NOVA SCHOOL OF
SCIENCE & TECHNOLOGY

Distributed MPC

Cyber-Physical Control Systems

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Course plan

Week	Subject	Assignment
1	M0 Course details and introduction	
2	M1.1 Discrete-time systems state model representation	P1 start
3	M1.2 Optimization and optimal control	HW1
4	M2.1 Introduction to model predictive control (MPC)	
5	M2.2 Constrained MPC design	HW2
6	M3.1 Nonlinear MPC design	
7	M3.2 Feasibility and Stability analysis of MPC design	HW3, P1 due
8	M4.1 Decentralized MPC design	P2 start
9	M4.2 Distributed MPC design	
10	M4.3 Networked control systems	HW4
11	Discussion and feedback of draft P2 paper	Draft P2 due
12	M5.1 Hybrid dynamic systems	
13	M5.2 MPC for hybrid dynamic systems	HW5
14	Final project 2 presentations and discussion	P2 due

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Outline

Centralized and decentralized two-player games

Distributed two-player MPC

Distributed two-player tracking MPC

Constrained distributed MPC and multi-player formulation

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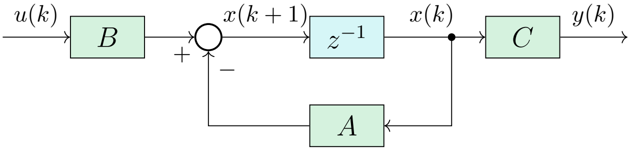
Discrete-time linear dynamic systems



- LTI state-space form:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \quad , x(0) = x_0 \\ y(k) &= Cx(k) \end{aligned}$$

  where $A \in \mathbb{R}^{n_x \times n_x}$ and $B \in \mathbb{R}^{n_x \times n_u}$.

- Yet another compact notation:

$$\begin{aligned} x^+ &= Ax + Bu \quad , x(0) = x_0 \\ y &= Cx \end{aligned}$$

Coupled two-player models

- Model for player 1:

$$x_1^+ = A_1 x_1 + B_{11} u_1 + B_{12} u_2 \quad , x_1(0) = x_{10}$$
$$y_1 = C_1 x_1$$

- Model for player 2:

$$x_2^+ = A_2 x_2 + B_{22} u_2 + B_{21} u_1 \quad , x_2(0) = x_{20}$$
$$y_2 = C_2 x_2$$

- Batch representations for an horizon N:

$$X_1 = F_1 x_1(0) + G_{11} U_1 + G_{12} U_2$$
$$X_2 = F_2 x_2(0) + G_{22} U_2 + G_{21} U_1$$

where F_i and G_{ij} are computed as introduced before in Module 3.
- Batch representation of the outputs is given by $Y_1 = H_1 X_1$ and $Y_2 = H_2 X_2$.

Joint coupled two-player model

- Joint model considering $x = \begin{bmatrix} x_1^T & x_2^T \end{bmatrix}^T$:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad , \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- Compact notation, with $u = \begin{bmatrix} u_1^T & u_2^T \end{bmatrix}^T$, and $y = \begin{bmatrix} y_1^T & y_2^T \end{bmatrix}^T$:

$$x^+ = Ax + Bu \quad , x(0) = x_0$$
$$y = Cx$$

- Batch state representation: $X = Fx(0) + GU$.
- Batch output representation: $Y = \bar{F}x(0) + \bar{G}U$.

Centralized regulation MPC

- Joint regulation MPC problem:

$$\min_{U} \quad J(x(0), U) = p(x(N)) + \textstyle\sum_{k=0}^{N-1} q(x(k), u(k))$$
$$s.t. \quad x^+ = Ax + Bu$$

with $p(x) = x^T P x$ and $q(x, u) = x^T Q x + u^T R u$.

- Equivalent batch formulation:

$$\min_{U} \quad J = U^T \tilde{R} U + 2U^T \tilde{S} x(0) + x(0)^T \tilde{Q} x(0)$$

- Optimal control sequence: $U^* = K^* x(0) = \{u^*(0), u^*(1), \dots, u^*(N-1)\}$
- Considering $K^* = -\tilde{R}^{-1}\tilde{S}$, the MPC control policy is

$$u(t_k) = u^*(0) = Kx(0) \quad , \quad K = EK^* \quad , \quad E = \begin{bmatrix} I_{n_u} & 0_{n_u \times (N-1)n_u} \end{bmatrix}$$

- For each player i, the control law is $u_i(t_k) = u_i^*(0) = K_{i1}x_1(0) + K_{i2}x_2(0)$

Decentralized regulation MPC

- No connection or communication between models.
- Implies model errors: inputs of player 2 ignored in player 1, and vice-versa.
- For each player i, the MPC problem is then

$$\min_{U_i} \quad J_i(x_i(0), U_i) = p_i(x_i(N)) + \sum_{k=0}^{N-1} q_i(x_i(k), u_i(k))$$

$$s.t. \quad x_i^+ = A_i x_i + B_{ii} u_i \cancel{+ B_{ij} u_j}$$

with $p_i(y) = y^T P_i y$ and $q_i(y, u) = y^T Q_i y + u^T R_i u$.

- Equivalent batch formulation for player i:

$$\min_{U_i} \quad J_i = U_i^T \tilde{R}_i U_i + 2U_i^T \tilde{S}_i x_i(0) + x_i(0)^T \tilde{Q}_i x_i(0)$$

- Considering $K_i^* = -\tilde{R}_i^{-1} \tilde{S}_i$, the MPC control policy for player i is

$$u_i(t_k) = u_i^*(0) = K_i x_i(0) \ , \quad K_i = E_i K_i^* \ , \quad E_i = \begin{bmatrix} I_{n_{ui}} & 0_{n_{ui} \times (N-1) n_{ui}} \end{bmatrix}$$

Centralized tracking MPC

- Joint tracking MPC problem:

$$\min_{U} \quad J(x(0),U) = p(y(N)-\bar{y}(N)) + \textstyle\sum_{k=0}^{N-1} q(y(k)-\bar{y}(k),u(k))$$
$$s.t. \quad x^+ = Ax+Bu\ , \quad y = Cx$$

- Equivalent batch formulation:

$$\min_{U} \quad J = U^T\tilde{R}U + 2U^T\tilde{S}(\bar{F}x(0)-\bar{Y}) + (\bar{F}x(0)-\bar{Y})^T\bar{Q}(\bar{F}x(0)-\bar{Y})$$

- Optimal control sequence: $U^* = K^*x(0) + K_y^*\bar{Y} = \{u^*(0),\dots,u^*(N-1)\}$
- Considering $K_y^* = \tilde{R}^{-1}\tilde{S}$ and $K^* = -K_y^*\bar{F}$, the MPC control policy is:

$$u(t_k) = u^*(0) = Kx(0) + K_y\bar{Y}\ , \quad K = EK^*\ , \quad K_y = EK_y^*$$

- For each player i, the control law is

$$u_i(t_k) = u_i^*(0) = K_{i1}x_1(0) + K_{i2}x_2(0) + K_{yi1}\bar{Y}_1 + K_{yi2}\bar{Y}_2$$

Decentralized tracking MPC

- For each player i, the tracking MPC problem is

$$\min_{U_i} \quad J_i(x_i(0), U_i) = p(y_i(N) - \bar{y}_i(N)) + \sum_{k=0}^{N-1} q(y(k) - \bar{y}(k), u_i(k))$$

$$s.t. \quad x_i^+ = A_i x_i + B_{ii} u_i \cancel{+ B_{ij} u_j}\ , \ y_i = C_i x_i$$

- Equivalent batch formulation for player i:

$$\min_{U_i} \quad J_i = U_i^T \tilde{R}_i U_i + 2U_i^T \tilde{S}_i (\bar{F}_i x_i(0) - \bar{Y}_i) + (\bar{F}x(0) - \bar{Y})^T \bar{Q} (\bar{F}x(0) - \bar{Y})$$

- Optimal control sequence: $U_i^* = K_i^* x_i(0) + K_{yi}^* \bar{Y}_i = \{u_i^*(0), \ldots, u_i^*(N-1)\}$
- Considering $K_{yi}^* = \tilde{R}_i^{-1} \tilde{S}_i$ and $K_i^* = -K_{yi}^* \bar{F}$, the MPC control policy is

$$u_i(t_k) = u_i^*(0) = K_i x_i(0) + K_{yi} \bar{Y}_i\ , \quad K_i = E_i K_i^*\ , \quad K_{yi} = E_i K_{yi}^*$$

- Using Δu_i instead of u_i follows the examples of the previous course modules, by augmenting the state-space model.

Exercise 4.1

Problem (Exercise 4.1 - two-input two-output system)

Consider the time-invariant discrete linear system $i = 1, 2$ *described by*

$$x_i^+ = A_i x_i + B_{ii} u_i + B_{ij} u_j \ , \quad y_i = x_i$$

with $A_1 = 0.7$, $A_2 = 0.8$, $B_{11} = 1.1$, $B_{12} = 0.9$, $B_{21} = -0.9$, *and* $B_{22} = 1.1$
Considering the quadratic cost for player $i = 1, 2$ *given by*

$$J_i = p_i(y_i(N) - \bar{y}_i(N)) + \sum_{k=0}^{N-1} q_i(y_i(k) - \bar{y}_i(k), u_i(k))$$

where the functions $p_i()$ *and* $q_i(.)$ *are defined above, compute and simulate the centralized and decentralized MPC policies, noting that* $J = J_1 + J_2$, *with* $N = 3$, $P_i = Q_i = 5$, $R_i = 1$, $x(0) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$, *and* $\bar{y}(k) = \begin{bmatrix} 1 & 1.5 \end{bmatrix}^T \varepsilon(k - 30)$.

Exercise 4.1

Exercise 4.1 - two-input two-output system (part 1)

For the centralized case, the joint system is $x^+ = Ax + Bu$ and $y = Cx$, with $A = \operatorname{diag}(A_1, A_2)$, $C = \operatorname{diag}(C_1, C_2) = I_2$, and $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$. For the cost function, let $P = Q = \operatorname{diag}(Q_1, Q_2) = 5I_2$ and $R = \operatorname{diag}(R_1, R_2) = I_2$.
Using the batch approach, the first step is to obtain the equivalent unconstrained quadratic optimization problem

$$\min_U \quad J = U^T \tilde{R} U + 2U^T \tilde{S}(\bar{F}x(0) - \bar{Y}) + (\bar{F}x(0) - \bar{Y})^T \bar{Q}(\bar{F}x(0) - \bar{Y})$$

which provides at each time step the optimal control sequence

$$U^* = Kx(0) + K_y \bar{Y} = \begin{bmatrix} u^*(0)^T & u^*(1)^T & u^*(2)^T \end{bmatrix}^T$$

Exercise 4.1

Exercise 4.1 - two-input two-output system (part 2)

Considering that $Y = \bar{F}x(0) + \bar{G}U$ and noting that $\bar{Q} = \operatorname{diag}(Q, Q, Q, P) = 5I_8$ and $\bar{R} = \operatorname{diag}(R, R, R) = I_6$, the final matrices are $\tilde{R} = \bar{G}^T\bar{Q}\bar{G} + \bar{R}$, $\tilde{S} = \bar{G}^T\bar{Q}$, whereas $K_y = \tilde{R}^{-1}\tilde{S}$ and $K = -K_y\bar{F}$. The state batched matrices are given by

$$\bar{F} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.7 & 0 \\ 0 & 0.8 \\ 0.49 & 0 \\ 0 & 0.64 \\ 0.34 & 0 \\ 0 & 0.51 \end{bmatrix} \quad \bar{G} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1.1 & 0.9 & 0 & 0 & 0 & 0 \\ -0.9 & 1.1 & 0 & 0 & 0 & 0 \\ 0.77 & 0.63 & 1.1 & 0.9 & 0 & 0 \\ -0.72 & 0.88 & -0.9 & 1.10 & 0 & 0 \\ 0.54 & 0.44 & 0.77 & 0.63 & 1.1 & 0.9 \\ -0.58 & 0.70 & -0.72 & 0.88 & -0.9 & 1.1 \end{bmatrix}$$

Exercise 4.1

Exercise 4.1 - two-input two-output system (part 3)

The control gains are

$$K_y = \begin{bmatrix} 0 & 0 & 0.48 & -0.39 & 0.03 & -0.03 & 0.00 & -0.00 \\ 0 & 0 & 0.39 & 0.47 & 0.02 & 0.03 & 0.00 & 0.00 \\ 0 & 0 & -0.30 & 0.28 & 0.46 & -0.37 & 0.03 & -0.03 \\ 0 & 0 & -0.25 & -0.34 & 0.37 & 0.45 & 0.02 & 0.03 \\ 0 & 0 & -0.02 & 0.02 & -0.30 & 0.28 & 0.48 & -0.39 \\ 0 & 0 & -0.02 & -0.02 & -0.25 & -0.34 & 0.39 & 0.47 \end{bmatrix} \quad K = \begin{bmatrix} -0.35 & 0.33 \\ -0.29 & -0.40 \\ -0.02 & 0.02 \\ -0.02 & -0.02 \\ -0.00 & 0.00 \\ -0.00 & -0.00 \end{bmatrix}$$

Exercise 4.1

Exercise 4.1 - two-input two-output system (part 4)

For the decentralized case, the system i is $x_i^+ = A_i x_i + B_{ii} u_i$ and $y_i = x_i$. Using the batch approach, the equivalent unconstrained optimization problem is

$$\min_{U_i} \quad J_i = U_i^T \tilde{R}_i U_i + 2U_i^T \tilde{S}_i(\bar{F}_i x_i(0) - \bar{Y}_i) + (\bar{F}_i x(0) - \bar{Y}_i)^T \bar{Q}_i (\bar{F}_i x_i(0) - \bar{Y}_i)$$

which provides at each time step the optimal control sequence

$$U_i^* = K_i x_i(0) + K_{yi} \bar{Y}_i = \begin{bmatrix} u_i^*(0)^T & u_i^*(1)^T & u_i^*(2)^T \end{bmatrix}^T$$

Considering that $Y_i = \bar{F}_i x_i(0) + \bar{G}_i U_i$ and noting that $\bar{Q}_i = \operatorname{diag}(Q_i, Q_i, Q_i, P_i) = 5I_4$ and $\bar{R}_i = \operatorname{diag}(R_i, R_i, R_i) = I_3$, we obtain $\tilde{R}_i = \bar{G}_i^T \bar{Q}_i \bar{G}_i + \bar{R}_i$, $\tilde{S}_i = \bar{G}_i^T \bar{Q}_i$, whereas $K_{yi} = \tilde{R}_i^{-1} \tilde{S}_i$ and $K_i = -K_{yi} \bar{F}_i$.

Exercise 4.1

Exercise 4.1 - two-input two-output system (part 5)

The state batched matrices are given by

$$\bar{F}_i = \begin{bmatrix} C_i \\ C_iA_i \\ C_iA_i^2 \\ C_iA_i^3 \end{bmatrix} \qquad \bar{F}_1 = \begin{bmatrix} 1.0000 \\ 0.7000 \\ 0.4900 \\ 0.3430 \end{bmatrix} \qquad \bar{F}_2 = \begin{bmatrix} 1.0000 \\ 0.8000 \\ 0.6400 \\ 0.5120 \end{bmatrix}$$

$$\bar{G}_i = \begin{bmatrix} 0 & 0 & 0 \\ C_iB_i & 0 & 0 \\ C_iA_iB_i & C_iB_i & 0 \\ C_iA_i^2B_i & C_iA_iB_i & C_iB_i \end{bmatrix} \quad \bar{G}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1.1 & 0 & 0 \\ 0.77 & 1.1 & 0 \\ 0.54 & 0.77 & 1.1 \end{bmatrix} \quad \bar{G}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1.1 & 0 & 0 \\ 0.88 & 1.1 & 0 \\ 0.71 & 0.88 & 1.1 \end{bmatrix}$$

Exercise 4.1

Exercise 4.1 - two-input two-output system (part 6)

The control gains are

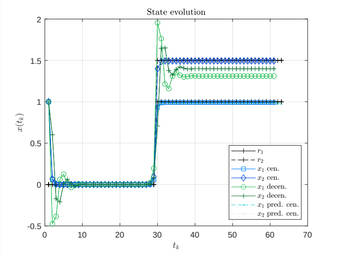
$$K_{y1} = \begin{bmatrix} 0 & 0.736 & 0.069 & 0.007 \\ 0 & -0.446 & 0.694 & 0.069 \\ 0 & -0.041 & -0.446 & 0.736 \end{bmatrix} \qquad K_{y2} = \begin{bmatrix} 0 & 0.723 & 0.076 & 0.009 \\ 0 & -0.502 & 0.671 & 0.076 \\ 0 & -0.052 & -0.502 & 0.723 \end{bmatrix}$$

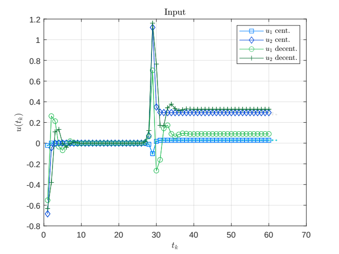
$$K_1 = \begin{bmatrix} -0.551 \\ -0.052 \\ -0.005 \end{bmatrix} \qquad K_2 = \begin{bmatrix} -0.632 \\ -0.066 \\ -0.007 \end{bmatrix}$$

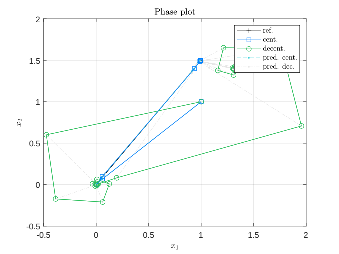
Exercise 4.1

Exercise 4.1 - two-input two-output system (part 7)

Simulation results for 60 sampling intervals:







Outline

Distributed two-player game model

- Connection or communication between models.
- Inputs of player 2 sent to player 1, and vice-versa.
- Model used for each player:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} u_1 + \begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix} u_2$$

- Cooperative goal for each player is a combination of each player goals:

$$\begin{aligned} J(x_1(0), x_2(0), U_1, U_2) = & \alpha_1 J_1(x_1(0), U_1, U_2) + \alpha_2 J_2(x_2(0), U_1, U_2) \\ = & \alpha_1 p(x_1(N)) + \alpha_2 p(x_2(N)) \\ & + \textstyle\sum_{k=0}^{N-1} \alpha_1 q(x_1(k), u_1(k), u_2(k))) + \alpha_2 q(x_2(k), u_1(k), u_2(k))) \end{aligned}$$

Distributed two-player game cost

- Cooperative quadratic cost for player 1:

$$J(x_1(0), x_2(0), U_1, U_2) = p_1(x_1(N), x_2(N)) \\ + \sum_{k=0}^{N-1} q_1(x_1(k), x_2(k), u_1(k), u_2(k)))$$

where

$$q_1(x_1, x_2, u_1, u_2) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} \alpha_1 Q_1 & 0 \\ 0 & \alpha_2 Q_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u_1^T \alpha_1 R_1 u_1 + \underbrace{\cancel{u_2^T \alpha_2 R_2 u_2}}_{\text{const.}}$$

$$p_1(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} \alpha_1 P_1 & 0 \\ 0 & \alpha_2 P_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- Similar cost for player 2.

Computing the batched cost (1)

- Considering $X_1 = F_1 x_1(0) + G_{11}U_1 + G_{12}U_2$

$$
\begin{aligned}
&\alpha_1 p(x_1(N)) + \textstyle\sum_{k=0}^{N-1} \alpha_1 q(x_1(k), u_1(k))) \\
&= x_1(N)^T \alpha_1 P_1 x_1(N) + \textstyle\sum_{k=0}^{N-1} x_1^T(k)\alpha_1 Q_1 x_1(k) + u_1^T(k)\alpha_1 R_1 u_1(k) \\
&= X_1^T \bar{Q}_1 X_1 + U_1^T \bar{R}_1 U_1 \\
&= (F_1x_1(0)+G_{11}U_1+G_{12}U_2)^T \bar{Q}_1 (F_1x_1(0)+G_{11}U_1+G_{12}U_2)+U_1^T \bar{R}_1 U_1 \\
&= U_1^T(\bar{R}_1 + G_{11}^T \bar{Q}_1 G_{11})U_1 + 2(F_1x_1(0) + G_{12}U_2)^T \bar{Q}_1 G_{11} U_1 \\
&\quad + (F_1x_1(0) + G_{12}U_2)^T \bar{Q}_1 (F_1x_1(0) + G_{12}U_2)
\end{aligned}
$$

where $\bar{Q}_1 = \alpha_1 \mathrm{diag}\,(Q_1, Q_1, \ldots, P_1)$ and $\bar{R}_1 = \alpha_1 \mathrm{diag}\,(R_1, \ldots, R_1)$.

Computing the batched cost (2)

- Considering $X_2 = F_2 x_2(0) + G_{22}U_2 + G_{21}U_1$

$$
\begin{aligned}
&\alpha_2 p(x_2(N)) + \textstyle\sum_{k=0}^{N-1} \alpha_2 q(x_2(k), u_2(k)))\\
&= x_2(N)^T \alpha_2 P_2 x_2(N) + \textstyle\sum_{k=0}^{N-1} x_2^T(k)\alpha_2 Q_2 x_2(k) + u_2^T(k)\alpha_2 R_2 u_2(k)\\
&= X_2^T \bar{Q}_2 X_2 + U_2^T \bar{R}_2 U_2\\
&= (F_2x_2(0)+G_{22}U_2+G_{21}U_1)^T \bar{Q}_2 (F_2x_2(0)+G_{22}U_2+G_{21}U_1)+U_2^T\bar{R}_2U_2\\
&= U_1^T G_{21}^T \bar{Q}_2 G_{21} U_1 + 2(F_2x_2(0) + G_{22}U_2)^T \bar{Q}_2 G_{21} U_1\\
&\quad + (F_2x_2(0) + G_{22}U_2)^T \bar{Q}_2 (F_2x_2(0) + G_{22}U_2) + U_2\bar{R}_2U_2
\end{aligned}
$$

where $\bar{Q}_2 = \alpha_2 \mathrm{diag}\left(Q_2, Q_2, \ldots, P_2\right)$ and $\bar{R}_2 = \alpha_2 \mathrm{diag}\left(R_2, \ldots, R_2\right)$.

Computing the batched cost (3)

- Combining the above terms, we can say that the batched cost is

$$J(x_1(0), x_2(0), U_1, U_2) = U_1^T \tilde{R}_1 U_1 + 2\tilde{S}_1^T U_1 + \tilde{Q}_1$$

where

$$\begin{aligned}
\tilde{R}_1 &= \bar{R}_1 + G_{11}^T \bar{Q}_1 G_{11} + G_{21}^T \bar{Q}_2 G_{21} \\
\tilde{S}_1 &= G_{11}^T \bar{Q}_1 (F_1 x_1(0) + G_{12} U_2) + G_{21}^T \bar{Q}_2 (F_2 x_2(0) + G_{22} U_2) \\
&= G_{11}^T \bar{Q}_1 F_1 x_1(0) + G_{21}^T \bar{Q}_2 F_2 x_2(0) + (G_{11}^T \bar{Q}_1 G_{12} + G_{21}^T \bar{Q}_2 G_{22}) U_2 \\
&= S_{11x} x_1(0) + S_{12x} x_2(0) + S_{12u} U_2 \\
\tilde{Q}_1 &= (F_1 x_1(0) + G_{12} U_2)^T \bar{Q}_1 (F_1 x_1(0) + G_{12} U_2) \\
&\quad + (F_2 x_2(0) + G_{22} U_2 +)^T \bar{Q}_2 (F_2 x_2(0) + G_{22} U_2) + U_2^T \bar{R}_2 U_2
\end{aligned}$$

Distributed two-player game problem

- For player 1, the batch distributed MPC problem is then

$$\min_{U_1} \quad J(x_1(0), x_2(0), U_1, U_2) = U_1^T \tilde{R}_1 U_1 + 2\tilde{S}_1^T U_1$$

- First order condition of optimality yields

$$\nabla_{U_1} J_0 = 2\tilde{R}_1 U_1 + 2\tilde{S}_1 = 0 \quad \Leftrightarrow \quad \tilde{R}_1 U_1 + S_{11x} x_1(0) + S_{12x} x_2(0) + S_{12u} U_2 = 0$$

- The optimal control sequence for player 1 is

$$U_1^* = K_{11} x_1(0) + K_{12} x_2(0) + L_1 U_2$$

where $K_{11} = -\tilde{R}_1^{-1} S_{11x}$, $K_{12} = -\tilde{R}_1^{-1} S_{12x}$, and $L_1 = -\tilde{R}_1^{-1} S_{12u}$.

- Similarly, we can compute the optimal control sequence for player 2, given by

$$U_2^* = K_{21} x_1(0) + K_{22} x_2(0) + L_2 U_1$$

Distributed convex step

- Notice that U_1^* depends on U_2, while U_2^* depends on U_1.
- The information from previous communication is represented by $U_i^p := U_i$.
- How to compute the new information to send to other players, U_i^{p+1}, while ensuring the distributed algorithm converges?
- Basic option of setting $U_i^{p+1} = U_i^*$ might drive the distributed system to instability, as no consensus might be possible.
- Solution: convex combination of previous and current solution $(0 < w_i < 1)$:

$$U_i^{p+1} = w_i U_i^* + (1 - w_i) U_i^p$$

- Convex step for player 1:

$$U_1^{p+1} = w_1 K_{11} x_1(0) + w_1 K_{12} x_2(0) + w_1 L_1 U_2^p + (1 - w_i) U_1^p$$

- Iterations on p might happen more than once per sampling time.
- Note: this is just one way to implement distributed MPC (see [2] for others).

Distributed control law

- Combining both optimal sequences we can write

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}^* = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 0 & L_1 \\ L_2 & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

- The convex combination $U_i^{p+1} = w_i U_i^* + (1 - w_i) U_i^p$ results in

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}^{p+1} = \begin{bmatrix} w_1 K_{11} & w_1 K_{12} \\ w_2 K_{21} & w_2 K_{22} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} (1 - w_1) I & w_1 L_1 \\ w_2 L_2 & (1 - w_2) I \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}^p$$

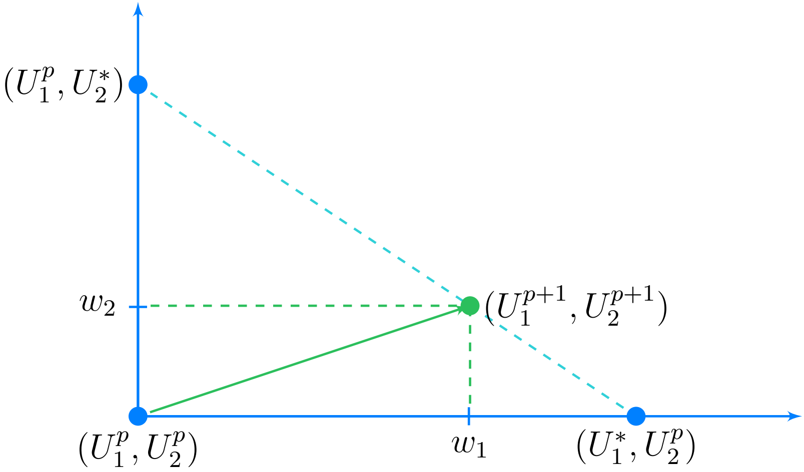
- Equivalently, defining $U = \begin{bmatrix} U_1^T & U_2^T \end{bmatrix}^T$ we can write

$U^{p+1} = \bar{K} x(0) + L U^p$ where

$$\bar{K} = \begin{bmatrix} w_1 K_{11} & w_1 K_{12} \\ w_2 K_{21} & w_2 K_{22} \end{bmatrix}, \qquad L = \begin{bmatrix} (1 - w_1) I & w_1 L_1 \\ w_2 L_2 & (1 - w_2) I \end{bmatrix}$$

- Noting that $0 < w_i < 1$ and $\sum_i w_i = 1$.

Distributed control law convex step



Solution of the distributed control law

- Consider p iterations of the distributed control law:

$$
\begin{aligned}
U^1 &= LU^0 + \bar{K}x(0) \\
U^2 &= LU^1 + \bar{K}x(0) = L(LU^0 + \bar{K}x(0)) + \bar{K}x(0) \\
&= L^2U^0 + (L+1)Kx(0) \\
U^3 &= LU^2 + \bar{K}x(0) = L(L^2U^0 + (L+1)\bar{K}x(0)) + \bar{K}x(0) \\
&= L^3U^0 + (L^2 + L + 1)\bar{K}x(0) \\
&\vdots
\end{aligned}
$$

- Solution of the distributed law for p iterations

$$U^p = L^pU^0 + \sum_{j=0}^{p-1} L^j\bar{K}x(0)$$

Convergence of the distributed control law

- The distributed convex step will converge iff matrix L is stable.
- When convergence is achieved, as $p \to \infty$, then

$$U^\infty = \bar{K}x(0) + LU^\infty$$
$$U^\infty = (I - L)^{-1}\bar{K}x(0)$$

- Converged solution does not depend of choice of w_i:

$$(I - L)^{-1}\bar{K} = \begin{bmatrix} w_1 I & -w_1 L_1 \\ -w_2 L_2 & w_2 I \end{bmatrix}^{-1} \begin{bmatrix} w_1 K_{11} & w_1 K_{12} \\ w_2 K_{21} & w_2 K_{22} \end{bmatrix}$$
$$= \begin{bmatrix} I & -L_1 \\ -L_2 & I \end{bmatrix}^{-1} \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

considering the partitioned matrix inversion result in Section A.6 of [1].

Distributed MPC policy

- Consider the converged distributed control sequences for player i

  $$U_i := U_i^{\infty} = \{u_i^*(0), \ldots, u_i^*(N-1)\}$$

- The MPC control policy for player i is then

  $$u_i(t_k) = u_i^*(0) = E_i U_i \quad , \quad E_i = \begin{bmatrix} I_{n_{ui}} & 0_{n_{ui} \times N n_{ui}} \end{bmatrix}$$

- Combining both MPC control policies, using $E = \operatorname{diag}(E_1, E_2)$ and defining $K = E(I - L)^{-1}\bar{K}$, then

  $$u(t_k) = K x(t_k)$$

- The resulting closed loop system is

  $$x(t_{k+1}) = (A + BK) x(t_k)$$

- Stability: if L and $A + BK$ are stable (with converged distributed law).
- Suboptimal MPC analysis when distributed solution may not converge.

Summary 4.1

The main learning outcomes of this class are:

- Define centralized and decentralized two-player MPC problems
- Define unconstrained two-player distributed MPC regulator
- Solve the distributed optimization using a convex step rule

Course plan

Week	Subject	Assignment
1	M0 Course details and introduction	
2	M1.1 Discrete-time systems state model representation	P1 start
3	M1.2 Optimization and optimal control	HW1
4	M2.1 Introduction to model predictive control (MPC)	
5	M2.2 Constrained MPC design	HW2
6	M3.1 Nonlinear MPC design	
7	M3.2 Feasibility and Stability analysis of MPC design	HW3, P1 due
8	M4.1 Decentralized MPC design	P2 start
9	M4.2 Distributed MPC design	
10	M4.3 Networked control systems	HW4
11	Discussion and feedback of draft P2 paper	Draft P2 due
12	M5.1 Hybrid dynamic systems	
13	M5.2 MPC for hybrid dynamic systems	HW5
14	Final project 2 presentations and discussion	P2 due

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Outline

Distributed two-player game tracking cost

- Cooperative quadratic tracking cost for player 1:

$$J(x_1(0), x_2(0), U_1, U_2) = p_1(y_1(N) - \bar{y}_1(N), y_2(N) - \bar{y}_2(N)) \\ + \sum_{k=0}^{N-1} q_1(y_1(k) - \bar{y}_1(k), y_2(k) - \bar{y}_2(k), u_1(k), u_2(k)))$$

where

$$q_1(y_1, y_2, u_1, u_2) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^T \begin{bmatrix} \alpha_1 Q_1 & 0 \\ 0 & \alpha_2 Q_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + u_1^T \alpha_1 R_1 u_1 + \underbrace{\cancel{u_2^T \alpha_2 R_2 u_2}}_{\text{const.}}$$

$$p_1(y_1, y_2) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^T \begin{bmatrix} \alpha_1 P_1 & 0 \\ 0 & \alpha_2 P_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

- Similar cost for player 2.

Computing the batched tracking cost (1)

- Considering $Y_1 = \bar{F}_1 x_1(0) + \bar{G}_{11} U_1 + \bar{G}_{12} U_2$

$$\alpha_1 p(y_1(N) - \bar{y}_1(N)) + \sum_{k=0}^{N-1} \alpha_1 q(y_1(k) - \bar{y}_1(k), u_1(k)))$$
$$= (y_1(N) - \bar{y}_1(N))^T \alpha_1 P_1 (y_1(N) - \bar{y}_1(N))$$
$$+ \sum_{k=0}^{N-1} (y_1(k) - \bar{y}_1(k))^T \alpha_1 Q_1 (y_1(k) - \bar{y}_1(k)) + u_1^T(k) \alpha_1 R_1 u_1(k)$$
$$= U_1^T \bar{R}_1 U_1 + (Y_1 - \bar{Y}_1)^T \bar{Q}_1 (Y_1 - \bar{Y}_1)$$
$$= U_1^T \bar{R}_1 U_1$$
$$+ (\bar{F}_1 x_1(0) + \bar{G}_{11} U_1 + \bar{G}_{12} U_2 - \bar{Y}_1)^T \bar{Q}_1 (\bar{F}_1 x_1(0) + \bar{G}_{11} U_1 + \bar{G}_{12} U_2 - \bar{Y}_1)$$
$$= U_1^T (\bar{R}_1 + \bar{G}_{11}^T \bar{Q}_1 \bar{G}_{11}) U_1 + 2(\bar{F}_1 x_1(0) + \bar{G}_{12} U_2 - \bar{Y}_1)^T \bar{Q}_1 \bar{G}_{11} U_1$$
$$+ (\bar{F}_1 x_1(0) + \bar{G}_{12} U_2 - \bar{Y}_1)^T \bar{Q}_1 (\bar{F}_1 x_1(0) + \bar{G}_{12} U_2 - \bar{Y}_1)$$

where $\bar{Q}_1 = \alpha_1 \mathrm{diag}(Q_1, Q_1, \ldots, P_1)$ and $\bar{R}_1 = \alpha_1 \mathrm{diag}(R_1, \ldots, R_1)$.

Computing the batched tracking cost (2)

- Considering $Y_2 = \bar{F}_2 x_2(0) + \bar{G}_{22} U_2 + \bar{G}_{21} U_1$

$$
\begin{aligned}
&\alpha_2 p(y_2(N) - \bar{y}_2(N)) + \textstyle\sum_{k=0}^{N-1} \alpha_2 q(y_2(k) - \bar{y}_2(k), u_2(k))) \\
&= (y_2(N) - \bar{y}_2(N))^T \alpha_2 P_2 (y_2(N) - \bar{y}_2(N)) \\
&\quad + \textstyle\sum_{k=0}^{N-1} (y_2(k) - \bar{y}_2(k))^T \alpha_2 Q_2 (y_2(k) - \bar{y}_2(k)) + u_2^T(k) \alpha_2 R_2 u_2(k) \\
&= (Y_2 - \bar{Y}_2)^T \bar{Q}_2 (Y_2 - \bar{Y}_2) + U_2^T \bar{R}_2 U_2 \\
&= (\bar{F}_2 x_2(0) + \bar{G}_{22} U_2 + \bar{G}_{21} U_1 - \bar{Y}_2)^T \bar{Q}_2 (\bar{F}_2 x_2(0) + \bar{G}_{22} U_2 + \bar{G}_{21} U_1 - \bar{Y}_2) \\
&\quad + U_2^T \bar{R}_2 U_2 \\
&= U_1^T \bar{G}_{21}^T \bar{Q}_2 \bar{G}_{21} U_1 + 2(\bar{F}_2 x_2(0) + \bar{G}_{22} U_2 - \bar{Y}_2)^T \bar{Q}_2 \bar{G}_{21} U_1 \\
&\quad + (\bar{F}_2 x_2(0) + \bar{G}_{22} U_2 - \bar{Y}_2)^T \bar{Q}_2 (\bar{F}_2 x_2(0) + \bar{G}_{22} U_2 - \bar{Y}_2) + U_2^T \bar{R}_2 U_2
\end{aligned}
$$

where $\bar{Q}_2 = \alpha_2 \mathrm{diag}\,(Q_2, Q_2, \ldots, P_2)$ and $\bar{R}_2 = \alpha_2 \mathrm{diag}\,(R_2, \ldots, R_2)$.

Computing the batched tracking cost (3)

- Combining the above terms, we can say that the batched cost is

$$J(x_1(0), x_2(0), U_1, U_2) = U_1^T \tilde{R}_1 U_1 + 2\tilde{S}_1^T U_1 + \tilde{Q}_1$$

where

$$\tilde{R}_1 = \bar{R}_1 + \bar{G}_{11}^T \bar{Q}_1 \bar{G}_{11} + \bar{G}_{21}^T \bar{Q}_2 \bar{G}_{21}$$

$$\begin{aligned}
\tilde{S}_1 &= \bar{G}_{11}^T \bar{Q}_1 (\bar{F}_1 x_1(0) + \bar{G}_{12} U_2 - \bar{Y}_1) + \bar{G}_{21}^T \bar{Q}_2 (\bar{F}_2 x_2(0) + \bar{G}_{22} U_2 - \bar{Y}_2) \\
&= \bar{G}_{11}^T \bar{Q}_1 \bar{F}_1 x_1(0) - \bar{G}_{11}^T \bar{Q}_1 Y_1 + \bar{G}_{21}^T \bar{Q}_2 \bar{F}_2 x_2(0) - \bar{G}_{21}^T \bar{Q}_2 Y_2 \\
&\quad + (\bar{G}_{11}^T \bar{Q}_1 \bar{G}_{12} + \bar{G}_{21}^T \bar{Q}_2 \bar{G}_{22}) U_2 \\
&= S_{11x} x_1(0) - S_{11y} \bar{Y}_1 + S_{12x} x_2(0) - S_{12y} \bar{Y}_2 + S_{12u} U_2
\end{aligned}$$

$$\begin{aligned}
\tilde{Q}_1 &= (F_1 x_1(0) + \bar{G}_{12} U_2 - \bar{Y}_1)^T \bar{Q}_1 (F_1 x_1(0) + \bar{G}_{12} U_2 - \bar{Y}_1) \\
&\quad + (F_2 x_2(0) + \bar{G}_{22} U_2 - \bar{Y}_2)^T \bar{Q}_2 (F_2 x_2(0) + \bar{G}_{22} U_2 - \bar{Y}_2) + U_2^T \bar{R}_2 U_2
\end{aligned}$$

Distributed two-player game tracking problem

- For player 1, the batch distributed MPC problem is then

$$\min_{U_1} \quad J(x_1(0), x_2(0), U_1, U_2) = U_1^T \tilde{R}_1 U_1 + 2\tilde{S}_1^T U_1$$

- First order condition of optimality yields

$$\nabla_{U_1} J_0 = 2\tilde{R}_1 U_1 + 2\tilde{S}_1 = 0$$

$$\tilde{R}_1 U_1 + S_{11x} x_1(0) - S_{11y} \bar{Y}_1 + S_{12x} x_2(0) - S_{12y} \bar{Y}_2 + S_{12u} U_2 = 0$$

- The optimal control sequence for player 1 is

$$U_1^* = K_{11} x_1(0) + K_{11y} \bar{Y}_1 + K_{12} x_2(0) + K_{12y} \bar{Y}_2 + L_1 U_2$$

where $K_{1i} = -\tilde{R}_1^{-1} S_{1ix}$, $K_{1iy} = \tilde{R}_1^{-1} S_{1iy}$, and $L_1 = -\tilde{R}_1^{-1} S_{12u}$.

- Similarly, we can compute the optimal control sequence for player 2, given by

$$U_2^* = K_{21} x_1(0) + K_{21y} \bar{Y}_1 + K_{22} x_2(0) + K_{22y} \bar{Y}_2 + L_2 U_1$$

Distributed tracking MPC control law

- Combining both optimal sequences we can write

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}^* = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} K_{11y} & K_{12y} \\ K_{21y} & K_{22y} \end{bmatrix} \begin{bmatrix} \bar{Y}_1 \\ \bar{Y}_2 \end{bmatrix} + \begin{bmatrix} 0 & L_1 \\ L_2 & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

- Each player applies the distributed step $U_i^{p+1} = w_i U_i^* + (1 - w_i) U_i^p$.
- Defining $U = \begin{bmatrix} U_1^T & U_2^T \end{bmatrix}^T$ and $\bar{Y} = \begin{bmatrix} \bar{Y}_1^T & \bar{Y}_2^T \end{bmatrix}^T$, the distributed step is

$$U^{p+1} = \bar{K} x(0) + \bar{K}_y \bar{Y} + L U^p \,, \quad \text{where} \quad \bar{K}_y = \begin{bmatrix} w_1 K_{11y} & w_1 K_{12y} \\ w_2 K_{21y} & w_2 K_{22y} \end{bmatrix}$$

- The MPC policy for player i after p steps is then $u_i(t_k) = u_i^*(0) = E_i U_i^p$.
- Considering the converged distributed sequence by defining $K = E(I - L)^{-1} \bar{K}$ and $K_y = E(I - L)^{-1} \bar{K}_y$, the MPC policy is

$$u(t_k) = K x(t_k) + K_y \bar{Y}$$

Exercise 4.3

Problem (Exercise 4.3 - coupled double integrators)

Consider two coupled double integrators, for which the time-invariant discrete linear joint system with $x = \begin{bmatrix} \theta_{12} & \omega_1 & \omega_2 \end{bmatrix}^T$ *is described by*

$$\begin{aligned} x^+ &= Ax + Bu \\ y &= Cx \end{aligned} \quad \textit{with} \;\; A = \begin{bmatrix} 1 & 0.1 & -0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Considering the quadratic cost for each player $i = 1, 2$, *given by* $J_i = p_i(y_i(N) - \bar{y}_i(N)) + \sum_{k=0}^{N-1} q_i(y_i(k) - \bar{y}_i(k), u_i(k))$, *where* $y_1 = \theta_{12}$ *and* $y_2 = -\theta_{12}$, *compute and simulate the distributed MPC policy, noting that* $J = J_1 + J_2$, *with* $N = 3$, $P_i = Q_i = 2$, $R_i = 1$, $x(0) = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}^T$, *and* $\bar{y}(k) = \begin{bmatrix} 2 & -2 \end{bmatrix}^T \varepsilon(k - 50)$.

Exercise 4.3

Exercise 4.3 - coupled double integrators (part 1)

For completion of this solution we will also include the centralized case, for which the joint system is given in the exercise, while for the cost function, let $P = Q = \operatorname{diag}(Q_1, Q_2) = 2I_2$ and $R = \operatorname{diag}(R_1, R_2) = I_2$.
Using the batch approach, the first step is to obtain the equivalent unconstrained quadratic optimization problem

$$\min_U \quad J = U^T \tilde{R} U + 2U^T \tilde{S}(\bar{F}x(0) - \bar{Y}) + (\bar{F}x(0) - \bar{Y})^T \bar{Q}(\bar{F}x(0) - \bar{Y})$$

which provides at each time step the optimal control sequence

$$U^* = Kx(0) + K_y \bar{Y} = \begin{bmatrix} u^*(0)^T & u^*(1)^T & u^*(2)^T \end{bmatrix}^T$$

Exercise 4.3

Exercise 4.3 - coupled double integrators (part 2)

Considering that $Y = \bar{F}x(0) + \bar{G}U$ and noting that $\bar{Q} = \operatorname{diag}(Q, Q, Q, P) = 2I_8$ and $\bar{R} = \operatorname{diag}(R, R, R) = I_6$, the final matrices are $\tilde{R} = \bar{G}^T\bar{Q}\bar{G} + \bar{R}$, $\tilde{S} = \bar{G}^T\bar{Q}$, whereas $K_y = \tilde{R}^{-1}\tilde{S}$ and $K = -K_y\bar{F}$, given by

$$K_y = \begin{bmatrix} & 0.15 & -0.15 & 0.27 & -0.27 \\ 0_{4\times4} & -0.15 & 0.15 & -0.27 & 0.27 \\ & -0.02 & 0.02 & 0.15 & -0.15 \\ & 0.02 & -0.02 & -0.15 & 0.15 \\ 0_{2\times4} & & 0_{2\times4} & & \end{bmatrix} \quad K = \begin{bmatrix} -0.83 & -0.22 & 0.22 \\ 0.83 & 0.22 & -0.22 \\ -0.25 & -0.08 & 0.08 \\ 0.25 & 0.08 & -0.08 \\ & 0_{2\times3} & \end{bmatrix}$$

Exercise 4.3

Exercise 4.3 - coupled double integrators (part 3)

For the distributed case, the system i must be in the form $x_i^+ = A_i x_i + B_{ii} u_i + B_{ij} u_j$, which will not be the case if we choose the individual states as $x_1 = \begin{bmatrix}\theta_{12} & \omega_1\end{bmatrix}^T$ and $x_2 = \begin{bmatrix}\theta_{12} & \omega_3\end{bmatrix}^T$. As the state-space equations can be written as

$$
\begin{aligned}
\theta_{12}(k+1) &= \theta_{12}(k) + 0.1(\omega_1(k) - \omega_2(k))\\
\omega_1(k+1) &= \omega_1(k) + u_1(k)\\
\omega_2(k+1) &= \omega_2(k) + u_2(k)
\end{aligned}
$$

we notice that the only way the states of each distributed model i will going to be independent from the states of model j, is by duplicating state variables, in particular, duplicating all of them.

Exercise 4.3

Exercise 4.3 - coupled double integrators (part 4)

As such, the state vector for each model $i = 1, 2$ will be $x_i = \begin{bmatrix}\theta_{12} & \omega_1 & \omega_2\end{bmatrix}^T$, implying that the state matrix for each model is $A_i = A$, yet a different output, $y_i = C_i x_i$, is used for each model, where $C_1 = \begin{bmatrix}1 & 0 & 0\end{bmatrix}$ and $C_2 = \begin{bmatrix}-1 & 0 & 0\end{bmatrix}$.
In this way, we can define the matrices $B_{11} = \begin{bmatrix}0 & 1 & 0\end{bmatrix}^T$, $B_{12} = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$, $B_{22} = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$, and $B_{21} = \begin{bmatrix}0 & 1 & 0\end{bmatrix}^T$.
Considering the batch version of each model $i = 1, 2$ as $Y_i = \bar{F}_i x_i(0) + \bar{G}_{ii} U_i + \bar{G}_{ij} U_j$ and noting that $\bar{Q}_i = \operatorname{diag}(Q_i, Q_i, Q_i, P_i) = 2I_4$ and $\bar{R}_i = \operatorname{diag}(R_i, R_i, R_i) = I_3$, the equivalent unconstrained optimization problem for model 1 is

$$\min_{U_1} \quad J(x_1(0), x_2(0), U_1, U_2) = U_1^T \tilde{R}_1 U_1 + 2\tilde{S}_1^T U_1$$

Exercise 4.3

Exercise 4.3 - coupled double integrators (part 5)

Thus, at each time step the optimal control sequence is

$$U_1^* = K_{11}x_1(0) + K_{11y}\bar{Y}_1 + K_{12}x_2(0) + K_{12y}\bar{Y}_2 + L_1 U_2$$

where $K_{1iy} = \tilde{R}_1^{-1} S_{1iy}$, $K_{1i} = -K_{1iy}\bar{F}_i$, and $L_1 = -K_{11y}\bar{G}_{12} - K_{12y}\bar{G}_{22}$, considering the matrices $\tilde{R}_1 = \bar{R}_1 + \bar{G}_{11}^T\bar{Q}_1\bar{G}_{11} + \bar{G}_{21}^T\bar{Q}_2\bar{G}_{21}$ and $S_{1iy} = \bar{G}_{i1}^T\bar{Q}_i$. A similar control law can be deduced for model 2, yielding

$$U_2^* = K_{21}x_1(0) + K_{21y}\bar{Y}_1 + K_{22}x_2(0) + K_{22y}\bar{Y}_2 + L_2 U_1$$

where $K_{2iy} = \tilde{R}_2^{-1} S_{2iy}$, $K_{2i} = -K_{2iy}\bar{F}_i$, and $L_2 = -K_{22y}\bar{G}_{21} - K_{21y}\bar{G}_{11}$, considering the matrices $\tilde{R}_2 = \bar{R}_2 + \bar{G}_{22}^T\bar{Q}_2\bar{G}_{22} + \bar{G}_{12}^T\bar{Q}_1\bar{G}_{12}$ and $S_{2iy} = \bar{G}_{i2}^T\bar{Q}_i$.

Exercise 4.3

Exercise 4.3 - coupled double integrators (part 6)

The gain matrices for both players are

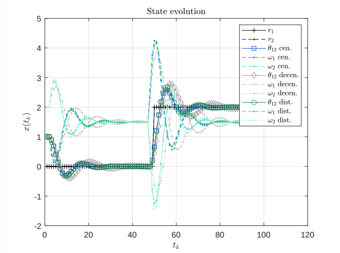
$$K_{11} = K_{12} = -K_{22} = -K_{21} = \begin{bmatrix} -0.49 & -0.13 & 0.13 \\ -0.15 & -0.05 & 0.05 \\ 0 & 0 & 0 \end{bmatrix} \quad L_1 = \begin{bmatrix} 0.16 & 0.06 & 0 \\ 0.06 & 0.03 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

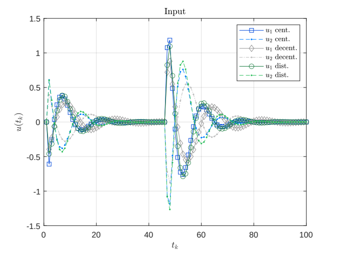
$$K_{11y} = -K_{12y} = K_{22y} = -K_{21y} = \begin{bmatrix} 0 & 0 & 0.17 & 0.32 \\ 0 & 0 & -0.01 & 0.17 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad L_2 = \begin{bmatrix} 0.16 & 0.06 & 0 \\ 0.06 & 0.03 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

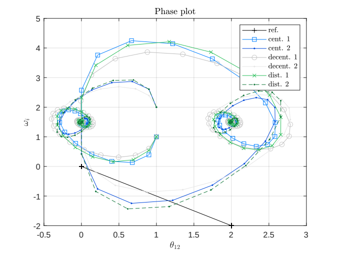
Exercise 4.3

Exercise 4.3 - coupled double integrators (part 7)

Simulation results for 100 sampling intervals and 2 communication iterations for each sampling time ($p = 2$):



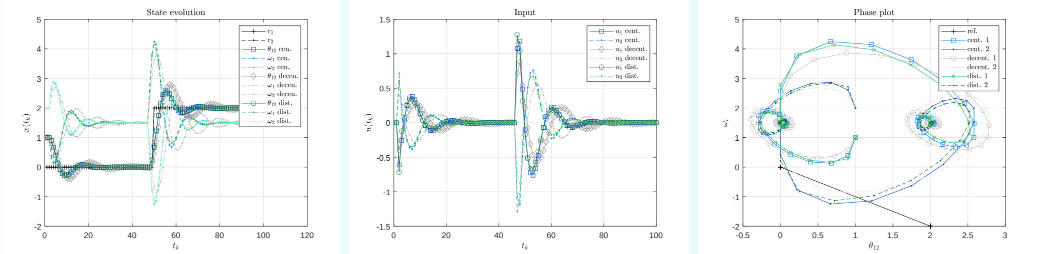




Exercise 4.3

Exercise 4.3 - coupled double integrators (part 8)

Simulation results for 100 sampling intervals and 10 communication iterations for each sampling time ($p = 10$):



Outline

Distributed MPC joint model and costs

- Connection or communication between systems.
- Inputs of player 2 sent to player 1, and vice-versa.
- Model used for each player:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} u_1 + \begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix} u_2$$

- Regulation cost for each player i:

$$J_i(x_i(0), U_i, U_j) = p_i(x_i(N)) + \sum_{k=0}^{N-1} q_i(y(k) - \bar{y}(k), u_i(k))$$

- Tracking cost for each player i:

$$J_i(x_i(0), U_i, U_j) = p_i(y_i(N) - \bar{y}_i(N)) + \sum_{k=0}^{N-1} q_i(y(k) - \bar{y}(k), u_i(k))$$

- Cooperative goal for each player is a combination of each player goals:

$$J(x_1(0), x_2(0), U_1, U_2) = \alpha_1 J_1(x_1(0), U_1, U_2) + \alpha_2 J_2(x_2(0), U_1, U_2)$$

Uncoupled constrained DMPC problem

- Uncoupled constrained optimal control problem for player i:

$$\min_{U_i} \quad J(x_1(0), x_2(0), U_1, U_2)$$

$$s.t. \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} u_1 + \begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix} u_2$$

$$x_i \in \mathcal{X}_i \ , \ x_i(N) \in \mathcal{X}_{fi} \ , \ U_i \in \mathcal{U}_i$$

- Using the batch approach, the cooperative goal for each player is

$$\begin{aligned} J(x_1(0), x_2(0), U_1, U_2) &= U_1^T \tilde{R}_1 U_1 + 2\tilde{S}_1^T U_1 + \tilde{Q}_1 \\ &= U_2^T \tilde{R}_2 U_2 + 2\tilde{S}_2^T U_2 + \tilde{Q}_2 \end{aligned}$$

- The (linear) state and input constraints can be reformulated as

$$M_i U_i \leq w_i$$

Uncoupled constrained DMPC solution

- For player i, the distributed MPC problem with decoupled input constraints is then

$$\begin{aligned} \min_{U_i} \quad & J(x_1(0), x_2(0), U_1, U_2) = U_i^T \tilde{R}_i U_i + 2\tilde{S}_i^T U_i + \tilde{Q}_i \\ s.t. \quad & M_i U_i \leq w_i \end{aligned}$$

- Constrained optimal control sequences for both players computed as a function of the initial conditions and the previous inputs of the other player:

$$\begin{aligned} U_1^*(x_1(0), x_2(0), U_2^p) &= \{u_1^*(0), u_1^*(1), \ldots, u_1^*(N-1)\} \\ U_2^*(x_1(0), x_2(0), U_1^p) &= \{u_2^*(0), u_2^*(1), \ldots, u_2^*(N-1)\} \end{aligned}$$

- Apply n_p times the convex step $U_i^{p+1} = w_i U_i^* + (1 - w_i) U_i^p$.
- From the new sequence, $U_i = U_i^{n_p}$, apply the first value $u_i(t_k) = u_i(0)$.

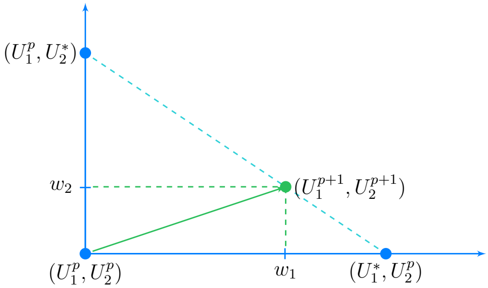
The convex step

- As $w_1 + w_2 = 1$ we can rewrite the convex step as

$$\begin{aligned}
\begin{bmatrix} U_1^{p+1} \\ U_2^{p+1} \end{bmatrix} &= \begin{bmatrix} w_1 I & 0 \\ 0 & w_2 I \end{bmatrix} \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix} + \begin{bmatrix} 0 & (1-w_1)I \\ (1-w_2)I & 0 \end{bmatrix} \begin{bmatrix} U_1^p \\ U_2^p \end{bmatrix} \\
&= \begin{bmatrix} w_1 I & 0 \\ 0 & w_2 I \end{bmatrix} \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix} + \begin{bmatrix} 0 & w_2 I \\ w_1 I & 0 \end{bmatrix} \begin{bmatrix} U_1^p \\ U_2^p \end{bmatrix} \\
&= \begin{bmatrix} w_1 U_1^* + w_2 U_2^p \\ w_2 U_2^* + w_1 U_1^p \end{bmatrix} \\
&= w_1 \begin{bmatrix} U_1^* \\ U_1^p \end{bmatrix} + w_2 \begin{bmatrix} U_2^p \\ U_2^* \end{bmatrix}
\end{aligned}$$

- Or simply:

$$(U_1, U_2)^{p+1} = w_1(U_1^*, U_2^p) + w_2(U_1^p, U_2^*)$$



Properties of the uncoupled constrained DMPC

- Assume:
  - Systems (A_i, B_i) are stabilizable;
  - Systems (A_i, C_i) and (A_i, Q_i) are detectable;
  - Matrices R_i are positive definite while Q_i and P_i are positive semi-definite;
  - The horizon N is long enough to zero the unstable modes of the system.
- Adding necessary stability constraints (see [1]), then:
  1. Iterates are feasible: $(U_1^p, U_2^p) \in (\mathcal{U}_1, \mathcal{U}_2)$ implies $(U_1^{p+1}, U_2^{p+1}) \in (\mathcal{U}_1, \mathcal{U}_2)$.
  2. Cost decreases: $J(x_1(0), x_2(0), (U_1, U_2)^{p+1}) \leq J(x_1(0), x_2(0), (U_1, U_2)^p)$.
  3. The converged solution of the cooperative distributed problem is equal to the centralized problem.
- Warm start: previous solution, $U_i = \{u_1^*(0), u_1^*(1), \ldots, u_1^*(N-1)\}$, is used as initial condition of the next optimization, $U_i^+ = \{u_1^*(1), \ldots, u_1^*(N-1), 0\}$
- It can be shown the distributed closed-loop system is exponentially stable.

Coupled constrained DMPC problem

- Coupled constrained optimal control problem for player i:

$$\min_{U_i} \quad J(x_1(0), x_2(0), U_1, U_2)$$

$$s.t. \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} u_1 + \begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix} u_2$$

$$(x_i, x_j) \in \mathcal{X} \ , \ (x_i(N), x_j(N)) \in \mathcal{X}_f \ , \ (U_i, U_j) \in \mathcal{U}$$

- Using the batch approach, the constraints can be reformulated as

$$M_1 U_1 + M_2 U_2 \leq w$$

- For the optimization problem of player i the other player inputs are constant.
- Thus, defining $w_i = w - M_j U_j$, we can use the equivalent constraint

$$M_i U_i \leq w_i$$

Coupled constrained DMPC solution

- For player i, the distributed MPC problem with coupled input constraints is then

$$\begin{aligned} \min_{U_i} \quad & J(x_1(0), x_2(0), U_1, U_2) = U_i^T \tilde{R}_i U_i + 2\tilde{S}_i^T U_i + \tilde{Q}_i \\ s.t. \quad & M_i U_i \leq w - M_j U_j \end{aligned}$$

- Constrained optimal control sequences for both players computed as a function of the initial conditions and the previous inputs of the other player:

$$\begin{aligned} U_1^*(x_1(0), x_2(0), U_2^p) &= \{u_1^*(0), u_1^*(1), \ldots, u_1^*(N-1)\} \\ U_2^*(x_1(0), x_2(0), U_1^p) &= \{u_2^*(0), u_2^*(1), \ldots, u_2^*(N-1)\} \end{aligned}$$

- Apply n_p times the convex step $U_i^{p+1} = w_i U_i^* + w_j U_j^p$.
- From the new sequence, $U_i = U_i^{n_p}$, apply the first value $u_i(t_k) = u_i(0)$.

Batch state and output constraints

- Input and input variation constraints are presented in Module 2.
- For state we have to consider the influence of the other players, as $X_i = F_i x_i(0) + G_{ii} U_i + G_{ij} U_j$, yielding

$$X_{i_{min}} \leq X_i \leq X_{i_{max}} \Leftrightarrow X_{i_{min}} \leq F_i x_i(0) + G_{ii} U_i + G_{ij} U_j \leq X_{i_{max}}$$

- Separating into two inequalities can be stated in matrix form as

$$\begin{bmatrix} -G_{ii} \\ G_{ii} \end{bmatrix} U_i \leq \begin{bmatrix} -X_{i_{min}} + F_i x_i(0) + G_{ij} U_j \\ X_{i_{max}} - F_i x_i(0) - G_{ij} U_j \end{bmatrix} \Leftrightarrow M_{xi} U_i \leq w_{xi}$$

- Similarly, as the output is $Y_i = \bar{F}_i x_i(0) + \bar{G}_{ii} U_i + \bar{G}_{ij} U_j$, results in

$$\begin{bmatrix} -\bar{G}_{ii} \\ \bar{G}_{ii} \end{bmatrix} U_i \leq \begin{bmatrix} -Y_{i_{min}} + \bar{F}_i x_i(0) + \bar{G}_{ij} U_j \\ Y_{i_{max}} - \bar{F}_i x_i(0) - \bar{G}_{ij} U_j \end{bmatrix} \Leftrightarrow M_{yi} U_i \leq w_{yi}$$

Properties of the coupled constrained DMPC

- Assume:
  - Systems (A_i, B_i) are stabilizable;
  - Systems (A_i, C_i) and (A_i, Q_i) are detectable;
  - Matrices R_i are positive definite while Q_i and P_i are positive semi-definite;
  - The horizon N is long enough to zero the unstable modes of the system.
- Adding necessary stability constraints (see [1]), then:
  1. Iterates are feasible: $(U_1^p, U_2^p) \in (\mathcal{U}_1, \mathcal{U}_2)$ implies $(U_1^{p+1}, U_2^{p+1}) \in (\mathcal{U}_1, \mathcal{U}_2)$.
  2. Cost decreases: $J(x_1(0), x_2(0), (U_1, U_2)^{p+1}) \leq J(x_1(0), x_2(0), (U_1, U_2)^p)$.
- The converged solution of the cooperative distributed problem may not be equal to the centralized problem.
- It can be shown the distributed closed-loop system is exponentially stable.
- The limitation is on the performance of the coupled distributed system, which is currently an active topic of research.

Constrained m-player DMPC problem

- Consider a set of m players, $\mathcal{I}_m = \{1, 2, \ldots, m\}$.
- Define the combined variables

$$x(0) = \begin{bmatrix} x_1(0) \\ \vdots \\ x_m(0) \end{bmatrix}, \; U = \begin{bmatrix} U_1 \\ \vdots \\ U_m \end{bmatrix}, \; A = \begin{bmatrix} A_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & A_m \end{bmatrix}, \; B_i = \begin{bmatrix} B_{1i} \\ \vdots \\ B_{mi} \end{bmatrix}$$

- Define the cooperative cost as

$$J(x(0), U) = \textstyle\sum_{j \in \mathcal{I}_m} \alpha_j J_j(x_j(0), U)$$

- The constrained optimal control problem for each player i is

$$\begin{aligned} \min_{U_i} \quad & J(x(0), U) \\ s.t. \quad & x^+ = Ax + \textstyle\sum_{j \in \mathcal{I}_m} B_j u_j \;, \; x \in \mathcal{X} \;, \; x(N) \in \mathcal{X}_f \;, \; U_i \in \mathcal{U}_i \end{aligned}$$

Constrained m-player DMPC batch problem

- Using the batch approach, the cooperative goal for each player i can be expressed as

$$J(x(0), U) = U_i^T \tilde{R}_i U_i + 2\tilde{S}_i^T U_i + \tilde{Q}_i$$

- The (linear) state and input constraints can be reformulated as

$$M_i U_i \leq w_i - \sum_{j \in \mathcal{I}_m, j \neq i} M_j U_j$$

- For player i, the distributed MPC problem is then

$$\begin{aligned} \min_{U_i} \quad & J(x(0), U) \\ s.t. \quad & M_i U_i \leq w_i - \sum_{j \in \mathcal{I}_m, j \neq i} M_j U_j \end{aligned}$$

Constrained m-player DMPC solution

- Constrained optimal control sequence for each players is computed as a function of the initial conditions and the previous inputs of the other players:

$$U_i^*(x(0), U_{j\neq i}^p) = \{u_i^*(0), u_i^*(1), \ldots, u_i^*(N-1)\}$$

- Apply n_p times the convex step

$$U_i^{p+1} = \sum_{j\in\mathcal{I}_m} w_j(U_1^p, \ldots, U_j^*, \ldots, U_m^p)$$

- From the new sequence, $U_i = U_i^{n_p}$, apply the first value, $u_i(t_k) = u_i(0)$.
- Prepare next warm start, with local information, defining

$$U_i^+ = \{u_i(1), \ldots, u_i(N-1), 0\}$$

Exercise 4.7

Problem (Exercise 4.7 - constrained coupled double integrators)

Consider two coupled double integrators, for which the time-invariant discrete linear joint system with $x = \begin{bmatrix}\theta_{12} & \omega_1 & \omega_2\end{bmatrix}^T$ *is described by*

$$\begin{aligned} x^+ &= Ax + Bu \\ y &= Cx \end{aligned} \quad \textit{with} \quad A = \begin{bmatrix}1 & 0.1 & -0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix}, \; B = \begin{bmatrix}0 & 0 \\ 1 & 0 \\ 0 & 1\end{bmatrix}, \; C = \begin{bmatrix}1 & 0 & 0 \\ -1 & 0 & 0\end{bmatrix}$$

Considering the quadratic cost for each player $i = 1, 2$, *given by* $J_i = p_i(y_i(N) - \bar{y}_i(N)) + \sum_{k=0}^{N-1} q_i(y_i(k) - \bar{y}_i(k), u_i(k))$, *where* $y_1 = \theta_{12}$ *and* $y_2 = -\theta_{12}$, *compute and simulate the distributed MPC policy, in the conditions of Exercise 4.3, enforcing the constraints* $-0.5 \leq u_i(k) \leq 0.5$.

Exercise 4.7

Exercise 4.7 - constrained coupled double integrators (part 1)

The main differences relative to Exercise 4.3 are the computation of the term $\tilde{S}_i$ and the linear constraints. The former is given by

$$\tilde{S}_i = S_{i1x}x_1(0) - S_{i1y}\bar{Y}_1 + S_{i2x}x_2(0) - S_{i2y}\bar{Y}_2 + S_{iju}U_j$$

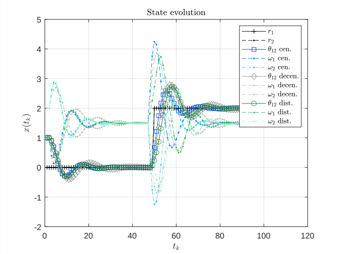
where $S_{ijy} = \bar{G}_{ji}^T\bar{Q}_j$, $S_{ijx} = S_{ijy}\bar{F}_j$, and $S_{iju} = \S_{iiy}\bar{G}_{ij} + S_{ijy}\bar{G}_{jj}$.
Considering $U_{max} = \begin{bmatrix}0.5 & 0.5 & 0.5\end{bmatrix}^T$ and $U_{max} = \begin{bmatrix}-0.5 & -0.5 & -0.5\end{bmatrix}^T$, the input constraint matrices are given by

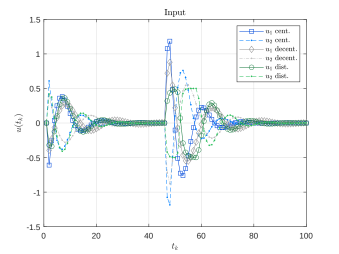
$$M_i = \begin{bmatrix}-I \\ I\end{bmatrix} \quad , \quad w_i = \begin{bmatrix}-U_{min} \\ U_{max}\end{bmatrix}$$

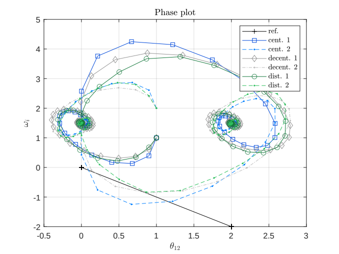
Exercise 4.7

Exercise 4.7 - constrained coupled double integrators (part 2)

Simulation results for 100 sampling intervals and 2 communication iterations for each sampling time ($p = 2$):



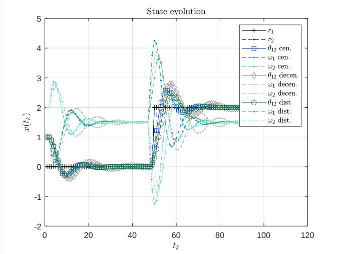


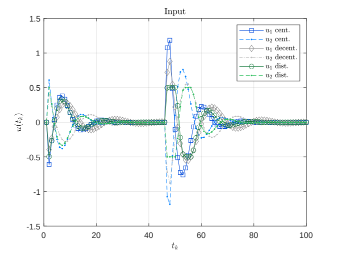


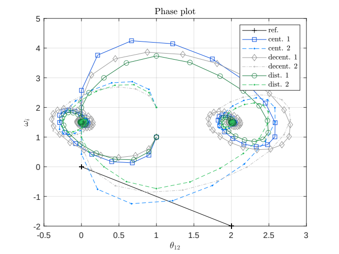
Exercise 4.7

Exercise 4.7 - constrained coupled double integrators (part 3)

Simulation results for 100 sampling intervals and 10 communication iterations for each sampling time ($p = 10$):







Summary 4.2

The main learning outcomes of this class are:

- Define and solve unconstrained two-player distributed MPC tracker
- Define and solve constrained two-player distributed MPC
- Define and solve multi-player distributed MPC

Example 4.8

Problem (Example 4.8 - constrained coupled double integrators)

Consider two coupled agents, for which the time-invariant discrete linear joint system with $x = \begin{bmatrix}\theta_{12} & \omega_1 & \omega_2\end{bmatrix}^T$ *is described by*

$$\begin{aligned} x^+ &= Ax + Bu \\ y &= Cx \end{aligned} \quad \text{with } A = \begin{bmatrix} 0.7 & 0.2 & -0.3 \\ 0 & 1 & 0 \\ 0 & 0 & 0.9 \end{bmatrix},\ B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Considering the quadratic cost for each player $i = 1, 2$*, given by* $J_i = p_i(y_i(N) - \bar{y}_i(N)) + \sum_{k=0}^{N-1} q_i(y_i(k) - \bar{y}_i(k), u_i(k))$*, where* $y_1 = \theta_{12}$ *and* $y_2 = -\theta_{12}$*, compute and simulate the distributed MPC policy, in the conditions of Exercise 4.4, enforcing the constraints* $-0.4 \leq u_i(k) \leq 0.5$*.*

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