long as N is greater than 20 or 30 and the signal is not overwhelmed by noise. There's nothing sacred about the absolute values of the curves in Figures 3–23(a) and 3–23(b). They were generated through a simulation of noise and a tone whose frequency was at a DFT bin center. Had the tone's frequency been between bin centers, the processing gain curves would have been shifted downward, but their shapes would still be the same;† that is, Eq. (3–33) is still valid regardless of the input tone's frequency.

3.12.2 Integration Gain Due to Averaging Multiple DFTs

Theoretically, we could get very large DFT processing gains by increasing the DFT size arbitrarily. The problem is that the number of necessary DFT multiplications increases proportionally to N^2, and larger DFTs become very computationally intensive. Because addition is easier and faster to perform than multiplication, we can average the outputs of multiple DFTs to obtain further processing gain and signal detection sensitivity. The subject of averaging multiple DFT outputs is covered in Section 11.3.

3.13 THE DFT OF RECTANGULAR FUNCTIONS

We continue this chapter by providing the mathematical details of two important aspects of the DFT. First, we obtain the expressions for the DFT of a rectangular function (rectangular window), and then we'll use these results to illustrate the magnitude response of the DFT. We're interested in the DFT's magnitude response because it provides an alternate viewpoint to understand the leakage that occurs when we use the DFT as a signal analysis tool.

One of the most prevalent and important computations encountered in digital signal processing is the DFT of a rectangular function. We see it in sampling theory, window functions, discussions of convolution, spectral analysis, and in the design of digital filters. As common as it is, however, the literature covering the DFT of rectangular functions can be confusing to the digital signal processing beginner for several reasons. The standard mathematical notation is a bit hard to follow at first, and sometimes the equations are presented with too little explanation. Compounding the problem, for the beginner, are the various expressions of this particular DFT. In the literature, we're likely to find any one of the following forms for the DFT of a rectangular function:

$$\mathrm{DFT}_{\mathrm{rect.function}} = \frac{\sin(x)}{\sin(x/N)}, \text{ or } \frac{\sin(x)}{x}, \text{ or } \frac{\sin(Nx/2)}{\sin(x/2)}. \qquad (3–34)$$

† The curves would be shifted downward, indicating a lower SNR, because leakage would raise the average noise-power level, and scalloping loss would reduce the DFT bin's output power level.

In this section we'll show how the forms in Eq. (3–34) were obtained, see how they're related, and create a kind of *Rosetta Stone* table allowing us to move back and forth between the various DFT expressions. Take a deep breath and let's begin our discussion with the definition of a rectangular function.

3.13.1 DFT of a General Rectangular Function

A general rectangular function $x(n)$ can be defined as N samples containing K unity-valued samples as shown in Figure 3–24. The full N-point sequence, $x(n)$, is the rectangular function that we want to transform. We call this the general form of a rectangular function because the K unity samples begin at an arbitrary index value of $-n_o$. Let's take the DFT of $x(n)$ in Figure 3–24 to get our desired $X(m)$. Using m as our frequency-domain sample index, the expression for an N-point DFT is

$$X(m) = \sum_{n=-(N/2)+1}^{N/2} x(n)e^{-j2\pi nm/N}. \tag{3–35}$$

With $x(n)$ being nonzero only over the range of $-n_o \le n \le -n_o + (K-1)$, we can modify the summation limits of Eq. (3–35) to express $X(m)$ as

$$X(m) = \sum_{n=-n_o}^{-n_o+(K-1)} 1 \cdot e^{-j2\pi nm/N}, \tag{3–36}$$

because only the K samples contribute to $X(m)$. That last step is important because it allows us to eliminate the $x(n)$ terms and make Eq. (3–36) easier to handle. To keep the following equations from being too messy, let's use the dummy variable $q = 2\pi m/N$.

OK, here's where the algebra comes in. Over our new limits of summation, we eliminate the factor of one and Eq. (3–36) becomes

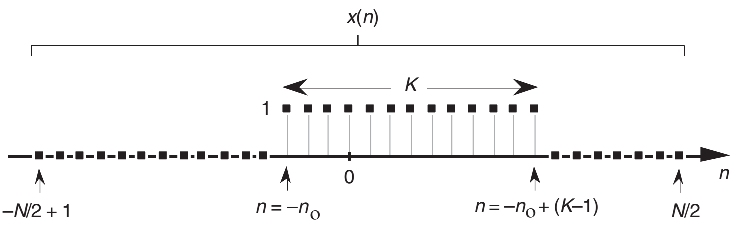

Figure 3–24 Rectangular function of width K samples defined over N samples where $K < N$.

$$X(q) = \sum_{n=-n_o}^{-n_o+(K-1)} e^{-jqn}$$

$$= e^{-jq(-n_o)} + e^{-jq(-n_o+1)} + e^{-jq(-n_o+2)} + \ldots + e^{-jq(-n_o+(K-1))}$$

$$= e^{-jq(-n_o)}e^{-j0q} + e^{-jq(-n_o)}e^{-j1q} + e^{-jq(-n_o)}e^{-j2q} + \ldots + e^{-jq(-n_o)}e^{-jq(K-1)}$$

$$= e^{jq(n_o)} \cdot \left[e^{-j0q} + e^{-j1q} + e^{-j2q} + \ldots + e^{jq(K-1)}\right]. \tag{3–37}$$

The series inside the brackets of Eq. (3–37) allows the use of a summation, such as

$$X(q) = e^{jq(n_o)} \sum_{p=0}^{K-1} e^{-jpq}. \tag{3–38}$$

Equation (3–38) certainly doesn't look any simpler than Eq. (3–36), but it is. Equation (3–38) is a *geometric series* and, from the discussion in Appendix B, it can be evaluated to the closed form of

$$\sum_{p=0}^{K-1} e^{-jpq} = \frac{1 - e^{-jqK}}{1 - e^{-jq}}. \tag{3–39}$$

We can now simplify Eq. (3–39)—here's the clever step. If we multiply and divide the numerator and denominator of Eq. (3–39)'s right-hand side by the appropriate half-angled exponentials, we break the exponentials into two parts and get

$$\sum_{p=0}^{K-1} e^{-jpq} = \frac{e^{-jqK/2}(e^{jqK/2} - e^{-jqK/2})}{e^{-jq/2}(e^{jq/2} - e^{-jq/2})}$$

$$= e^{-jq(K-1)/2} \cdot \frac{(e^{jqK/2} - e^{-jqK/2})}{(e^{jq/2} - e^{-jq/2})}. \tag{3–40}$$

Let's pause for a moment here to remind ourselves where we're going. We're trying to get Eq. (3–40) into a usable form because it's part of Eq. (3–38) that we're using to evaluate $X(m)$ in Eq. (3–36) in our quest for an understandable expression for the DFT of a rectangular function.

Equation (3–40) looks even more complicated than Eq. (3–39), but things can be simplified inside the parentheses. From Euler's equation, $\sin(\o) = (e^{j\o} - e^{-j\o})/2j$, Eq. (3–40) becomes

$$\sum_{p=0}^{K-1} e^{-jpq} = e^{-jq(K-1)/2} \cdot \frac{2j\sin(qK/2)}{2j\sin(q/2)}$$

$$= e^{-jq(K-1)/2} \cdot \frac{\sin(qK/2)}{\sin(q/2)}. \tag{3–41}$$

Substituting Eq. (3–41) for the summation in Eq. (3–38), our expression for $X(q)$ becomes

$$X(q) = e^{jq(n_o)} \cdot e^{-jq(K-1)/2} \cdot \frac{\sin(qK/2)}{\sin(q/2)}$$

$$= e^{jq(n_o-(K-1)/2)} \cdot \frac{\sin(qK/2)}{\sin(q/2)}. \tag{3–42}$$

Returning our dummy variable q to its original value of $2\pi m/N$,

$$X(m) = e^{j(2\pi m/N)(n_o-(K-1)/2)} \cdot \frac{\sin(2\pi mK/2N)}{\sin(2\pi m/2N)}, \text{ or}$$

General form of the Dirichlet kernel:→

$$X(m) = e^{j(2\pi m/N)(n_o-(K-1)/2)} \cdot \frac{\sin(\pi mK/N)}{\sin(\pi m/N)}. \tag{3–43}$$

So there it is (whew!). Equation (3–43) is the general expression for the DFT of the rectangular function as shown in Figure 3–24. Our $X(m)$ is a complex expression (pun intended) where a ratio of sine terms is the amplitude of $X(m)$ and the exponential term is the phase angle of $X(m)$.† The ratio of sines factor in Eq. (3–43) lies on the periodic curve shown in Figure 3–25(a), and like all N-point DFT representations, the periodicity of $X(m)$ is N. This curve is known as the *Dirichlet kernel* (or the *aliased sinc function*) and has been thoroughly described in the literature[10,13,14]. (It's named after the nineteenth-century German mathematician Peter Dirichlet [pronounced dee-ree-'klay],

† N was an even number in Figure 3–24 depicting the $x(n)$. Had N been an odd number, the limits on the summation in Eq. (3–35) would have been $-(N-1)/2 \le n \le (N-1)/2$. Using these alternate limits would have led us to exactly the same $X(m)$ as in Eq. (3–43).

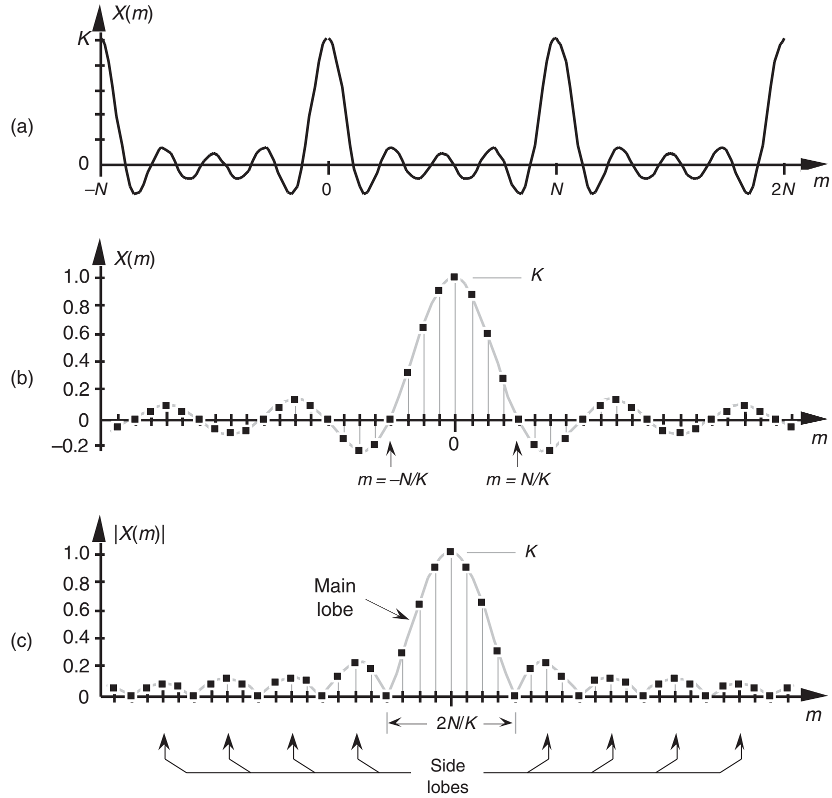

Figure 3–25 The Dirichlet kernel of $X(m)$: (a) periodic continuous curve on which the $X(m)$ samples lie; (b) $X(m)$ amplitudes about the $m = 0$ sample; (c) $|X(m)|$ magnitudes about the $m = 0$ sample.

who studied the convergence of trigonometric series used to represent arbitrary functions.)

We can zoom in on the curve at the $m = 0$ point and see more detail in Figure 3–25(b). The dots are shown in Figure 3–25(b) to remind us that the DFT of our rectangular function results in discrete amplitude values that lie on the curve. So when we perform DFTs, our discrete results are sampled values of the continuous sinc function's curve in Figure 3–25(a). As we'll show later, we're primarily interested in the absolute value, or magnitude, of the Dirichlet kernel in Eq. (3–43). That magnitude, $|X(m)|$, is shown in Figure 3–25(c). Although we first saw the sinc function's curve in Figure 3–9 in Section 3.8, where we introduced the topic of DFT leakage, we'll encounter this curve often in our study of digital signal processing.

For now, there are just a few things we need to keep in mind concerning the Dirichlet kernel. First, the DFT of a rectangular function has a main lobe, centered about the $m = 0$ point. The peak amplitude of the main lobe is K. This peak value makes sense, right? The $m = 0$ sample of a DFT $X(0)$ is the sum of the original samples, and the sum of K unity-valued samples is K. We can show this in a more substantial way by evaluating Eq. (3–43) for $m = 0$. A difficulty arises when we plug $m = 0$ into Eq. (3–43) because we end up with sin(0)/sin(0), which is the indeterminate ratio 0/0. Well, hardcore mathematics to the rescue here. We can use L'Hopital's Rule to take the derivative of the numerator and the denominator of Eq. (3–43), and *then* set $m = 0$ to determine the peak value of the magnitude of the Dirichlet kernel.† We proceed as

$$|X(m)|_{m\to 0} = \frac{d}{dm}X(m) = \frac{d[\sin(\pi mK/N)]/dm}{d[\sin(\pi m/N)]/dm}$$

$$= \frac{\cos(\pi mK/N)}{\cos(\pi m/N)} \cdot \frac{d(\pi mK/N)/dm}{d(\pi m/N)/dm}$$

$$= \frac{\cos(0)}{\cos(0)} \cdot \frac{\pi K/N}{\pi/N} = 1 \cdot K = K, \tag{3–44}$$

which is what we set out to show. (We could have been clever and evaluated Eq. (3–35) with $m = 0$ to get the result of Eq. (3–44). Try it, and keep in mind that $e^{j0} = 1$.) Had the amplitudes of the nonzero samples of $x(n)$ been other than unity, say some amplitude A_o, then, of course, the peak value of the Dirichlet kernel would be A_oK instead of just K. The next important thing to notice about the Dirichlet kernel is the main lobe's width. The first zero crossing of Eq. (3–43) occurs when the numerator's argument is equal to π, that is, when $\pi mK/N = \pi$. So the value of m at the first zero crossing is given by

$$m_{\text{first zero crossing}} = \frac{\pi N}{\pi K} = \frac{N}{K} \tag{3–45}$$

as shown in Figure 3–25(b). Thus the main lobe width $2N/K$, as shown in Figure 3–25(c), is inversely proportional to K.††

Notice that the main lobe in Figure 3–25(a) is surrounded by a series of oscillations, called *sidelobes*, as in Figure 3–25(c). These sidelobe magnitudes decrease the farther they're separated from the main lobe. However, no matter how far we look away from the main lobe, these sidelobes never reach zero magnitude—and they cause a great deal of heartache for practitioners in

† L'Hopital is pronounced 'lō-pē-tȯl, like baby doll.

†† This is a fundamental characteristic of Fourier transforms. The narrower the function in one domain, the wider its transform will be in the other domain.