

MATH 200 - Elementary Linear Algebra

MATH 201 - Intermediate Linear Algebra

Section 1.2

Gaussian Elimination and Gauss-Jordan Elimination

Devotional Thought

Matrices

Due to the reliability of Gaussian elimination in solving linear systems, we will look into a way to streamline the process.

Definition If m and n are positive integers, then an $m \times n$ (read “ m by n ”) **matrix** is a rectangular array

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

in which each **entry**, a_{ij} of the matrix is a number. An $m \times n$ matrix has m **rows** and n **columns**. Matrices are usually denoted by capital letters.

The entry a_{ij} is located in the i th row and the j th column. The index i is called the **row subscript** because it identifies the row in which the entry lies, and the index j is called the **column subscript** because it identifies the column in which the entry lies.

A matrix with m rows and n columns is of **size** $m \times n$. When $m = n$, the matrix is **square** of **order** n and the entries $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ are the **main diagonal** entries.

Example Determine the size of each matrix.

a. [2]

b. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

c. $\begin{bmatrix} e & 2 \\ -7 & \pi \\ \sqrt{2} & 4 \end{bmatrix}$

d. $\begin{bmatrix} 1 & -2 & 4 & -8 & 16 & 1/12 \end{bmatrix}$

The matrix derived from the coefficients and constant terms of a system of linear equations is the **augmented matrix** of the system. The matrix containing the coefficients of the system is the **coefficient matrix** of the system.

Example

$$3x - y + z = 5$$

$$x + 2y + z = 0$$

$$x + z = 2$$

Elementary Row Operations

Last section, we studied three operations that produce equivalent systems of linear equations.

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of one equation to another equation.

In matrix terminology, these three operations correspond to **elementary row operations**.

Elementary Row Operations

1. Interchange two rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of a row to another row.

An elementary row operation on an augmented matrix produces a new augmented matrix corresponding to a new (but equivalent) system of linear equations. Two matrices are **row-equivalent** when one can be obtained from the other by a finite sequence of elementary row operations.

Same warning as before: although elementary row operations are relatively simple to perform, they can involve a lot of arithmetic, so it is easy to make a mistake. Noting the elementary row operations performed in each step can make checking your work easier.

Additionally, solving some systems involves many steps, so it is helpful to use a short-hand method of notation to keep track of each elementary row operation you perform.

Examples [Exercises 7 and 9]

Identify the elementary row operation(s) being performed to obtain the new row-equivalent matrix.

Original Matrix

$$\begin{bmatrix} -2 & 5 & 1 \\ 3 & -1 & -8 \end{bmatrix}$$

New Row-Equivalent Matrix

$$\begin{bmatrix} 13 & 0 & -39 \\ 3 & -1 & -8 \end{bmatrix}$$

Elementary Row Operation(s)

$$\begin{bmatrix} 0 & -1 & -7 & 7 \\ -1 & 5 & -8 & 7 \\ 3 & -2 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 5 & -8 & 7 \\ 0 & -1 & -7 & 7 \\ 0 & 13 & -23 & 23 \end{bmatrix}$$

Last section we also used Gaussian elimination with back-substitution to solve a system of linear equations. The next example demonstrates the matrix version of Gaussian elimination. The two methods are essentially the same. The basic difference is that with matrices you do not need to keep writing the variables.

Example [Exercise 32]

Solve the system using Gaussian elimination with back-substitution.

$$3x_1 - 2x_2 + 3x_3 = 22$$

$$3x_2 - x_3 = 24$$

$$6x_1 - 7x_2 = -22$$

Note We put the *matrix* into row-echelon form before back-substituting in the *system*.

Definition A matrix in **row-echelon form (r.e.f)** has the properties below.

1. Any rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is (called a leading 1).
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

A matrix in row-echelon form is in **reduced row-echelon form (r.r.e.f.)** when every column that has a leading 1 has zeros in every position above and below its leading 1.

Examples Determine whether each matrix is in r.e.f. If it is, determine whether the matrix is also in r.r.e.f.

20.
$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 \end{bmatrix}$$

22.
$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

21.
$$\begin{bmatrix} -2 & 0 & 1 & 5 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

24.
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Gauss-Jordan Elimination

With Gaussian elimination, we apply elementary row operations to a matrix to obtain a (row-equivalent) row-echelon form. A second method of elimination, called **Gauss-Jordan elimination** (named after Carl Friedrich Gauss and Wilhelm Jordan), continues the reduction process until a *reduced* row-echelon form is obtained.

Example [Exercise 38]

Use Gauss-Jordan elimination to solve the system.

$$2x + y - z + 2w = -6$$

$$3x + 4y + w = 1$$

$$x + 5y + 2z + 6w = -3$$

$$5x + 2y - z - w = 3$$

Example, continued

Bottom Line: More row-reduction, no back-substitution.

Class Discussion

1. Without performing any row operations, explain why the system of linear equations below is consistent.

$$2x_1 + 3x_2 + 5x_3 = 0$$

$$-5x_1 + 6x_2 - 17x_3 = 0$$

$$7x_1 - 4x_2 + 3x_3 = 0$$

2. Without performing any row operations, explain why the system of linear equations below must have infinitely many solutions.

$$2x_1 + 3x_2 + 5x_3 + 2x_4 = 0$$

$$-5x_1 + 6x_2 - 17x_3 - 3x_4 = 0$$

$$7x_1 - 4x_2 + 3x_3 + 13x_4 = 0$$

Homogeneous Systems of Linear Equations

Systems of linear equations in which each of the constant terms is zero are called **homogeneous**. A homogeneous system of m equations in n variables would thus have the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

A *homogeneous system of linear equations must have at least one solution.*

Specifically, if all variables in a homogeneous system of linear equations have the value zero, then each of the equations is satisfied. Such a solution is called **trivial** (or obvious)...

...Let me repeat that:

The trivial solution $(0, 0, 0, \dots, 0)$ is *always* a solution to a homogeneous system of linear equations!

Theorem 1.1 Every homogeneous system of linear equations is consistent. Moreover, if the system has fewer equations than variables, then it must have infinitely many solutions.

A proof for Theorem 1.1 can be constructed as a $m \times n$ generalization of the following example (with any coefficients).

Example Solve the system of linear equations.

$$x_1 - x_2 + 3x_3 = 0$$

$$2x_1 + x_2 + 3x_3 = 0$$

More Exercises (for extra practice as time allows)

Example [Exercise 52]

Find values of a , b and c (if possible) such that the system of equations has (a) a unique solution, (b) no solution, and (c) infinitely many solutions.

$$x + y = 0$$

$$y + z = 0$$

$$x + z = 0$$

$$ax + by + cz = 0$$

Example [Exercise 66]

Find all values of λ (the Greek letter lambda) for which the homogeneous linear equation has nontrivial solutions.

$$(2\lambda + 9)x - 5y = 0$$

$$x - \lambda y = 0$$