MATH 200 - Elementary Linear Algebra

MATH 201 - Intermediate Linear Algebra

Section 3.2 Notes

Determinants and Elementary Row Operations

Devotional Thought

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Which of the determinants below is easier to evaluate?

$$|A| = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 4 & -6 & 3 & 2 \\ -2 & 4 & -9 & -3 \\ 3 & -6 & 9 & 2 \end{vmatrix} \quad \text{or} \quad |B| = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 2 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

Given what you know about the determinant of a triangular matrix, it should be clear that the second determinant is much easier to evaluate. Its determinant is simply the product of the entries on the main diagonal. That is,

$|B| =$

Using expansion by cofactors (the only technique discussed so far) to evaluate the first determinant is messy. For example, when you expand by cofactors in the first row, you have

$|A| =$

Evaluating the determinants of these four 3×3 matrices produces

$|A| =$

Note that $|A|$ and $|B|$ have the same value. Also note that you can obtain matrix B from matrix A by adding multiples of the first row to the second, third, and fourth rows.

Class Discussion

For each pair of matrices, determine a single row operation that takes A to B and calculate the determinants of both matrices.

a. $A = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 \\ 2 & -3 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & -3 \\ 2 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix}$

c. $A = \begin{bmatrix} 2 & -8 \\ -2 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -4 \\ -2 & 9 \end{bmatrix}$

Elementary Row Operations and Determinants

Theorem 3.3 Let A and B be square matrices.

1. When B is obtained from A by interchanging two rows of A,
$$\det(B) = -\det(A).$$
2. When B is obtained from A by adding a multiple of a row of A to another row of A,
$$\det(B) = \det(A).$$
3. When B is obtained from A by multiplying a row of A by a nonzero constant c,
$$\det(B) = c\det(A).$$

Note that the third property is usually thought of as a factoring out from a row, e.g.,

$$\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}.$$

In a similar way, we can think of interchanging two rows as factoring out a negative (property 1 above). Ironically, property 2 tells us that the most complicated (and, in our experience, most useful) row operation of adding a multiple of a row to another has no effect on the determinant.

<u>Example</u> [Exercise 34] Find the determinant.

$$\begin{vmatrix} 0 & -4 & 9 & 3 \\ 9 & 2 & -2 & 7 \\ -5 & 7 & 0 & 11 \\ -8 & 0 & 0 & 16 \end{vmatrix}$$

<u>Warning</u> When adding multiples of rows to other rows, be sure that you only perform the operation as

$$R_i + c\,R_j \to R_i.$$

Namely, if you were to begin by performing the operation $5R_2 + 9R_3 \to R_3$ in the example above, then the value of the determinant *would* be affected. (How so?)

Determinants and Elementary Column Operations

Although Theorem 3.3 is stated in terms of elementary row operations, the theorem remains valid when the word "column" replaces the word "row." Operations performed on the columns (rather than on the rows) of a matrix are **elementary column operations**, and two matrices are **column-equivalent** when one can be obtained from the other by elementary column operations.

$$\begin{vmatrix} 2 & 1 & -3 \\ 4 & 0 & 1 \\ 0 & 0 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & -3 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{vmatrix}$$

Interchange the first two columns.

$$\begin{vmatrix} 2 & 3 & -5 \\ 4 & 1 & 0 \\ -2 & 4 & -3 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 & -5 \\ 2 & 1 & 0 \\ -1 & 4 & -3 \end{vmatrix}$$

Factor 2 out of the first column.

Example [Exercise 28] Find the determinant.

$$A = \begin{vmatrix} 3 & 0 & 6 \\ 2 & -3 & 4 \\ 1 & -2 & 2 \end{vmatrix}$$

Matrices and Zero Determinants

Theorem 3.4 If A is a square matrix and any one of the conditions below is true, then $\det(A) = 0$.

1. An entire row (or an entire column) consists of zeros.

2. Two rows (or columns) are equal.

3. One row (or column) is a multiple of another row (or column).

$$\begin{vmatrix} 0 & 0 & 0 \\ 2 & 4 & -5 \\ 3 & -5 & 2 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & -2 & 4 \\ 0 & 1 & 2 \\ 1 & -2 & 4 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & -6 \\ -2 & 0 & 6 \end{vmatrix} = 0.$$

The first row has all zeros.

The first and third rows are the same.

The third column is a multiple of the first column.

Question Considering row and column operations, what would be the most general condition for determining whether a matrix has a zero determinant?

Example Find the determinant.

$$\begin{vmatrix} 1 & 5 & -3 \\ 2 & -1 & 0 \\ 0 & 22 & -12 \end{vmatrix}$$

A Computational Comparison of Cofactor Expansion and Row Reduction

You have now studied two methods for evaluating determinants. Of these, the method of using elementary row operations to reduce the matrix to triangular form is usually faster than cofactor expansion along a row or column. If the matrix is large, then the number of arithmetic operations needed for cofactor expansion can become extremely large. For this reason, most computer and calculator algorithms use the method involving elementary row operations. The table below shows the maximum numbers of additions (plus subtractions) and multiplications (plus divisions) needed for each of these two methods for matrices of orders 3, 5, and 10.

	Cofactor Expansion		*Row Reduction*	
Order n	*Additions*	*Multiplications*	*Additions*	*Multiplications*
3	5	9	8	10
5	119	205	40	44
10	3,628,799	6,235,300	330	339

In fact, the maximum number of additions alone for the cofactor expansion of an $n \times n$ matrix is $n! - 1$. The factorial 30! is approximately equal to 2.65×10^{32}, so even a relatively small 30×30 matrix could require an extremely large number of operations. If a computer could do one trillion operations per second, it would still take more than 22 trillion years to compute the determinant of this matrix using cofactor expansion. Yet, row reduction would take only a fraction of a second.

Example [Exercise 36] Find the determinant.

$$\begin{vmatrix} 3 & -2 & 4 & 3 & 1 \\ -1 & 0 & 2 & 1 & 0 \\ 5 & -1 & 0 & 3 & 2 \\ 4 & 7 & -8 & 0 & 0 \\ 1 & 2 & 3 & 0 & 2 \end{vmatrix}$$