Problem Set 1

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This assignment is designed to develop your skills with formal definitions and to provide some practice with inductive reasoning. Write your answers in the ps1.tex LaTeX template. You will submit your solutions in GradeScope as a PDF file with your answers to the questions in this template.

Collaboration Policy: You may discuss the problems with anyone you want. You are permitted to use any resources you find for this assignment **other than solutions from previous/concurrent CS3120 courses**. You should write up your own solutions and understand everything in them, and submit only your own work. You should note in the *Collaborators and Resources* box below the people you collaborated with and any external resources you used (you do not need to list resources you used for help with LaTeX).

Collaborators and Resources: *None*)

To do this assignment:

1. Visit https://www.overleaf.com and register for an Overleaf account (if you don't already have one). UVA has a site license to Overleaf, so if you register with your @virginia.edu email address you will have full access to all the Overleaf features for free.
2. After logging in, open this read-only Overleaf project located at https://www.overleaf.com/read/qjqncgcjmvgw#6a8554, then select the "Menu" button at the top-left, and then select "Copy Project". You will have an opportunity to rename the project, and then Overleaf will create a new copy of the project which you can edit.
3. Open your copy of the project and in the left side of the browser, you should see a file directory containing ps1.tex. Click on ps1.tex to see the LaTeX source for this file, and enter your solutions in the marked places. (You will also see the uvatoc.sty file, a "style" file that defines some useful macros. You are welcome to look at this file but should not need to modify it.)
4. The first thing you should do in ps1.tex is set up your name as the author of the submission by replacing the line, `\submitter{TODO: your name}`, with your name and UVA id, e.g., `\submitter{Haolin Liu (srs8rh)}`.
5. Write insightful and clear answers to all of the questions in the marked spaces provided.
6. Before submitting your ps1.pdf file, also remember to:
   - List your collaborators and resources, replacing the TODO in `\collaborators{TODO: replace ...}` with your collaborators and resources. (Remember to update this before submitting if you work with more people.)
   - Replace the second line in ps1.tex, `\usepackage{uvatoc}` with `\usepackage[response]{uvatoc}` so the directions do not appear in your final PDF.

Problem 1 *Induction Practice:* Prove that for any natural number $n \geq 1$, $2^n \leq \binom{2n}{n}$.

Note: For this problem (and any other problems where we don't explicitly state that you should use a particular definition), you can use the intuitive informal definition of natural numbers, and assume all of the familiar operations are defined and behave as expected.

Answer: *Proof.*

Prove $\forall n \in \mathbb{N}, 2^n \leq \binom{2n}{n}$ Proof by induction:

Base case:
when $n = 1$,
$2^{(1)} \leq \binom{2(1)}{(1)}$
$2 \leq 2$
when n = 2,
$2(2) \leq \binom{2(2)}{(2)}$
$4 \leq 6$
Inductive step:
assume P(n) holds $P(n+1) - > 2^{n+1} \leq \binom{2n}{n}$
because,
using pascals identity, we know that $\binom{n}{k}$ = $\binom{n-1}{k}$+ $\binom{n-1}{k-1}$
$2^{n+1} \leq \binom{2(n+1)}{n+1}$
$2(2^n) \leq \binom{2(n+1)}{n+1}$ = $\binom{2n}{n}$+$\binom{2n}{n+1}$
becomes,
$2(2^n) \leq 2$ $\binom{2n}{n}$
$2 \leq 2$ thus, $p(n+1)$ holds,
Therefore, by induction, 2^n $\leq\binom{2n}{n}$ $\forall n \in \mathbb{N}$ $\square$

1. **Problem 2** *Higher Induction Practice:* Prove that any binary tree of height h has at most 2^{h-1} leaves.

Note: We haven't defined a *binary tree* (and the book doesn't). An adequate answer to this question will use the informal understanding of a binary tree which we expect you have entering this class, but an excellent answer will include a definition of a binary tree and connect your proof to that definition.

Answer: *Proof.*

1.
2. proof by induction
3. base case: a tree of height 1 (just the root node), has at most one leaf node (again, just the root).
4. for h –> h+ 1
5. the number of leaf nodes doubles the number of leaf nodes for a binary tree of height h. Consider a tree, with all possible leaf node positions filled ie. complete. The height of this tree h has 2^{h-1} leaf nodes. Adding one more leaf node will result in a binary tree of height h+1. Therefore the the number of leaf node of a binary can be no more than 2^{h-1}.

□

Problem 3 *Addition is Commutative:* For this problem, we will use the successor definition of Natural Numbers (from Class 2) as follows.

Definition 1 (Natural Numbers) We define the *Natural Numbers* as:

1. **0** is a Natural Number.
2. If n is a Natural Number, **S**(n) is a Natural Number.

We will use this definition of addition:

Definition 2 (Sum) The *sum* of two Natural Numbers a and b (denoted as $a + b$) is defined as:

1. If a is **0**, then $a + b$ is b.
2. Otherwise, a is **S**(p) for some Natural Number p, and $a + b$ is **S**$(p + b)$.

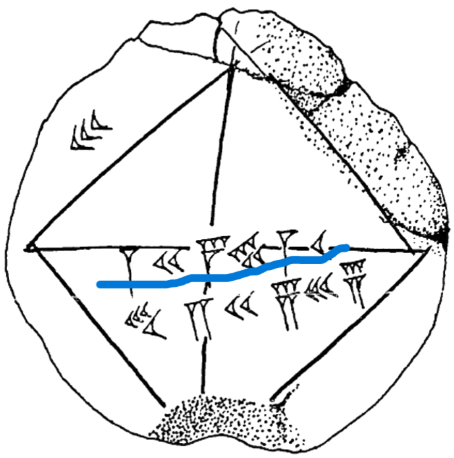
Prove that addition as defined above is *commutative*: for all Natural Numbers a and b, $a + b$ is equal to $b + a$. (we defined equality for our Natural Number representation in Class 2).

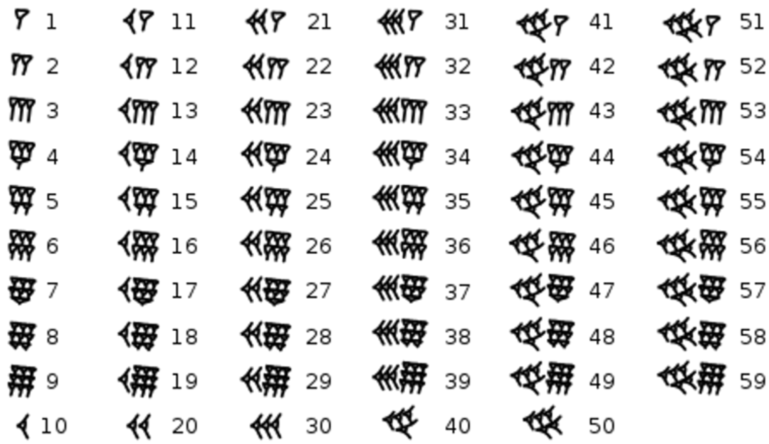
Answer: using the WOP make a the min(a,b) and b the max(b,a) induct a –> a+1 until a = 0 in which case the sum of a and b = b equality proven

using the WOP make a the max(a,b) and b the min(b,a) induct b –> b+1 until b = 0 in which case the sum of a and b = a equality proven

since equality holds for both orderings of a,b we can conclude that the Sum definition is commutative $\forall n \in \mathbb{N}$

Problem 4 *Babylonian Pebble.*





In Class 2, we mentioned that on the above pebble, the middle Babylonian number (with blue underline) is an approximation of $\sqrt{2} \approx 1.414213$.

1. Interpret and calculate the other two numbers on the pebble.
2. Explain the potential meaning of this pebble.

Clearly, the Babylonian can not tell us their real purpose, and we modern people can only guess with proper evidence. It is not hard to search for an answer. Please write down your answer before searching! That would be more fun.

Answer:

1. the numbers are 42,25,35
2. as a base 60 number, $42(60^0) + 25(60^-1) + 35(60^-2)$ = 42.426388...

   this quantity relates to the number in the upper left of the pebble marking the leg of the triangle (30)

   It is likely that the Babylonians used a measure of 30 units as the leg of the right triangle with a 42.426388 unit hypotenuse to find the quantity of $\sqrt{2}$, for which, the Babylonians would have great use in construction and other fields dependent on precise geometry.
3. $\frac{42.426388}{30} = 1.414212933 \approx \sqrt{2}$

Problem 5 *Claim: All cows have the same number of spots.*

From our experience (from farms or videos or other), the claim is false.[1] However, here is a proof by induction.

Proof.

1. Predicate $P(n)$: any $n + 1$ cows have the same number of spots.
2. Base case, $P(0)$: any 1 cow have the same number of spots. This holds because there is only 1 cow, and its number of spots is the same.
3. Inductive case, let $n \in \mathbb{N}$.
   (a) By induction hypothesis, suppose $P(n)$ holds. That is, any $n + 1$ cows have the same number of spots.
   (b) Consider any $n + 2$ cows. Let $c_1, c_2, ..., c_{n+2}$ be (the names of) the cows.
   (c) By $P(n)$, the first $n + 1$ cows, $c_1, c_2, ..., c_{n+1}$, have the same number of spots. Let x be the number.
   (d) By $P(n)$, the last $n + 1$ cows, $c_2, ..., c_{n+2}$, have the same number of spots. Let y be the number.
   (e) Let z be the number of spots of the cow c_2.
   (f) We have $x = z$ because c_2 is in the first $n + 1$.
   (g) We have $y = z$ because c_2 is in the last $n + 1$.
   (h) By $x = y = z$, all $n + 2$ cows have the same number of spots. That is, $P(n + 1)$ holds.

□

Please briefly explain why the proof is wrong (maybe by pointing out some incorrect steps). Notice there are many concepts and symbols that are not defined in class, such as $x = z$ and $y = z$ implies $x = y = z$. Please interpret them as standard math concepts you learned from other courses, such as DMT1.

Answer:

I believe the mistake of this proof happens in step *d and by extension step* g. The "last n + 1" cows would include c_{n+2} which is part of the inductive step. This leads to a circular proof.

[1] https://thedairyalliance.com/blog/udderly-unusual-cow-facts

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This is the end of the problems for PS1. Remember to follow the last step in the directions on the first page to prepare your PDF for submission.

