MATH111852021-2SS1SEM2 IDA Assignment 2

Johnny Lee

TOTAL POINTS

89 / 100

QUESTION 1

Question 1 23 pts

1.1 Question 1 a) 5 / 7

- **0 pts** Correct
- **7 pts** Wrong likelihood
- ✓ - **2 pts** **Not checking the second order condition (i.e. second derivative evaluated at the MLE < 0).**
- **1 pts** No reference to the fact that derivative needs to be equal to 0 so to obtain the MLE
- **2 pts** Minor issues with derivations / justification
- **3 pts** Major issues with derivations / justification
- **4 pts** Fundamental issues with derivations / justification

1.2 Question 1 b) 8 / 13

- **0 pts** Correct
- ✓ - **3 pts** **Not providing enough details about properties used (ex: law of total expectation, or simply writing E(RX^2) = int_0^C without almost no details or justification.)**
- ✓ - **2 pts** **Not mentioning that C is a constant when computing expectations**
- **3 pts** Minor issues with derivations / justification
- **6 pts** Major issues with derivations / justification
- **9 pts** Fundamental issues with derivations / justification

1.3 Question 1 c) 3 / 3

- ✓ - **0 pts** **Correct**
- **1 pts** Presenting Fisher information evaluated at a general theta (rather than at the estimate).
- **1 pts** Issues with justification / interval
- **2 pts** Major issues with justification / interval provided
- **3 pts** Fundamental issues with justification / interval provided
- **3 pts** No attempt to solve

QUESTION 2

Question 2 12 pts

2.1 Question 2 a) 6 / 6

- ✓ - **0 pts** **Correct**
- **1 pts** Very minor issue with derivation / justification
- **2 pts** Minor issues with derivation / justification
- **3 pts** Major issues with derivation / justification
- **6 pts** Fundamental issues with derivation / justification

2.2 Question 2 b) 6 / 6

- ✓ - **0 pts** **Correct**
- **2 pts** Minor issues with code
- **3 pts** Major issues with code
- **5 pts** Optimizing over both mu and sigma
- **6 pts** Fundamental issues with code
- **3 pts** Understanding sigma^2 as sigma

QUESTION 3

Question 3 15 pts

3.1 Question 3 a) 5 / 5

- ✓ - **0 pts** **Correct**
- **2 pts** Minor issues with justification (despite correct answer on ignorability)
- **3 pts** Major issues with justification (despite correct answer on ignorability)
- **4 pts** Fundamental issues with justification (despite correct answer on ignorability)
- **5 pts** Wrong answer on ignorability

3.2 Question 3 b) 1 / 5

- **0 pts** Correct

✓ - **4 pts** **Fundamental issues (e.g. not noticing that data can be MCAR if psi_1 = 0).**

- **4 pts** Other fundamental issues
- **1 pts** Minor issues
- **5 pts** Wrong answer.

1 Only if psi_1 \neq 0!

3.3 Question 3 c) 5 / 5

✓ - **0 pts** **Correct**

- **1 pts** Minor issues with justification (despite correct answer on ignorability)
- **3 pts** Major issues with justification (despite correct answer on ignorability)
- **4 pts** Fundamental issues with justification (despite correct answer on ignorability)
- **5 pts** Wrong answer.

QUESTION 4

4 Question 4 25 / 25

✓ - **0 pts** **Correct**

- **15 pts** Wrong E-step
- **20 pts** Wrong likelihood
- **5 pts** Minor issues with E-step or M-step
- **10 pts** Major issues with E-step or M-step
- **15 pts** Fundamental issues with E-step or M-step
- **20 pts** Not applying EM
- **5 pts** Minor issues with justifications / derivations
- **10 pts** Major issues with justifications / derivations
- **15 pts** Fundamental issues with justifications / derivations
- **20 pts** Just presenting code
- **0 pts** Correct

QUESTION 5

Question 5 25 pts

5.1 Question 5 a) 13 / 13

✓ - **0 pts** **Correct**

- **8 pts** Wrong Q function
- **5 pts** Wrong updating rule obtained from M-step
- **3 pts** Minor issues with justifications / derivations
- **7 pts** Major issues with justifications / derivations
- **10 pts** Fundamental issues with justifications / derivations

5.2 Question 5 b) 12 / 12

✓ - **0 pts** **Correct**

- **3 pts** Minor issues with justifications / code
- **7 pts** Major issues with justifications / code
- **10 pts** Fundamental issues with justifications / code
- **4 pts** Not presenting histogram / density estimator

IDA Assignment 2

Johnny Lee, s1687781

25th March 2022

Q1.

Suppose $Y_1, \cdots, Y_n$ are independent and identically distributed with cumulative distribution function given by

$$F(y;\theta) = 1 - e^{-y^2/(2\theta)}, \quad y \geq 0, \quad \theta > 0.$$

Further suppose that observations are (right) censored if $Y_i > C$, for some known $C > 0$, and let

$$X_i = \begin{cases} Y_i & \text{if} \quad Y_i \leq C, \\ C & \text{if} \quad Y_i > C, \end{cases} \qquad R_i = \begin{cases} 1 & \text{if} \quad Y_i \leq C \\ 0 & \text{if} \quad Y_i > C \end{cases}$$

a)

Show that the maximum likelihood estimator based on the observed data $\{(x_i, r_i)\}_{i=1}^n$ is given by

$$\hat{\theta} = \frac{\sum_{i=1}^n X_i^2}{2\sum_{i=1}^n R_i}.$$

Answer :

We first define the Survival function (from **Workshop 3**)as

$$S(C;\theta) = \mathbb{P}(Y_i > C;\theta) = 1 - F(y_i;\theta)$$

which also represents the censored observations. For the uncensored observations, we have

$$f(y_i;\theta) = \frac{d}{dy_i}F(y_i;\theta) = \frac{ye^{-y^2/2\theta}}{\theta}$$

Given that $Y_1, \ldots, Y_n$ are independent and identically distributed, we have the likelihood function as,

$$\begin{aligned} L(\theta|\boldsymbol{y}, \boldsymbol{r}) &= \prod_{i=1}^n \left([f(y_i;\theta)]^{r_i}[S(C;\theta)]^{1-r_i} \right) \\ &= \prod_{i=1}^n \left([\frac{y_i e^{-y_i^2/2\theta}}{\theta}]^{r_i}[e^{-C^2/\theta}]^{1-r_i} \right) \\ &= \left(\prod_{i=1}^n y_i^{r_i} \right) \left(\frac{1}{\theta} \right)^{\sum_{i=1}^n r_i} \exp\left(\frac{\sum_{i=1}^n (r_i y_i^2 + (1-r_i)C^2)}{2\theta} \right) \end{aligned} \tag{1}$$

Now we can rewrite the expression inside the exponential in terms of X_i. X_i can be expressed as

$$x_i = r_i y_i + C(1 - r_i)$$

Then by taking square on both sides we have,

$$x_i^2 = r_i^2 y_i^2 + (1-r_i)^2 C^2 + 2r_i y_i C(1-r_i)$$

Noting that R_i is binary, we can then conclude with the expression as

$$x_i^2 = r_i y_i^2 + (1-r_i)C^2 \tag{2}$$

Now we substitute (2) into (1) to have,

$$\begin{aligned}
L(\theta|\boldsymbol{y},\boldsymbol{r}) &= \left(\prod_{i=1}^{n} y_i^{r_i}\right)\left(\frac{1}{\theta}\right)^{\sum_{i=1}^{n} r_i} \exp\left(\frac{\sum_{i=1}^{n}(r_i y_i^2 + (1-r_i)C^2)}{2\theta}\right) \\
\implies \log(L(\theta|\boldsymbol{y},\boldsymbol{r})) &= \sum_{i=1}^{n} r_i \log(y_i) - \log\theta \sum_{i=1}^{n} r_i - \frac{\sum_{i=1}^{n} x_i^2}{2\theta} \qquad (3) \\
\implies \frac{d}{d\theta}\log L(\theta|\boldsymbol{y},\boldsymbol{r}) &= \frac{1}{\theta}\sum_{i=1}^{n} r_i + \frac{1}{2\theta^2}\sum_{i=1}^{n} x_i^2
\end{aligned}$$

By equating the derivative (3) to 0, we can obtain the maximum likelihood estimate of θ as below.

$$\hat{\theta}_{MLE} = \frac{\sum_{i=1}^{n} x_i^2}{2\sum_{i=1}^{n} r_i} \quad \text{(shown)}$$

.

1.1 Question 1 a) **5 / 7**

- **- 0 pts** Correct
- **- 7 pts** Wrong likelihood
- ✓ **- 2 pts Not checking the second order condition (i.e. second derivative evaluated at the MLE < 0).**
- **- 1 pts** No reference to the fact that derivative needs to be equal to 0 so to obtain the MLE
- **- 2 pts** Minor issues with derivations / justification
- **- 3 pts** Major issues with derivations / justification
- **- 4 pts** Fundamental issues with derivations / justification

b)

Show that the expected Fisher Information for the observed data likelihood is

$$I(\theta) = \frac{n}{\theta^2}(1 - e^{-C^2/(2\theta)})$$

Note: $\int_0^C y^2 f(y;\theta)dy = -C^2 e^{-C^2/(2\theta)} + 2\theta(1 - e^{-C^2/(2\theta)})$, where $f(y;\theta)$ is the density function corresponding to the cumulative distribution function $F(y;\theta)$ defined above.

Answer :

From (3), we take another derivative of it and thus obtain as below

$$\frac{d^2}{d\theta^2} \log L(\theta) = \frac{1}{\theta^2} \sum_{i=1}^{n} r_i - \frac{x_i^2}{\theta^3}$$

Then, the Fisher Information for the observed data likelihood is,

$$\begin{aligned} I(\theta) &= -\mathbb{E}\left(\frac{\sum_{i=1}^n r_i}{\theta^2} - \frac{x_i^2}{\theta^3} \right) \\ &= -\frac{n\mathbb{E}(R)}{\theta^2} + \frac{n\mathbb{E}(X^2)}{\theta^3} \\ &= -\frac{n\mathbb{E}(R)}{\theta^2} + \frac{1}{\theta^3}\left(n\mathbb{E}(RY^2) + nC^2\mathbb{E}(1-R) \right) \end{aligned} \tag{4}$$

Again, noting that R_i is binary,

$$\begin{aligned} \mathbb{E}(R) &= 1 \cdot \mathbb{P}(R=1) + 0 \cdot \mathbb{P}(R=0) \\ &= \mathbb{P}(R=1) = \mathbb{P}(Y \leq C) \\ &= F(C;\theta) = 1 - e^{-C^2/2\theta} \\ \implies \mathbb{E}(1-R) &= e^{-C^2/2\theta} \end{aligned} \tag{5}$$

With the given equation, $\mathbb{E}(RY^2) = \int_0^C y^2 f(y;\theta)dy = -C^2 e^{-C^2/(2\theta)} + 2\theta(1 - e^{-C^2/(2\theta)})$, we can combine all the above equations as express the expected Fisher Information again,

$$\begin{aligned} I(\theta) &= \frac{n\mathbb{E}(R)}{\theta^2} + \frac{1}{\theta^3}\left(n\mathbb{E}(RY^2) + nC^2\mathbb{E}(1-R) \right) \\ &= \frac{-n}{\theta^2}(1 - e^{-C^2/2\theta}) - \frac{n}{\theta^3}(C^2 e^{-C^2/2\theta}) + \frac{n}{\theta^3}(2\theta(1 - e^{-C^2/2\theta})) + \frac{n}{\theta^3}(C^2 e^{-C^2/2\theta}) \\ &= \frac{n}{\theta^2}(1 - e^{-C^2/2\theta}) \quad \text{(shown)} \end{aligned} \tag{6}$$

1.2 Question 1 b) **8 / 13**

- **- 0 pts** Correct

✓ **- 3 pts Not providing enough details about properties used (ex: law of total expectation, or simply writing E(RX^2) = int_0^C without almost no details or justification.)**

✓ **- 2 pts Not mentioning that C is a constant when computing expectations**

- **- 3 pts** Minor issues with derivations / justification
- **- 6 pts** Major issues with derivations / justification
- **- 9 pts** Fundamental issues with derivations / justification

c)

Appealing to the asymptotic normality of the maximum likelihood estimator, provide a 95% confidence interval for θ.

Answer :

By the Central Limit Theorem, asymptotic normality of the maximum likelihood estimator is given as,

$$\hat{\theta}_{MLE} \sim N_p(\theta, I(\theta)^{-1})$$

Thus, with 0 and $\frac{1}{I(\theta)}$ as the asymptotic mean and variance respectively, we can obtain the 95% confidence interval as below,

$$\hat{\theta}_{MLE} \pm \frac{1.96}{\sqrt{I(\theta)}} = \hat{\theta}_{MLE} \pm \frac{1.96 \cdot \theta_{MLE}}{\sqrt{n(1 - e^{-C^2/2\theta_{MLE}})}}$$

1.3 Question 1 c) 3 / 3

✓ **- 0 pts** **Correct**

- 1 pts Presenting Fisher information evaluated at a general theta (rather than at the estimate).

- 1 pts Issues with justification / interval

- 2 pts Major issues with justification / interval provided

- 3 pts Fundamental issues with justification / interval provided

- 3 pts No attempt to solve

Q2.

Suppose that a dataset consists of 100 subjects and 10 variables. Each variable contains 10% of missing values. What is the largest possible subsample under a complete case analysis? What is the smallest? Justify.

Suppose that $Y_i \overset{\text{iid}}{\sim} N(\mu, \sigma^2)$ are iid for $i = 1, ..., n$. Further suppose that now observations are (left) censored if $Y_i < D$, for some known D and let

$$X_i = \begin{cases} Y_i & \text{if } Y_i \geq D, \\ D & \text{if } Y_i < D, \end{cases} \qquad R_i = \begin{cases} 1 & \text{if } Y_i \geq D \\ 0 & \text{if } Y_i < D \end{cases}$$

a)

Show that the log-likelihood of the observed data $\{(x_i, r_i)\}_{i=1}^n$ is given by

$$\log L(\mu, \sigma^2 | \boldsymbol{x}, \boldsymbol{r}) = \sum_{i=1}^{n} \left\{ r_i \log \phi(x_i; \mu, \sigma^2) + (1 - r_i) \log \Phi(x_i; \mu, \sigma^2) \right\}$$

where $\phi(\cdot; \mu, \sigma^2)$ and $\Phi(\cdot; \mu, \sigma^2)$ stands, respectively, for the density function and cumulative distribution function of the normal distribution with mean μ and variance σ^2.

Answer :

We first define the case for censored observations (from **Workshop 3**)as

$$\mathbb{P}(Y_i < D; \mu, \sigma^2) = F(D; \mu, \sigma^2) = \Phi(x_i; \mu, \sigma^2)$$

For the uncensored observation, it is the density function and we have

$$\phi(x_i; \mu, \sigma^2)$$

Given that $X_1, \ldots, X_n$ are independent and identically distributed, we have the likelihood function as,

$$\begin{aligned} L(\mu, \sigma^2 | \boldsymbol{x}, \boldsymbol{r}) &= \prod_{i=1}^{n} \left([\phi(x_i; \mu, \sigma^2)]^{r_i} [\Phi(x_i; \mu, \sigma^2)]^{1-r_i} \right) \\ \implies l(\mu, \sigma^2 | \boldsymbol{x}, \boldsymbol{r}) &= \log \prod_{i=1}^{n} \left(\phi(x_i; \mu, \sigma^2)]^{r_i} [\Phi(x_i; \mu, \sigma^2)]^{1-r_i} \right) \qquad (7) \\ &= \sum_{i=1}^{n} \left(r_i \log \phi(x_i; \mu, \sigma^2) + (1 - r_i) \log \Phi(x_i; \mu, \sigma^2) \right) \quad \text{(shown)} \end{aligned}$$

2.1 Question 2 a) 6 / 6

✓ **- 0 pts** **Correct**

- 1 pts Very minor issue with derivation / justification

- 2 pts Minor issues with derivation / justification

- 3 pts Major issues with derivation / justification

- 6 pts Fundamental issues with derivation / justification

b)

Determine the maximum likelihood estimate of μ based on the data available in the file `dataex2.Rdata`. Consider σ^2 known and equal to 1.5^2. **Note**: You can use a built in function such as `optim` or the `maxLik` package in your implementation.

Answer :

```
#defining a function to simulate the log likelihood
log.lik <- function(mu, data){
  x <- data[, 1]
  r <- data[, 2]
  sum((r*dnorm(x, mu, 1.5, log = TRUE) +
         (1-r)*pnorm(x, mu, 1.5, log = TRUE)))
}

#computing the maximum likelihood estimate of mu
mle <- maxLik(logLik = log.lik, data = dataex2, start = c(mu = mean(dataex2$X)))
summary(mle)
```

```
## --------------------------------------------
## Maximum Likelihood estimation
## Newton-Raphson maximisation, 2 iterations
## Return code 8: successive function values within relative tolerance limit (reltol)
## Log-Likelihood: -336.3821
## 1  free parameters
## Estimates:
##    Estimate Std. error t value Pr(> t)
## mu   5.5328     0.1075   51.48  <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
## --------------------------------------------
```

We built a function `log.lik()` that produces the log likelihood and then used `maxLik()` to simulate μ based on the data. With Newton-Raphson method, we estimated $\hat{\mu} = 5.5328$ and standard error of 0.1075

2.2 Question 2 b) **6 / 6**

✓ **- 0 pts** Correct

- 2 pts Minor issues with code

- 3 pts Major issues with code

- 5 pts Optimizing over both mu and sigma

- 6 pts Fundamental issues with code

- 3 pts Understanding sigma^2 as sigma

Q3.

Consider a bivariate normal sample (Y_1, Y_2) with parameters $\theta = (\mu_1, \mu_2, \sigma_1^2, \sigma_{12}, \sigma_2^2)$. The variable Y_1 is fully observed, while some values of Y_2 are missing. Let R be the missingness indicator, taking the value 1 for observed values and 0 for missing values. For the following missing data mechanisms state, justifying, whether they are ignorable for likelihood-based estimation.

a)

$\text{logit}\{\mathbb{P}(R = 0|y_1, y_2, \theta, \psi)\} = \psi_0 + \psi_1 y_1, \quad \psi = (\psi_0, \psi_1)$ distinct from θ.

Answer :

Referring to the ignorability assumption (from **Lecture 6.1**), the missing in Y_2 is either **MAR** or **MCAR** and its model parameters, $\theta = (\mu_1, \mu_2, \sigma_1^2, \sigma_{12}, \sigma_2^2)$ and missing mechanism parameter, ψ.

First, the missing mechanism is **MAR**. This is because the missingness is only dependent on Y_1 which is a fully observed variable. The parameters, $\{\theta, \psi\}$ are also distinct. Therefore, the ignorability assumption holds here and (a) is ignorable for likelihood-based estimation.

b)

$\text{logit}\{\mathbb{P}(R = 0|y_1, y_2, \theta, \psi)\} = \psi_0 + \psi_1 y_2, \quad \psi = (\psi_0, \psi_1)$ distinct from θ.

Answer :

The missing mechanism is **MNAR** as the mechanism is only dependent on Y_2. Therefore, the missing value is depending on itself and possibly other factors. Hence, by referring to the ignorability assumption (from **Lecture 6.1**), we conclude that (b) is not ignorable for likelihood-based estimation.

c)

$\text{logit}\{\mathbb{P}(R = 0|y_1, y_2, \theta, \psi)\} = 0.5(\mu_1 + \psi y_1)$, scalar ψ distinct from θ.

Answer :

The missing mechanism here is dependent on both μ_1 and Y_1 thus **MAR**. We can observe similarity to (a). Distinctness of the parameters means that the parameter space of $\{\theta, \psi\}$ is equal to the Cartesian product of their individual product spaces. However, the μ_1 also exists in the parameter space. This violates the ignorability assumption. Hence, (c) is not ignorable for likelihood-based estimation.

3.1 Question 3 a) **5 / 5**

- ✓ **- 0 pts** **Correct**
- **- 2 pts** Minor issues with justification (despite correct answer on ignorability)
- **- 3 pts** Major issues with justification (despite correct answer on ignorability)
- **- 4 pts** Fundamental issues with justification (despite correct answer on ignorability)
- **- 5 pts** Wrong answer on ignorability

Q3.

Consider a bivariate normal sample (Y_1, Y_2) with parameters $\theta = (\mu_1, \mu_2, \sigma_1^2, \sigma_{12}, \sigma_2^2)$. The variable Y_1 is fully observed, while some values of Y_2 are missing. Let R be the missingness indicator, taking the value 1 for observed values and 0 for missing values. For the following missing data mechanisms state, justifying, whether they are ignorable for likelihood-based estimation.

a)

$\text{logit}\{\mathbb{P}(R = 0|y_1, y_2, \theta, \psi)\} = \psi_0 + \psi_1 y_1, \quad \psi = (\psi_0, \psi_1)$ distinct from θ.

Answer :

Referring to the ignorability assumption (from **Lecture 6.1**), the missing in Y_2 is either **MAR** or **MCAR** and its model parameters, $\theta = (\mu_1, \mu_2, \sigma_1^2, \sigma_{12}, \sigma_2^2)$ and missing mechanism parameter, ψ.

First, the missing mechanism is **MAR**. This is because the missingness is only dependent on Y_1 which is a fully observed variable. The parameters, $\{\theta, \psi\}$ are also distinct. Therefore, the ignorability assumption holds here and (a) is ignorable for likelihood-based estimation.

b)

$\text{logit}\{\mathbb{P}(R = 0|y_1, y_2, \theta, \psi)\} = \psi_0 + \psi_1 y_2, \quad \psi = (\psi_0, \psi_1)$ distinct from θ.

Answer :

The missing mechanism is **MNAR** as the mechanism is only dependent on Y_2. Therefore, the missing value is depending on itself and possibly other factors. Hence, by referring to the ignorability assumption (from **Lecture 6.1**), we conclude that (b) is not ignorable for likelihood-based estimation.

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Answer :

The missing mechanism here is dependent on both μ_1 and Y_1 thus **MAR**. We can observe similarity to (a). Distinctness of the parameters means that the parameter space of $\{\theta, \psi\}$ is equal to the Cartesian product of their individual product spaces. However, the μ_1 also exists in the parameter space. This violates the ignorability assumption. Hence, (c) is not ignorable for likelihood-based estimation.

3.2 Question 3 b) **1 / 5**

- **- 0 pts** Correct
- ✓ **- 4 pts Fundamental issues (e.g. not noticing that data can be MCAR if psi_1 = 0).**
- **- 4 pts** Other fundamental issues
- **- 1 pts** Minor issues
- **- 5 pts** Wrong answer.

1 Only if psi_1 \neq 0!

Q3.

Consider a bivariate normal sample (Y_1, Y_2) with parameters $\theta = (\mu_1, \mu_2, \sigma_1^2, \sigma_{12}, \sigma_2^2)$. The variable Y_1 is fully observed, while some values of Y_2 are missing. Let R be the missingness indicator, taking the value 1 for observed values and 0 for missing values. For the following missing data mechanisms state, justifying, whether they are ignorable for likelihood-based estimation.

a)

$\text{logit}\{\mathbb{P}(R = 0 | y_1, y_2, \theta, \psi)\} = \psi_0 + \psi_1 y_1, \quad \psi = (\psi_0, \psi_1)$ distinct from θ.

Answer :

Referring to the ignorability assumption (from **Lecture 6.1**), the missing in Y_2 is either **MAR** or **MCAR** and its model parameters, $\theta = (\mu_1, \mu_2, \sigma_1^2, \sigma_{12}, \sigma_2^2)$ and missing mechanism parameter, ψ.

First, the missing mechanism is **MAR**. This is because the missingness is only dependent on Y_1 which is a fully observed variable. The parameters, $\{\theta, \psi\}$ are also distinct. Therefore, the ignorability assumption holds here and (a) is ignorable for likelihood-based estimation.

b)

$\text{logit}\{\mathbb{P}(R = 0 | y_1, y_2, \theta, \psi)\} = \psi_0 + \psi_1 y_2, \quad \psi = (\psi_0, \psi_1)$ distinct from θ.

Answer :

The missing mechanism is **MNAR** as the mechanism is only dependent on Y_2. Therefore, the missing value is depending on itself and possibly other factors. Hence, by referring to the ignorability assumption (from **Lecture 6.1**), we conclude that (b) is not ignorable for likelihood-based estimation.

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$\text{logit}\{\mathbb{P}(R = 0 | y_1, y_2, \theta, \psi)\} = 0.5(\mu_1 + \psi y_1)$, scalar ψ distinct from θ.

Answer :

The missing mechanism here is dependent on both μ_1 and Y_1 thus **MAR**. We can observe similarity to (a). Distinctness of the parameters means that the parameter space of $\{\theta, \psi\}$ is equal to the Cartesian product of their individual product spaces. However, the μ_1 also exists in the parameter space. This violates the ignorability assumption. Hence, (c) is not ignorable for likelihood-based estimation.

3.3 Question 3 c) 5 / 5

✓ **- 0 pts** **Correct**

- 1 pts Minor issues with justification (despite correct answer on ignorability)

- 3 pts Major issues with justification (despite correct answer on ignorability)

- 4 pts Fundamental issues with justification (despite correct answer on ignorability)

- 5 pts Wrong answer.

Q4.

$$Y_i \overset{\text{ind.}}{\sim} \text{Bernoulli}(p_i(\boldsymbol{\beta}))$$

$$p_i(\boldsymbol{\beta}) = \frac{exp(\beta_0 + x_i\beta_1)}{1 + exp(\beta_0 + x_i\beta_1)},$$

for $i = 1, \cdots, n$ and $\boldsymbol{\beta} = (\beta_0, \beta_1)'$. Although the covariate x is fully observed, the response variable Y has missing values. Assuming ignorability, derive and implement the EM algorithm to compute the MLE of $\boldsymbol{\beta}$ based on the data available in `dataex4.Rdata`. **Note**: 1) For simplicity, and without loss of generality because we have a univariate pattern of missingness, when writing down your expressions, you can assume that the first m values of Y are observed and the remaining $n - m$ are missing. 2) You can use a built in function such as `optim` or the `maxLik` package for the M-step.

Answer :

```
head(dataex4)
```

```
##            X  Y
## 1 -0.4689827  1
## 2 -0.2557522  1
## 3  0.1457067  1
## 4  0.8164156 NA
## 5 -0.5966361  1
## 6  0.7967794 NA
```

```
cat("Number of missing values in Y:", sum(is.na(dataex4)))
```

```
## Number of missing values in Y: 95
```

Scrutinising on the dataset, we can observe that the missing value only occurs in Y and there are 95 missing values occurring in a univariate pattern

We first derive the likelihood function to implement the EM algorithm given that $\boldsymbol{y_{obs}} = y_1, \ldots, y_m$ and $\boldsymbol{y_{mis}} = y_{m+1}, \ldots, y_n$.

$$\begin{aligned}
L(\beta_0, \beta_1 | \boldsymbol{x}, \boldsymbol{y_{obs}}, \boldsymbol{y_{mis}}) &= \prod_{i=1}^{n} \left([p_i(\beta_0, \beta_1)]^{y_i} [1 - p(\beta_0, \beta_1)]^{1-y_i} \right) \\
&= \prod_{i=1}^{n} \left(\frac{e^{\beta_0 + x_i\beta_1}}{1 + e^{\beta_0 + x_i\beta_1}} \right)^{y_i} \left(\frac{1}{1 + e^{\beta_0 + x_i\beta_1}} \right)^{1-y_i} \\
\implies \log L(\beta_0, \beta_1 | \boldsymbol{x}, \boldsymbol{y_{obs}}, \boldsymbol{y_{mis}}) &= \sum_{i=1}^{n} \left(y_i \log \left(\frac{e^{\beta_0 + x_i\beta_1}}{1 + e^{\beta_0 + x_i\beta_1}} \right) + (1 - y_i) \log \left(\frac{1}{1 + e^{\beta_0 + x_i\beta_1}} \right) \right) \\
&= \sum_{i=1}^{n} \left(y_i \log(e^{\beta_0 + x_i\beta_1}) - \log(1 + e^{\beta_0 + x_i\beta_1}) - y_i \log(1 + e^{\beta + x_i\beta_1}) + y_i \log(1 + e^{\beta + x_i\beta_1}) \right) \\
&= \sum_{i=1}^{n} \left(y_i(\beta_0 + x_i\beta_1) - \log(1 + e^{\beta_0 + x_i\beta_1}) \right) \\
&= l(\boldsymbol{\beta} | \boldsymbol{x}, \boldsymbol{y_{obs}}, \boldsymbol{y_{mis}})
\end{aligned} \tag{8}$$

Now we proceed to implement the EM algorithm by calculating $Q(\boldsymbol{\beta}|\boldsymbol{\beta^{(t)}})$

$$
\begin{aligned}
Q(\boldsymbol{\beta}|\boldsymbol{\beta^{(t)}}) &= \mathbb{E}_{\boldsymbol{y_{mis}}}[l(\boldsymbol{\beta}|\boldsymbol{x}, \boldsymbol{y_{obs}}, \boldsymbol{y_{mis}})|\boldsymbol{y_{obs}}, \boldsymbol{x}, \boldsymbol{\beta^{(t)}}] \\
&= \sum_{i=1}^{m}\Big(y_i(\beta_0 + x_i\beta_i)\Big) - \sum_{i=1}^{n}\Big(\log(1+e^{\beta_0+x_i\beta_1})\Big) + \sum_{i=m+1}^{n}\Big((\beta_0 + x_i\beta_1)\mathbb{E}_{\boldsymbol{y_{mis}}}[y_i|\boldsymbol{x}, \boldsymbol{y_{obs}}, \boldsymbol{\beta^{(t)}}]\Big) \\
&= \sum_{i=1}^{m}\Big(y_i(\beta_0 + x_i\beta_i)\Big) - \sum_{i=1}^{n}\Big(\log(1+e^{\beta_0+x_i\beta_1})\Big) + \sum_{i=m+1}^{n}\Big((\beta_0 + x_i\beta_1)p_i(\boldsymbol{\beta^{(t)}})\Big) \\
&(\mathbb{E}(Y_i) = p_i(\boldsymbol{\beta}) \text{ as } Y_i \sim \text{Bernoulli}(p_i(\boldsymbol{\beta})))
\end{aligned} \tag{9}
$$

Now we differentiate Q with respect to β_0 and β_1 for the M-Step

$$
\begin{aligned}
\frac{d}{d\beta_0}Q(\boldsymbol{\beta}|\boldsymbol{\beta^{(t)}}) &= \sum_{i=1}^{m} y_i - \sum_{i=1}^{n}\left(\frac{e^{\beta_0+x_i\beta_1}}{1+e^{\beta_0+x_i\beta_1}}\right) \\
&+ \sum_{i=m+1}^{n}\left(\frac{e^{\beta_0^{(t)}+x_i\beta_1^{(t)}}}{1+e^{\beta_0^{(t)}+x_i\beta_1^{(t)}}} + x_i\beta_1\frac{e^{\beta_0^{(t)}+x_i\beta_1^{(t)}}}{(1+e^{\beta_0^{(t)}+x_i\beta_1^{(t)}})^2} + \beta_0\frac{e^{\beta_0^{(t)}+x_i\beta_1^{(t)}}}{(1+e^{\beta_0^{(t)}+x_i\beta_1^{(t)}})^2}\right) \\
\frac{d}{d\beta_1}Q(\boldsymbol{\beta}|\boldsymbol{\beta^{(t)}}) &= \sum_{i=1}^{m} y_i x_i - \sum_{i=1}^{n}\left(x_i\frac{e^{\beta_0+x_i\beta_1}}{1+e^{\beta_0+x_i\beta_1}}\right) \\
&+ \sum_{i=m+1}^{n}\left(\beta_0\frac{x_i e^{\beta_0^{(t)}+x_i\beta_1^{(t)}}}{(1+e^{\beta_0^{(t)}+x_i\beta_1^{(t)}})^2} + x_i\frac{e^{\beta_0^{(t)}+x_i\beta_1^{(t)}}}{1+e^{\beta_0^{(t)}+x_i\beta_1^{(t)}}} + x_i\beta_1\frac{x_i e^{\beta_0^{(t)}+x_i\beta_1^{(t)}}}{(1+e^{\beta_0^{(t)}+x_i\beta_1^{(t)}})^2}\right)
\end{aligned} \tag{10}
$$

However, the solutions of the derivatives have no closed form expressions and thus we need to resort to numerical methods. Before proceeding to the implementation, we first need to preprocess `dataex4` due to the `NA` values by rearranging it.

```r
#rearranging the order where the NA values go the last
dataex4 <- dataex4[order(dataex4$Y),]
row.names(dataex4) <- NULL
head(dataex4,5)
```

```
##              X Y
## 1  0.3215956 0
## 2  0.2582281 0
## 3  0.4352370 0
## 4 -0.2277718 0
## 5 -0.3193020 0
```

```r
tail(dataex4,5)
```

```
##              X  Y
## 496 0.6989142 NA
## 497 0.8936356 NA
## 498 0.7551561 NA
## 499 0.8782734 NA
## 500 0.6924915 NA
```

We can confirm that the order of `dataex4` is changed where the `NA` values are at the last. Now we proceed to the implementation of the EM algorithm. In the code, we have used for the stopping criterion as below

$$|\beta_0^{(t+1)} - \beta_0^{(t)}| + |\beta_1^{(t+1)} - \beta_1^{(t)}| < \varepsilon$$

```r
#function to compute the Q function
qfn <- function(param, data){
  beta0 <- param[1]; beta1 <- param[2]
  x <- data[, 1]; y <- data[, 2]
  express <- beta0 + x[1:405]*beta1
  express.na <- beta[1] + x[406:500]*beta[2]
  express.all <- beta0 + x*beta1
  express.fix <- beta0 + x[406:500]*beta1
  #expression of the Q function
  sum(y[1:405]*express) - sum(log(1 + exp(express.all))) +
    sum(express.fix*exp(express.na)/(1 + exp(express.na)))
}

#using maxlik
beta <- c(0,0)
diff <- 1
while(diff > 0.000001){
    mle <- maxLik(logLik = qfn, data = dataex4, start = beta)
    diff <- sum(abs(mle$estimate - beta))
    beta <- mle$estimate
}
cat("The value for beta0:", beta[1], "\n",
    "The value for beta1:", beta[2])
```

```
## The value for beta0: 0.9755263
##  The value for beta1: -2.480383
```

```r
#using optim
beta <- c(0,0)
diff <- 1
while(diff > 0.000001){
    mle <- optim(beta, qfn, data = dataex4,
                 control = list(fnscale = -1), hessian = TRUE)
    diff <- sum(abs(mle$par - beta))
    beta <- mle$par
}
cat("The value for beta0:", beta[1], "\n",
    "The value for beta1:", beta[2])
```

```
## The value for beta0: 0.9757079
##  The value for beta1: -2.479987
```

Using two different methods, `maxLik()` and `optim()` with $\varepsilon = 1e-6$, we can see that we both obtained similar results by a small difference (0.001). Thus, we conclude that the MLE of $\hat{\beta}_{\text{MLE}} = (\hat{\beta}_{0\text{MLE}}, \hat{\beta}_{1\text{MLE}})$ is $(0.976, -2.48)$ respectively.

4 Question 4 **25 / 25**

- ✓ **- 0 pts** Correct
- **- 15 pts** Wrong E-step
- **- 20 pts** Wrong likelihood
- **- 5 pts** Minor issues with E-step or M-step
- **- 10 pts** Major issues with E-step or M-step
- **- 15 pts** Fundamental issues with E-step or M-step
- **- 20 pts** Not applying EM
- **- 5 pts** Minor issues with justifications / derivations
- **- 10 pts** Major issues with justifications / derivations
- **- 15 pts** Fundamental issues with justifications / derivations
- **- 20 pts** Just presenting code
- **- 0 pts** Correct

Q5

Consider a random sample $Y_1, ..., Y_n$ from the mixture distribution with density

$$f(y) = pf_{\text{LogNormal}}(y; \mu, \sigma^2) + (1-p)f_{\text{Exp}}(y; \lambda),$$

with

$$f_{\text{LogNormal}}(y; \mu, \sigma^2) = \frac{1}{y\sqrt{2\pi\sigma^2}} \exp\left\{\frac{1}{2\sigma^2}(\log y - \mu)^2\right\}, \quad y > 0, \quad \mu \in \mathbb{R},\ \sigma > 0$$
$$f_{\text{Exp}}(y; \lambda) = \lambda e^{-\lambda y}, \quad y \geq 0, \quad \lambda > 0$$

and $\boldsymbol{\theta} = (p, \mu, \sigma^2, \lambda)$

a)

Derive the EM algorithm to find the updating equations for $\boldsymbol{\theta}^{(t+1)} = (p^{(t+1)}, \mu^{(t+1)}, (\sigma^{(t+1)})^2, \lambda^{(t+1)})$.

Answer :

Let us consider a mixture model of Log-Normal and Exponential distributions.

$$\mathbb{P}(Y \leq y) = p \cdot \frac{1}{y\sqrt{2\pi\sigma^2}} \exp\left\{\frac{1}{2\sigma^2}(\log y - \mu)^2\right\} + (1-p) \cdot \lambda e^{-\lambda y}$$

Let z_i be the binary latent variables indicating component membership, i.e.

$$z_i = \begin{cases} 1 & \text{if } y_i \text{ belong to } f_{\text{LogNormal}}(y; \mu, \sigma^2) \\ 0 & \text{if } y_i \text{ belong to } f_{\text{Exp}}(y; \lambda) \end{cases}$$

The observed data in this context is $\boldsymbol{y} = (y_1 \dots y_n)$ and the missing data is $\boldsymbol{z} = (z_1 \dots z_n)$. The likelihood of the complete data $(\boldsymbol{y}, \boldsymbol{z})$ is

$$\begin{aligned} L(\theta; \boldsymbol{y}, \boldsymbol{z}) &= \prod_{i=1}^{n} \left(p \cdot \frac{1}{y_i\sqrt{2\pi\sigma^2}} \exp\left\{\frac{1}{2\sigma^2}(\log y_i - \mu)^2\right\}\right)^{z_i} \left((1-p) \cdot \lambda e^{-\lambda y_i}\right)^{1-z_i} \\ \implies \log L(\theta; \boldsymbol{y}, \boldsymbol{z}) &= \sum_{i=1}^{n} z_i \log\left(p \cdot \frac{1}{y_i\sqrt{2\pi\sigma^2}} \exp\left\{\frac{1}{2\sigma^2}(\log y_i - \mu)^2\right\}\right) + \sum_{i=1}^{n}(1-z_i)\log\left((1-p) \cdot \lambda e^{-\lambda y_i}\right) \end{aligned} \tag{11}$$

with the corresponding log likelihood (11), we proceed to E-Step,

$$\begin{aligned} Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(t)}) &= \mathbb{E}_Z(\log L(\theta; \boldsymbol{y}, \boldsymbol{z})|\boldsymbol{y}, \boldsymbol{\theta}^{(t)}) \\ &= \sum_{i=1}^{n} \mathbb{E}(Z_i|y_i, \theta^{(t)}) \log\left(p \cdot \frac{1}{y_i\sqrt{2\pi\sigma^2}} \exp\left\{\frac{1}{2\sigma^2}(\log y_i - \mu)^2\right\}\right) \\ &+ \sum_{i=1}^{n}(1 - \mathbb{E}(Z_i|y_i, \theta^{(t)})) \log\left((1-p) \cdot \lambda e^{-\lambda y_i}\right) \end{aligned} \tag{12}$$

We know that $\mathbb{E}(Z_i|\boldsymbol{y}, \boldsymbol{\theta}^{(t)}) = \mathbb{P}(Z_i = 1|y_i, \theta^{(t)})$, and applying Bayes Theorem and the Law of Total Probability, we obtain,

$$
\begin{aligned}
\mathbb{E}(Z_i|\boldsymbol{y},\boldsymbol{\theta^{(t)}}) &= \mathbb{P}(Z_i=1|y_i,\theta^{(t)}) \\
&= \frac{\left(p^{(t)}\cdot\frac{1}{y_i\sqrt{2\pi(\sigma^2)^{(t)}}}\exp\left\{\frac{1}{2(\sigma^2)^{(t)}}(\log y_i-\mu^{(t)})^2\right\}\right)}{\left(p\cdot\frac{1}{y_i\sqrt{2\pi(\sigma^2)^{(t)}}}\exp\left\{\frac{1}{2(\sigma^2)^{(t)}}(\log y_i-\mu^{(t)})^2\right\}\right)\left((1-p^{(t)})\cdot\lambda^{(t)}e^{-\lambda^{(t)}y_i}\right)} \\
&= \tilde{p}_i^{(t)}, \quad i=1,\ldots,n
\end{aligned} \tag{13}
$$

Therefore, we substitute 13 into 12

$$
\begin{aligned}
Q(\boldsymbol{\theta}|\boldsymbol{\theta^{(t)}}) = &\sum_{i=1}^{n}\tilde{p}_i^{(t)}\log\left(p\cdot\frac{1}{y_i\sqrt{2\pi\sigma^2}}\exp\left\{\frac{1}{2\sigma^2}(\log y_i-\mu)^2\right\}\right) \\
&+\sum_{i=1}^{n}(1-\tilde{p}_i^{(t)})\log\left((1-p)\cdot\lambda e^{-\lambda y_i}\right)
\end{aligned} \tag{14}
$$

For the M-step, we only need to compute the partial derivatives

$$
\begin{aligned}
\frac{\partial}{\partial p}Q(\boldsymbol{\theta}|\boldsymbol{\theta^{(t)}}) = 0 &\implies p^{(t+1)} = \frac{\sum_{i=1}^{n}\tilde{p}_i^{(t)}}{n} \\
\frac{\partial}{\partial \mu}Q(\boldsymbol{\theta}|\boldsymbol{\theta^{(t)}}) = 0 &\implies \mu^{(t+1)} = \frac{\sum_{i=1}^{n}\tilde{p}_i^{(t)}\log(y_i)}{\sum_{i=1}^{n}\tilde{p}_i^{(t)}} \\
\frac{\partial}{\partial \sigma^2}Q(\boldsymbol{\theta}|\boldsymbol{\theta^{(t)}}) = 0 &\implies (\sigma^{(t+1)})^2 = \frac{\sum_{i=1}^{n}\tilde{p}_i^{(t)}(\log(y_i)-\mu^{(t+1)})^2}{\sum_{i=1}^{n}\tilde{p}_i^{(t)}y_i} \\
\frac{\partial}{\partial \lambda}Q(\boldsymbol{\theta}|\boldsymbol{\theta^{(t)}}) = 0 &\implies \lambda^{(t+1)} = \frac{\sum_{i=1}^{n}(1-\tilde{p}_i^{(t)})}{\sum_{i=1}^{n}y_i(1-\tilde{p}_i^{(t)})}
\end{aligned} \tag{15}
$$

5.1 Question 5 a) 13 / 13

- ✓ **- 0 pts** **Correct**
- **- 8 pts** Wrong Q function
- **- 5 pts** Wrong updating rule obtained from M-step
- **- 3 pts** Minor issues with justifications / derivations
- **- 7 pts** Major issues with justifications / derivations
- **- 10 pts** Fundamental issues with justifications / derivations

b)

Using the dataset datasetex5.Rdata implement the EM algorithm and find the MLEs for each component of θ. As starting values, you might want to consider $\theta^{(0)} = (p^{(0)}, \mu^{0)}, (\sigma^{(0)})^2, \lambda^{(0)}) = (0.1, 1, 0.5^2, 2)$. Draw the histogram of the data with the estimated density superimposed.

Answer :

In the code, we have used for the stopping criterion

$$|p^{(t+1)} - p^{(t)}| + |\mu^{(t+1) - \mu(t)}| + |(\sigma^{(t+1)})^2 - (\sigma^{(t)})^2| + |\lambda^{(t+1)} - \lambda^{(t)}| < \varepsilon$$

with $\varepsilon = 0.00001$. For the starting values we use $\theta^{(0)} = (p^{(0)}, \mu^{0)}, (\sigma^{(0)})^2, \lambda^{(0)}) = (0.1, 1, 0.5^2, 2)$ as given.

```
mixture.model <- function(y, theta, eps){
  n <- length(y)
  #initialising the parameters
  p <- theta[1]; mu <- theta[2]; sigma <- theta[3]; lambda <- theta[4]
  diff <- 1
  while(diff > eps){
  theta.old <- theta
  #E-step: computing ptilde
  ptilde1 <- p*dlnorm(y, mean = mu, sd = sigma)
  ptilde2 <- (1 - p)*dexp(y, lambda)
  ptilde <- ptilde1/(ptilde1 + ptilde2)

  #M-step: computing each parameter
  p <- mean(ptilde)
  mu <- sum(log(y)*ptilde)/sum(ptilde)
  sigma <- sqrt(sum(((log(y) - mu)^2)*ptilde)/sum(ptilde))
  lambda <- sum(1 - ptilde)/sum(y*(1 - ptilde))

  #Checking with stopping criterion
  theta <- c(p, mu, sigma, lambda)
  diff <- sum(abs(theta - theta.old))
  }
  return(theta)
}
#Performing the EM algorithm on Mixture Modelz
res <- mixture.model(y = dataex5, c(0.1, 1, 0.5, 2), 0.000001)
p <- res[1]; mu <- res[2]; sigma <- res[3]^2; lambda <- res[4]
cat("The value for p is", p)
```

```
## The value for p is 0.47955
```

```
cat("The value for mu is", mu)
```

```
## The value for mu is 2.013262
```

```
cat("The value for sigma^2 is", sigma)
```

```
## The value for sigma^2 is 0.8637414
```

```
cat("The value for lambda is", lambda)
```

```
## The value for lambda is 1.033019
```

```
B <- 500
pb <- mub <- sigmab <- lambdab <- numeric(B)
```

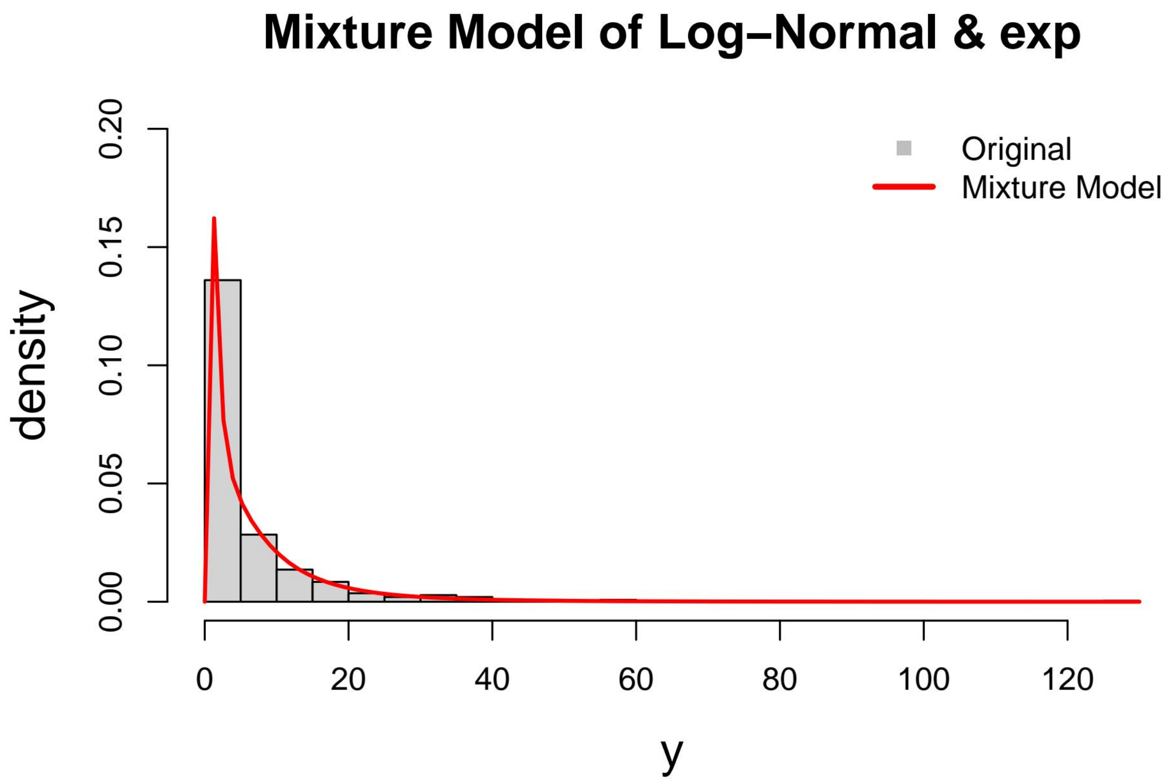
```r
set.seed(1)
for(l in 1:B){
yb <- sample(dataex5, size = length(dataex5), replace = TRUE)
fitb <- mixture.model(y = yb, theta = c(p, mu, sigma, lambda),
eps = 0.000001)
pb[l] <- fitb[1]
mub[l] <- fitb[2]
sigmab[l] <- fitb[3]
lambdab[l] <- fitb[4]
}

ql <- function(x){quantile(x,0.025)}
qh <- function(x){quantile(x,0.975)}
pl <- ql(pb); ph <- qh(pb)
mul <- ql(mub); muh <- qh(mub)
sigmal <- ql(sigmab); sigmah <- qh(sigmab)
lambdal <- ql(lambdab); lambdah <- qh(lambdab)
df <- data.frame("Estimate" = c(p,mu, sigma, lambda),
"lq" = c(pl, mul, sigmal, lambdal),
"uq" = c(ph, muh, sigmah, lambdah))
rownames(df) <- c("$p$", "$\\mu$","$\\sigma^2$", "$\\lambda$")
colnames(df) <- c("Estimate", "2.5% quantile", "97.5% quantile")
knitr::kable(df, escape = FALSE, digits = 4)
```

	Estimate	2.5% quantile	97.5% quantile
p	0.4796	0.3574	0.6642
μ	2.0133	1.4947	2.3554
σ^2	0.8637	0.7237	1.1906
λ	1.0330	0.7443	1.7217

Using the **R** code provided, we obtained $\hat{p} = 0.480$, $\hat{\mu} = 2.01$, $\hat{\sigma}^2 = 0.864$, $\hat{\lambda} = 1.03$. Also, we further performed the nonparametric bootstrap to quantify uncertainty about the parameters as shown on the table above. The plot of the observed counts against the expected counts under this mixture model is shown below. We can see that the generated mixture model represents the dataset well.

```r
#plotting the histogram of the mixture model
hist(dataex5, breaks=45, main = "Mixture Model of Log-Normal & exp",
     xlab = "y", ylab = "density", ylim = c(0,0.2),
     cex.main = 1.5, cex.lab = 1.5, freq = F)
curve(p*dlnorm(x, mu, sigma)+(1-p)*dexp(x, lambda), add = TRUE, lwd = 2, col = "red")
legend("topright", c("Original", "Mixture Model"), col=c("gray", "red"),
       bty="n",lty = c(NA, 1), pch = c(15, NA),lwd=c(100,3))
```


Mixture Model of Log−Normal & exp
Original
Mixture Model
density
0.00
0.05
0.10
0.15
0.20
0
20
40
60
80
100
120
y

5.2 Question 5 b) 12 / 12

✓ **- 0 pts** **Correct**

- 3 pts Minor issues with justifications / code

- 7 pts Major issues with justifications / code

- 10 pts Fundamental issues with justifications / code

- 4 pts Not presenting histogram / density estimator