MATH 200 - Elementary Linear Algebra

MATH 201 - Intermediate Linear Algebra

Section 3.1 Notes

The Determinant of a Matrix

Devotional Thought

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The Determinant of a 2×2 Matrix

Historically, the use of determinants arose from the recognition of special patterns that occur in the solutions of systems of linear equations. For example, the system

$$\begin{array}{rcrcl} a_{11}x_1 & + & a_{12}x_2 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & = & b_2 \end{array}$$

has the solution

$$x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \quad \text{and} \quad x_2 = \frac{b_2 a_{11} - b_1 a_{21}}{a_{11}a_{22} - a_{21}a_{12}}$$

when $a_{11}a_{22} - a_{21}a_{12} \neq 0$.

Note that both fractions have the same denominator, $a_{11}a_{22} - a_{21}a_{12} \neq 0$. For similar reasons, this quantity also showed up as the denominator of the scalar multiple in our formula for the inverse of a 2×2 matrix; namely

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Definition The **determinant** of the 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is $\det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}$.

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Remark The determinant of a matrix of order 1 is simply the entry of the matrix.

You Try [Exercises 2, 8, and 12]
Find the determinant of each matrix.

$$A = \begin{bmatrix} -3 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{1}{3} & 5 \\ 4 & -9 \end{bmatrix}$$

$$C = \begin{bmatrix} \lambda - 2 & 0 \\ 4 & \lambda - 4 \end{bmatrix}$$

What values of λ make this determinant zero?

Remark Notice that the determinant of a matrix can be positive, zero, or negative.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

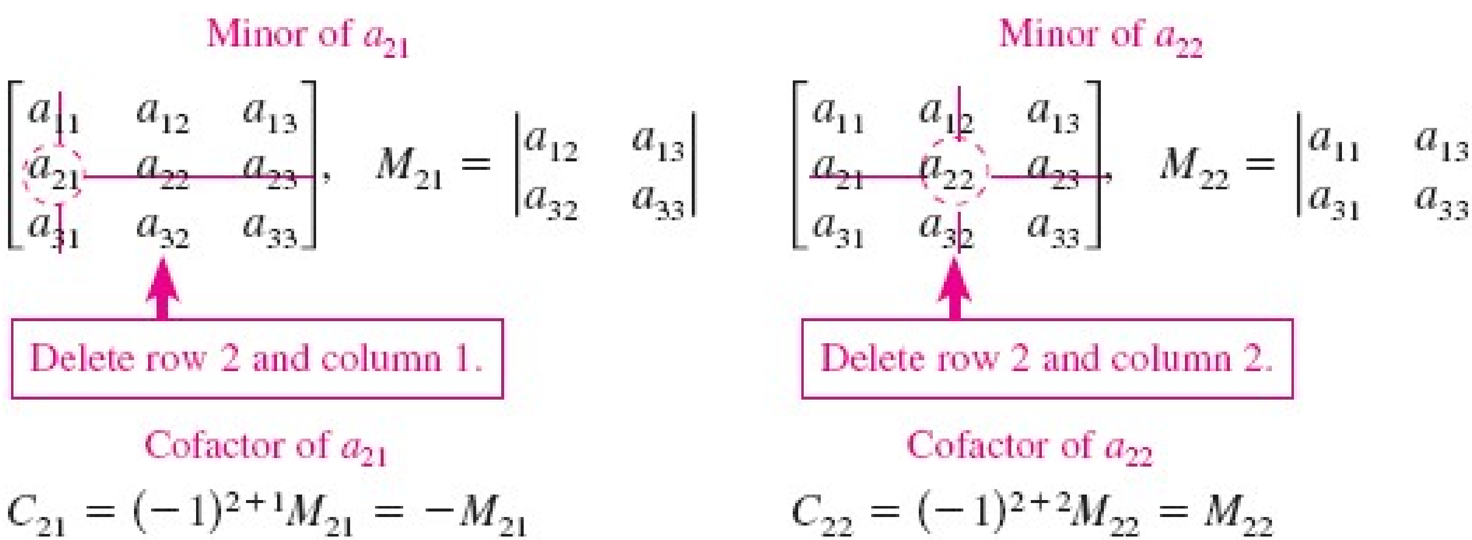
$$x = \det \begin{pmatrix} -b & \sqrt{b^2 - 4ac} \\ \mp 1/2a & 1/2a \end{pmatrix}$$

Minors and Cofactors

Every *square* matrix has a determinant. To calculate the determinant of a square matrix of order greater than two we utilize a recursive procedure that involves the minors and cofactors of a matrix.

<u>**Definition**</u> If A is a square matrix, then the **minor** M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The cofactor C_{ij} of the entry a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$.

For example, if A is a 3×3 matrix, then the minors and cofactors of a_{21} and a_{22} are as shown below.



Example [Exercise 20] Find all the minors of

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 1 & 4 \\ -2 & 0 & 1 \end{bmatrix}.$$

The cofactors and minors of a matrix can differ only in sign: $C_{ij} = (-1)^{i+j} M_{ij}$.

To obtain the cofactors of a matrix, first find the minors and then apply the checkerboard pattern of $+$'s and -'s shown below.

By doing this we are accounting for the sign change given by the factor $(-1)^{i+j}$. Note that odd positions (where $i + j$ is odd) have negative signs, and even positions (where is even) have positive signs.

<u>Example</u> [Exercise 20, continued] Find all the cofactors of

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 1 & 4 \\ -2 & 0 & 1 \end{bmatrix}.$$

Determinant of a Square Matrix

<u>**Definition**</u> The determinant of a square matrix of order 1 is simply the entry of the matrix.

If A is a square matrix of order $n \geq 2$, then the determinant of A is the sum of the entries in the first row of A multiplied by their respective cofactors. That is,

$$\det(A) = |A| = \sum_{j=1}^{n} a_{1j} C_{1j} = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

The definition above is **inductive** because it uses the determinant of a square matrix of order $n-1$ to define the determinant of a square matrix of order n.

TEST CASE: Can we derive our old determinant definition for a 2×2 matrix from the above definition?

When you use the above definition to evaluate a determinant, you are **expanding by cofactors in the first row.**

Example [Exercise 20, concluded]

Find the determinant by expanding by cofactors in the first row.

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 1 & 4 \\ -2 & 0 & 1 \end{bmatrix}.$$

What if we were to expand in the second row?

What if we were to expand in the second column?

Theorem 3.1 Let A be a square matrix of order n. Then the determinant of A is

$$\det(A) = |A| = \sum_{j=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

or

$$\det(A) = |A| = \sum_{i=1}^{n} a_{ij}C_{ij} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

This is known as Laplace's Expansion of a Determinant.

You Try [Exercise 26] Find the determinant of the matrix.

$$A = \begin{bmatrix} x & y & 1 \\ -2 & -2 & 1 \\ 1 & 5 & 1 \end{bmatrix}.$$

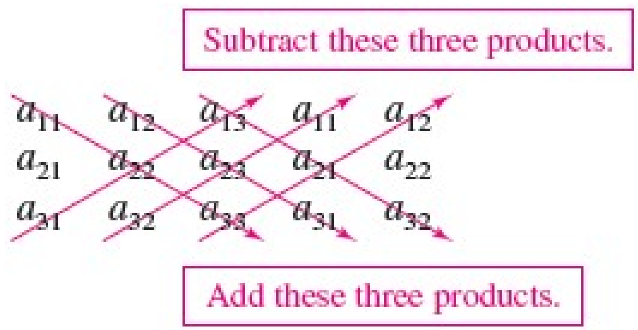
One of the key tricks with determinants is deciding which row (or column) to expand along (or down).

Example [Exercise 34 from Section 3.2]
Find the determinant of

$$A = \begin{bmatrix} 0 & -4 & 9 & 3 \\ 9 & 2 & -2 & 7 \\ -5 & 7 & 0 & 11 \\ -8 & 0 & 0 & 16 \end{bmatrix}.$$

A Trick for Finding the Determinant of a 3×3 Matrix

An alternative method is commonly used to evaluate the determinant of a 3×3 matrix A. To apply this method, copy the first and second columns of A to form fourth and fifth columns. Then obtain the determinant of A by adding (or subtracting) the products of the six diagonals, as shown in the diagram below.



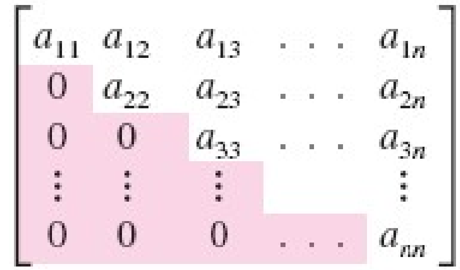
Example [Exercise 34] Find the determinant of the matrix.

$$A = \begin{bmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 8 & 1 & 6 \end{bmatrix}$$

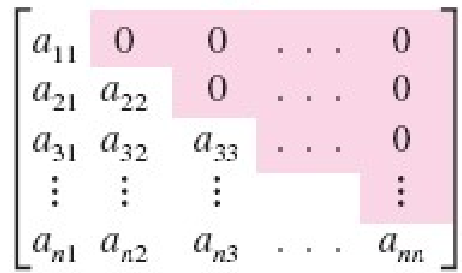
Triangular Matrices

A square matrix is **upper triangular** when it has all zero entries below its main diagonal, and **lower triangular** when it has all zero entries above its main diagonal, as shown in the diagram below. A matrix that is both upper and lower triangular is a **diagonal matrix**. That is, a diagonal matrix is one in which all entries above and below the main diagonal are zero.

Upper Triangular Matrix



Lower Triangular Matrix



You Try [Exercise 40]
Find the determinant of the given diagonal matrix. What do you notice?

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

<u>Theorem 3.2</u> If A is a triangular matrix of order n, then its determinant is the product of the entries on the main diagonal. That is,

$$\det(A) = |A| = a_{11}a_{22}a_{33}\cdots a_{nn}.$$

The proof follows using *mathematical induction* on the order n.

<u>You Try</u> [Exercises 42 and 32] Find the determinant of the given matrix.

$$A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & 0 \\ 3 & 5 & 3 & 0 \\ -8 & 7 & 0 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} -4 & 3 & 2 & -1 & -2 \\ 1 & -2 & 7 & -13 & -12 \\ -6 & 2 & -5 & -6 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & -2 & 0 & -9 \end{bmatrix}$$

More Exercises (for extra practice as time allows)

Example [Exercise 52]

Find the values of λ for which the determinant is zero.

$$\begin{vmatrix} \lambda & 0 & 1 \\ 0 & \lambda & 3 \\ 2 & 2 & \lambda - 2 \end{vmatrix}.$$

Remark This sort of calculation occurs when solving for eigenvalues (Chapter 7).

Example [Exercise 62]
Evaluate the following determinant in which the entries are functions.

$$\begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}.$$

Remark Determinants of the above form occur when performing a change of variables in multiple integrals (think u-substitution in more than one variable).

Determinants involving functions come up a few times when studying differential equations. Besides the aforementioned eigenvalues (which may appear when solving a system of differential equations), there is also the determination of the fundamental solution set (showing linear independence of solutions) and the solution method known of variation of parameters (using Cramer's Rule to solve a system involving functions).

Example [Exercise 70]

The determinant of a 2×2 matrix involves two products. The determinant of a 3×3 matrix involves six triple products. How many quadruple products are involved in finding the determinant of a 4×4 matrix?

Question How many n-tuple products would be involved in finding the determinant of an order n matrix?