Master in Computer Vision Barcelona

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Module: **M2. Optimization and inference techniques for Computer Vision**
nal exam
Date: 4rth December, 2014
Teachers: Coloma Ballester, Juan Fco. Garamendi, Oriol Ramos, Joan Serra **Time: 2h30min**

- Books, lecture notes, calculators, phones, etc. are not allowed.
- All sheets of paper should have your name.
- Answer each problem in a separate sheet of paper.
- All results should be demonstrated or justified.

Problem 1 *0.75 Points*

Consider the function $f : R^n \to R$ defined by $f(x) = \|Ax - b\|^2$, for $x \in R^n$, where A is a $m \times n$ matrix, and $b \in R^m$ $(m, n \in N)$. (The notation $\| \cdot \|$ stands for the Euclidean norm, and $\langle \cdot, \cdot \rangle$ stands for the scalar product.)

(a) Let $v \in R^n, v \neq 0$. Compute $D_v f(x)$, the directional derivative of f at the point x in the direction v.

(b) Use the result in (a) to compute $\nabla f(x)$.

(c) Which is the equation satisfied by a minimum of $f(x)$? (it is called the Euler-Lagrange equation).

Problem 2 *1 Point*

Let A be a $m \times n$ matrix, and $b \in R^m$. Consider, for $x \in R^n$, the function $f(x) = \langle x, x \rangle$. Consider the problem (P) defined as

$$\begin{array}{l} \min f(x) \\ \text{subject to } Ax = b. \end{array}$$

(a) Write problem (P) as a min-max problem and define the duality gap.

(b) Define and compute the dual function of problem (P).

(c) Write down the dual problem.

Problem 3 *0.75 Points*

Consider the following data fitting (or regression) problem: we are given a data set of N pairs $(t_1, y_1), \ldots, (t_n, y_n)$, where $t_i \in R$, $y_i \in R$, $i = 1, \ldots, N$, with $N > 10$. We would like to fit a function f to this known data set and assume that we know that there exist a functional relationship $y = f(t)$, with f modeled as the following parametric function

$$f(t) = x_1 + x_2 t + x_3 t^2$$

where $x_1, x_2, x_3 \in R$ are unknows.

(a) Explain the least squares solution to this problem of data fitting, used to determine the best parameters $\mathbf{x} = (x_1, x_2, x_3)$. Write down the expression of the energy $E(\mathbf{x})$ (or $E(x_1, x_2, x_3)$) to be minimized.

(b) Write down the normal equations associated to this problem.

(c) How could you determine the parameters $\mathbf{x} = (x_1, x_2, x_3)$ using the SVD or the pseudoinverse of the matrix associated to your problem? (Recall that SVD stands for Singular Value Decomposition of a matrix).

Problem 4 *1.5 Points*

Let

$$\begin{aligned} J:\ \mathcal{V} &\to \mathbb{R}, \\ u &\to J(u) = \int_\Omega \mathcal{F}\left(x, u(x), \nabla u(x)\right) dx \end{aligned}$$

be a convex energy functional over functions u, where

- $\mathcal{V}$ is a suitable space of functions.
- $\Omega \in \mathbb{R}^d$ is a bounded open domain of the d dimensional space $\mathbb{R}^d$.
- $u \in \mathcal{V}$, $u : \Omega \to \mathbb{R}$ is a scalar function defined on Ω.
- ∇ is the gradient operator and $x \in \Omega$ is the spatial variable.

(a) (0.15 points) Say in few words which is the fundamental problem in calculus of variations.

(b) (0.15 points) Write the definition of the Gâteaux derivative of $J(u)$ (the directional derivative of J at u in direction h)

(c) (0.15 points) Let

$$\frac{dJ}{du}$$

be the derivative of J at u. Which is the necessary condition for extremality of J?

(d) (1.05 points) Apply the definition of Gâteaux derivative for finding the minimum at u of the following energy

$$J(u) = \frac{1}{2\lambda} \int_\Omega (u - f)^2\, dx$$

where $f(x) \in \mathcal{V}, f : \Omega \to \mathbb{R}$ is a fixed function.

Problem 5 *0.5 Points*

Let $f : \Omega \to \mathbb{R}$ be a given noisy image, where Ω is a bounded open subset of $\mathbb{R}^2$ and $f \in L^\infty(\Omega)$. Consider the following two minimization problems
P1:

$$\underset{u \in W^{1,2}(\Omega)}{\arg\min} \left\{ \int_\Omega |\nabla u|^2 dx + \frac{1}{2\lambda_1} \int_\Omega |u - f|^2 dx \right\}$$

Where

- $W^{1,2}(\Omega) = \{u \in L^2(\Omega);\ \nabla u \in L^2(\Omega)^2\}$.
- $\lambda_1 \in \mathbb{R}^+$ is a given parameter.

P2:

$$\underset{u \in BV(\Omega)}{\arg\min} \left\{ \int_\Omega |Du| + \frac{1}{2\lambda_2} \int_\Omega |u - f|^2 dx \right\}$$

Where

- Du is the distributional gradient of u.
- $\int_\Omega |Du|$ is the total variation of u. If $u \in C^1(\Omega)$ (u smooth), $\int_\Omega |Du| = \int_\Omega |\nabla u| dx$.
- $BV(\Omega)$ is the space of bounded variation such as

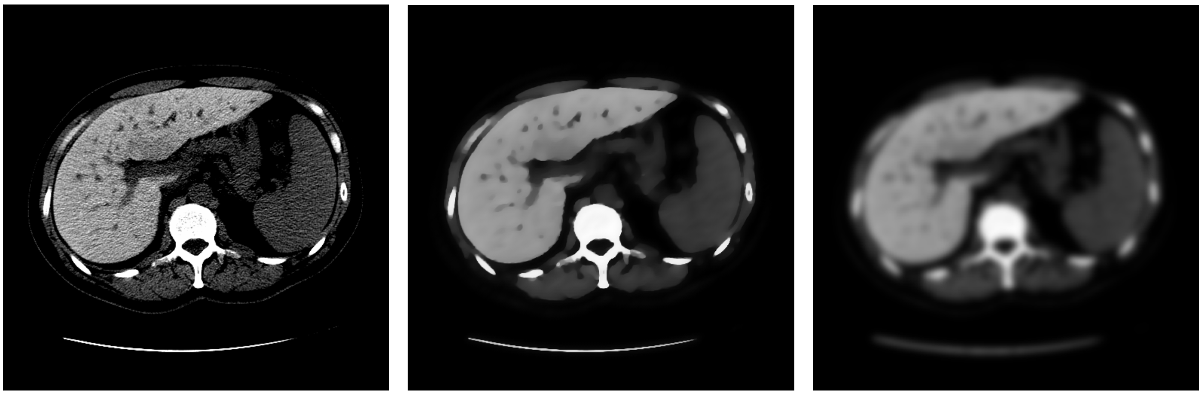
$$BV(\Omega) = \left\{ u \in L^1(\Omega);\ \int_\Omega |Du| < \infty \right\}$$

- $\lambda_2 \in \mathbb{R}^+$ is a given parameter.

Now, let $\lambda_1 = \lambda_2$ and consider the axial slice of a abdominal computed tomography shown at figure 1a as the noisy image f.

(a) (0.25 points) Say which image, figure 1b or figure 1c, corresponds to the solution of which problem P1 or P2. Justify your answer.

Hint: With abuse of notation, problem P2 can be written as $\underset{u \in BV(\Omega)}{\arg\min} \left\{ \int_\Omega |\nabla u| dx + \frac{1}{2\lambda_2} \int_\Omega |u - f|^2 dx \right\}$



(a) Given noisy image f (b) Solution (c) Solution

Figura 1: Figures for Problem 2

(b) (0.25 points) Which numerical and theoretical issue has the problem P2? Write at least one possible solution to overcome the issue.

Problem 6 *0.5 Points*

For binary segmentation, the model problem can be defined as follows: Let $\omega \subset \Omega$ be an open, positive measured sub-region of the original domain (eventually not connected). If the curve Γ represents the boundary of such a segmentation ω then, in the level set formulation, the (free) moving boundary Γ is the zero level set of a Lipschitz function $\phi : \Omega \to \mathbb{R}$, that is:

$$\Gamma = \{(x, y) \in \Omega : \phi(x, y) = 0\}$$

where

$$\omega = \{(x, y) \in \Omega : \phi(x, y) > 0\}$$

and

$$\Omega \backslash \overline{\omega} = \{(x, y) \in \Omega : \phi(x, y) < 0\}$$

The level set function ϕ can be characterized as a minimum of the following energy functional,

$$J(\bar{c},\phi) = \int_\Omega |DH(\phi)| + \lambda \int_\Omega (f-c_1)^2 H(\phi) \ + (f-c_2)^2(1-H(\phi))dx$$

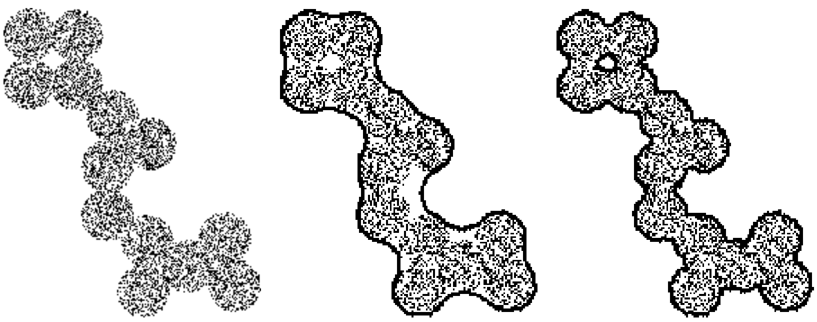
where $DH(\phi)$ is the distributional gradient of Heaviside function $H(\phi)$, f is a bounded function representing the image (the data), $\bar{c} \in \mathbb{R}^2$, $\bar{c} = (c_1, c_2)$ and $\lambda \in \mathbb{R}^+$ is a given parameter. The function $H(x)$ represents the Heaviside function, i.e.: $H(x) = 1$ if $x \geq 0$ and $H(x) = 0$ otherwise, and it allows to express the length of Γ by

$$|\Gamma| = \text{Length}(\phi = 0) = \int_\Omega |DH(\phi)|$$

where the term $\int_\Omega |DH(\phi)|$ denotes, properly, the total variation of the discontinuous function $H(\phi)$ in Ω.

(a) (0.25 points) We want to segment the image at figure 2a. For doing this, we minimize $J(\bar{c},\phi)$ with two differents values of λ, named it λ_1 and λ_2, obtaing the segmentations at figure 2b and 2c. Consider $\lambda_1 >> \lambda_2$. Which λ corresponds with which result? Justify your answer.

(b) (0.25 points) Why c_1 and c_2 are the mean values of the inside and outside regions?



(a) Image f to segment

(b) Solution. $\phi = 0$ is drawn over f

(c) Solution. $\phi = 0$ is drawn over f

Figura 2: Figures for Problem 3.

Problem 7 *2.5 Points*

A consultancy hires us to solve one of its problems. After some discussions, we have arrived to the following definitions of binary random variables (1=true, 0=false):

- x_1: A businessman corrupts a political leader,

$$p(x_1 = 1) = 0.30$$

- x_2: A political leader wins lottery,

$$p(x_2 = 1) = 0.01$$

- x_3: A businessman is in jail,

x_1	$p(x_3 = 1 \vert x_1)$
0	0.05
1	0.3

- x_4: A political leader is rich

x_1	x_2	$p(x_4 = 1 \vert x_1, x_2)$
0	0	0.1
0	1	0.2
1	0	0.6
1	1	0.9

- x_5: A political leader invests in a nightclub

x_2	$p(x_5 = 1 \vert x_2)$
0	0.01
1	0.1

- x_6: A political leader has a Swiss bank account,

x_4	$p(x_6 = 1 \vert x_4)$
0	0.05
1	0.1

Moreover, we have the following observations:

- a businessman is in jail and
- a political leader has a Swiss bank account

a) Draw the Bayes Net of this problem and write the associated joint distribution (0.5 point).

b) Convert the above Bayes net to a factor graph and define the corresponding factors, given the joint distribution defined in a) (0.5 point).

c) Which is the complexity of the Belief Propagation (BP) algorithm when it is applied to any chain? and for the factor graph defined in b)? (0.5 point)

d) Estimate the probability that a political leader won a lottery given the above observations (1 point).

Hints for d): remember that the messages functions in Sum-Prod BP are

$$m_{i_k \leftarrow s}(x_{i_k}) = \int \phi_s(x_s) \prod_{j \in s \setminus \{i_k\}} m_{j \to s}(x_j) dx_{s \setminus \{i_k\}} \text{ and } m_{i_k \to u}(x_{i_k}) = \prod_{\substack{t \ni i_k \\ t \neq u}} m_{i_k \leftarrow t}(x_{i_k})$$

Problem 8 — *1.5 Points*

When posing the problem of binary image denoising as inference on a graphical model, we arrived at the following formulation:

$$x^\star = \arg\max_x \underbrace{\prod_i \exp^{-\alpha x_i} \prod_{j \text{ neighbor of } i} \exp^{-\beta x_i x_j}}_{a} \underbrace{\prod_i \exp^{-\gamma x_i y_i}}_{b}$$

where x was the clean image, y the observed noisy image, x_i, y_i the value of pixels at i, j and $x_i, y_i \in [+1, -1]$.

(a) where does this expression come from (develop probabilistic origin) ? Also, what's the name of parts (a) and (b) ?

(b) what's the meaning of α, β, γ ? what does it mean they are small or large ?

(c) which where the two assumptions made to arrive there ?

Problem 9 *0.5 Point*

Can the former problem be solved with the min-cut/max-flow algorithm ? why or why not ?

Problem 10 *0.5 Point*

What's false with respect to the graph-cuts algorithm ?

(a) does MAP inference on a graphical model

(b) repeatedly runs the max-flow/min-cut algorithm

(c) performs local "moves", eventually changing the label of many variables at each one

(d) can deal with any kind of potential functions

(e) it is iterative