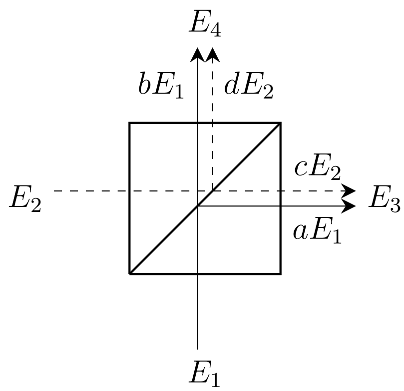
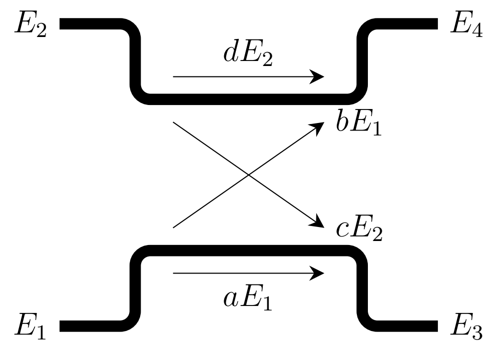
1 Beam Splitters

In the most general sense, a beam splitter can be written as relating two input fields to two output fields by the transformation

$$\begin{bmatrix} E_3 \\ E_4 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \tag{1}$$

where the incoming and outgoing fields are represented by the equivalent diagrams of a bulk optical beam splitter and an integrated optical directional coupler





Expanding these equations out and taking their magnitude gives

$$|E_3|^2 = |a|^2|E_1|^2 + |c|^2|E_2|^2 + ac^*E_1E_2^* + a^*cE_1^*E_2, \tag{2}$$
$$|E_4|^2 = |b|^2|E_1|^2 + |d|^2|E_2|^2 + bd^*E_1E_2^* + b^*dE_1^*E_2, \tag{3}$$

the sum of which is

$$|E_3|^2 + |E_4|^2 = (|a|^2 + |b|^2)|E_1|^2 + (|c|^2 + |d|^2)|E_2|^2 + 2\Re\{E_1E_2^*(ac^* + bd^*)\}. \tag{4}$$

For a lossless beamsplitter, it must be true that $|E_1|^2 + |E_2|^2 = |E_3|^2 + |E_4|^2$. When $E_2 = 0$, (4) reduces to

$$\begin{aligned} |E_3|^2 + |E_4|^2 &= (|a|^2 + |b|^2)|E_1|^2, \\ 1 &= |a|^2 + |b|^2, \end{aligned} \tag{5}$$

and when $E_1 = 0$, (4) reduces to

$$\begin{aligned} |E_3|^2 + |E_4|^2 &= (|c|^2 + |d|^2)|E_2|^2 \\ 1 &= |c|^2 + |d|^2. \end{aligned} \tag{6}$$

Applying these results to (4), and noting that in general E_1 and E_2 can take any value, gives the relation

$$\begin{aligned} |E_3|^2 + |E_4|^2 - |E_1|^2 - |E_2|^2 &= \Re\{E_1E_2^*(ac^* + bd^*)\}, \\ 0 &= ac^* + bd^* \end{aligned} \tag{7}$$

Now writing the coefficients with a magnitude and phase (e.g. $a = |a|e^{i\phi_a}$) gives the equation

$$\frac{|a||c|}{|b||d|} = -e^{i(-\phi_a+\phi_b+\phi_c-\phi_d)}, \tag{8}$$

which as the left side is strictly real and positive implies

$$-\phi_a + \phi_b + \phi_c - \phi_d = \pi + 2n\pi,\ n \in \mathbb{R}, \tag{9}$$

and the squared magnitude gives

$$\frac{|a|^2|c|^2}{|b|^2|d|^2} = 1. \tag{10}$$

Eliminating b and c using (5) and (6) gives

$$\begin{aligned} |a|^2(1-|d|^2) &= |d|^2(1-|a|^2), \\ |a| &= |d|, \end{aligned} \tag{11}$$

and therefore

$$|b| = |c|. \tag{12}$$

Using (11), (12), and (9), and pulling a constant phase factor $\phi_0 \triangleq (\phi_a + \phi_d)/2$ outside, the beamsplitter relation (1) becomes

$$\begin{bmatrix} E_3 \\ E_4 \end{bmatrix} = e^{i\phi_0} \begin{bmatrix} |a|e^{i(\phi_a-\phi_d)/2} & |b|e^{-i(\phi_b-\phi_c-(2n+1)\pi)/2} \\ |b|e^{i(\phi_b-\phi_c+(2n+1)\pi)/2} & |a|e^{-i(\phi_a-\phi_d)/2} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}. \tag{13}$$

By defining reflection and transmission phases $\phi_r \triangleq (\phi_a - \phi_d)/2$ and $\phi_t \triangleq (\phi_b - \phi_c + \pi + 2\pi n)/2$, this reduces to

$$\begin{bmatrix} E_3 \\ E_4 \end{bmatrix} = e^{i\phi_0} \begin{bmatrix} |a|e^{i\phi_r} & -|b|e^{-i\phi_t} \\ |b|e^{i\phi_t} & |a|e^{-i\phi_r} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}. \tag{14}$$

Defining the transmission coefficient $\kappa = |b|e^{i\phi_t}$ and the reflection coefficient $\sigma = |a|e^{i\phi_r}$, a typical convention when talking about directional couplers, this reduces to the general beamsplitter relationship

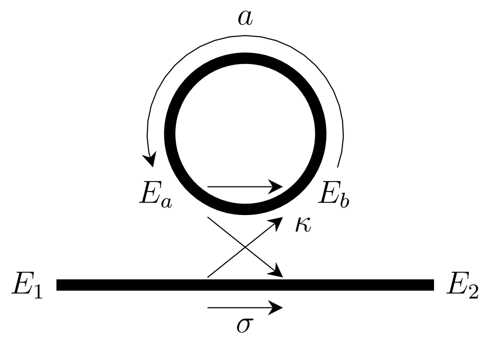
$$\begin{bmatrix} E_3 \\ E_4 \end{bmatrix} = e^{i\phi_0} \begin{bmatrix} \sigma & -\kappa^* \\ \kappa & \sigma^* \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \tag{15}$$

leaving free choice of the magnitude $|\sigma|$ as well as the phases $\angle\sigma$ and $\angle\kappa$ and the global phase ϕ_0 leading to different sources using different conventions for the form of this matrix.

2 Ring Resonator Transmission

A simple model of a ring resonator consists of a beamsplitter with one of the output paths routed back to one of the inputs, as represented by the diagram below, where the beamsplitter can be represented by the matrix

$$\begin{bmatrix} E_2 \\ E_b \end{bmatrix} = e^{i\phi_0} \begin{bmatrix} \sigma & -\kappa^* \\ \kappa & \sigma^* \end{bmatrix} \begin{bmatrix} E_1 \\ E_a \end{bmatrix}, \tag{16}$$



and the loss and phase in the ring can be represented by the coefficient a such that

$$E_a = aE_b. \tag{17}$$

The fields relative to the input can then be found as

$$\begin{aligned} E_b &= e^{i\phi_0}[\kappa E_1 + \sigma^* E_a], \\ E_b &= e^{i\phi_0}[\kappa E_1 + \sigma^* a E_b], \\ \frac{E_b}{E_1} &= \frac{\kappa}{e^{-i\phi_0} - \sigma^* a}, \end{aligned} \tag{18}$$

$$\begin{aligned} \frac{E_a}{E_1} &= a\frac{E_b}{E_1}, \\ \frac{E_a}{E_1} &= \frac{a\kappa}{e^{-i\phi_0} - \sigma^* a}, \end{aligned} \tag{19}$$

and

$$\begin{aligned} \frac{E_2}{E_1} &= e^{i\phi_0}\sigma - e^{i\phi_0}\kappa^* \frac{E_a}{E_1}, \\ \frac{E_2}{E_1} &= \frac{\sigma - ae^{i\phi_0}}{e^{-i\phi_0} - \sigma^* a}. \end{aligned} \tag{20}$$

The transmission through the ring can then be found as the ratio of ratio of $|E_2|^2$ to $|E_1|^2$

$$\begin{aligned} T = \frac{|E_2|^2}{|E_1|^2} &= \Big(\frac{\sigma - ae^{i\phi_0}}{e^{-i\phi_0} - \sigma^* a}\Big)\Big(\frac{\sigma^* - a^* e^{-i\phi_0}}{e^{i\phi_0} - \sigma a^*}\Big), \\ &= \frac{|\sigma|^2 + |a|^2 - 2|\sigma||a|\cos(\phi_0 + \phi_a - \phi_\sigma)}{1 + |\sigma|^2|a|^2 - 2|\sigma||a|\cos(\phi_0 + \phi_a - \phi_\sigma)}, \\ &= \frac{(|\sigma| + |a|)^2 - 4|\sigma||a|\cos^2((\phi_0 + \phi_a - \phi_\sigma)/2)}{(1 + |\sigma||a|)^2 - 4|\sigma||a|\cos^2((\phi_0 + \phi_a - \phi_\sigma)/2)}, \end{aligned} \tag{21}$$

where $\sigma = |\sigma|e^{i\phi_\sigma}$ and $a = |a|e^{i\phi_a}$. The global phase ϕ_0 can be freely chosen by changing the exact location of E_1 and E_2, but the phases ϕ_a and ϕ_σ can be determined by the ring cavity size, as

$$\phi_a - \phi_\sigma = kL, \tag{22}$$

where $k = 2\pi/\lambda$ is the wavenumber. Choosing $\phi_0 = 0$, the transmission then becomes

$$T = \frac{(|\sigma| + |a|)^2 - 4|\sigma||a|\cos^2(kL/2)}{(1 + |\sigma||a|)^2 - 4|\sigma||a|\cos^2(kL/2)}. \tag{23}$$

3 Linewidth

In a material with no gain, $|\sigma| \leq 1$ and $|a| \leq 1$, the numerator is strinctly smaller than the denominator, so the maximum value will occur when $\cos^2(kL/2) = 0$, and the minimum on resonance when $\cos^2(kL/2) = 1$, or

$$kL = 2\pi m, \ m \in \mathbb{Z}, \tag{24}$$

where $\mathbb{Z}$ is the set of integers. The resonances therefore have spacing (free spectral range)

$$\Delta k_{FSR} = \frac{2\pi}{L}. \tag{25}$$

Evaluating the transmission at these points gives minimum and maximum transmissions

$$T_{max} = \frac{(|\sigma| + |a|)^2}{(1 + |\sigma||a|)^2}, \tag{26}$$

$$T_{min} = \frac{(|\sigma| + |a|)^2 - 4|\sigma||a|}{(1 + |\sigma||a|)^2 - 4|\sigma||a|} = \frac{(|\sigma| - |a|)^2}{(1 - |\sigma||a|)^2}. \tag{27}$$

The extinction ratio will then be the ratio

$$ER = \frac{T_{max}}{T_{min}} = \frac{(1 - |\sigma||a|)^2(|\sigma| + |a|)^2}{(1 + |\sigma||a|)^2(|\sigma| - |a|)^2}. \tag{28}$$

As one resonance will occur at $k = 0$, the half width at half delta can be found as the k value which causes the transmission to rise to the average of the minumum and maximum

$$\frac{(|\sigma| + |a|)^2 - 4|\sigma||a|\cos^2(\Delta k_{HWHD}L/2)}{(1 + |\sigma||a|)^2 - 4|\sigma||a|\cos^2(\Delta k_{HWHD}L/2)} = \frac{1}{2}\left[\frac{(|\sigma| + |a|)^2}{(1 + |\sigma||a|)^2} + \frac{(|\sigma| - |a|)^2}{(1 - |\sigma||a|)^2}\right], \tag{29}$$

which can be rearranged to give

$$\cos^2(\Delta k_{HWHD}L/2) = \frac{(1 + |a||\sigma|)^2}{2(1 + |a|^2|\sigma|^2)}. \tag{30}$$

The full width at half delta can then be found as

$$\begin{aligned} \Delta k_{FWHD} &= 2\Delta k_{HWHD}, \\ &= \frac{4}{L}\arccos\left(\frac{1 + |a||\sigma|}{\sqrt{2(1 + |a|^2|\sigma|^2)}}\right), \\ &= \frac{2\Delta k_{FSR}}{\pi}\arccos\left(\frac{1 + |a||\sigma|}{\sqrt{2(1 + |a|^2|\sigma|^2)}}\right), \end{aligned} \tag{31}$$

or equivalently, using $\sin^2(x) + \cos^2(x) = 1$,

$$\Delta k_{FWHD} = \frac{2\Delta k_{FSR}}{\pi}\arcsin\left(\frac{1 - |a||\sigma|}{\sqrt{2(1 + |a|^2|\sigma|^2)}}\right), \tag{32}$$

and the linewidth in frequency can be approximated as to first order

$$\frac{\Delta\omega_{FWHD}}{\Delta\omega_{FSR}} \approx \frac{\Delta k_{FWHD}}{\Delta k_{FSR}} = \frac{2}{\pi}\arcsin\left(\frac{1 - |a||\sigma|}{\sqrt{2(1 + |a|^2|\sigma|^2)}}\right). \tag{33}$$

4 Quality Factors

The loaded quality factor (the quality factor of the entire structure) of resonance m is defined as

$$Q_{load,m} = \frac{\omega_m}{\Delta\omega_{FWHD}} = \frac{\pi}{2}\frac{\omega_m/\Delta\omega_{FSR}}{\arcsin\left(\frac{1-|a||\sigma|}{\sqrt{2(1+|a|^2|\sigma|^2)}}\right)}, \tag{34}$$

the extrinsic quality factor (excluding the ring) can be found by setting $|a| = 1$

$$Q_{ext,m} = \frac{\omega_m}{\Delta\omega_{FWHD}} = \frac{\pi}{2}\frac{\omega_m/\Delta\omega_{FSR}}{\arcsin\left(\frac{1-|\sigma|}{\sqrt{2(1+|\sigma|^2)}}\right)}, \tag{35}$$

and the intrinsic (ring) quality factor can not be directly measured, but can be approximated from the other two assuming the amplitude damping is exponential in time, i.e.

$$\frac{1}{Q_{load,m}} = \frac{1}{Q_{int,m}} + \frac{1}{Q_{ext,m}}, \tag{36}$$

which can be rearranged to give

$$Q_{int,m} = \frac{\pi}{2}\frac{\omega_m/\Delta\omega_{FSR}}{\arcsin\left(\frac{1-|a||\sigma|}{\sqrt{2(1+|a|^2|\sigma|^2)}}\right) - \arcsin\left(\frac{1-|\sigma|}{\sqrt{2(1+|\sigma|^2)}}\right)}. \tag{37}$$

The escape efficiency can also be calculated from the ratio

$$\eta_{esc} = \frac{Q_{load}}{Q_{ext}} = \frac{\arcsin\left(\frac{1-|\sigma|}{\sqrt{2(1+|\sigma|^2)}}\right)}{\arcsin\left(\frac{1-|a||\sigma|}{\sqrt{2(1+|a|^2|\sigma|^2)}}\right)} \tag{38}$$

5 Estimating a and σ

The parameters a and σ can be estimated from measureable parameters $\Delta\omega_{FWHD}$, $\Delta\omega_{FSR}$, and ER. By defining the parameters parameters X, Y, and Z as

$$X = \sin^2\left(\frac{\pi}{2}\frac{\Delta\omega_{FWHD}}{\Delta\omega_{FSR}}\right), \tag{39}$$

$$Y = |\sigma||a| = \frac{1-2\sqrt{X(1-X)}}{1-2X}, \tag{40}$$

$$Z = ER\frac{(1+Y)^2}{(1-Y)^2}, \tag{41}$$

where Y was solved for by rearranging (33) to give

$$X = \frac{(1-Y)^2}{2(1+Y)}, \tag{42}$$

which can be solved using the quadratic equation to give

$$Y = \frac{1 \pm 2\sqrt{X(1-X)}}{1-2X}, \tag{43}$$

and by noting that $Y \leq 1$, and therefore $X \leq 1/2$, only the negative branch will give a valid solution.

Now the parameters σ and a, here denoted p_1 and p_2 in some order, can be solved for by rearranging equation (28) as

$$Z = \frac{(p_1 + p_2)^2}{(p_1 - p_2)^2} \tag{44}$$

from which p_1 can be solved using the quadratic equation

$$\frac{p_1}{p_2} = \frac{(1 \pm \sqrt{Z})^2}{Z-1} \tag{45}$$

which can then be combined with (40), giving

$$\begin{aligned} \frac{Y p_1}{p_2} = p_{1,2}^2 &= Y\frac{(1 \pm \sqrt{Z})^2}{Z-1}, \\ p_{1,2} &= \pm\sqrt{Y}\frac{(1 \pm \sqrt{Z})}{\sqrt{Z-1}}, \\ p_{1,2} &= \sqrt{Y}\frac{(\sqrt{Z} \pm 1)}{\sqrt{Z-1}}, \end{aligned} \tag{46}$$

The smaller of a and σ will then correspond to the negative branch, and the larger will correspond to the positive. It is however impossible to distinguish which is which due to their degeneracy in equation (23).