Principle Vectors

39. For a matrix A with linear elementary divisors there exist n eigenvectors spanning the whole n-space. If A has a non-linear divisor, however, this is not true since there are then fewer than n independent eigenvectors. It is convenient nevertheless to have a set of vectors which span the whole n-space and to choose these in such a way that they reduce to the n eigenvectors when A has linear divisors. Now we have seen that in the latter case the eigenvectors may be taken as the columns of a matrix X, such that

$$X^{-1}AX = \operatorname{diag}(\lambda_i). \tag{39.1}$$

A natural extension when the matrix has non-linear divisors is to take, as base vectors the n columns of a matrix X which reduces A to the Jordan canonical form.

These vectors satisfy important relations. It will suffice to demonstrate them for a simple example of order 8. Suppose the matrix A is such that

$$AX = A\begin{bmatrix} C_3(\lambda_1) & & & \\ & C_2(\lambda_1) & & \\ & & C_2(\lambda_2) & \\ & & & C_1(\lambda_3) \end{bmatrix}, \tag{39.2}$$

then if $x_1, x_2, \ldots, x_8$ are the columns of X, we have equating columns

$$\left.\begin{array}{llll} Ax_1 = \lambda_1 x_1 + x_2, & Ax_4 = \lambda_1 x_4 + x_5, & Ax_6 = \lambda_2 x_6 + x_7, & Ax_8 = \lambda_3 x_8 \\ Ax_2 = \lambda_1 x_2 + x_3, & Ax_5 = \lambda_1 x_5, & Ax_7 = \lambda_2 x_7 & \\ Ax_3 = \lambda x_1 & & & \end{array}\right\}, \tag{39.3}$$

from which we deduce

$$\left.\begin{array}{llll} (A-\lambda_1 I)^3 x_1 = 0, & (A-\lambda_1 I)^2 x_4 = 0, & (A-\lambda_2 I)^2 x_6 = 0, & (A-\lambda_3 I)x_8 = 0 \\ (A-\lambda_1 I)^2 x_2 = 0, & (A-\lambda_1 I)x_5 = 0, & (A-\lambda_2 I)x_7 = 0 & \\ (A-\lambda_1 I)x_3 = 0 & & & \end{array}\right\}. \tag{39.4}$$

Each of these vectors therefore satisfies a relation of the form

$$(A-\lambda_i I)^j x_k = 0. \tag{39.5}$$

A vector which satisfies equation (39.5) but does not satisfy a relation of lower degree in $(A-\lambda_i I)$ is called a *principal vector of grade j* corresponding to λ_i.

From J. H. Wilkinson, **The Algebraic Eigenvalue Problem**, Oxford University Press, Oxford, 1965.