

MATH 200 - Elementary Linear Algebra

MATH 201 - Intermediate Linear Algebra

## Section 2.3 Notes

### The Inverse of a Matrix

#### Devotional Thought

## Matrices and Their Inverses

Section 2.2 discussed some of the similarities between the algebra of real numbers and the algebra of matrices. This section further develops the algebra of matrices to include the solutions of matrix equations involving matrix multiplication.

As motivation, recall that any system of equations can be written as  $A\mathbf{x} = \mathbf{b}$ . The real number analog would be  $ax = b$ . To solve this equation for  $x$ , we would multiply both sides of the equation by  $a^{-1}$  (provided  $a \neq 0$ ):

$$\begin{aligned} ax &= b \\ a^{-1}ax &= a^{-1}b \\ 1x &= a^{-1}b \\ x &= a^{-1}b \end{aligned}$$

Is it possible to do something similar to the above in the matrix case? If so, we would end up with a solution to our system as being  $\mathbf{x} = A^{-1}\mathbf{b}$ .

Some caution should be exercised here, though. The number  $a^{-1}$  is the *multiplicative inverse* of  $a$  because  $a^{-1}a = 1 = aa^{-1}$ . A similar property would need to hold for  $A$  and  $A^{-1}$ .

We also know that  $a$  has no multiplicative inverse if and only if  $a = 0$ . Does a similar result hold for matrices?

**Definition** An  $n \times n$  matrix  $A$  is **invertible** (or **nonsingular**) when there exists an  $n \times n$  matrix  $B$  such that

$$AB = BA = I_n$$

where  $I_n$  is the identity matrix of order  $n$ . The matrix  $B$  is the (multiplicative) **inverse** of  $A$ . A matrix that does not have an inverse is **noninvertible** (or **singular**).

**Example** [Exercise 4] Show that  $B$  is an inverse of  $A$ , where

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}.$$

*Nonsquare matrices do not have inverses.* To see this, note that if  $A$  is of size  $m \times n$  and  $B$  is of size  $n \times m$  (where  $n \neq m$ ), then the products  $AB$  and  $BA$  are of different sizes and cannot be equal to each other.

It has already been noted that the identity matrix of a particular order must be unique. (See the “Sidenote” slide in section 2.2).

### **Theorem 2.7 Uniqueness of an Inverse Matrix**

If  $A$  is an invertible matrix, then its inverse is unique. The inverse of  $A$  is denoted by  $A^{-1}$ . That is,  $A^{-1}$  is *the* unique matrix (for a given  $A$ ) where  $AA^{-1} = A^{-1}A = I$ .

The proof is similar to the one given for the uniqueness of the identity.

**Proof.** Suppose  $A$  is an invertible matrix with inverses  $B$  and  $C$ . Then

$$AB = I = BA \quad \text{and} \quad AC = I = CA.$$

Thus

$$B = BI = B(AC) = (BA)C = IC = C.$$

Therefore  $B = C$  meaning that the inverse of  $A$  must be unique. ■

**Remark** It is possible to show even more: If  $A$  and  $B$  are square matrices with  $AB = I$ , then it must be that  $BA = I$  as well.

## Finding the Inverse of a Matrix

Example [Exercise 10] Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}.$$

## Example (continued)

### Finding the Inverse of a Matrix by Gauss-Jordan Elimination

Let  $A$  be a square matrix of order  $n$ .

1. Write the  $n \times 2n$  matrix that consists of  $A$  on the left and the  $n \times n$  identity matrix  $I$  on the right to obtain  $[A | I]$ . This process is called **adjoining** matrix  $I$  to matrix  $A$ .
2. If possible, row reduce  $A$  to  $I$  using elementary row operations on the entire matrix  $[A | I]$ . The result will be the matrix  $[I | A^{-1}]$ . If this is not possible, then  $A$  is noninvertible (or singular).
3. Check your work by multiplying to see that  $AA^{-1} = I = A^{-1}A$ .

Example [Exercise 16] Find the inverse of the matrix (if it exists).

$$A = \begin{bmatrix} 10 & 5 & -7 \\ -5 & 1 & 4 \\ 3 & 2 & -2 \end{bmatrix}$$

**Example** [Exercise 24] Find the inverse of the matrix (if it exists).

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 5 & 5 \end{bmatrix}$$

## Sidenote: A Formula for the Inverse of a $2 \times 2$ Matrix

If  $A$  is a  $2 \times 2$  matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then  $A$  is \_\_\_\_\_ if and only if \_\_\_\_\_. Moreover, if \_\_\_\_\_, then the inverse is \_\_\_\_\_.

$$A^{-1} =$$

.

# Class Discussion

Let  $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ ,  $B = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$ , and  $C = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$ .

1. Find  $A^{-1}$ ,  $B^{-1}$ , and  $C^{-1}$ .
  2. Find  $(AB)^{-1}$ ,  $A^{-1}B^{-1}$ , and  $B^{-1}A^{-1}$ . What do you notice?  
(Does this feel familiar)?

## Properties of Inverses

### Theorem 2.8 Properties of Inverse Matrices

If  $A$  is an invertible matrix,  $k$  is a positive integer, and  $c$  is a nonzero scalar, then  $A^{-1}$ ,  $A^k$ ,  $cA$ , and  $A^T$  are invertible and the statements below are true.

1.  $(A^{-1})^{-1} = A$
2.  $(A^k)^{-1} = (A^{-1})^k$
3.  $(cA)^{-1} = \frac{1}{c}A^{-1}$
4.  $(A^T)^{-1} = (A^{-1})^T$

All of these can be verified using the fact that an inverse is unique.  
("If it quacks like an inverse...")

**Proof of 4.** Suppose  $A$  is invertible. Then, by Theorem 2.6 part 4,

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I.$$

Additionally,

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I.$$

So  $(A^{-1})^T$  acts as an inverse for  $A^T$ . Therefore (by uniqueness)  $(A^T)^{-1} = (A^{-1})^T$ . ■

**Example** [Exercise 38] Compute  $A^{-2}$  two different ways and show that the results are equal.

$$A = \begin{bmatrix} 2 & 7 \\ -5 & 6 \end{bmatrix}$$

## Theorem 2.9 The Inverse of a Matrix Product

If  $A$  and  $B$  are invertible matrices of order  $n$ , then  $AB$  is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

**Proof.** For invertible matrices  $A$  and  $B$  (of a common order),

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

Similarly  $(B^{-1}A^{-1})(AB) = I$ . Therefore  $AB$  is invertible with  $(AB)^{-1} = B^{-1}A^{-1}$ . ■

What this proof helps to make clear is that the lack of commutativity for matrix multiplication causes on-going issues, e.g.,  $(AB)(A^{-1}B^{-1}) \neq I$  because  $BA^{-1} \neq A^{-1}B$ .

**Example** [Exercise 42] Use the inverse matrices to find (a)  $(AB)^{-1}$ , (b)  $(A^T)^{-1}$ , and (c)  $(2A)^{-1}$ .

$$A^{-1} = \begin{bmatrix} -\frac{2}{7} & \frac{1}{7} \\ \frac{3}{7} & \frac{2}{7} \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} \frac{5}{11} & \frac{2}{11} \\ \frac{3}{11} & -\frac{1}{11} \end{bmatrix}$$

Having a multiplicative inverse also allows us to cancel in multiplication: For real numbers  $a$ ,  $b$ , and  $c$ , if  $ac = bc$  then  $a = b$  provided that  $c \neq 0$ . This last stipulation is important as zero is the only real number without a multiplicative inverse.

### Theorem 2.10 Cancellation Properties

If  $C$  is an invertible matrix, then the properties below are true.

1. If  $AC = BC$ , then  $A = B$ .
2. If  $CA = CB$ , then  $A = B$ .

The proofs of 1 and 2 are similar. The book proves 1. Below is the proof for 2.

**Proof of 2.** Since  $C$  is invertible,  $C^{-1}$  exists and thus

$$\begin{aligned} CA &= CB \\ C^{-1}CA &= C^{-1}CB \\ IA &= IB \\ A &= B. \end{aligned}$$

■

**Remark** Be sure that you always apply the inverse on the same side (left or right) when multiplying both sides of an equation by an inverse. The same goes for canceling.

# Systems of Equations Revisited

As eluded to at the start of the section, for square systems of equations (those having the same number of equations as variables), you can use the theorem below to determine whether the system has a unique solution.

## Theorem 2.11

If  $A$  is an invertible matrix, then the system of linear equations  $A\mathbf{x} = \mathbf{b}$  has a unique solution  $\mathbf{x} = A^{-1}\mathbf{b}$ .

*Proof.* The matrix  $A$  is nonsingular, meaning that we can solve out as we normally would.

$$\begin{aligned}A\mathbf{x} &= \mathbf{b} \\ A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} \\ I\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b}\end{aligned}$$

This solution is unique because if  $\mathbf{x}_1$  and  $\mathbf{x}_2$  were two solutions, then we could apply the cancellation property to the equation  $A\mathbf{x}_1 = \mathbf{b} = A\mathbf{x}_2$  to conclude that  $\mathbf{x}_1 = \mathbf{x}_2$ .  $\square$

On the face of it, Theorem 2.11 is not super useful; to find  $A^{-1}$  we would need to perform Gauss-Jordan elimination which would be sufficient for solving the system  $A\mathbf{x} = \mathbf{b}$ . The above statement says that we would need to perform matrix multiplication as an additional step.

There is, however, one scenario where Theorem 2.11 could save us some work: What if we had the same coefficient matrix for multiple systems?

**Example** [Exercise 46] Use an inverse matrix to solve each system.

$$(a) \begin{array}{rcrcr} 2x & - & y & = & -3 \\ 2x & + & y & = & 7 \end{array}$$

$$(b) \begin{array}{rcrcr} 2x & - & y & = & 1 \\ 2x & + & y & = & -3 \end{array}$$

## Closing Note: Why can we not divide matrices?

For real numbers, **division** is defined as

$$a \div b = ab^{-1} = \frac{a}{b}$$

provided that  $b \neq 0$ . More fundamentally, for integers we say that  $a$  is a **divisor** of  $b$  if there exists a *unique*  $c$  such that  $b = ac$ .

Why does a similar thing not work for matrices?

Issue #1 For a given nonzero matrix, the uniqueness property of divisors fails:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

Based on this,  $\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \div \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = ?$

(This issue is fixed if we assume all matrices involved must be invertible; like the real zero, singular matrices create problems for division).

Issue #2 Even assuming invertibility, there is a problem relating to commutativity.

For real number  $a \div b$  can be unambiguously thought of as  $ab^{-1} = \frac{a}{b} = b^{-1}a$ . That is, right division and left division are the same. This is not true of matrices, however:

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 & -2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}, \quad \text{but} \quad \begin{bmatrix} -1 & -2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix}$$

## More Exercises (for extra practice as time allows)

**Example** [Exercise 54] Find  $x$  such that the matrix is equal to its own inverse.

$$A = \begin{bmatrix} 2 & x \\ -1 & -2 \end{bmatrix}$$

**Example** [Exercise 60] Show that the matrix is invertible and find its inverse.

$$A = \begin{bmatrix} \sec \theta & \tan \theta \\ \tan \theta & \sec \theta \end{bmatrix}$$