Classical Mechanics (320)

7.15, 7.20, 7.21, 7.24

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7.15

We begin with the Lagrangian

$$\mathcal{L} = T - U.$$

Where T is the kinetic energy and U is the potential energy of the system. Since the two masses are connected by a string we can then say that both of the masses will have the same velocity.

$$\dot{x_1} = \dot{x_2} = \dot{x}$$

This also works well since the problem told us to use a generalized coordinate of x. So, the total kinetic energy will be a sum of the two masses.

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2$$

The author tells us that we need to have the generalized coordinate of distance x below the top. Therefore, we should set the potential equal to zero at the top of the table. This means that the potential energy of m_1 will be 0 and we will only have

$$U = -m_2gx$$

If we combine the two terms then our Lagrangian will be

$$\mathcal{L} = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + m_2gx$$

So we can then use the Euler-Lagrange equation to solve for the acceleration of the blocks.

$$\frac{\partial\mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) = 0$$

We'll work piecewise working on the first term. We get

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + m_2gx\right)\\ &= m_2g\end{aligned}$$

And then for the second term we get

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) &= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + m_2 g x\right)\right)\\ &= \frac{d}{dt}\left(m_1\dot{x} + m_2\dot{x}\right)\\ &= (m_1 + m_2)\ddot{x}\end{aligned}$$

So if we then combine the two expression we have

$$m_2 g - (m_1 + m_2)\ddot{x} = 0$$

Not only is it traditional to solve for $\ddot{x}$, but the problem demands we solve for acceleration. Since $\ddot{x}$ is acceleration I think we can say

$$\boxed{a = \frac{m_2 g}{m_1 + m_2}}$$

7.20 We always start with trying to form the Lagrangian of the system

$$\mathcal{L} = T - U$$

First we will find the position of the bead on the string. In cylindrical polar coordinates it would be defined as

$$\vec{r} = (\rho, \phi, z)$$

The relationships were given in the problem. Solving for ϕ. We can plug them in to get

$$\vec{r} = (R, \frac{z}{\lambda}, z)$$

Our aim is to find the kinetic energy expression in generalized coordinates. To do this we need to find an expression for the velocity. Well it will just be d/dt of each term in r. However, we know R and λ are constants. So our velocity will be

$$\vec{v} = (0, \frac{\dot{z}}{\lambda}, \dot{z})$$

However, we need v^2 for our kinetic energy term. That will be

$$\begin{aligned}v^2 &= R^2\dot{\phi}^2 + \dot{z}^2\\ &= (1 + \frac{R^2}{\lambda^2})\dot{z}^2\end{aligned}$$

We can then plug this into our kinetic energy and find

$$\begin{aligned}T &= \frac{1}{2}mv^2\\ &= \frac{1}{2}m(1 + \frac{R^2}{\lambda^2})\dot{z}^2\end{aligned}$$

Gravity will point down along the z coordinate so the potential energy will simply be

$$U = mgz$$

If we combine T and U we form the Lagrangian.

$$\mathcal{L} = \frac{1}{2}m(1+\frac{R^2}{\lambda^2})\dot{z}^2 - mgz$$

We now apply the Euler-Lagrange equation.

$$\frac{\partial \mathcal{L}}{\partial z} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}\right) = 0$$

If we start with the first term.

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial z} &= \frac{\partial}{\partial z}\left(\frac{1}{2}m(1+\frac{R^2}{\lambda^2})\dot{z}^2 - mgz\right)\\ &= -mg\end{aligned}$$

If we evaluate the second term we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}\right) &= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{z}}\left(\frac{1}{2}m(1+\frac{R^2}{\lambda^2})\dot{z}^2 - mgz\right)\right)\\ &= \frac{d}{dt}\left(m(1+\frac{R^2}{\lambda^2})\dot{z}\right)\\ &= m(1+\frac{R^2}{\lambda^2})\ddot{z}\end{aligned}$$

We then recombine the two terms of our Euler-Lagrange.

$$-mg - m(1+\frac{R^2}{\lambda^2})\ddot{z} = 0$$

We can then solve for $\ddot{z}$

$$\boxed{\ddot{z} = \frac{-g}{1+\frac{R^2}{\lambda^2}}}$$

We examine when $R \to 0$ and notice that $\ddot{z}$ will be $-g$. This would make sense because we would approach being completely on the z axis and gravity is completely on along that coordinate.

7.21 We begin with the Lagrangian

$$L = T - U$$

We note that the frictionless rod is in a horizontal plane. So it seems logical to choose the potential energy to zero at that point. Therefore we won't have to worry about a potential energy term in our Lagrangian. Since we have two coordinates of r and ϕ. So the derivative of those terms will be needed. We'll get

$$v = \sqrt{\dot{r}^2 + (r\omega)^2}$$

So we can then say that our kinetic energy term T will be

$$T = \frac{1}{2}(m\dot{r}^2 + (r\omega)^2)$$

So, our Lagrangian will just be

$$\mathcal{L} = \frac{1}{2}(m\dot{r}^2 + (r\omega)^2)$$

We can then use the Euler-Lagrange equation.

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = 0$$

We begin with the first term to be

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial r} &= \frac{\partial}{\partial r}\left(\frac{1}{2}(m\dot{r}^2 + (r\omega)^2)\right)\\ &= mr\omega^2\end{aligned}$$

The second term will be

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) &= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{r}}\left(\frac{1}{2}(m\dot{r}^2 + (r\omega)^2)\right)\right)\\ &= \frac{d}{dt}(m\dot{r})\\ &= m\ddot{r}\end{aligned}$$

We then combine the two terms which yields

$$mr\omega^2 - m\ddot{r} = 0$$

We then solve for $\ddot{r}$ to get

$$\boxed{\ddot{r} = r\omega^2} \tag{1}$$

Now, if we analyze when the bead is initially at rest at the origin then that means $r = 0$ and $\dot{r} = 0$. This would lead to the bead never leaving the origin because $r = 0$, $\dot{r} = 0$ and $\ddot{r} = 0$. So then equation (1) will just be zero too.

However, if we have $r_0 > 0$ then we know the solution to this differential equation is the familiar

$$r(t) = Ae^{\omega t} + Be^{-\omega t}$$

We can then use the initial conditions to find the constants A and B.

$$\begin{aligned}r_0 &= Ae^{\omega 0} + Be^{-\omega 0}\\ &= A + B\end{aligned}$$

We need another initial condition to be able to solve for A and B so we take the derivative since we know that $\dot{r}(0) = 0$.

$$\dot{r}(t) = A\omega Ae^{\omega t} + B\omega e^{-\omega t}$$

If we use the initial condition to solve for the constants we have

$$\begin{aligned}0 &= A\omega - B\omega\\ A &= B\end{aligned}$$

We can then use both results of the initial conditions to find

$$A = B = \frac{r_0}{2}$$

If we plug these constants back into our general solution we have our equation of motion to be.

$$\boxed{r(t) = \frac{r_0}{2}e^{\omega t} + \frac{r_0}{2}e^{-\omega t}}$$

By inspection we can see that this equation will eventually grow exponentially as the $e^{-\omega t}$ will go to zero and the $e^{\omega t}$ will dominate, growing exponentially.

This results corresponds to a centrifugal force because if you were in the inertial reference frame then $r = 0$ and $\dot{r} = 0$ so wouldn't seem to be moving. Which is just like the first case that we looked at. However, if outside the inertial reference frame then you find a radially outward force like in the second case.

7.24 From equation (7.54) we know that the Lagrangian of the Atwood machine is

$$\mathcal{L} = T - U = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_1 + m_2)gx$$

We can then apply the Euler-Lagrange equation to find generalized force and generalized momentum.

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0$$

If we start with the first term we have

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2}(m_1 + m_2)\dot{x}_2^{\,2} + (m_1 + m_2)gx\right)\\ &= (m_1 - m_2)g\end{aligned}$$

And then we can carry out the derivatives of the second term

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) &= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}(m_1 + m_2)x\dot{}^2 + (m_1 + m_2)gx\right)\right)\\ &= \frac{d}{dt}((m_1 + m_2)\dot{x})\\ &= (m_1 + m_2)\ddot{x}\end{aligned}$$

So we know that

$$\begin{gathered}\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)\\ \text{generalized force} = \text{rate of change of generalized momentum}\\ (m_1 - m_2)g = (m_1 + m_2)\ddot{x}\end{gathered}$$

So, we have found the expected result. We can then solve for $\ddot{x}$ to get

$$\boxed{\ddot{x} = \frac{(m_1 - m_2)g}{m_1 + m_2}}$$