MATH 200 - Elementary Linear Algebra

MATH 201 - Intermediate Linear Algebra

Section 1.1

Introduction to Systems of Linear Equations

Devotional Thought

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What do you need to know starting off?

The study of linear algebra demands familiarity with algebra, analytic geometry, and trigonometry. Occasionally, we will encounter examples and exercises requiring a knowledge of calculus (MATH 191 is a prerequisite for this course).

What will we be doing in this course?

Much of what we will be doing starting off is building efficient processes (e.g. Gaussian elimination, matrix multiplication, cofactor expansion) for solving systems of equations. Many of these solution methods involve multiple arithmetic steps, so it is essential that you check your work. You can use software, internet applets, or a calculator to check your work and perform routine computations. However, you need to be confident in your abilities to perform said computations as only a limited amount of technology will be available during the test; you are only allowed a non-graphing calculator.

Later on, we will use the developed processes to investigate more abstract questions regarding representation in a coordinate system. This material will be much more theoretical in nature (subspaces, bases, linear transformations, eigenvalues and eigenvectors), but is actually much more useful in application as background mathematical content for STEM courses.

A Word of Caution

Although you will be familiar (perhaps *very* familiar) with some of this material starting off, you should carefully study the methods presented. This will both cultivate an intuition while clarifying your understanding for the more advanced material to come.

Something to Start With...

I will often pose examples for you to work out during class. While no points will be assigned based on these exercises, it is important for both your engagement and understanding to put forth your best effort when said questions are presented.

Disclaimer I will try to use examples other than those fully worked out in the textbook. It is assumed that students will look over the section before attending the class lecture.

Example [Exercises 34 and 48]
Solve each system of equations using whatever method you like.

$$\begin{aligned} 9x - 4y &= 5 \\ \tfrac{1}{2}x + \tfrac{1}{3}y &= 0 \end{aligned}$$

$x = 1/3 \quad y = -1/2$

Check: $9\left(\frac{1}{3}\right) - 4\left(-\frac{1}{2}\right) = 3 + 2 = 5$ ✓

$\frac{1}{2}\left(\frac{1}{3}\right) + \frac{1}{3}\left(-\frac{1}{2}\right) = \frac{1}{6} - \frac{1}{6} = 0$ ✓

$$\begin{aligned} x + y + z &= 2 \\ -x + 3y + 2z &= 8 \\ 4x + y &= 4 \end{aligned}$$

$(x, y, z) = (0, 4, -2)$

One Sort of Application: Unlocking and Interacting with More Mathematics

Example [Exercise 74]

Solve the system of equations by first letting $A = 1/x$ and $B = 1/y$.

$$\begin{aligned} \frac{3}{x} - \frac{2}{y} &= -1 \\ \frac{2}{x} - \frac{3}{y} &= 3 \end{aligned}$$

$$\begin{aligned} 3A - 2B &= -1 \\ 2A - 3B &= 3 \end{aligned}$$

$$\begin{aligned} 6A - 6B &= -2 \\ -6A + 9B &= -9 \\ \hline 5B &= -11 \end{aligned}$$

$B = -\frac{11}{5}$

$$2A - 3\left(-\frac{11}{5}\right) = 3$$

$$2A = 3 - \frac{33}{5} = -\frac{18}{5}$$

$$A = -\frac{9}{5}$$

$$\boxed{(x, y) = \left(-\frac{5}{9}, -\frac{5}{11}\right)}$$

Check: $\frac{3}{(-5/9)} - \frac{2}{(-5/11)} = -\frac{27}{5} + \frac{22}{5} = -\frac{5}{5} = -1$ ✓

$$\frac{2}{(-5/9)} - \frac{3}{(-5/11)} = -\frac{18}{5} + \frac{33}{5} = \frac{15}{5} = 3 \quad ✓$$

Example [Exercise 78] Solve the system of linear equations for x and y.

$$\begin{cases}(\cos\theta)\,x + (\sin\theta)\,y = 1\\(-\sin\theta)\,x + (\cos\theta)\,y = 1\end{cases}\begin{matrix}\cdot\sin\theta\\\cdot\cos\theta\end{matrix}$$

$$\cancel{\sin\theta\cos\theta\, x} + \sin^2\theta\, y = \sin\theta$$
$$\cancel{-\sin\theta\cos\theta\, x} + \cos^2\theta\, y = \cos\theta$$

$$\underbrace{\{\sin^2\theta + \cos^2\theta\}}_{=1} y = \sin\theta + \cos\theta$$

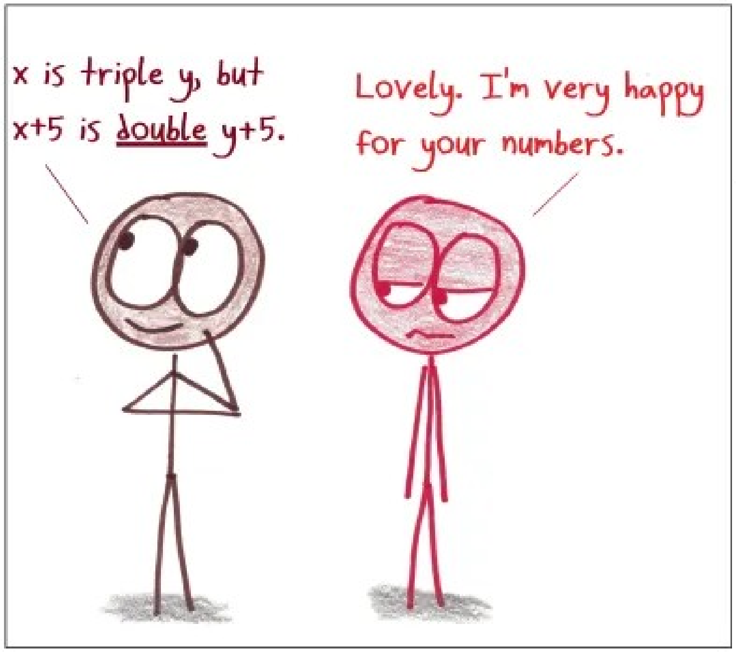
$$\boxed{y = \sin\theta + \cos\theta}$$

$$\cos\theta\, x + \sin\theta\{\sin\theta + \cos\theta\} = 1$$

$$\cos\theta\, x = \underbrace{1 - \sin^2\theta}_{\cos^2\theta} - \sin\theta\cos\theta$$

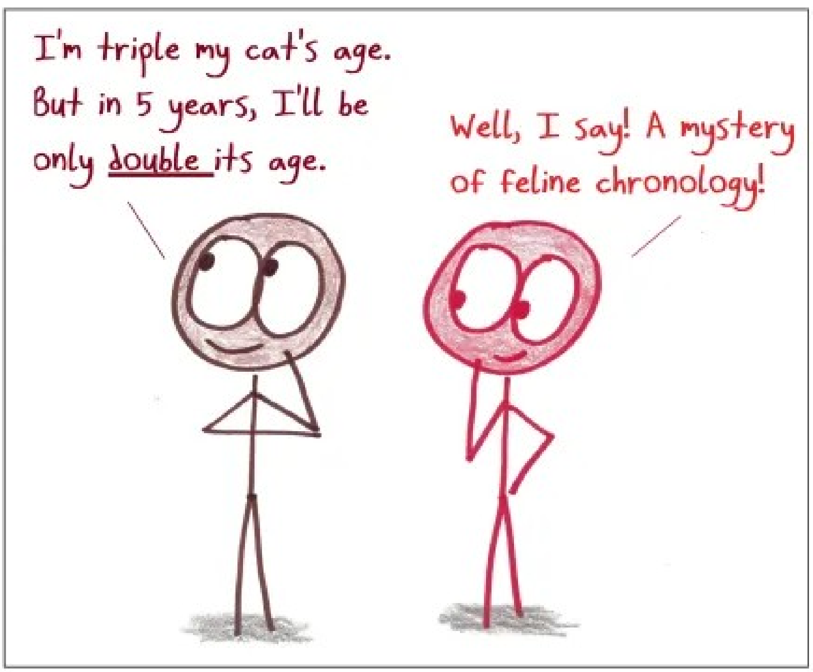
$$\boxed{x = \cos\theta - \sin\theta}$$

The More Traditional Application



Version #1

Version #2



Example [Exercise 67] One eight-ounce glass of apple juice and one eight-ounce glass of orange juice contain a total of 227 milligrams of vitamin C. Two eight-ounce glasses of apple juice and three eight-ounce glasses of orange juice contain a total of 578 milligrams of vitamin C. How much vitamin C is in an eight-ounce glass of each type of juice?

$\underline{A}$ for apple juice

$\underline{B}$ for orange juice

$$A + B = 227$$
$$2A + 3B = 578$$

$$\begin{array}{r} \cancel{-2A} - 2B = -454 \\ \underline{\cancel{2A} + 3B = 578} \\ B = 124 \end{array}$$

$$\begin{aligned} A &= 227 - B \\ &= 227 - 124 \\ &= 103 \end{aligned}$$

$$\boxed{\begin{array}{l} 103 \text{ mg in apple juice} \\ 124 \text{ mg in orange juice} \end{array}}$$

Linear Equations in n Variables

Recall from past courses that the *equation of a line* in two-dimensional space has the form

$$a_1x + a_2y = b, \qquad a_1, a_2, \text{ and } b \text{ are constants.}$$

This is also called a **linear equation in two variables** x and y. Similarly, the **equation of a plane** in three-dimensional space has the form

$$a_1x + a_2y + a_3z = b, \qquad a_1, a_2, a_3, \text{ and } b \text{ are constants.}$$

This is a **linear equation in three variables** x, y, and z. In general, a **linear equation in n variables** $x_1, x_2, x_3, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b.$$

The **coefficients** $a_1, a_2, a_3, \ldots, a_n$ are real numbers, and the **constant term** b is a real number. The number a_1 is called the **leading coefficient**, and x_1 is the **leading variable.**

<u>**Example**</u> [Exercises 4 and 6] Classify the following equations as linear or nonlinear.

a. $x^2 + y^2 = 4$

b. $(\cos 3)x + y = -16$

Notice that linear equations have no products or roots of variables and no variables involved in trigonometric, exponential, or logarithmic functions. Also, all variables appear only to the first power.

Solutions and Solution Sets for Linear Equations

A **solution** of a linear equation in n variables is a sequence of n real numbers $s_1, s_2, \ldots, s_n$ that satisfy the equation when you substitute the values

$$x_1 = s_1, \quad x_2 = s_2, \quad x_3 = s_3, \quad \ldots, \quad x_n = s_n$$

into the equation.

<u>**Example**</u> [Exercise 7] One solution to the equation $2x - 4y = 0$ is $x = 2$ and $y = 1$.

Are there others? If so, how many?

The set of *all* solutions of a linear equation is its **solution set**, and when you have found this set, you have **solved** the equation.

To describe the entire solution set of a linear equation, we can use a **parametric representation** (also called a **parameterization**). This process involves identifying a **free variable(s)**, which may take on any real value. All other variables should depend upon the free variable(s). We can then **parameterize** the solution set (which is infinite) by introducing a **parameter** for each free variable.

Example Solve the equation $2x - 4y = 0$

Example [Exercise 10] Solve the linear equation $12x_1 + 24x_2 - 36x_3 = 12$.

It is important to note that the choice of free variables and parameters determine the parameterization of the solution set.

What are some other parameterizations for the solution set found in the example above?

Systems of Linear Equations

A **system of m linear equations in n variables** is a set of m equations, each of which is linear in the same n variables:

$$\begin{array}{ccccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \cdots & + & a_{3n}x_n & = & b_3 \\ & & & & & & & & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

(The double-subscript notation indicates a_{ij} is the coefficient of x_i in the jth equation.)

A system of linear equations is also called a **linear system**, or when there is no ambiguity, simply a **system**.* A **solution** of a linear system is a sequence of numbers $s_1, s_2, \ldots, s_n$ that is a solution of each equation in the system.

As an example, one solution of the system

$$\begin{array}{ccccc} 3x_1 & + & 2x_2 & = & 3 \\ -x_1 & + & x_2 & = & 4 \end{array}$$

is $x_1 = -1$ and $x_2 = 3$.

On the other hand, $x_1 = 1$ and $x_2 = 0$ is *not* a solution. (Why?)

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*We will not discuss general methods for solving *nonlinear* systems in this course. While a useful skill in its own right, the solution process tends to be messier algebraically, with an overemphasis on graphing software for definitive answers.

Class Exercise

1. Solve the system.
$$\begin{array}{rcrcl} 3x & - & y & = & 1 \\ 2x & - & y & = & 0 \end{array}$$
How many solutions does this system of linear equations have?

2. Graph the two lines $y = 3x - 1$ and $y = 2x$ in the xy-plane. Where do they intersect?

3. Repeat 1 and 2 for the systems
$$\begin{array}{rcrcl} 3x & - & y & = & 1 \\ 3x & - & y & = & 0 \end{array}$$
and
$$\begin{array}{rcrcl} 3x & - & y & = & 1 \\ 6x & - & 2y & = & 2 \end{array}$$

4. Based on these three examples, what basic types of solution sets are possible for a system of two linear equations in two variables?

Example [Exercise 82] Determine the value(s) of k such that the system of linear equations has no solution.

$$\begin{array}{rcrcrcr} x & + & 2y & + & kz & = & 6 \\ 3x & + & 6y & + & 8z & = & 4 \end{array}$$

Solving a System of Linear Equations

In summary, for a system of linear equations, precisely one of the statements below is true.

1. The system has exactly one solution (consistent system).

2. The system has infinitely many solutions (consistent system).

3. The system has no solution (inconsistent system).

This result will be justified much more thoroughly later on...

BUT ... *how* do we actually solve a system of equations?

More specifically, is there is a step-by-step process (algorithm) that always works no matter the size of the system?

We say two systems are **equivalent** when they have the same solution set.

The second system from our starting example is equivalent to a much nicer system:

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 2 \\ -x & + & 3y & + & 2z & = & 8 \\ 4x & + & y & & & = & 4 \end{array} \qquad\qquad \begin{array}{rcrcrcr} x & + & y & + & z & = & 2 \\ & & y & - & z & = & 6 \\ & & & & z & = & -2 \end{array}$$

The second system is easier to solve because it is in *row-echelon form*.

When a system of equation is in **row-echelon form** (**r.e.f.**), which means that it has a "stair-step" pattern with leading coefficients of 1, it is much easier to solve as we may use **back-substitution**.

Example Solve the system.

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 2 \\ & & y & - & z & = & 6 \\ & & & & z & = & -2 \end{array}$$

As noted earlier, the following systems are equivalent.

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 2 \\ -x & + & 3y & + & 2z & = & 8 \\ 4x & + & y & & & = & 4 \end{array} \qquad \begin{array}{rcrcrcr} x & + & y & + & z & = & 2 \\ & & y & - & z & = & 6 \\ & & & & z & = & -2 \end{array}$$

But how do we show this? How does one transform a linear system without changing its solution set?

Operations that Produce Equivalent Systems

Each of these operations on a system of linear equations produces an equivalent system.

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of an equation to another equation.

Rewriting a system of linear equations into row-echelon form usually involves a *chain* of equivalent systems, using one of the three basic operations to obtain several new (but equivalent) systems along the way. This process is called **Gaussian elimination**, after the German mathematician Carl Friedrich Gauss (1777–1855).

Example [Exercise 48] Solve the system.

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 2 \\ -x & + & 3y & + & 2z & = & 8 \\ 4x & + & y & & & = & 4 \end{array}$$

Note Because of the prevalence for arithmetic errors, you should develop the habit of checking your solutions:

As mentioned earlier, a system can have infinitely many solutions or no solution. Also recall that in the former case, the solution set is given by a parametric representation.

Example [Exercise 52] Solve the system.

$$\begin{array}{rcrcrcr} x_1 & & & + & 4x_3 & = & 13 \\ 4x_1 & - & 2x_2 & + & x_3 & = & 7 \\ 2x_1 & - & 2x_2 & - & 7x_3 & = & -19 \end{array}$$

What happens in the Gaussian elimination process when there is no solution?

Example [Exercise 50] Solve the system.

$$\begin{array}{rcrcrcl} 5x_1 & - & 3x_2 & + & 2x_3 & = & 3 \\ 2x_1 & + & 4x_2 & - & x_3 & = & 7 \\ x_1 & - & 11x_2 & + & 4x_3 & = & 3 \end{array}$$

Closing Note Each of the three equations in our past three examples represents a plane in a three-dimensional coordinate system, just as the two equations of a two variable system represent lines in a two-dimensional coordinate system.

Some Important Notes Before Moving On...

In the next section, we will streamline the Gaussian elimination approach even further, leading us to consider Gauss-Jordan elimination. The idea behind both processes is the same and both are extensively used for the rest of the course. As such, **you need to master this process as soon as possible.**

The biggest hurdles initially with solving linear systems of any size with Gaussian elimination can be overcome by remembering the following "three 3's."

- There are only three possibilities for the number of solutions:
  1. a solution exists and is unique (one solution),
  2. a solution exists but is not unique (infinitely many solutions), or
  3. a solution does not exist (no solution).

  In the latter case, the system is called *inconsistent*.

- There are only three operations that produce equivalent systems:
  1. interchange two equations,
  2. scale an equation by a nonzero constant, and
  3. add a multiple of one equation to another.

- Lastly, **here are three pro-tips that will help you avoid the most common pitfalls when performing this elimination process:**
  1. **Avoid Fractions.** Scale equations from the start to get rid of fractions and decimals; avoid fractions for as long as possible throughout the process (this will simplify your calculations.)
  2. There is a strategy that will produce r.e.f. every time: **get a leading-one on the first variable in the top equation** (using one or more of the three operations), **then use it to zero-out that same variable in the equations below** (by adding a multiple of the leading-one row). **Repeat this process for the remaining variables until the system is in r.e.f.**
  3. **Always check your solution(s) at the end.**