Black Holes Do Not Exist For The Anti-Gravity Effect? Unified Field Theory Between The Fifth Force And The Gravitational Force In General Relativity

Abstract

If we consider the "Fifth force" as an external force of the classical newtonian gravitational field introduced in the Theory of General Relativity, we will find the modified time metric tensor and so obtain the new modified Einstein's field equations.

We unify the "Fifth Force" with the "Gravitational Force" applying the Theory of General Relativity. The result is that probably the "new modified Einstein's field equations" will not present any kind of singularity, with the presence of the anti-gravity effect of the "Fifth Force" in the nucleus of every astrophysical object; no existence of black holes? Peculiar phenomenology on stars and galaxies nucleus, the expansion or not of the universe can be fairly treated with the new modified Einstein's field equations.

It is well known that General Theory of Relativity is a geometric theory of gravitation and the curvature of space-time is related to the energy and momentum. This relation is specified by Einstein's equations. The experimental tests are numerous and, therefore, it is impossible to question its validity but this does not prevent us from proposing new ideas. In this work, we want to introduce the Fifth Force, not considering the universal gravitational constant. In this way we obtain modified field equations. The aim of the paper is to observe that, if we introduce this fifth force, the principle of the equivalence is violated and, probably, there are never singularities in the metric when we solve the field equations.

Maybe the Schwarzschild radius for a symmetrical static spherical star doesn't compare: to be proved with "computational mathematics".

Throughout the paper, the symbols refer to the textbook of Weinberg , considering even the speed of light as c=1.

Resuming the Principle of Equivalence

The principle of equivalence is the fundamental hypothesis for the theory of general relativity. But if we consider the fifth force it must be considered the following two points:

1) A potential breakdown of the principle of equivalence has a remarkable geometrical explanation in the framework of extended theories of gravity, if one assumes an explicit coupling between an arbitrary function of the scalar curvature, R, and the Lagrangian density of matter, [1]. On the other hand, one must be very careful when assuming potential deviations from the principle of equivalence, which has today unchallengeable empiric evidence, at least on Earth, [2]. Thus, one must be able to argue that such deviations could, eventually, work at astrophysical and/or cosmological scales. An interesting mechanism which can permit this approach has been developed, for example, in [3].
2) Otherwise, if we try to include the fifth force in some equations of general relativity theory, as an external force, and find out how to find the solutions of the metric tensors g_{ij}, probably there aren't any *singularity (as we shall do in a future paper).

The principle of equivalence means that the inertial mass m_i of an object is equal to its gravitational mass m_g: $m_i=m_g$. Different from formula (1).

Introduction of the fifth force

What is the fifth force?

If we consider two nucleons of mass m_1 and m_2, they will exert a gravitational potential energy of the following form ([5]-[8], formula of Gibbons and Fischbach without singularity):

(1) $$U_{12} = -G\, m_1 m_2 (1+ \alpha\, e^{-\mu\, r1,2})/ r_{1,2}$$

where G is the universal gravitational constant ($G=6.673*10^{-11}\ Nm^2/kg^2$), corrected by α=0.01:0.001, which is the intensity of the fifth force, called ipercharge, *that depends on the relative amount of neutrons upon number of protons per nucleon*, in range μ^{-1} =100:1000 meters, (if α can be a positive or a negative quantity).

The property of this phenomenological formula (1) in confront of Newton's gravity law is that it hasn't any *singularity. In fact:

(2) $$\lim_{r1,2 \dashrightarrow 0}(1+\alpha\, e^{-\mu\, r1,2})/ r_{1,2} = \alpha\mu \neq \infty.$$ (theorem of De Hopital for limits).

We know that in General Relativity Theory the Einstein Field Equations derived from the Newtons formula, (see [9] (7.1.3) and (7.1.12)), have the presence of singularity for propagation. Instead the Newtonian formula of gravity for point masses has singularity at r=0: or better $F_{1,2} = \lim_{r1,2 \dashrightarrow 0} (G\, m_1 m_2/ r^2_{1,2}) = \infty$. If the radius of the star reaches the Schwarzschild radius (1=2Gm(R)/R), the metric tensor $A(r)=g_{rr}=1/(1-2Gm(r)/r)$, (see [9] (11.1.11)) gives the presence of black holes.

But if we use the corrected gravitational potential [5] without *singularity, modifying the Einstein Field Equations; probably the new Einstein Field Equations shall become without the presence of *singularity; it is amazing.

We know that the gravitational potential $\phi(\mathbf{r_2})$ at point $\mathbf{r_2}$, with distribution of matter density $\rho(\mathbf{r_1})$, as a spherical star, is (see [10] (3.1a)):

(3) $$\phi (\mathbf{r_2}) = -G \int d^3r_1 \frac{\rho(\mathbf{r_1})}{|\mathbf{r_2} - \mathbf{r_1}|}$$

and the fifth force potential $\phi_5(\mathbf{r_2})$ at point $\mathbf{r_2}$, **that depends on the amount of neutrons upon the number of protons per nucleon in the star matter** (or depends even from antimatter properties to be investigated), is (See [10] (3.1b)):

(4) $$\phi_5(\mathbf{r_2}) = -G\, \alpha \int d^3r_1 \frac{\rho(\mathbf{r_1})}{|\mathbf{r_2} - \mathbf{r_1}|} e^{-\mu|\mathbf{r2} - \mathbf{r1}|}$$

The Laplacian equations of both the fields above are respectively (see [10] (3.2a) and (3.2b)):

(5) $$\nabla^2\phi=4\pi G\rho$$ (Newtonian field equation)

(6) $$(\nabla^2 - \mu^2)\phi_5=4\pi G\rho\alpha$$ (Fifth force field equation)

Where the Laplacian operator is defined $\nabla^2\phi= (\partial^2\phi/\partial x^2)+ (\partial^2\phi/\partial y)+ (\partial^2\phi/\partial z^2)$. These two Poissan's field equations shall be used after to find the new modified Einstein's field equations.

The geodetic equation inserting the fifth force

We follow the same calculations of Chapter 3.2 Weinberg 1972, to find the geodesic equation adding the fifth force: considering it as an "external force", acting upon the gravitational field. We need to find this type of geodesic equation to modify the Einstein field equation, acting with metric tensor, g_{oo}, (see its formula (24)).

Considering the fifth force f_5 acting on a particle of mass m, immersed in the gravitational field, detected by a free falling coordinate system ξ^α, where its equation of motion is a straight line in space-time (see [9] (3.2.1), (3.2.2) and [11] chapter 8, n°58):

$$(7a) \qquad f_5^\alpha(\xi)/m = g^{\alpha\beta}(\xi)\,[f_{5\beta}(\xi)/m] = \frac{d^2\xi^\alpha}{d\tau^2} \qquad \text{with } d\tau^2 = -\eta_{\mu\nu} d\xi^\mu\, d\xi^\nu$$

With $\eta_{\mu\nu}$ the *Minkowski tensor*: $\eta_{\mu\nu}$= +1 for $\mu=\nu$=1,2 or 3 else $\eta_{\mu\nu}$= -1 for $\mu=\nu$=0 else $\eta_{\mu\nu}$= 0 for $\mu\neq\nu$. Where the metric tensor $g^{\alpha\beta}(\xi)$ function of ξ is (see [9] (2.5.6)):

$$(7b) \qquad g^{\alpha\beta}(\xi) = \eta^{\alpha\beta}, \quad g_{\alpha\beta}(\xi) = \eta_{\alpha\beta} \quad \text{and satisfies } g_{\alpha\beta}(\xi) g^{\gamma\alpha}(\xi) = \delta^\gamma{}_\beta$$

with $\delta^\gamma{}_\beta$ the *Kroenecker tensor*: $\delta^\gamma{}_\beta$=+1 for $\gamma=\beta$, and $\delta^\gamma{}_\beta$= -1 for $\gamma\neq\beta$.

So using any other coordinate system x^μ, the free falling coordinates ξ^α are functions of the x^μ, and from (7a) we have the geodetic equations as (see [9] (3.2.3),(5.1.11) and (5.1.12); and [11] chapter 8, n°68-69; [12] (10.24)):

$$(8) \qquad [\,f_5^\alpha(\xi)/m]\,\frac{\partial x^\lambda}{\partial\xi^\alpha} = \frac{d^2\xi^\alpha}{d\tau^2}\frac{\partial x^\lambda}{\partial\xi^\alpha} = \frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

with x^λ which are the four coordinates of the particle moving along its trajectory, and the first member appears with variable ξ; and where the proper time $d\tau^2 = -g_{\mu\nu}(x)dx^\mu dx^\nu$ and $\Gamma^\lambda{}_{\mu\nu}$ is the Christoffel symbol with the metric tensor $g_{\mu\nu}(x)$ function of x (see [9] (3.2.4) ,(3.2.6), (3.3.7)):

$$(9a) \qquad \Gamma^\lambda{}_{\mu\nu} = \frac{\partial x^\lambda}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu\,\partial x^\nu} = (g^{\alpha\lambda}/2)\{(\partial g_{\nu\alpha}/\partial x^\mu) + (\partial g_{\mu\alpha}/\partial x^\nu) - (\partial g_{\nu\mu}/\partial x^\alpha)\}$$

where (see [9] (3.2.7) or (4.2.6)):

$$(9b) \qquad g_{\mu\nu}(x) = \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}\eta_{\alpha\beta}$$

The fifth force f_5 is the gradient of it's potential ϕ_5, valid for a conservative field and depending on the coordinate x^λ, (see [9] (4.7.1), (4.2.4) and [11] chapter 5- n°10, chapter 8-n°58):

$$(10) \qquad f_{5\beta}(\xi)/m = -(\partial\phi_5/\partial\,\xi^\beta)$$

"valid for a conservative field of f_5".

But the coordinate transformation implies (see [9] (4.2.4)):

$$(11) \qquad \partial\phi_5/\partial\xi^\beta = \frac{\partial\phi_5}{\partial x^\nu}\frac{\partial x^\nu}{\partial\xi^\beta}$$

So the first member, in variable ξ, of (8) becomes with (7a), (10) and (11) with the new variable x:

(12) $$[f_5^{\alpha}(\xi)/m] \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} = g^{\alpha\beta}(\xi)\,[f_{5\beta}(\xi)/m] \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} = -\,g^{\alpha\beta}(\xi)\,(\partial\phi_5/\partial \xi^{\beta}) \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} = -\,g^{\alpha\beta}(\xi)\left(\frac{\partial\phi_5}{\partial x^{\nu}} \frac{\partial x^{\nu}}{\partial \xi^{\beta}}\right) \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} =$$

$$= -\, g^{\nu\lambda}(x) \frac{\partial \phi_5}{\partial x^{\nu}}$$

because:

$g^{\alpha\beta}(\xi) \frac{\partial x^{\nu}}{\partial \xi^{\beta}} \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} = g^{\nu\lambda}(x)$ as *contravariant tensor* (see (7a) and [9] (4.2.7)).

The geodetic equation (8), with the fifth force considered as an external force upon the gravitational field, all in coordinates x, becomes, using (12):

(13) $$\boxed{-\, g^{\nu\lambda}(x) \frac{\partial \phi_5}{\partial x^{\nu}} = \frac{d^2 x^{\lambda}}{d\tau^2} + \Gamma^{\lambda}_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau}}$$

(see [9] (5.1.11)).
We will use it to find the metric tensor of time, g_{oo}, in Newtonian limit to modify the Einstein field equation with the fifth force.
And knowing that $U^{\nu} = dx^{\nu} / d\tau$ is the four-vector velocity of a falling object; multiplying U_{λ}, in both sides of (13) we have:

(13a) $$-\, g^{\nu\lambda}(x)\ (\partial\phi_5 / \partial x^{\nu})\, U_{\lambda} = (dU^{\lambda} / d\tau)\, U_{\lambda} + \Gamma^{\lambda}_{\mu\nu}\, U^{\mu}\, U^{\nu}\, U_{\lambda}\,,$$

U^{ν}, satisfying the condition:

(13b) $$1 = -\, g_{\mu\nu}(x)\, U^{\mu} U^{\nu},$$

useful to find the potential function of the fifth force, ϕ_5, (knowing $g^{\nu\lambda}(x)$), instead of using the phenomenological Yukawa-Newtonian expression (4).

<u>The Newtonian limit with the fifth force to find the metric tensor of time g_{oo}</u>

We want to find the dependency of the metric tensor g_{oo} upon the potential of the gravitational field ϕ and the potential of the fifth force ϕ_5. This shall be useful to obtain the Einstein's field equations with the fifth force, following the same calculations of Chapter 3.4 Weinberg 1972, but violating the principle of equivalence.
In the Newtonian limit considering a particle moving slowly in a weak stationary gravitational field in presence of the fifth force, neglecting $d\mathbf{x}/d\tau$ respect to $dt/d\tau$ and using (13), we have the equation of motion:

(14) $$\frac{d^2 x^{\lambda}}{d\tau^2} + \Gamma^{\lambda}_{oo}\ (dt/ d\tau)^2 = -(\partial\phi_5/\partial x^{\nu})\, g^{\nu\lambda}(x)$$

(15) $$d^2t/ d\tau^2 = 0 \quad \text{which implies } dt/d\tau = \text{constant1} = a.$$

In the "nearly" Cartesian coordinate system, we adopt the metric tensor in a weak field as (see [9] (3.4.1)):

(16) $$g_{\mu\alpha} = \eta_{\mu\alpha} + h_{\mu\alpha} \quad , \qquad | h_{\mu\alpha}| << 1$$

where $\eta_{\mu\alpha}$= +1 for $\alpha=\mu$=1,2 or 3 else $\eta_{\mu\alpha}$= -1 for $\alpha=\mu=0$ else $\eta_{\mu\alpha}$= 0 for $\alpha\neq\mu$.
Since the field is stationary and putting the first order of $h_{\mu\alpha}$, we have (see [13] (6.15) and [14] (17.18)):

(17) $$\Gamma^{\alpha}_{oo} = (-1/2)\, g^{\alpha\beta}\, (\partial g_{oo} / \partial x^{\beta}) = (-1/2)\, \eta^{\alpha\beta}\, (\partial h_{oo} / \partial x^{\beta})$$

Substituting this in equation (14) gives (see [9] (10.1.7a),(3.3.6)):

(18) $$d^2 x^{\lambda}/d\tau^2 + (-1/2)\, \eta^{\lambda\beta}\, (\partial h_{oo} / \partial x^{\beta})\, (dt/ d\tau)^2 = -(\partial\phi_5/\partial x^{\alpha})\, (\eta^{\lambda\alpha} - h^{\lambda\alpha})$$

but $g_{\alpha\beta}\, g^{\gamma\alpha} \sim (\eta_{\alpha\beta} + h_{\alpha\beta})(\eta^{\gamma\alpha} - h^{\gamma\alpha}) = \delta^{\gamma}_{\beta}$ implies $g_{\alpha\alpha} g^{\alpha\alpha} = \delta^{\alpha}_{\alpha} = 1$ or better $(1+ h_{\alpha\alpha})(1- h^{\alpha\alpha})=1$ which gives $h^{\alpha\alpha}= h_{\alpha\alpha}/(1+ h_{\alpha\alpha})$ so $h^{\lambda\alpha} \sim h_{\lambda\alpha} << 1$ for condition (16). And (18) becomes the equation of motion:

(19) $$\frac{d^2 \mathbf{x}}{d\tau^2} = (1/2)\, (dt/ d\tau)^2\, \nabla h_{oo} - \nabla \phi_5$$

so dividing this equation by $(dt/ d\tau)^2$ we have (see [9] (3.4.2)):

(20) $$\frac{d^2 \mathbf{x}}{dt^2} = (1/2)\, \nabla h_{oo} - (d\tau/ dt)^2\, \nabla \phi_5$$

The corresponding newtonian result is (see [9] (3.4.3)):

(21) $$\frac{d^2 \mathbf{x}}{dt^2} = - \nabla\phi - \nabla \phi_5$$

where ϕ is the Newtonian potential, (3), (5). And comparing (20) and (21) we find:

(22) $$(1/2)\, \nabla h_{oo} - (d\tau/ dt)^2\, \nabla \phi_5 = - \nabla\phi - \nabla \phi_5$$

for these equalities and with (15),

(23) $$h_{oo} = -2\, k\, \phi_5 - 2\phi + \text{constant2},$$

where $k = 1- (d\tau/ dt)^2 = 1 - (1/a)^2$.

Constant2=0 because the coordinate system becomes Minkowskian at infinity: so $h_{oo}=0$ for $r \rightarrow \infty$, as even $\phi=0$ and $\phi_5=0$ from (3) and (4).
So the metric tensor of time g_{oo} becomes with (16), (see [9] (3.4.5)):

(24) $$\boxed{g_{oo} = - (1 + 2\phi + 2\, k\, \phi_5)}$$

It's added a potential term as, $2\, k\, \phi_5$, which represents the presence of the fifth force. We shall see in the next chapter the necessary use of formula (24) to obtain the new Einstein's field equations.

Einstein's field equations modified with the fifth force

Now following Chapter 7.1 Weinberg 1972 and it's discussions, violating the principle of equivalence and introducing the fifth force in an opportune way in the same equations; we have summing the members of the two Laplacian equations (5) and (6) as:

(25) $$\nabla^2 (\phi + k\phi_5) = 4\pi G\rho + k\, [\, 4\pi G\rho\alpha + \mu^2\phi_5] \,,$$

in system with equation (6), whose variable functions are ϕ and ϕ_5.

And using equation (24) and the approximation for non-relativistic matter $T_{oo} \approx \rho$, for a weak static gravitational field, we obtain from (25), (see [9] (7.1.3)(7.1.4)):

(26) $$\nabla^2 g_{oo} = -8\pi G\, T_{oo}\, (1 + \alpha k) \; - 2\, k\, [\, \mu^2\phi_5]$$

We see that the last term of equation (26) is only a scalar function, $2\, k\, [\, \mu^2\phi_5]$, without two indexes 00, as the terms g_{oo} and T_{oo}, so it cannot transform as a tensor of rank 2, with two indexes, to be written in a covariant form. This tells us to go beyond the calculus to substitute this scalar function in some way.

So we will apply again the Laplacian operator, ∇^2, in both members of equation (26), having:

(27) $$2\, k\, \mu^2\, \nabla^2\, \phi_5 \; = (1 + \alpha k) \;\; \nabla^2\, [(-8\pi G)\, T_{oo}\,] - \nabla^2\, [\nabla^2\, g_{oo}],$$

In which are considered constant the terms k, μ, α (see formula (1)); instead the term, $(-8\pi G)$, is to be considered not as the famous universal gravitational constant, but as a scalar function of the coordinates, x^{ν}, influenced by the presence of the "fifth force" (see (1), and in a further chapter here we will mention about it).

But from (6), substituting the density of mass with the energy-momentum tensor, $T_{oo} \approx \rho$, becomes:

(28) $$\mu^2\, \phi_5 \; = \nabla^2\, \phi_5 \; - (4\pi G)\, \alpha\, T_{oo}$$

Substituting (28) to equation (26):

(29) $$\nabla^2\, g_{oo} = (-8\pi G)\, (1 + \alpha k)\, T_{oo} - 2\, k\, [\nabla^2\, \phi_5 \; - (4\pi G)\, \alpha\, T_{oo}]$$

and finally substituting (27) to (29), we have:

(30) $$\nabla^2 g_{oo} = (-8\pi G)(1+\alpha k)T_{oo} - \{[(1 + \alpha k)/\mu^2]\, \nabla^2\, [(-8\pi G)T_{oo}] - \nabla^2\, [\nabla^2 g_{oo}]/\mu^2 + (-8\pi G)\alpha k\, T_{oo}]\}$$

which instead of a weak stationary gravitational field, we consider the general case of relativity, to have the new Einstein's field equations modified by the presence of the fifth force, with $\nabla^2\, g_{oo}$ associated to the *"Einstein's tensor"* $G_{ij} = R_{ij} - (1/2)\, g_{ij}\, R$, (see [9] chapter 7.1, (7.1.8), (7.1.13)).

So transforming the term $\nabla^2 g_{oo} \approx G_{00} \approx G_{ij} = R_{ij} - (1/2)\, g_{ij} R$, the term $g_{oo} \approx g_{ij}$, the $T_{oo} \approx T_{ij}$, and the classical Laplacian operator ∇^2, (used for a space of 3 dimensions), to a d'Alembertian operator $\Box^2$ (necessary for a space of 4 dimensions, instead of 3), in $\nabla^2 \approx \Box^2$, (see Weinberg method passing from the Newtonian limit to general relativistic gravitation field [9] using (7.2.4)&(7.2.5)), and even considering the transformation of term $\nabla^2 [\nabla^2 g_{oo}]/\mu^2 \approx \Box^2 [R_{ij} -(1/2)g_{ij}R]/\mu^2$; so the equation (30) from its Newtonian limit becomes in the general case:

$$(31) \qquad R_{ij} -(1/2)g_{ij}R =(-8\pi G)(1+\alpha k)T_{ij} + \\ - \{[(1+\alpha k)/\mu^2]\, \Box^2\, [(-8\pi G)T_{ij}] - \Box^2 [R_{ij} -(1/2)g_{ij}R]/\mu^2 \ +(-8\pi G)\alpha k\, T_{ij}]\}$$

Where the d'Alembertian operator $\Box^2$ is defined as, (see [15] (53.07); see [11] chapter 8, page 174; and [9] (4.4.1)):

$$(31a) \qquad \Box^2 f= (1/\sqrt{|g|})\, \{\partial[(\sqrt{|g|})(g^{bc})(\partial f/\partial x^c)]/\partial x^b\} = [(g^{bc})(\partial^2 f/\partial x^b \partial x^c)] - g^{bc}\, \Gamma^{a}{}_{bc}\, (\partial f/\partial x^a)\,,$$

where $g \equiv$ - Determinant(g_{ij}).

Note: in a justified opinion and discussion from external researchers and in a future paper, instead of the d'Alembertian operator of Fock V. 1959 (53.07) used for curvature space, it could be used Weinbergs d'Alembertian, applied especially in special relativtity flat space, which is defined as, (see [9] (2.5.12)):

$$(31b) \qquad \Box^2 = \eta^{jk}(\partial/\partial x^k)(\partial/\partial x^j) = \nabla^2 - \partial^2/\partial^2 t.$$

There is an ambiguity in using the d'Alembertian (31a) or (31b) on various relativistic papers that we consulted, especially in the Weinberg's book, instead of Fock V. book (1959) "The theory of space, time and gravitation", (53.07). In my point of view and opinion it's better to use operator (31a) for curvature space, transforming the classic Laplacian, ∇^2*, (see (5) and (6)) to a d'Alembertian operator,* $\Box^2$*:* $\nabla^2 \approx \Box^2$.

In a "complete and general" form the field equations, considering ($-8\pi G$) a scalar function, looks as:

$$(32) \qquad R_{ij} -(1/2)g_{ij}R - \Box^2 [R_{ij} -(1/2)g_{ij}R]/\mu^2 = (-8\pi G)T_{ij} - \{[(1+\alpha k)/\mu^2]\, \Box^2\, [(-8\pi G)T_{ij}]\}$$

For the considerations of point (B) of [9] chapter 7.1, where G_{ij} must have the dimensions of a second derivative, (see formulas (5) and (6)); the term,
$\nabla^2[\nabla^2 g_{oo}]/\mu^2 \approx \Box^2 [R_{ij} -(1/2)g_{ij}R]/\mu^2$, is of fourth derivative of the metric components, g_{ij}, and will become negligible for gravitational fields of sufficiently large space-time scale:
$R_{ij} -(1/2)g_{ij}R >> \Box^2 [R_{ij} -(1/2)g_{ij}R]/\mu^2$.
So for this approximation, and considering $(-8\pi G) = (-8\pi G/c^4$ for velocity of light, c=1), not a constant, but a scalar function, (32) becomes finally the ***new Einstein's field equations modified*** by the presence of the fifth force:

$$\boxed{(33) \qquad R_{ij} -(1/2)g_{ij}R = (-8\pi G)T_{ij} - \{ [(1+\alpha k)/\mu^2]\, \Box^2\, [(-8\pi G)\, T_{ij}]\}= \\ =\{(-8\pi G) - [(1+\alpha k)/\mu^2]\, \Box^2\, [(-8\pi G)]\}T_{ij} - [(1+\alpha k)/\mu^2](-8\pi G)\Box^2 T_{ij}}$$

where T_{ij} is the energy-momentum tensor (see [9] chapter 5.3), and R_{ij} is the "*Ricci tensor*", (see [9] (6.2.4),(6.1.5),(8.1.12)):

(34) $$R_{\mu\kappa} = \partial\Gamma^{\lambda}_{\mu\lambda}/\partial x^{\kappa} - \partial\Gamma^{\lambda}_{\mu\kappa}/\partial x^{\lambda} + \Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\lambda\eta}$$

For the presence of the "fifth force" in the gravitational field, (1), the universal gravitational constant, ($-8\pi G$), shall be assumed as a scalar function of the space-time coordinates, x^{λ}, and no more considered a constant: see the next chapters.

<u>Using another independent equation the density of force $\mathcal{G}^{\nu}$</u>

If we take the following **independent equation** of general relativity theory (see [9] (5.3.2), (2.8.6)):

(35) $$T^{\mu\nu}{}_{;\mu} = \mathcal{G}^{\nu}$$

Where $\mathcal{G}^{\nu}$ is the *"density of force",* f_5, acting externally on the system, it can be found as expression of a function of the fifth force potential: $\mathcal{G}^{\nu} = f^{\nu}(\phi_5)$. For an isolated system, $\mathcal{G}^{\nu}=0$. But in our case the gravitational field doesn't seem an isolated system, because the "Fifth force" was treated as an external force upon the field, to find the geodetic equation (13) and then g_{oo} from formula (24). About this statement, the energy-momentum tensor is not considered conserved, so, $T^{\mu\nu}{}_{;\mu} \neq 0$.
The covariant derivative of the energy-momentum tensor is (see [9] 1972 (4.7.9)):

(36) $$T^{\mu\nu}{}_{;\mu} = (\partial T^{\mu\nu}/\partial x^{\mu}) + \Gamma^{\mu}_{\mu\lambda} T^{\lambda\nu} + \Gamma^{\nu}_{\mu\lambda} T^{\mu\lambda}$$

The covariant differentiation of the first member of (33) must be zero for the "*Bianchi identity"*, (see [9] (6.8.4) or (7.1.6), or because, with careful logic, the energy-momentum tensor, if its conserved with the fifth force, it must be $T^{\mu\nu}{}_{;\mu} = 0$), it means that the following condition must be satisfied:

(37) $$0 = G^{ij}{}_{;i} = (R^{ij} - (1/2)\, g^{ij} R)_{;i}$$

Where G^{ij} is the "*Einstein tensor*". This last step must be found to arrive to a compact calculus of the 16 metric tensors, g_{ij}, and of the gravitational constant or scalar, ($-8\pi G$). As we shall show now.

<u>Consequences considering G as a scalar</u>

So we consider, now, the gravitational constant, ($-8\pi G/c^4$, with velocity of light, c=1), as a scalar, because we know that with the presence of the fifth force, as an external force on the gravitational field, may modify the gravitational "constant" G, see the factor $G\,(1+ \alpha\, e^{-\mu\, r1,2})$ varying in the phenomenological formula (1); then assuming:

(38) $$\hat{G} = \hat{G}(x^{\nu}) = (-8\pi G) = \text{scalar function of } x^{\nu}.$$

we take *contravariant* equation (33), in convariant differentiation, with condition (37), so:

(39) $$0 = (R^{ij} - (1/2)\, g^{ij} R)_{;i} = (-8\pi G)\, T^{ij}{}_{;i} + (-8\pi G)_{;i}\, T^{ij} - \{[(1+\alpha k)/\mu^2]\, \Box^2\, [(-8\pi G) T^{ij}]\}_{;i}$$

Here (39) is a differential equation where it shall be found by "computational mathematics" the scalar function ($-8\pi G$), in a "general" form, (instead who wants to prosecute in the "particular" case, with the energy-momentum conservation, putting $T^{ij}{}_{;i}=0$, about other meaningful physical reasons, can do it); knowing that the covariant derivative of a scalar is an ordinary gradient:

(39a) $$(-8\pi G)_{;\,i} = \partial(-8\pi G)/\partial x^i,$$

instead of four equations with index j=0,1,2,3 in equation (39), we multiply both members of it by the term, four-vector velocity, U_j, becoming one equation of scalar ($-8\pi G$), as follows:

(40) $$0 = (-8\pi G)\, T^{ij}{}_{;\,i}\, U_j + (\partial(-8\pi G)/\partial x^i) T^{ij}\, U_j - [(1+\alpha k)/\mu^2]\{\Box^2\, [(-8\pi G)T^{ij}]\}_{;\,i}\, U_j$$

Or using in this equation (40), the d'Alembertian operator (31a), where necessary, and applying the covariant derivative after (39a) and (36), becomes in an extended, spreading formalism, useful for who wants to find the scalar function $\hat{G} = (-8\pi G)$ with **"computational mathematics" or "numerical relativity"**, (see full calculations below in appendix (a)); which becomes the following:

(40a) $$0 = \hat{G}\, U_j \frac{\partial T^{ij}}{\partial x^i} + \hat{G}\, U_j\, \Gamma^i{}_{in} T^{nj} + \hat{G}\, U_j\, \Gamma^j{}_{in} T^{in} + U_j\, T^{ij} \frac{\partial \hat{G}}{\partial x^i} - \acute{K}\, U_j\, T^{ij} \frac{\partial}{\partial x^i}\left[g^{ko} \frac{\partial^2 \hat{G}}{\partial x^k\, \partial x^o} \right] +$$
$$+ \acute{K}\, U_j\, T^{ij} \frac{\partial}{\partial x^i}\left[g^{ko}\, \Gamma^m{}_{ko} \frac{\partial \hat{G}}{\partial x^m} \right] - \acute{K}\, U_j\, \hat{G} \frac{\partial}{\partial x^i}\left[g^{ko} \frac{\partial^2 T^{ij}}{\partial x^k\, \partial x^o} \right] - \acute{K}\, U_j\, \hat{G}\, \Gamma^i{}_{in} \left[g^{ko} \frac{\partial^2 T^{nj}}{\partial x^k\, \partial x^o} \right] -$$
$$- \acute{K}\, U_j\, \hat{G}\, \Gamma^j{}_{in} \left[g^{ko} \frac{\partial^2 T^{io}}{\partial x^k\, \partial x^o} \right] - \acute{K}\, U_j\, \hat{G} \frac{\partial}{\partial x^i}\left[g^{ko}\, \Gamma^m{}_{ko} \frac{\partial T^{ij}}{\partial x^m} \right] - \acute{K}\, U_j\, \hat{G}\, \Gamma^i{}_{in} \left[g^{ko}\, \Gamma^m{}_{ko} \frac{\partial T^{nj}}{\partial x^m} \right] -$$
$$- \acute{K}\, U_j\, \hat{G}\, \Gamma^j{}_{in} \left[g^{ko}\, \Gamma^m{}_{ko} \frac{\partial T^{in}}{\partial x^m} \right]$$

where $\hat{G} = \hat{G}(x^\nu) = (-\,8\pi G)$, (to be found explicitly), and $\acute{K} = (1+\alpha k)/\mu^2$.

So the general system of the **new Einstein's field equations,** modified for the presence of the "fifth force" in the gravitational field, where, by "***computational calculus***", we can find the 16 values of the metric tensors, g_{ij}, and the value of the new "gravitational scalar variable", ($-8\pi G$), is the following:

SYSTEM (41) **New modified Einstein's Field equations with the Fifth Force**, (considering G a gravitational scalar variable, and not a universal constant):

$$\begin{cases} R_{ij} - (1/2)g_{ij}R = (-8\pi G)T_{ij} - \{[(1+\alpha k)/\mu^2]\,\Box^2\,[(-8\pi G)T_{ij}]\} & \textbf{(equations to find } g_{ij}\textbf{)} \\ 0 = (-8\pi G)\,T^{ij}{}_{;\,i}\,U_j + (\partial(-8\pi G)/\partial x^i)T^{ij}\,U_j - [(1+\alpha k)/\mu^2]\{\Box^2\,[(-8\pi G)T^{ij}]\}_{;\,i}\,U_j & \textbf{(equation to find } (-8\pi G)\textbf{)} \end{cases}$$

System (41) permits to find in totally 11 variables by "*computational mathematics*": the 10 variables, g_{ij}, (instead of 16 variables, because g_{ij} is symmetric), joining the gravitational variable as a scalar function, $(-8\pi G)$. Where α, μ and k are three constants: α=0.01:0.001, is the intensity of the fifth force, called ipercharge, in range μ^{-1} =100:1000 meters; and $k = 1- (d\tau/ dt)^2$ = constant, see (23). With anti-gravity effect we have $\alpha<0$.
It would be more elegant to put the field equations (41), in one new compact Einstein's field equation, in a more simplified way, or leaving it to a good researcher as a skilful calculus.

Conclusion

In this paper we have obtained a modified Einstein's field equations by introducing the fifth force of Gibbons-Fischbach. We invite to look for numerical solutions in the case of a static field with spherical symmetry. It would be interesting to understand if there are *singularities or not.
It seems that if there is "anti-gravity effect" of the fifth force in the nucleus of astrophysical objects, (for example compact and massive as neutron stars), the ipercharge is negative, $\alpha<0$, and this means probably from our modified new Einstein's field equations, system (41) or from (33), that black holes do not exist.

The system (41), must be resolved with "computational mathematics", to verify if there are present or not *singularities in the metric tensors g_{ij}.
It is well known that in the standard Schwarzschild space-time there is a *singularity in the metric and it would be interesting if our approach leads to a *singularity-free metric or not. Indeed, this would have consequences about the existence of black holes.

We think that it's possible to arrive, in another way, to a complete compact Einstein's field equation, (33), about its second member with the energy-momentum tensor, introducing the "fifth force" in General Relativity, by using another approach, instead of using "Newtonian limit": the theorems, properties and methods of the "Continuum Mechanics Theory" (because the formula (1) of Gibbons and Fischbach has "elastic" properties as a stretching and squeezing spring holding two masses, that is a repulsive and an attractive gravitational force). And after we can apply the "Principle of Minimum

Action", trying to find the opportune Langragian of this "elastic" continuum ensemble of celestial masses, (considering the universe as a many body, n-body system).

Nevertheless it's necessary and it must be verified the orbitals of the planets and stars, for the consistency of the new modified Einstein's field equations (33), found by introducing the fifth force in the general relativity theory.

Notes for who can use "computational mathematics" or even "numerical relativity":
*Good researchers, who has the software of "**computational mathematics**", (I haven't), can analyse if the metric tensors, g_{ij}, have *singularities or not, in the following system of equations (9a), (13b), (34) and (41); considering **a spherically symmetric static star** of radius R and mass M, in the **spherical polar coordinates** of x^ν, (see Weinberg S. 1972 chapter 8.2 or better 11.1).*
Just to see if black holes exist or not.

*singularity of *metric tensors* g_{ij}, means if its values go to infinite varying radius r to r_o: $\lim_{(r \to ro)} g_{ij}(r) = \infty$, where r_o=2Gm(R)="Schwarzschild radius", or for another singularity where, r_o=0, at the center of the star.

Appendix (a):

To find the scalar function, $\hat{G} = \hat{G}(x^\nu) = (-8\pi G)$ "implicitly", we start from equation (40), using the Leibniz property of covariant derivative (see [9] (4.6.14)), which becomes:

(40b) $0 = \hat{G}\, T^{ij}{}_{;i} + \hat{G}_{;i}\, T^{ij} - \acute{K}\, [\Box^2 \hat{G}]_{;i}\, T^{ij} - \acute{K}\, \hat{G}\, [\Box^2 T^{ij}]_{;i}$, where $\acute{K} = (1+\alpha k)/\mu^2$.

Applying first the II d'Alembertian operator, (31a), on equation (40b), we obtain:

$$(40b) \quad 0 = \hat{G}\, T^{ij}{}_{;i} + \hat{G}_{;i}\, T^{ij} - \acute{K}\, T^{ij} \left[g^{ko} \frac{\partial^2 \hat{G}}{\partial x^k\, \partial x^o} - g^{ko}\, \Gamma^m{}_{ko} \frac{\partial \hat{G}}{\partial x^m}\right]_{;i} - \acute{K}\, \hat{G} \left[g^{ko} \frac{\partial^2 T^{ij}}{\partial x^k\, \partial x^o} - g^{ko}\, \Gamma^m{}_{ko} \frac{\partial T^{ij}}{\partial x^m}\right]_{;i}$$

Now, we do the covariant derivative, substituting (36) and (39a):

$$(40c) \quad 0 = \hat{G}\, [(\partial T^{ij}/\partial x^i) + \Gamma^i{}_{in} T^{nj} + \Gamma^j{}_{in} T^{in}] + (\partial \hat{G}/\partial x^i)\, T^{ij} -$$

$$- \acute{K}\, T^{ij} \left\{ \frac{\partial}{\partial x^i} \left[g^{ko} \frac{\partial^2 \hat{G}}{\partial x^k\, \partial x^o}\right] - \frac{\partial}{\partial x^i} \left[g^{ko}\, \Gamma^m{}_{ko} \frac{\partial \hat{G}}{\partial x^m}\right] \right\} -$$

$$- \acute{K}\, \hat{G} \left\{ \left[\frac{\partial}{\partial x^i} \left(g^{ko} \frac{\partial^2 T^{ij}}{\partial x^k\, \partial x^o}\right) + \Gamma^i{}_{in} \left(g^{ko} \frac{\partial^2 T^{nj}}{\partial x^k\, \partial x^o}\right)\right] + \Gamma^j{}_{in} \left(g^{ko} \frac{\partial^2 T^{in}}{\partial x^k\, \partial x^o}\right)\right] -$$

$$- \left[\frac{\partial}{\partial x^i} \left(g^{ko}\, \Gamma^m{}_{ko} \frac{\partial T^{ij}}{\partial x^m}\right) + \Gamma^i{}_{in} \left(g^{ko}\, \Gamma^m{}_{ko} \frac{\partial T^{nj}}{\partial x^m}\right) + \Gamma^j{}_{in} \left(g^{ko}\, \Gamma^m{}_{ko} \frac{\partial T^{in}}{\partial x^m}\right)\right] \right\}$$

Multiplying each term of (40c) by the four-vector velocity $U^\nu = dx^\nu / d\tau$, we obtain (40a) as expected in an implicit aspect; useful to obtain the scalar function $\hat{G} = \hat{G}(x^\nu) = (-8\pi G)$, together with the field equation (33), by using "computational mathematics" or "numerical relativity". In the future if some researcher can use resolve the differential equation, (40a), (of third degree), to obtain explicitly, $\hat{G} = \hat{G}(x^\nu) = (-8\pi G)$, is welcome.

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