5) The due date is Wednesday, March 6 at 11:59 pm. If you submit after Wednesday, March 6 at 11:59 pm, you will begin consuming your late days. The late day policy can be found in the course website: Course Logistics and FAQ.

All students must submit an electronic PDF version of the written question including plots generated from the codes. We highly recommend typesetting your solutions via L<sup>A</sup>T<sub>E</sub>X. All students must also submit a zip file of their source code to Gradescope, which should be created using the `make_zip.py` script. You should make sure to (1) restrict yourself to only using libraries included in the `environment.yml` file, and (2) make sure your code runs without errors. Your submission may be evaluated by the auto-grader using a private test set, or used for verifying the outputs reported in the writeup. Please make sure that your PDF file and zip file are submitted to the corresponding Gradescope assignments respectively. We reserve the right to not give any points to the written solutions if the associated code is not submitted.

**Honor code:** We strongly encourage students to form study groups. Students may discuss and work on homework problems in groups. However, each student must write down the solution independently, and without referring to written notes from the joint session. Each student must understand the solution well enough in order to reconstruct it by him/herself. It is an honor code violation to copy, refer to, or look at written or code solutions from a previous year, including but not limited to: official solutions from a previous year, solutions posted online, and solutions you or someone else may have written up in a previous year. Furthermore, it is an honor code violation to post your assignment solutions online, such as on a public git repo. We run plagiarism-detection software on your code against past solutions as well as student submissions from previous years. Please take the time to familiarize yourself with the Stanford Honor Code<sup>1</sup> and the Stanford Honor Code<sup>2</sup> as it pertains to CS courses.

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<sup>1</sup><https://communitystandards.stanford.edu/policies-and-guidance/honor-code>

<sup>2</sup><https://web.stanford.edu/class/archive/cs/cs106b/cs106b.1164/handouts/honor-code.pdf>

**Regarding Notation:** The notation used in this problem set matches the notation used in the lecture notes. Some notable differences from lecture notation:

- The superscript “ $(i)$ ” represents an index into the training set – for example,  $x^{(i)}$  is the  $i$ -th feature vector and  $y^{(i)}$  is the  $i$ -th output variable. In lecture notation, these would instead be expressed as  $x_i$  and  $y_i$ .
- The subscript  $j$  represents an index in a vector – in particular,  $x_j^{(i)}$  represents the  $j$ -th feature in the feature vector  $x^{(i)}$ . In lecture notation,  $x_j^{(i)}$  would be  $h_j(x_i)$  or  $x_i[j]$ .
- The vector that contains the weights parameterizing a linear regression is expressed by the variable  $\theta$ , whereas lectures use the variable  $\mathbf{w}$ . As such,  $\theta_0 = w_0$ ,  $\theta_1 = w_1$ , and so on.

An overview of this notation is also given at the beginning of the lecture notes (pages 6-7).

1. **[8 points] Model Calibration.** In this question, we will discuss the uncertainty qualification of the machine learning models, and, in particular, a specific notion of uncertainty quantification called calibration. In many real-world applications such as policy-making or health care, prediction by machine learning models is not the end goal. Such predictions are used as the input to decision making process which needs to balance risks and rewards, both of which are often uncertain. Therefore, machine learning models not only need to produce a single prediction for each example, but also to measure the uncertainty of the prediction.

In this question we consider the binary classification setting. Let  $\mathcal{X}$  be the input space and  $\mathcal{Y} = \{0, 1\}$  be the label space. For simplicity, we assume that  $\mathcal{X}$  is a discrete space of finite number of elements. Let  $X$  and  $Y$  be random variables denoting the input and label, respectively, over  $\mathcal{X}$  and  $\mathcal{Y}$ , respectively. Let  $X, Y$  have a joint distribution  $\mathcal{P}$ . Suppose we have a model  $h : \mathcal{X} \rightarrow [0, 1]$  (here we dropped the dependency on the parameters  $\theta$  because the parameterization is not relevant to the discussion.) Recall that in logistic regression, we assume that there exists some model  $h^*(\cdot)$  (parameterized by a certain form, e.g., linear functions or neural networks) such that the output of the model  $h^*$  represents the conditional probability of  $Y$  being 1 given  $X$ :

$$P[Y = 1 | X = x] = h^*(x) \quad (1)$$

Under this assumption, we can derive the logistic loss to train and obtain some model  $h(\cdot)$ . Recall that the decision boundary is typically set to correspond to  $h(x) = 1/2$ , which means that the prediction of  $x$  is 1 if  $h(x) \geq 1/2$ , and 0 if  $h(x) < 1/2$ . To quantify the uncertainty of the prediction, it's tempting to just use the value of  $h(x)$  — the closer  $h(x)$  is to 0 or 1, the more confident we are with our prediction.

How do we know whether the learned model  $h(\cdot)$  indeed outputs a reliable and truthful probability  $h(x)$ ? We note that we shouldn't take it for granted because a) the assumption (1) may not exactly be satisfied by the data, and b) even if the assumption (1) holds perfectly, the learned model  $h(x)$  may be different from the true model  $h^*$  that satisfies (1).

We will introduce a metric to evaluate how reliably the probabilities output by a model  $h$  capture the confidence of the model. In order for these probabilities to be useful as confidence measures, we would ideally like them to be *calibrated* in the following sense. Calibration intuitively requires that among all the examples for which the model predicts the value 0.7, indeed 70% of them should have label 1.

Formally, we say that a model  $h$  is perfectly calibrated if for all possible probabilities  $p \in [0, 1]$  such that  $P[h(X) = p] > 0$ ,

$$P[Y = 1 | h(X) = p] = p. \quad (2)$$

Recall that  $(X, Y)$  is a random draw from the (population) data distribution.

In the example in Table 1, the model  $h$  is not perfectly calibrated, because when  $p = 0.3$ ,  $P[Y = 1 | h(X) = p] \neq p$ ,

$x$	$P[X = x]$	$P[Y = 1   X = x]$	$h(x)$	$T(x) = \mathbb{E}[Y   h(X) = h(x)]$
0	0.25	0.2	0.3	0.1
1	0.25	0.0	0.3	0.1
2	0.25	1.0	0.9	0.9
3	0.25	0.8	0.9	0.9

Table 1: An example of a model  $h : \mathcal{X} = \{0, 1, 2, 3\} \rightarrow [0, 1]$ . For calculating  $T(x)$ , we look at all data points with the same score  $h(x)$ , and compute the probability of  $Y = 1$  for these data points.

$$\begin{aligned}
P[Y = 1 | h(X) = 0.3] &= \frac{P[Y = 1 \text{ and } h(X) = 0.3]}{P[h(X) = 0.3]} \\
&= \frac{P[Y = 1 \text{ and } (X = 0 \text{ or } 1)]}{P[X = 0 \text{ or } 1]} \\
&= \frac{P[Y = 1 \text{ and } X = 0] + P[Y = 1 \text{ and } X = 1]}{P[X = 0 \text{ or } 1]} \\
&= \frac{P[Y = 1 | X = 0]P[X = 0] + P[Y = 1 | X = 1]P[X = 1]}{P[X = 0 \text{ or } 1]} \\
&= \frac{0.2 \times 0.25 + 0.0 \times 0.25}{0.5} = 0.1 \neq 0.3
\end{aligned}$$

- (a) [3 points] Show that the perfect calibration does not necessarily imply that the model achieves perfect accuracy. Is the converse necessarily true? Justify your answers by providing either a proof or a counterexample. (Note that perfect accuracy means that  $P[\mathbb{I}[h(X) \geq 0.5] = Y] = 1$ <sup>3</sup>.)

**Answer:** Even if a model is perfectly calibrated, this does *not* necessarily mean it correctly classifies all the training examples. If the dataset has an equal number of positive and negative examples, a model that simply outputs 0.5 probability for all examples is perfectly calibrated, as useless as it may be. Likewise, a model that outputs 0.01 for all negative examples and 0.51 for all positive examples is poorly calibrated, but perfectly accurate. Thus accuracy and calibration can sometimes be unrelated.

- (b) [5 points] As you showed in the last part, calibration by itself does not necessarily guarantee good accuracy. Good models must also be sharp, i.e., the probabilities output by the model should be close to 0 or 1. Mean squared error (MSE) is a common measure for evaluating the quality of a model.

$$\text{MSE}(h) = \mathbb{E}[(Y - h(X))^2] \quad (3)$$

In this part, we will show that the MSE can be decomposed to two parts, such that one part corresponds to the calibration error, and the other part corresponds to the sharpness of the model.

Formally, let  $T(x) = P[Y = 1 | h(X) = h(x)]$  denote the true probability of  $Y = 1$  given that the prediction is equal to  $h(x)$ . Intuitively, for a data point  $x$ ,  $T(x)$  is the probability of  $Y = 1$  for all the data points with the same score as  $x$ . See Table 1 for an example.

<sup>3</sup> $\mathbb{I}$  is the indicator function.  $\mathbb{I}[h(x) \geq 0.5]$  is equal to 1 if  $h(x) \geq 0.5$  and is equal to 0 if  $h(x) < 0.5$ .

Define calibration error CE to be<sup>4</sup>:

$$\text{CE}(h) = \mathbb{E}[(T(X) - h(X))^2] \quad (4)$$

The calibration error here is a quantitative instantiation of the notion of perfect calibration in equation (2). Indeed, zero calibration error implies perfect calibration: zero calibration means the model perfectly predicts the true probability, i.e.,  $h(X) = T(X)$  w.p. 1. This in turns implies that the model is perfectly calibrated because for any  $p$  such that  $P[h(X) = p] > 0$ , we can take some  $x_0$  such that  $h(x_0) = p$  and  $h(x_0) = T(x_0)$ , and conclude

$$\begin{aligned} P[Y = 1 \mid h(X) = p] &= P[Y = 1 \mid h(X) = h(x_0)] & (5) \\ &= T(x_0) = h(x_0) & \text{(by calibration error = 0)} \\ &= p & \text{(by the assumption that } h(x_0) = p) \end{aligned}$$

As explained before, we want our model prediction to be sharp as well. One way to define sharpness of a model is to look at the variance of  $T(X)$ . Let's define sharpness of model  $h$  as follows:

$$\text{SH}(h) = \text{Var}(T(X)) \quad (6)$$

The sharpness term measures how much variation there is in the true probability across model prediction. It is small when  $T(x)$  is similar across all data points. To get a better intuition for sharpness, let's group data points according to the model prediction; i.e., all data points with the same prediction ( $h(X)$ ) are in the same group. In each group we want the true underlying probability of labels  $T(X) = \mathbb{E}[Y|h(X)]$  to be close to  $h(X)$  (calibration). For sharpness, we want the value of  $T(X) = \mathbb{E}[Y|h(X)]$  to be spread out, i.e., not all the groups have similar values.  $\text{Var}[T(X)]$  is one way to measure how spread out  $T(X)$  is.

MSE can be decomposed as follows:

$$\text{MSE}(h) = \underbrace{\text{Var}[Y]}_{\text{Intrinsic uncertainty}} - \underbrace{\text{Var}[T(X)]}_{\text{Sharpness}} + \underbrace{\mathbb{E}[(T(X) - h(X))^2]}_{\text{Calibration error}} \quad (7)$$

This decomposition states that by minimizing MSE we try to find a sharp model with small calibration error. In other words, when we choose a model with minimum MSE it means between two models with the same calibration error we prefer the one that is sharper and between two models with the same sharpness we prefer the one with lower calibration error. Note that the uncertainty term does not depend on the model and can be mostly ignored.

**Prove that the decomposition in Equation (7) is correct.**

**Hint.** Recall that using the law of total variance, we can decompose the variance of  $Y$  onto  $h(X)$ :

$$\text{Var}[Y] = \mathbb{E}[\text{Var}[Y \mid h(X)]] + \text{Var}[\mathbb{E}[Y \mid h(X)]]$$

**Answer:**

For proving that the decomposition is correct, we are using the law of total expectation and the law of total variance. Recall that for any two random variables  $X_1$  and  $X_2$  we have:

$$\mathbb{E}[X_1] = \mathbb{E}[\mathbb{E}[X_1 \mid X_2]] \quad (8)$$

$$\text{Var}[X_1] = \text{Var}[\mathbb{E}[X_1 \mid X_2]] + \mathbb{E}[\text{Var}[X_1 \mid X_2]] \quad (9)$$

<sup>4</sup>There are other definition of calibration errors, e.g., the one introduced in the next part.

Using these two laws, we can decompose the variance of  $Y$  onto  $h(X)$ :

$$\text{Var}[Y] = \mathbb{E}[\text{Var}[Y | h(X)]] + \text{Var}[\mathbb{E}[Y | h(X)]] \quad (10)$$

$$= \mathbb{E}[\mathbb{E}[(Y - \mathbb{E}[Y | h(X)])^2 | h(X)]] + \text{Var}[T(X)] \quad (11)$$

$$= \mathbb{E}[(Y - T(X))^2] + \text{Var}[T(X)] \quad (12)$$

Therefore, we have:

$$\mathbb{E}[(Y - T(X))^2] = \text{Var}[Y] - \text{Var}[T(X)] \quad (13)$$

Using 13, and also noting  $\mathbb{E}[Y - T(X)] = 0$  we can decompose MSE as follows:

$$\text{MSE}(h) = \mathbb{E}[(Y - h(X))^2] \quad (14)$$

$$= \mathbb{E}[(Y - T(X) + T(X) - h(X))^2] \quad (15)$$

$$= \mathbb{E}[(Y - T(X))^2] + \mathbb{E}[(T(X) - h(X))^2] \quad (16)$$

$$+ 2\mathbb{E}[\mathbb{E}[(Y - T(X))(T(X) - h(X)) | h(X)]] \quad (17)$$

$$= \mathbb{E}[(Y - T(X))^2] + \mathbb{E}[(T(X) - h(X))^2] \quad (18)$$

$$= \underbrace{\text{Var}[Y]}_{\text{Uncertainty}} - \underbrace{\text{Var}[T(X)]}_{\text{Sharpness}} + \underbrace{\mathbb{E}[(T(X) - h(X))^2]}_{\text{Calibration error}} \quad (19)$$

**More intuition regarding sharpness.** To get a better intuition for sharpness, let's group data points according to the model prediction; i.e., all data points with the same prediction ( $h(X)$ ) are in the same group. In each group we want the true underlying probability of labels  $T(X) = \mathbb{E}[Y|h(X)]$  to be close to  $h(X)$  (calibration). For sharpness, we want the value of  $T(X) = \mathbb{E}[Y|h(X)]$  to be spread out, i.e., not all the groups have similar values.  $\text{Var}[T(X)]$  is one way to measure how spread out  $T(X)$  is. Note that  $\text{Var}[Y]$  is always greater than  $\text{Var}[T(X)]$  (use the law of total variance).  $\text{Var}[Y]$  is not dependent of the model; therefore, it can be ignored. However,  $\text{Var}[T(X)]$  is dependent to the model and we want it to be as large as possible. One way to think about the difference between  $\text{Var}[Y]$  and  $\text{Var}[T(X)]$  is as follows:

$$\text{Var}[Y] - \text{Var}[T(x)] = \mathbb{E}[\text{Var}[Y|X]] + \text{Var}[\mathbb{E}[Y|X]] - \text{Var}[\mathbb{E}[Y|h(X)]], \quad (20)$$

the first term in RHS ( $\mathbb{E}[\text{Var}[Y|X]]$ ) is uncertainty in each data point and is not avoidable. However,  $\text{Var}[\mathbb{E}[Y|X]] - \text{Var}[\mathbb{E}[Y|h(X)]]$  has smaller value if  $h(X)$  is different for data points with different  $\mathbb{E}[Y|X]$ .

- (c) [0 points] **[This part is optional and does not have any points]** In the previous part, we studied MSE and decomposed it to two terms corresponding to the sharpness and calibration error. But as we explained, there are other different ways to measure the calibration and sharpness of a model. In this part we focus on logistic regression models. In particular, we are going to show logistic loss can also be decomposed to two terms; where one term can be interpreted as the sharpness of the prediction (which we call log-sharpness) and the other can be interpreted as the calibration error (which we call log-calibration-error). Recall that the logistic loss (on population) of the model  $h$  is defined as:

$$\text{Log-Loss}(h) = \mathbb{E}[-Y \log(h(X)) - (1 - Y) \log(1 - h(X))] \quad (21)$$

Prove that logistic loss can be decomposed to two terms as follows:

$$\begin{aligned} \text{Log-Loss}(h) &= \mathbb{E} \left[ \underbrace{T(X) \log \left( \frac{T(X)}{h(X)} \right) + (1 - T(X)) \log \left( \frac{1 - T(X)}{1 - h(X)} \right)}_{\text{log-calibration-error}} \right] \\ &\quad - \underbrace{\mathbb{E} [T(X) \log(T(X)) + (1 - T(X)) \log(1 - T(X))]}_{\text{log-sharpness}} \end{aligned} \quad (22)$$

Discuss why the log-calibration-error term in (22) is a meaningful term for measuring the calibration error and specify when it attains its minimum. Similarly discuss why log-sharpness term in (22) is a meaningful term for measuring the sharpness of a model and specify when it attains its maximum.

**Hint.** For showing that log-calibration-error and log-sharpness are meaningful terms, you should use your information theory knowledge (there is a section about information theory in question 4). For each data point  $x$ , both model prediction ( $h(x)$ ) and underlying probability ( $T(x)$ ) define a distribution over the label set  $\mathcal{Y} = \{0, 1\}$ . In particular, define distribution  $P_1$  on  $\mathcal{Y} = \{0, 1\}$  as follows:  $P_1(Y = 1) = T(x)$  and  $P_1(Y = 0) = 1 - T(x)$ . Similarly, define distribution  $P_2$  on  $\mathcal{Y} = \{0, 1\}$  as follows:  $P_2(Y = 1) = h(x)$  and  $P_2(Y = 0) = 1 - h(x)$ . You can interpret the log-calibration-error in (22) as KL-divergence distance between these two distributions. Recall that the KL divergence distance between these two distributions is:

$$D_{\text{KL}}(P_1 \parallel P_2) = P_1(Y = 0) \log \left( \frac{P_1(Y = 0)}{P_2(Y = 0)} \right) + P_1(Y = 1) \log \left( \frac{P_1(Y = 1)}{P_2(Y = 1)} \right) \quad (23)$$

The log-sharpness term in (22) can be expressed as the negative entropy of the distribution corresponding to  $T(x)$ . Recall that entropy of distribution  $P_1$  is:

$$H(P_1) = -P_1(Y = 0) \log(P_1(Y = 0)) - P_1(Y = 1) \log(P_1(Y = 1)) \quad (24)$$

**Remark:** The decomposition suggests that minimizing the logistic loss (on the population) tends to minimize the calibration error as well, since the calibration error is upper bounded by the logistic loss. In practice, when the train and test sets are from the same distribution and when the model has not overfit or underfit, logistic regression tends to be well calibrated on the test data as well. In contrast, modern large-scale deep learning models trained with the logistic loss are typically not well-calibrated, likely since the population logistic loss suffers from overfitting (the test loss is much higher than the train loss), even when there is little overfitting in terms of the accuracy. As such, often people use other recalibration methods to adjust the outputs of a deep learning model to be better calibrated.

**Answer:**

$$\text{Log-Loss}(h) = \mathbb{E} [-Y \log(h(X)) - (1 - Y) \log(1 - h(X))] \quad (25)$$

$$= \mathbb{E} [\mathbb{E} [-Y \log(h(X)) - (1 - Y) \log(1 - h(X)) \mid h(X)]] \quad (26)$$

$$= \mathbb{E} [-T(X) \log(h(X)) - (1 - T(X)) \log(1 - h(X))] \quad (27)$$

$$\begin{aligned} &= \mathbb{E} \left[ \underbrace{T(X) \log \left( \frac{T(X)}{h(X)} \right) + (1 - T(X)) \log \left( \frac{1 - T(X)}{1 - h(X)} \right)}_{\text{log-calibration-error}} \right] \\ &\quad - \underbrace{\mathbb{E} [T(X) \log(T(X)) + (1 - T(X)) \log(1 - T(X))]}_{\text{log-sharpness}} \end{aligned} \quad (28)$$

We define distribution  $P_1$  on  $\mathcal{Y} = \{0, 1\}$  as follows:  $P_1(Y = 1) = T(x)$  and  $P_1(Y = 0) = 1 - T(x)$ . Similarly, we define distribution  $P_2$  on  $\mathcal{Y} = \{0, 1\}$  as follows:  $P_2(Y = 1) = h(x)$  and  $P_2(Y = 0) = 1 - h(x)$ . The log-calibration-error is the expected KL-divergence between these two distributions and attains its minimum when for all  $x$ ,  $h(x) = T(x)$ . The log-sharpness term is the expected negative entropy of  $P_1$  and attains its maximum when  $T(x) = 0$  or  $T(x) = 1$  for all  $x$ . This is aligned with our intuition regarding sharpness an calibration error. For sharpness we want  $T(x)$  to be more close to 0 and 1 and for calibration we want  $h(x)$  be close to  $T(x)$ .



## 2. [10 points] PCA

In class, we showed that PCA finds the “variance maximizing” directions onto which to project the data. In this problem, we find another interpretation of PCA.

Suppose we are given a set of points  $\{x^{(1)}, \dots, x^{(n)}\}$ . Let us assume that we have as usual preprocessed the data to have zero-mean and unit variance in each coordinate. For a given unit-length vector  $u$ , let  $f_u(x)$  be the projection of point  $x$  onto the direction given by  $u$ . I.e., if  $\mathcal{V} = \{\alpha u : \alpha \in \mathbb{R}\}$ , then

$$f_u(x) = \arg \min_{v \in \mathcal{V}} \|x - v\|^2.$$

Show that the unit-length vector  $u$  that minimizes the mean squared error between projected points and original points corresponds to the first principal component for the data. I.e., show that

$$\arg \min_{u: \|u\|=1} \sum_{i=1}^n \|x^{(i)} - f_u(x^{(i)})\|_2^2 = \arg \max_{u: \|u\|=1} u^T \left( \sum_{i=1}^n x^{(i)} x^{(i)T} \right) u.$$

**Remark.** If we are asked to find a  $k$ -dimensional subspace onto which to project the data so as to minimize the sum of squares distance between the original data and their projections, then we should choose the  $k$ -dimensional subspace spanned by the first  $k$  principal components of the data. This problem shows that this result holds for the case of  $k = 1$ .

**Answer:** First note we have  $f_u(x^{(i)}) = u^T x^{(i)} u$  (why?)<sup>5</sup>. So we have to solve the following problem:

$$\begin{aligned} \arg \min_{u: \|u\|=1} \sum_{i=1}^m \|x^{(i)} - f_u(x^{(i)})\|_2^2 &= \arg \min_{u: \|u\|=1} \sum_{i=1}^m \|x^{(i)} - u^T x^{(i)} u\|_2^2 \\ &= \arg \min_{u: \|u\|=1} \sum_{i=1}^m \left( (x^{(i)} - u^T x^{(i)} u)^T (x^{(i)} - u^T x^{(i)} u) \right) \\ &= \arg \min_{u: \|u\|=1} \sum_{i=1}^m \left( x^{(i)T} x^{(i)} - 2(u^T x^{(i)})^2 + u^T u (u^T x^{(i)})^2 \right) \\ &= \arg \min_{u: \|u\|=1} \sum_{i=1}^m \left( x^{(i)T} x^{(i)} - 2(u^T x^{(i)})^2 + (u^T x^{(i)})^2 \right) \\ &= \arg \min_{u: \|u\|=1} \sum_{i=1}^m -(u^T x^{(i)})^2 \\ &= \arg \max_{u: \|u\|=1} u^T \left( \sum_{i=1}^m x^{(i)} x^{(i)T} \right) u \end{aligned}$$

And the last line corresponds to the optimization problem that defines the first principal component.

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<sup>5</sup>To see why, observe that

$$f_u(x) = u \cdot \left( \arg \min_{\alpha} \|x - \alpha u\|^2 \right) = u \cdot \left( \arg \min_{\alpha} (x^T x - 2\alpha x^T u + \alpha^2 u^T u) \right) = u \cdot \left( \frac{2x^T u}{2u^T u} \right) = u x^T u$$

where the third equality follows from the fact that the minimum of a convex quadratic function  $ax^2 + bx + c$  is given by  $x = -\frac{b}{2a}$ , and the last equality follows from the fact that  $u$  is a unit-length vector.

### 3. [20 points] K-means for compression

In this problem, we will apply the K-means algorithm to lossy image compression, by reducing the number of colors used in an image.

We will be using the files `src/k_means/peppers-small.tiff` and `src/k_means/peppers-large.tiff`.

The `peppers-large.tiff` file contains a  $512 \times 512$  image of peppers represented in 24-bit color. This means that, for each of the 262,144 pixels in the image, there are three 8-bit numbers (each ranging from 0 to 255) that represent the red, green, and blue intensity values for that pixel. The straightforward representation of this image therefore takes about  $262144 \times 3 = 786432$  bytes (a byte being 8 bits). To compress the image, we will use K-means to reduce the image to  $k = 16$  colors. More specifically, each pixel in the image is considered a point in the three-dimensional  $(r, g, b)$ -space. To compress the image, we will cluster these points in color-space into 16 clusters, and replace each pixel with the closest cluster centroid.

Follow the instructions below. Be warned that some of these operations can take a while (several minutes even on a fast computer)!

- (a) [15 points] **[Coding Problem] K-Means Compression Implementation.** First let us *look* at our data. From the `src/k_means/` directory, open an interactive Python prompt, and type

```
from matplotlib.image import imread; import matplotlib.pyplot as plt;
```

and run `A = imread('peppers-large.tiff')`. Now, `A` is a “three dimensional matrix,” and `A[:, :, 0]`, `A[:, :, 1]` and `A[:, :, 2]` are  $512 \times 512$  arrays that respectively contain the red, green, and blue values for each pixel. Enter `plt.imshow(A); plt.show()` to display the image.

Since the large image has 262,144 pixels and would take a while to cluster, we will instead run vector quantization on a smaller image. Repeat (a) with `peppers-small.tiff`.

Next we will implement image compression in the file `src/k_means/k_means.py` which has some starter code. Treating each pixel's  $(r, g, b)$  values as an element of  $\mathbb{R}^3$ , implement K-means with 16 clusters on the pixel data from this smaller image, iterating (preferably) to convergence, but in no case for less than 30 iterations. For initialization, set each cluster centroid to the  $(r, g, b)$ -values of a randomly chosen pixel in the image.

Take the image of `peppers-large.tiff`, and replace each pixel's  $(r, g, b)$  values with the value of the closest cluster centroid from the set of centroids computed with `peppers-small.tiff`. Visually compare it to the original image to verify that your implementation is reasonable.

**Include in your write-up a copy of this compressed image alongside the original image. Answer:**

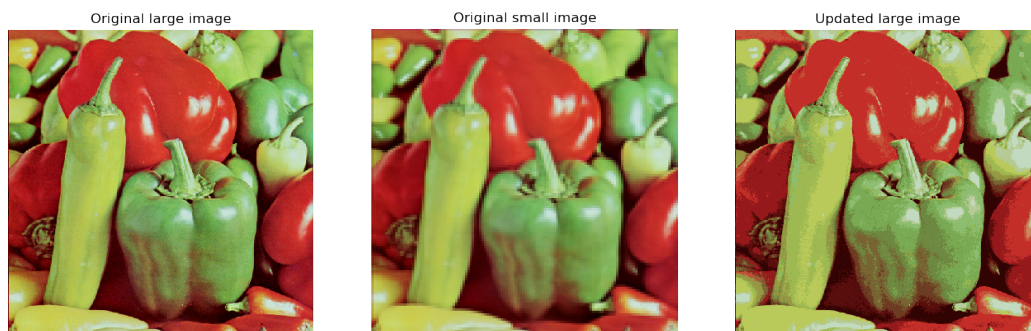


Figure 1: Original and Compressed Images

(b) [5 points] **Compression Factor.**

If we represent the image with these reduced (16) colors, by (approximately) what factor have we compressed the image?

**Answer:** The original image used 24 bits per pixel. To represent one of 16 colors requires  $\log_2 16 = 4$  bits per pixel. We have therefore achieved a compression factor of about  $24/4 = 6$  for the image.

#### 4. [15 points] KL divergence and Maximum Likelihood

The Kullback-Leibler (KL) divergence is a measure of how much one probability distribution is different from a second one. It is a concept that originated in Information Theory, but has made its way into several other fields, including Statistics, Machine Learning, Information Geometry, and many more. In Machine Learning, the KL divergence plays a crucial role, connecting various concepts that might otherwise seem unrelated.

In this problem, we will introduce KL divergence over discrete distributions, practice some simple manipulations, and see its connection to Maximum Likelihood Estimation.

The *KL divergence* between two discrete-valued distributions  $P(X), Q(X)$  over a finite sample space  $\mathcal{X}$  is defined as follows<sup>6</sup>:

$$D_{KL}(P\|Q) = \sum_{x \in \mathcal{X}} P(x) \log \frac{P(x)}{Q(x)}.$$

For notational convenience, we assume  $P(x) > 0, \forall x$ . (One other standard thing to do is to adopt the convention that “ $0 \log 0 = 0$ .”) Sometimes, we also write the KL divergence more explicitly as  $D_{KL}(P\|Q) = D_{KL}(P(X)\|Q(X))$ .

##### *Background on Information Theory*

Before we dive deeper, we give a brief (optional) Information Theoretic background on KL divergence. While this introduction is not necessary to answer the assignment question, it may help you better understand and appreciate why we study KL divergence, and how Information Theory can be relevant to Machine Learning.

We start with the *entropy*  $H(P)$  of a probability distribution  $P(X)$ , which is defined as

$$H(P) = - \sum_{x \in \mathcal{X}} P(x) \log P(x).$$

Intuitively, entropy measures how dispersed a probability distribution is. For example, a uniform distribution is considered to have very high entropy (i.e., a lot of uncertainty), whereas a distribution that assigns all its mass to a single point is considered to have zero entropy (i.e., no uncertainty). Notably, it can be shown that among continuous distributions over  $\mathbb{R}$ , the Gaussian distribution  $\mathcal{N}(\mu, \sigma^2)$  has the highest entropy (highest uncertainty) among all possible distributions that have the given mean  $\mu$  and variance  $\sigma^2$ .

To further solidify our intuition, we present motivation from communication theory. Suppose we want to communicate from a source to a destination, and our messages are always (a sequence of) discrete symbols over space  $\mathcal{X}$  (for example,  $\mathcal{X}$  could be letters  $\{a, b, \dots, z\}$ ). We want to construct an encoding scheme for our symbols in the form of sequences of binary bits that are transmitted over the channel. Further, suppose that in the long run the frequency of occurrence of symbols follow a probability distribution  $P(X)$ . This means, in the long run, the fraction of times the symbol  $x$  gets transmitted is  $P(x)$ .

A common desire is to construct an encoding scheme such that the average number of bits per symbol transmitted remains as small as possible. Intuitively, this means we want very frequent symbols to be assigned to a bit pattern having a small number of bits. Likewise, because we are

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<sup>6</sup>If  $P$  and  $Q$  are densities for continuous-valued random variables, then the sum is replaced by an integral, and everything stated in this problem works fine as well. But for the sake of simplicity, in this problem we'll just work with this form of KL divergence for probability mass functions/discrete-valued distributions.

interested in reducing the average number of bits per symbol in the long term, it is tolerable for infrequent words to be assigned to bit patterns having a large number of bits, since their low frequency has little effect on the long term average. The encoding scheme can be as complex as we desire, for example, a single bit could possibly represent a long sequence of multiple symbols (if that specific pattern of symbols is very common). The entropy of a probability distribution  $P(X)$  is its optimal bit rate, i.e., the lowest average bits per message that can possibly be achieved if the symbols  $x \in \mathcal{X}$  occur according to  $P(X)$ . It does not specifically tell us *how* to construct that optimal encoding scheme. It only tells us that no encoding can possibly give us a lower long term bits per message than  $H(P)$ .

To see a concrete example, suppose our messages have a vocabulary of  $K = 32$  symbols, and each symbol has an equal probability of transmission in the long term (i.e, uniform probability distribution). An encoding scheme that would work well for this scenario would be to have  $\log_2 K$  bits per symbol, and assign each symbol some unique combination of the  $\log_2 K$  bits. In fact, it turns out that this is the most efficient encoding one can come up with for the uniform distribution scenario.

It may have occurred to you by now that the long term average number of bits per message depends only on the frequency of occurrence of symbols. The encoding scheme of scenario A can in theory be reused in scenario B with a different set of symbols (assume equal vocabulary size for simplicity), with the same long term efficiency, as long as the symbols of scenario B follow the same probability distribution as the symbols of scenario A. It might also have occurred to you, that reusing the encoding scheme designed to be optimal for scenario A, for messages in scenario B having a *different probability* of symbols, will always be suboptimal for scenario B. To be clear, we do not need to know *what* the specific optimal schemes are in either scenarios. As long as we know the distributions of their symbols, we can say that the optimal scheme designed for scenario A will be suboptimal for scenario B if the distributions are different.

Concretely, if we reuse the optimal scheme designed for a scenario having symbol distribution  $Q(X)$ , into a scenario that has symbol distribution  $P(X)$ , the long term average number of bits per symbol achieved is called the *cross entropy*, denoted by  $H(P, Q)$ :

$$H(P, Q) = - \sum_{x \in \mathcal{X}} P(x) \log Q(x).$$

To recap, the entropy  $H(P)$  is the best possible long term average bits per message (optimal) that can be achieved under a symbol distribution  $P(X)$  by using an encoding scheme (possibly unknown) specifically designed for  $P(X)$ . The cross entropy  $H(P, Q)$  is the long term average bits per message (suboptimal) that results under a symbol distribution  $P(X)$ , by reusing an encoding scheme (possibly unknown) designed to be optimal for a scenario with symbol distribution  $Q(X)$ .

Now, KL divergence is the penalty we pay, as measured in average number of bits, for using the optimal scheme for  $Q(X)$ , under the scenario where symbols are actually distributed as  $P(X)$ . It is straightforward to see this

$$\begin{aligned} D_{KL}(P\|Q) &= \sum_{x \in \mathcal{X}} P(x) \log \frac{P(x)}{Q(x)} \\ &= - \sum_{x \in \mathcal{X}} P(x) \log Q(x) + \sum_{x \in \mathcal{X}} P(x) \log P(x) \\ &= H(P, Q) - H(P). \quad (\text{difference in average number of bits.}) \end{aligned}$$

If the cross entropy between  $P$  and  $Q$  is  $H(P)$  (and hence  $D_{KL}(P||Q) = 0$ ) then it necessarily means  $P = Q$ . In Machine Learning, it is a common task to find a distribution  $Q$  that is “close” to another distribution  $P$ . To achieve this, it is common to use  $D_{KL}(Q||P)$  as the loss function to be optimized. As we will see in this question below, Maximum Likelihood Estimation, which is a commonly used optimization objective, turns out to be equivalent to minimizing the KL divergence between the training data (i.e., the empirical distribution over the data) and the model.

Now, we get back to showing some simple properties of KL divergence.

- (a) [5 points] **Nonnegativity.**

Prove the following:

$$\forall P, Q. \quad D_{KL}(P||Q) \geq 0$$

and

$$D_{KL}(P||Q) = 0 \quad \text{if and only if} \quad P = Q.$$

[Hint: You may use the following result, called **Jensen’s inequality**. If  $f$  is a convex function, and  $X$  is a random variable, then  $E[f(X)] \geq f(E[X])$ . Moreover, if  $f$  is strictly convex ( $f$  is convex if its Hessian satisfies  $H \geq 0$ ; it is *strictly* convex if  $H > 0$ ; for instance  $f(x) = -\log x$  is strictly convex), then  $E[f(X)] = f(E[X])$  implies that  $X = E[X]$  with probability 1; i.e.,  $X$  is actually a constant.]

**Answer:**

$$\begin{aligned} -D_{KL}(P||Q) &= -\sum_x P(x) \log \frac{P(x)}{Q(x)} \\ &= \sum_x P(x) \log \frac{Q(x)}{P(x)} \\ &\leq \log \sum_x P(x) \frac{Q(x)}{P(x)} \\ &= \log \sum_x Q(x) \\ &= \log 1 \\ &= 0 \end{aligned}$$

Where all equalities follow from straight forward algebraic manipulation. The inequality follows from Jensen’s inequality.

To show the second part of the claim, note that  $\log t$  is a strictly concave function of  $t$ . Using the form of Jensen’s inequality given in the lecture notes, we have equality if and only if  $\frac{Q(x)}{P(x)} = \mathbb{E}_P \left[ \frac{Q(x)}{P(x)} \right]$  for all  $x$ . But since  $\mathbb{E}_P \left[ \frac{Q(x)}{P(x)} \right] = \sum_x P(x) \frac{Q(x)}{P(x)} = \sum_x Q(x) = 1$ , it follows that  $P(x) = Q(x)$ . Hence we have  $D_{KL}(P||Q) = 0$  if and only if  $P(x) = Q(x)$  for all  $x$ .

- (b) [5 points] **Chain rule for KL divergence.**

The KL divergence between 2 conditional distributions  $P(X|Y), Q(X|Y)$  is defined as follows:

$$D_{KL}(P(X|Y)||Q(X|Y)) = \sum_y P(y) \left( \sum_x P(x|y) \log \frac{P(x|y)}{Q(x|y)} \right)$$

This can be thought as the expected KL divergence between the corresponding conditional distributions on  $x$  (that is, between  $P(X|Y = y)$  and  $Q(X|Y = y)$ ), where the expectation is taken over the random  $y$ .

Prove the following chain rule for KL divergence:

$$D_{KL}(P(X, Y) \| Q(X, Y)) = D_{KL}(P(X) \| Q(X)) + D_{KL}(P(Y|X) \| Q(Y|X)).$$

**Answer:**

$$\begin{aligned} D_{KL}(P(X, Y) \| Q(X, Y)) &= \sum_{x, y} P(x, y) \log \frac{P(x, y)}{Q(x, y)} \\ &= \sum_{x, y} P(x, y) \log \frac{P(x)P(y|x)}{Q(x)Q(y|x)} \\ &= \sum_{x, y} P(x, y) \log \frac{P(x)}{Q(x)} + \sum_{x, y} P(x, y) \log \frac{P(y|x)}{Q(y|x)} \\ &= \sum_{x, y} P(x, y) \log \frac{P(x)}{Q(x)} + \sum_{x, y} P(x)P(y|x) \log \frac{P(y|x)}{Q(y|x)} \\ &= \sum_x P(x) \log \frac{P(x)}{Q(x)} + \sum_x P(x) \sum_y P(y|x) \log \frac{P(y|x)}{Q(y|x)} \\ &= D_{KL}(P(X) \| Q(X)) + D_{KL}(P(Y|X) \| Q(Y|X)). \end{aligned}$$

Where we applied (in order): definition of KL, definition of conditional probability, log of product is sum of logs, splitting the summation,  $\sum_y P(x, y) = P(x)$ , definition of KL.

(c) [5 points] **KL and maximum likelihood.**

Consider a density estimation problem, and suppose we are given a training set  $\{x^{(i)}; i = 1, \dots, n\}$ . Let the empirical distribution be  $\hat{P}(x) = \frac{1}{n} \sum_{i=1}^n 1\{x^{(i)} = x\}$ . ( $\hat{P}$  is just the uniform distribution over the training set; i.e., sampling from the empirical distribution is the same as picking a random example from the training set.)

Suppose we have some family of distributions  $P_\theta$  parameterized by  $\theta$ . (If you like, think of  $P_\theta(x)$  as an alternative notation for  $P(x; \theta)$ .) Prove that finding the maximum likelihood estimate for the parameter  $\theta$  is equivalent to finding  $P_\theta$  with minimal KL divergence from  $\hat{P}$ , that is, prove:

$$\arg \min_{\theta} D_{KL}(\hat{P} \| P_\theta) = \arg \max_{\theta} \sum_{i=1}^n \log P_\theta(x^{(i)})$$

**Remark.** Consider the relationship between parts (b-c) and multi-variate Bernoulli Naive Bayes parameter estimation. In the Naive Bayes model we assumed  $P_\theta$  is of the following form:  $P_\theta(x, y) = p(y) \prod_{i=1}^d p(x_i|y)$ . By the chain rule for KL divergence, we therefore have:

$$D_{KL}(\hat{P} \| P_\theta) = D_{KL}(\hat{P}(y) \| p(y)) + \sum_{i=1}^d D_{KL}(\hat{P}(x_i|y) \| p(x_i|y)).$$

This shows that finding the maximum likelihood/minimum KL-divergence estimate of the parameters decomposes into  $2n + 1$  independent optimization problems: One for the class priors  $p(y)$ , and one for each of the conditional distributions  $p(x_i|y)$  for each feature  $x_i$

given each of the two possible labels for  $y$ . Specifically, finding the maximum likelihood estimates for each of these problems individually results in also maximizing the likelihood of the joint distribution. (If you know what Bayesian networks are, a similar remark applies to parameter estimation for them.)

**Answer:**

$$\begin{aligned}
 \arg \min_{\theta} D_{KL}(\hat{P} \| P_{\theta}) &= \arg \min_{\theta} \sum_x \hat{P}(x) \log \hat{P}(x) - \hat{P}(x) \log P_{\theta}(x) \\
 &= \arg \min_{\theta} \sum_x -\hat{P}(x) \log P_{\theta}(x) \\
 &= \arg \max_{\theta} \sum_x \hat{P}(x) \log P_{\theta}(x) \\
 &= \arg \max_{\theta} \sum_x \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{x^{(i)} = x\} \log P_{\theta}(x) \\
 &= \arg \max_{\theta} \frac{1}{n} \sum_{i=1}^n \sum_x \mathbb{1}\{x^{(i)} = x\} \log P_{\theta}(x) \\
 &= \arg \max_{\theta} \frac{1}{n} \sum_{i=1}^n \log P_{\theta}(x^{(i)}) \\
 &= \arg \max_{\theta} \sum_{i=1}^n \log P_{\theta}(x^{(i)})
 \end{aligned}$$

where we used in order: definition of KL, leaving out terms independent of  $\theta$ , flip sign and correspondingly flip min-max, definition of  $\hat{P}$ , switching order of summation, definition of the indicator and simplification since  $1/n$  is non-negative.

Another possible proof without switching sums is as follows. Without loss of generality, let's write  $\mathcal{X} = \{x_1, \dots, x_{|\mathcal{X}|}\}$ . The function being optimized over in the fourth equation above can then be re-written as:

$$\begin{aligned}
 \sum_{x \in \mathcal{X}} \sum_{i=1}^n \mathbb{1}\{x^{(i)} = x\} \log p_{\theta}(x) &= \sum_{i=1}^n \mathbb{1}\{x^{(i)} = x_1\} \log p_{\theta}(x_1) + \dots + \sum_{i=1}^n \mathbb{1}\{x^{(i)} = x_{|\mathcal{X}|}\} \log p_{\theta}(x_{|\mathcal{X}|}) \\
 &= \sum_{k=1}^{|\mathcal{X}|} \sum_{x \in S_k} \log p_{\theta}(x) \quad \text{where } S_k = \{x^{(i)} \mid x^{(i)} = x_k, i = 1, \dots, n\} \\
 &= \sum_{i=1}^n \log p_{\theta}(x^{(i)})
 \end{aligned}$$

since:

$$\bigcup_{k=1}^{|\mathcal{X}|} S_k = \{x^{(i)} \mid i = 1, \dots, n\}$$

and this union is disjoint because each  $x_k$  (from the set  $\mathcal{X}$ ) is distinct. This proof can easily be extended in the case of  $\mathcal{X}$  infinite (but still countable).



### 5. [35 points] Semi-supervised EM

Expectation Maximization (EM) is a classical algorithm for unsupervised learning (*i.e.*, learning with hidden or latent variables). In this problem we will explore one of the ways in which EM algorithm can be adapted to the semi-supervised setting, where we have some labeled examples along with unlabeled examples.

In the standard unsupervised setting, we have  $n \in \mathbb{N}$  unlabeled examples  $\{x^{(1)}, \dots, x^{(n)}\}$ . We wish to learn the parameters of  $p(x, z; \theta)$  from the data, but  $z^{(i)}$ 's are not observed. The classical EM algorithm is designed for this very purpose, where we maximize the intractable  $p(x; \theta)$  indirectly by iteratively performing the E-step and M-step, each time maximizing a tractable lower bound of  $p(x; \theta)$ . Our objective can be concretely written as:

$$\begin{aligned}\ell_{\text{unsup}}(\theta) &= \sum_{i=1}^n \log p(x^{(i)}; \theta) \\ &= \sum_{i=1}^n \log \sum_{z^{(i)}} p(x^{(i)}, z^{(i)}; \theta)\end{aligned}$$

Now, we will attempt to construct an extension of EM to the semi-supervised setting. Let us suppose we have an *additional*  $\tilde{n} \in \mathbb{N}$  labeled examples  $\{(\tilde{x}^{(1)}, \tilde{z}^{(1)}), \dots, (\tilde{x}^{(\tilde{n})}, \tilde{z}^{(\tilde{n})})\}$  where both  $x$  and  $z$  are observed. We want to simultaneously maximize the marginal likelihood of the parameters using the unlabeled examples, and full likelihood of the parameters using the labeled examples, by optimizing their weighted sum (with some hyperparameter  $\alpha$ ). More concretely, our semi-supervised objective  $\ell_{\text{semi-sup}}(\theta)$  can be written as:

$$\begin{aligned}\ell_{\text{sup}}(\theta) &= \sum_{i=1}^{\tilde{n}} \log p(\tilde{x}^{(i)}, \tilde{z}^{(i)}; \theta) \\ \ell_{\text{semi-sup}}(\theta) &= \ell_{\text{unsup}}(\theta) + \alpha \ell_{\text{sup}}(\theta)\end{aligned}$$

We can derive the EM steps for the semi-supervised setting using the same approach and steps as before. You are *strongly encouraged* to show to yourself (no need to include in the write-up) that we end up with:

#### E-step (semi-supervised)

For each  $i \in \{1, \dots, n\}$ , set

$$Q_i^{(t)}(z^{(i)}) := p(z^{(i)} | x^{(i)}; \theta^{(t)})$$

#### M-step (semi-supervised)

$$\theta^{(t+1)} := \arg \max_{\theta} \left[ \sum_{i=1}^n \left( \sum_{z^{(i)}} Q_i^{(t)}(z^{(i)}) \log \frac{p(x^{(i)}, z^{(i)}; \theta)}{Q_i^{(t)}(z^{(i)})} \right) + \alpha \left( \sum_{i=1}^{\tilde{n}} \log p(\tilde{x}^{(i)}, \tilde{z}^{(i)}; \theta) \right) \right]$$

- (a) [5 points] **Convergence.** First we will show that this algorithm eventually converges. In order to prove this, it is sufficient to show that our semi-supervised objective  $\ell_{\text{semi-sup}}(\theta)$  monotonically increases with each iteration of E and M step. Because log likelihood is bounded

above, the monotonically increasing objective will converge. Specifically, let  $\theta^{(t)}$  be the parameters obtained at the end of  $t$  EM-steps. Show that  $\ell_{\text{semi-sup}}(\theta^{(t+1)}) \geq \ell_{\text{semi-sup}}(\theta^{(t)})$ .

**Answer:**

$$\begin{aligned}
 \ell_{\text{semi-sup}}(\theta^{(t+1)}) &= \alpha \ell_{\text{sup}}(\theta^{(t+1)}) + \ell_{\text{unsup}}(\theta^{(t+1)}) && \text{Definition} \\
 &\geq \alpha \ell_{\text{sup}}(\theta^{(t+1)}) + \sum_{i=1}^n \sum_{z^{(i)}} Q_i^{(t)}(z^{(i)}) \log \frac{p(x^{(i)}, z^{(i)}; \theta^{(t+1)})}{Q_i^{(t)}(z^{(i)})} && \text{Jensen's inequality} \\
 &\geq \alpha \ell_{\text{sup}}(\theta^{(t)}) + \sum_{i=1}^n \sum_{z^{(i)}} Q_i^{(t)}(z^{(i)}) \log \frac{p(x^{(i)}, z^{(i)}; \theta^{(t)})}{Q_i^{(t)}(z^{(i)})} && \text{M-step} \\
 &= \alpha \ell_{\text{sup}}(\theta^{(t)}) + \sum_{i=1}^n \sum_{z^{(i)}} Q_i^{(t)}(z^{(i)}) \log \frac{p(z^{(i)}|x^{(i)}; \theta^{(t)}) p(x^{(i)}; \theta^{(t)})}{Q_i^{(t)}(z^{(i)})} && \text{Conditional probability} \\
 &= \alpha \ell_{\text{sup}}(\theta^{(t)}) + \sum_{i=1}^n \sum_{z^{(i)}} Q_i^{(t)}(z^{(i)}) \log p(x^{(i)}; \theta^{(t)}) && Q^{(t)}(z) := p(z|x; \theta^{(t)}) \\
 &= \alpha \ell_{\text{sup}}(\theta^{(t)}) + \ell_{\text{unsup}}(\theta^{(t)}) && \sum_{z^{(i)}} Q_i^{(t)}(z^{(i)}) = 1 \\
 &= \ell_{\text{semi-sup}}(\theta^{(t)})
 \end{aligned}$$

### Semi-supervised GMM

Now we will revisit the Gaussian Mixture Model (GMM), to apply our semi-supervised EM algorithm. Let us consider a scenario where data is generated from  $k \in \mathbb{N}$  Gaussian distributions, with unknown means  $\mu_j \in \mathbb{R}^d$  and covariances  $\Sigma_j \in \mathbb{S}_+^d$  where  $j \in \{1, \dots, k\}$ . We have  $n$  data points  $x^{(i)} \in \mathbb{R}^d, i \in \{1, \dots, n\}$ , and each data point has a corresponding latent (hidden/unknown) variable  $z^{(i)} \in \{1, \dots, k\}$  indicating which distribution  $x^{(i)}$  belongs to. Specifically,  $z^{(i)} \sim \text{Multinomial}(\phi)$ , such that  $\sum_{j=1}^k \phi_j = 1$  and  $\phi_j \geq 0$  for all  $j$ , and  $x^{(i)}|z^{(i)} \sim \mathcal{N}(\mu_{z^{(i)}}, \Sigma_{z^{(i)}})$  i.i.d. So,  $\mu$ ,  $\Sigma$ , and  $\phi$  are the model parameters.

We also have additional  $\tilde{n}$  data points  $\tilde{x}^{(i)} \in \mathbb{R}^d, i \in \{1, \dots, \tilde{n}\}$ , and an associated *observed* variable  $\tilde{z}^{(i)} \in \{1, \dots, k\}$  indicating the distribution  $\tilde{x}^{(i)}$  belongs to. Note that  $\tilde{z}^{(i)}$  are known constants (in contrast to  $z^{(i)}$  which are unknown *random* variables). As before, we assume  $\tilde{x}^{(i)}|\tilde{z}^{(i)} \sim \mathcal{N}(\mu_{\tilde{z}^{(i)}}, \Sigma_{\tilde{z}^{(i)}})$  i.i.d.

In summary we have  $n + \tilde{n}$  examples, of which  $n$  are unlabeled data points  $x$ 's with unobserved  $z$ 's, and  $\tilde{n}$  are labeled data points  $\tilde{x}^{(i)}$  with corresponding observed labels  $\tilde{z}^{(i)}$ . The traditional EM algorithm is designed to take only the  $n$  unlabeled examples as input, and learn the model parameters  $\mu$ ,  $\Sigma$ , and  $\phi$ .

Our task now will be to apply the semi-supervised EM algorithm to GMMs in order to also leverage the additional  $\tilde{n}$  labeled examples, and come up with semi-supervised E-step and M-step update rules specific to GMMs. Whenever required, you can cite the lecture notes for derivations and steps.

- (b) [5 points] **Semi-supervised E-Step.** Clearly state which are all the latent variables that need to be re-estimated in the E-step. Derive the E-step to re-estimate all the stated

latent variables. Your final E-step expression must only involve  $x, z, \mu, \Sigma, \phi$  and universal constants.

**Answer:** The latent variables are all the  $z^{(i)}, i \in \{1, \dots, n\}$ . The E-step remains unchanged from the lecture notes:

$$\begin{aligned} w_j^{(i)} = Q_i(z^{(i)} = j) &= \frac{p(x^{(i)}|z^{(i)} = j; \mu, \Sigma)p(z^{(i)} = j)}{\sum_{l=1}^k p(x^{(i)}|z^{(i)} = l; \mu, \Sigma)p(z^{(i)} = l)} \quad (\text{from notes}) \\ &= \frac{\frac{1}{\sqrt{(2\pi)^d |\Sigma_j|}} \exp\left(-\frac{1}{2}(x^{(i)} - \mu_j)^T \Sigma_j^{-1} (x^{(i)} - \mu_j)\right) \phi_j}{\sum_{l=1}^k \frac{1}{\sqrt{(2\pi)^d |\Sigma_l|}} \exp\left(-\frac{1}{2}(x^{(i)} - \mu_l)^T \Sigma_l^{-1} (x^{(i)} - \mu_l)\right) \phi_l} \\ &= \frac{|\Sigma_j|^{-1/2} \exp\left(-\frac{1}{2}(x^{(i)} - \mu_j)^T \Sigma_j^{-1} (x^{(i)} - \mu_j)\right) \phi_j}{\sum_{l=1}^k |\Sigma_l|^{-1/2} \exp\left(-\frac{1}{2}(x^{(i)} - \mu_l)^T \Sigma_l^{-1} (x^{(i)} - \mu_l)\right) \phi_l}. \end{aligned}$$

- (c) [10 points] **Semi-supervised M-Step.** Clearly state which are all the parameters that need to be re-estimated in the M-step. Derive the M-step to re-estimate all the stated parameters. Specifically, derive closed form expressions for the parameter update rules for  $\mu^{(t+1)}$ ,  $\Sigma^{(t+1)}$  and  $\phi^{(t+1)}$  based on the semi-supervised objective.

**Answer:**

In order to simplify derivation, we denote

$$w_j^{(i)} = Q_i^{(t)}(z^{(i)} = j),$$

and

$$\tilde{w}_j^{(i)} = \begin{cases} \alpha & \tilde{z}^{(i)} = j \\ 0 & \text{otherwise.} \end{cases}$$

We further denote  $S = \Sigma^{-1}$ , and note that because of chain rule of calculus,  $\nabla_S \ell = 0 \Rightarrow \nabla_{\Sigma} \ell = 0$ . So we choose to rewrite the M-step in terms of  $S$  and maximize it w.r.t  $S$ , and re-express the resulting solution back in terms of  $\Sigma$ .

Based on this, the M-step becomes:

$$\begin{aligned} \phi^{(t+1)}, \mu^{(t+1)}, S^{(t+1)} &= \arg \max_{\phi, \mu, S} \sum_{i=1}^n \sum_{j=1}^k Q_i^{(t)}(z^{(i)}) \log \frac{p(x^{(i)}, z^{(i)}; \phi, \mu, S)}{Q_i^{(t)}(z^{(i)})} + \sum_{i=1}^{\tilde{n}} \log p(\tilde{x}^{(i)}, \tilde{z}^{(i)}; \phi, \mu, S) \\ &= \arg \max_{\phi, \mu, S} \sum_{i=1}^n \sum_{j=1}^k w_j^{(i)} \log \frac{\frac{|S_j|^{1/2}}{(2\pi)^{d/2}} \exp\left(-\frac{1}{2}(x^{(i)} - \mu_j)^T S_j (x^{(i)} - \mu_j)\right) \phi_j}{w_j^{(i)}} \\ &\quad + \sum_{i=1}^{\tilde{n}} \sum_{j=1}^k \tilde{w}_j^{(i)} \log \frac{\frac{|S_j|^{1/2}}{(2\pi)^{d/2}} \exp\left(-\frac{1}{2}(\tilde{x}^{(i)} - \mu_j)^T S_j (\tilde{x}^{(i)} - \mu_j)\right) \phi_j}{\tilde{w}_j^{(i)}} \end{aligned}$$

First we start by maximizing w.r.t  $\phi_j$ . We construct the Lagrangian including the constraint that  $\sum_{j=1}^k \phi_j = 1$ , and absorbing all irrelevant terms into constant  $C$ :

$$\mathcal{L}(\phi, \beta) = C + \sum_{i=1}^n \sum_{j=1}^k w_j^{(i)} \log \phi_j + \sum_{i=1}^{\tilde{n}} \sum_{j=1}^k \tilde{w}_j^{(i)} \log \phi_j + \beta \left( \sum_{j=1}^k \phi_j - 1 \right)$$

$$\begin{aligned}
\nabla_{\phi_j} \mathcal{L}(\phi, \beta) &= \sum_{i=1}^n w_j^{(i)} \frac{1}{\phi_j} + \sum_{i=1}^{\tilde{n}} \tilde{w}_j^{(i)} \frac{1}{\phi_j} + \beta = 0 \\
\Rightarrow \phi_j &= \frac{\sum_{i=1}^n w_j^{(i)} + \sum_{i=1}^{\tilde{n}} \tilde{w}_j^{(i)}}{-\beta} \\
\nabla_{\beta} \mathcal{L}(\phi, \beta) &= \sum_{j=1}^k \phi_j - 1 = 0 \\
\Rightarrow \sum_{j=1}^k \frac{\sum_{i=1}^n w_j^{(i)} + \sum_{i=1}^{\tilde{n}} \tilde{w}_j^{(i)}}{-\beta} &= 1 \\
\Rightarrow -\beta &= \sum_{j=1}^k \left( \sum_{i=1}^n w_j^{(i)} + \sum_{i=1}^{\tilde{n}} \tilde{w}_j^{(i)} \right) \\
\Rightarrow \phi_j^{(t+1)} &= \frac{\sum_{i=1}^n w_j^{(i)} + \sum_{i=1}^{\tilde{n}} \tilde{w}_j^{(i)}}{\sum_{j=1}^k \left( \sum_{i=1}^n w_j^{(i)} + \sum_{i=1}^{\tilde{n}} \tilde{w}_j^{(i)} \right)} \\
&= \frac{\sum_{i=1}^n w_j^{(i)} + \sum_{i=1}^{\tilde{n}} \tilde{w}_j^{(i)}}{n + \alpha \tilde{n}}
\end{aligned}$$

Next we derive the update for  $\mu_j$  (absorbing irrelevant constants into  $C$ ):

$$\begin{aligned}
0 &= \nabla_{\mu_j} (\dots) \\
&= -\nabla_{\mu_j} \left( C + \sum_{i=1}^n w_j^{(i)} (x^{(i)} - \mu_j)^T S_j (x^{(i)} - \mu_j) + \sum_{i=1}^{\tilde{n}} \tilde{w}_j^{(i)} (\tilde{x}^{(i)} - \mu_j)^T S_j (\tilde{x}^{(i)} - \mu_j) \right) \\
&= -\nabla_{\mu_j} \left( \sum_{i=1}^n w_j^{(i)} (-2x^{(i)T} S_j \mu_j + \mu_j^T S_j \mu_j) + \sum_{i=1}^{\tilde{n}} \tilde{w}_j^{(i)} (-2\tilde{x}^{(i)T} S_j \mu_j + \mu_j^T S_j \mu_j) \right) \\
&= -2 \left( \sum_{i=1}^n w_j^{(i)} (-S_j x^{(i)} + S_j \mu_j) + \sum_{i=1}^{\tilde{n}} \tilde{w}_j^{(i)} (-S_j \tilde{x}^{(i)} + S_j \mu_j) \right) \\
&= 2S_j \left( \sum_{i=1}^n w_j^{(i)} x^{(i)} + \sum_{i=1}^{\tilde{n}} \tilde{w}_j^{(i)} \tilde{x}^{(i)} \right) - 2S_j \left( \sum_{i=1}^n w_j^{(i)} + \sum_{i=1}^{\tilde{n}} \tilde{w}_j^{(i)} \right) \mu_j \\
\Rightarrow \mu_j^{(t+1)} &= \frac{\sum_{i=1}^n w_j^{(i)} x^{(i)} + \sum_{i=1}^{\tilde{n}} \tilde{w}_j^{(i)} \tilde{x}^{(i)}}{\sum_{i=1}^n w_j^{(i)} + \sum_{i=1}^{\tilde{n}} \tilde{w}_j^{(i)}}
\end{aligned}$$

Finally we derive the update for  $\Sigma_j$  via  $S_j$ :

$$\begin{aligned}
0 &= \nabla_{S_j} (\dots) \\
&= \nabla_{S_j} \left( C + \sum_{i=1}^n w_j^{(i)} \left( \log |S_j| - (x^{(i)} - \mu_j)^T S_j (x^{(i)} - \mu_j) \right) + \sum_{i=1}^{\tilde{n}} \tilde{w}_j^{(i)} \left( \log |S_j| - (\tilde{x}^{(i)} - \mu_j)^T S_j (\tilde{x}^{(i)} - \mu_j) \right) \right) \\
&= \sum_{i=1}^n w_j^{(i)} \left( S_j^{-1} - (x^{(i)} - \mu_j)(x^{(i)} - \mu_j)^T \right) + \sum_{i=1}^{\tilde{n}} \tilde{w}_j^{(i)} \left( S_j^{-1} - (\tilde{x}^{(i)} - \mu_j)(\tilde{x}^{(i)} - \mu_j)^T \right)
\end{aligned}$$

$$\begin{aligned}
&= \left( \sum_{i=1}^n w_j^{(i)} + \sum_{i=1}^{\tilde{n}} \tilde{w}_j^{(i)} \right) S_j^{-1} - \left( \sum_{i=1}^n w_j^{(i)} (x^{(i)} - \mu_j)(x^{(i)} - \mu_j)^T + \sum_{i=1}^{\tilde{n}} \tilde{w}_j^{(i)} (\tilde{x}^{(i)} - \mu_j)(\tilde{x}^{(i)} - \mu_j)^T \right) \\
\Rightarrow \Sigma_j^{(t+1)} &= \frac{\sum_{i=1}^n w_j^{(i)} (x^{(i)} - \mu_j)(x^{(i)} - \mu_j)^T + \sum_{i=1}^{\tilde{n}} \tilde{w}_j^{(i)} (\tilde{x}^{(i)} - \mu_j)(\tilde{x}^{(i)} - \mu_j)^T}{\sum_{i=1}^n w_j^{(i)} + \sum_{i=1}^{\tilde{n}} \tilde{w}_j^{(i)}}
\end{aligned}$$

This results in the final set of update expressions:

$$\begin{aligned}
\phi_j &:= \frac{\sum_{i=1}^n w_j^{(i)} + \alpha \sum_{i=1}^{\tilde{n}} \mathbf{1}\{\tilde{z}^{(i)} = j\}}{n + \alpha \tilde{n}} \\
\mu_j &:= \frac{\sum_{i=1}^n w_j^{(i)} x^{(i)} + \alpha \sum_{i=1}^{\tilde{n}} \mathbf{1}\{\tilde{z}^{(i)} = j\} \tilde{x}^{(i)}}{\sum_{i=1}^n w_j^{(i)} + \alpha \sum_{i=1}^{\tilde{n}} \mathbf{1}\{\tilde{z}^{(i)} = j\}} \\
\Sigma_j &:= \frac{\sum_{i=1}^n w_j^{(i)} (x^{(i)} - \mu_j)(x^{(i)} - \mu_j)^T + \alpha \sum_{i=1}^{\tilde{n}} \mathbf{1}\{\tilde{z}^{(i)} = j\} (\tilde{x}^{(i)} - \mu_j)(\tilde{x}^{(i)} - \mu_j)^T}{\sum_{i=1}^n w_j^{(i)} + \alpha \sum_{i=1}^{\tilde{n}} \mathbf{1}\{\tilde{z}^{(i)} = j\}}
\end{aligned}$$

- (d) [5 points] **Classical (Unsupervised) EM Implementation.** For this sub-question, we are only going to consider the  $n$  unlabelled examples. Follow the instructions in `src/semi_supervised_em/gmm.py` to implement the traditional EM algorithm, and run it on the unlabelled data-set until convergence.

Run three trials and use the provided plotting function to construct a scatter plot of the resulting assignments to clusters (one plot for each trial). Your plot should indicate cluster assignments with colors they got assigned to (*i.e.*, the cluster which had the highest probability in the final E-step).

**Submit the three plots obtained above in your write-up.**

**Answer:**

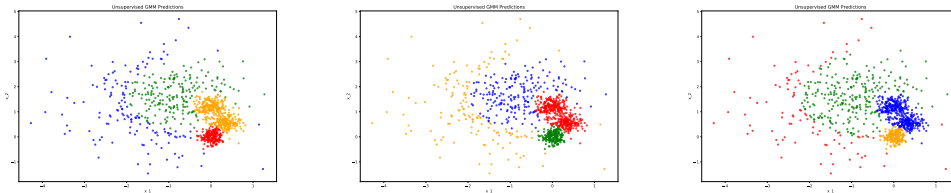


Figure 2: Predictions made by GMM model with unsupervised EM. See `src/semi_supervised_em/gmm.py` for source code.

- (e) [7 points] **Semi-supervised EM Implementation.** Now we will consider both the labelled and unlabelled examples (a total of  $n + \tilde{n}$ ), with 5 labelled examples per cluster. We have provided starter code for splitting the dataset into matrices `x` and `x_tilde` of unlabelled and labelled examples respectively. Add to your code in `src/semi_supervised_em/gmm.py` to implement the modified EM algorithm, and run it on the dataset until convergence.

Create a plot for each trial, as done in the previous sub-question.

**Submit the three plots obtained above in your write-up.**

**Answer:**

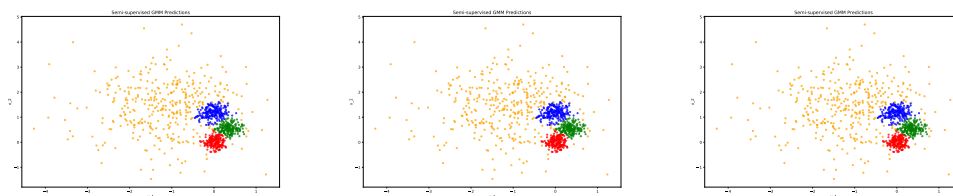


Figure 3: Predictions made by GMM model with semi-supervised EM. See `src/semi_supervised_em/gmm.py` for source code.

- (f) [3 points] **Comparison of Unsupervised and Semi-supervised EM.** Briefly describe the differences you saw in unsupervised *vs.* semi-supervised EM for each of the following:
- Number of iterations taken to converge.
  - Stability (*i.e.*, how much did assignments change with different random initializations?)
  - Overall quality of assignments.

**Note:** The dataset was sampled from a mixture of three low-variance Gaussian distributions, and a fourth, high-variance Gaussian distribution. This should be useful in determining the overall quality of the assignments that were found by the two algorithms.

**Answer:** Even with just 5 labeled examples per cluster, semi-supervised EM outperforms classical EM in all three categories.

- Semi-supervised is much faster: Unsupervised EM takes roughly 150 iterations to converge. Semi-supervised EM takes about 25.
- Semi-supervised is much more stable: Assignments change for unsupervised, but they are almost exactly the same for semi-supervised EM.
- Semi-supervised finds a higher quality assignment overall: Unsupervised EM suffers from a few types of errors (*e.g.*, failing to discover there is just a single relatively high-variance Gaussian distribution in the mixture). Semi-supervised EM almost exactly uncovers the underlying distribution.