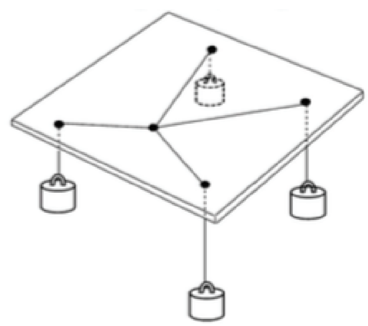
Exercise 0. (The Fermat point of a set of points)

Given set of points $\boldsymbol{y}_1,\ldots,\boldsymbol{y}_m$ in the plane, we want to find a point $\boldsymbol{x}^*$ whose sum of weighted distances to the given set of points is minimized. Mathematically, the problem is

$$\min \sum_{i=1}^{m} w_i \|\boldsymbol{x}^* - \boldsymbol{y}_i\|, \quad \textit{subject to } \boldsymbol{x}^* \in \mathbb{R}^2,$$

where $w_1, \ldots, w_m$ are given positive real numbers.

1. *Show that there exists a global minimum for this problem, and that it can be realized by means of the mechanical model shown in the following figure:*



2. *Is the optimal solution always unique?*

3. *Show that an optimal solution minimizes the potential energy of the mechanical model defined as*

$$\sum_{i=1}^{m} w_i h_i,$$

where h_i is the height of the i-th weight measured from some reference level.

Solution

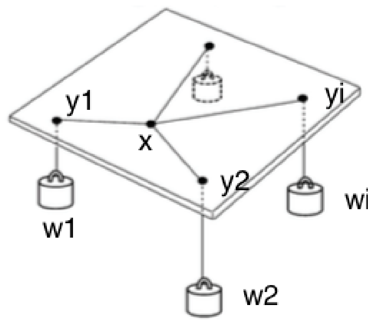
1. Let us see first that the function to be minimized is

$$f(\boldsymbol{x}) = \sum_{i=1}^{m} w_i \|\boldsymbol{x} - \boldsymbol{y}_i\|, \quad w_i \geq 0.$$

Since $f(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^2$, if we find a point $\boldsymbol{x}^*$ such that $f(\boldsymbol{x}^*) = 0$, it will be a minimum of $f(\boldsymbol{x})$.

If $\boldsymbol{x}^*$ is an equilibrium point of the system displayed in the figure, the sum of the forces made by the weights w_i at each point $\boldsymbol{y}_i$ must be zero, this is

$$\sum_{i=1}^{m} w_i \|\boldsymbol{x}^* - \boldsymbol{y}_i\| = 0.$$



So, we need to find a point $\boldsymbol{x}^*$ such that the above equlity holds and, according to what has already been said, it will be a minimum of $f(\boldsymbol{x})$.

Note that the function $f(\boldsymbol{x})$ is convex, because the norm is a convex function, and $f(\boldsymbol{x})$ is the weighted sum of convex functions. Since the function is convex, if there is a local minimum it will be a global one. (See Lecture 2 for the above claims.)

Instead of using the function $f(\boldsymbol{x})$, we will use

$$F(\boldsymbol{x}) = \sum_{i=1}^{m} w_i \|\boldsymbol{x} - \boldsymbol{y}_i\|^2.$$

Note that if $\boldsymbol{x}^*$ minimizes $F(\boldsymbol{x})$ it also minimizes $f(\boldsymbol{x})$. Let us see that $F(\boldsymbol{x})$ achieves its minimum at

$$\boldsymbol{x}^* = \frac{\sum_{i=1}^{m} w_i \boldsymbol{y}_i}{\sum_{i=1}^{m}},$$

this is

$$\nabla F(\boldsymbol{x}^*) = 0, \quad \nabla^2 F(\boldsymbol{x}^*) \text{ is positive definite.}$$

Introducing coordinates $\boldsymbol{x} = (x^1, x^2)^T$ and $\boldsymbol{y}_i = (y_i^1, y_i^2)^T$, we can write

$$\begin{aligned}
\frac{\partial F}{\partial x_j}(\boldsymbol{x}) &= \sum_{i=1}^{m} w_i \frac{\partial g}{\partial x_j} \|\boldsymbol{x} - \boldsymbol{y}_i\|^2 = 2 \sum_{i=1}^{m} w_i (x^j - y_i^j), \quad j = 1, 2, \\
\frac{\partial F}{\partial x_j}(\boldsymbol{x}^*) &= 2 \sum_{i=1}^{m} w_i (x^{*j} - y_i^j), \quad j = 1, 2,
\end{aligned}$$

and, since $\sum_{i=1}^{m} w_i \|\boldsymbol{x}^* - \boldsymbol{y}_i\| = 0$, it follows that

$$\nabla F(\boldsymbol{x}^*) = \left(\frac{\partial g}{\partial x_j}(\boldsymbol{x}^*), \frac{\partial g}{\partial x_j}(\boldsymbol{x}^*) \right)^T = 0.$$

Furthermore

$$\begin{aligned} \frac{\partial^2 F}{\partial x^j \partial x^j}(\boldsymbol{x}) &= 2\sum_{i=1}^{m} w_i > 0, \quad j = 1, 2, \\ \frac{\partial^2 F}{\partial x^1 \partial x^2}(\boldsymbol{x}) &= 0, \end{aligned}$$

so, the Hessian of $F(\boldsymbol{x})$ is a positive defined matrix at any point $\boldsymbol{x} \in \mathbb{R}^2$, so $\boldsymbol{x}^*$ is a local minimum, and since the function is convex it is also global one.

2. Consider the case with only two points $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ with the same weight $w_1 = w_2 = w$. Then, all the points in the segment between $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ minimize the function under consideration, since if $\boldsymbol{x}$ is a point in this segment

$$f(\boldsymbol{x}) = w\left(\|\boldsymbol{x} - \boldsymbol{y}_1\| + \|\boldsymbol{x} - \boldsymbol{y}_2\|\right) = w\|\boldsymbol{y}_1 - \boldsymbol{y}_2\| = \mathit{constant}.$$

3. We take the plane defined by the points $\boldsymbol{y}_i$ as the $z = 0$ plane, and fix the potential for all the points in this plane equal to zero (reference lebel), so the potential of the system is

$$\sum_{i=1}^{m} w_i h_i(\boldsymbol{x}).$$

For any point $\boldsymbol{x}$ in the convex hull defined by $\boldsymbol{y}_1, ..., \boldsymbol{y}_m$, the lenght of the line that joins $\boldsymbol{x}$ with the weight w_i is equal to a certain constant value l_i, and since $h_i(\boldsymbol{x}) < 0$ is the height of w_i, we can write

$$l_i = \|\boldsymbol{x} - \boldsymbol{y}_i\| - h_i(\boldsymbol{x}).$$

According to this equality

$$\sum_{i=1}^{m} w_i h_i(\boldsymbol{x}) = \sum_{i=1}^{m} w_i\left(\|\boldsymbol{x} - \boldsymbol{y}_i\| - l_i\right) = \sum_{i=1}^{m} w_i \|\boldsymbol{x} - \boldsymbol{y}_i\| - \sum_{i=1}^{m} w_i l_i.$$

The fisrt term of the potential has a minimum at $\boldsymbol{x}^*$, and $\sum_{i=1}^{m} w_i l_i$ is constant, so $\boldsymbol{x}^*$ minimizes the potencial $\sum_{i=1}^{m} w_i h_i(\boldsymbol{x})$.