Master in Computer Vision
Barcelona

UAB UOC UPC upf.

Module: M2. Optimization and inference techniques for Computer Vision **Final exam**
Date: December 3rd, 2015
Teachers: Juan Fco Garamendi, Coloma Ballester, Oriol Ramos, Joan Serra **Time: 2h30min**

- Books, lecture notes, calculators, phones, etc. are not allowed.
- All sheets of paper should have your name.
- Answer each problem in a separate sheet of paper.
- All results should be demonstrated or justified.

Problem 1

Juan F. Garamendi, 0.5 Points

Let

$$\begin{aligned} J: \ \mathcal{V} &\to \mathbb{R}, \\ u &\mapsto J(u) = \int_{\Omega} \mathcal{F}\left(\mathbf{x}, u(\mathbf{x}), \nabla u(\mathbf{x})\right) d\mathbf{x} \end{aligned}$$

be a convex energy functional over functions u, where

- $\mathcal{V}$ is a suitable space of functions.
- $\Omega \in \mathbb{R}^d$ is a bounded open domain of the d dimensional euclidean space $\mathbb{R}^d$.
- $u \in \mathcal{V}$, $u : \Omega \to \mathbb{R}$ is a scalar function defined on Ω.
- $\mathbf{x} \in \Omega$ such that $\mathbf{x} = (x_1, .., x_d)$ is the spatial variable and ∇ is the gradient operator such that $\nabla u(\mathbf{x}) = (u_{x_1}, .., u_{x_d})$

(a) (0.25 points) Say in few words which is the fundamental problem in calculus of variations.

(b) (0.25 points) Say in a few words what are the following expresions:

(i)

$$-\mathrm{div}_{\mathbf{x}}\left(\nabla_{\bar{g}} \mathcal{F}\right) + \frac{\partial \mathcal{F}}{\partial u} = 0$$

where

- The divergence $\mathrm{div}_{\mathbf{x}}$ is computed with respect to variable $\mathbf{x}$
- The gradient $\nabla_{\bar{g}}$ is computed with respect to the components of $\bar{g} = \nabla u(\mathbf{x})$

(ii)

$$-\sum_{i=1}^{d} \frac{\partial}{\partial x_i} \frac{\partial \mathcal{F}}{\partial u_{x_i}} + \frac{\partial \mathcal{F}}{\partial u} = 0$$

Problem 2 *Juan F. Garamendi, 1.5 Points*

Let $I_0 : \Omega \to \mathbb{R}$ and $I_1 : \Omega \to \mathbb{R}$ be two given (probably noisy) images, where Ω is a bounded open subset of $\mathbb{R}^2$ and $I_0, I_1 \in L^\infty(\Omega)$. Consider the following minimization problems

$$\arg\min_{\mathbf{u}} \left\{ \int_\Omega |\nabla u_1|^2 + |\nabla u_2|^2 d\mathbf{x} + \lambda \int_\Omega \left(I_1(\mathbf{x} + \mathbf{u}(\mathbf{x})) - I_0(\mathbf{x})\right)^2 d\mathbf{x} \right\}$$

Where

- $u_i \in W^{1,2}(\Omega)$.
- $W^{1,2}(\Omega) = \{u \in L^2(\Omega);\ \nabla u \in L^2(\Omega)^2\}$.
- $\mathbf{u} : \Omega \to \mathbb{R}^2$ such that $\mathbf{u} = (u_1(\mathbf{x}), u_2(\mathbf{x}))^T$ is a vector field.
- $|\cdot|$ is the usual Euclidean norm.
- $\lambda \in \mathbb{R}^+$ is a given parameter.

(a) (0.5 points) Describe in a few words what image problem solves the given minimization problem.

(b) (1 point) This energy functional can be locally linearized using a first order Taylor approximation of the data fidelity term:

$$\arg\min_{\mathbf{u}} \left\{ \int_\Omega |\nabla u_1|^2 + |\nabla u_2|^2 d\mathbf{x} + \lambda \int_\Omega \left(\langle \nabla I_0, \mathbf{u} \rangle + I_1 - I_0\right)^2 d\mathbf{x} \right\} \tag{1}$$

where the product $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product.

Let $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ be the Laplace operator. Prove that

$$\lambda \frac{\partial I_0}{\partial x_1} \left(u_1 \frac{\partial I_0}{\partial x_1} + u_2 \frac{\partial I_0}{\partial x_2} + I_1 - I_0 \right) - \Delta u_1 \quad = \quad 0$$

is a necessary condition for the minimization problem (1) with respect to u_1.

Problem 3 *Juan F. Garamendi, 0.5 Points*

Consider the following iterative scheme

- $\bar{x}^0 = \bar{0}$
- $\bar{x}^{k+1} \leftarrow S(\bar{x}^k, \bar{x}^{k+1}, \bar{b})$

applied to the algebraic problem $\mathbf{A}\bar{x} = \bar{b}$, where $\mathbf{A}$ is a known matrix, $\bar{b}$ is a known vector, $\bar{x}$ is an unknown vector , S some function and super-index represents iteration number.

(a) (0.5 points) Which is the most fundamental property that S must have for being a smoother component in the Multigrid context. In a few words explain why.

Problem 4 Coloma Ballester *1. Points*

Consider the function $f : R^n \to R$ defined by

$$f(x) = \frac{1}{2}\langle Ax, x \rangle - \langle x, b \rangle + c,$$

for $x \in R^n$, where A is a $n \times n$ symmetric matrix, $n \in N$, $b \in R^n$ and $c \in R$. (The notation $\langle \cdot, \cdot \rangle$ stands for the scalar product.)

(a) What codition on the matrix A implies that f is a convex function?

(b) Let $v \in R^n, v \neq 0$. Compute $D_v f(x)$, the directional derivative of f at the point x in the direction v.

(c) Use the result in (a) to compute $\nabla f(x)$. Which is the equation satisfied by a minimum of $f(x)$? (it is called the Euler-Lagrange equation).

(d) Give a codition on the matrix A implying that a solution x_0 of the Euler-Lagrange equation is a unique minimum of $f(x)$.

Problem 5 Coloma Ballester *0.75 Points*

Let A be a $m \times n$ matrix, and $b \in R^m$. Consider the following constrained optimization problem (P) defined as

$$\min \tfrac{1}{2}\langle x, x\rangle \\ \text{subject to } Ax = b.$$

(a) Write problem (P) as a min-max problem and define the duality gap.

(b) Define and compute the dual function of problem (P).

(c) Write down the dual problem.

Problem 6 Coloma Ballester *0.75 Points*

Working on a data fitting (or regression) problem where we were interested in fitting a function to a given set of data, we have transformed our data fitting problem to the following least squares problem:

$$\min_x \|A\mathbf{x} - \mathbf{b}\|^2$$

where A and $\mathbf{b}$ are a fixed matrix and vector, respectively, obtained from the given data, and $\mathbf{x}$ is a vector of unknowns.

(a) Write down the normal equations associated to this problem.

(b) How could you determine the solution $\mathbf{x}$ using the SVD or the pseudoinverse of the matrix associated to your problem? (Recall that SVD stands for Singular Value Decomposition of a matrix).

Problem 7 J.Serrat *0.5 Points*

The starting point for the probabilistic learning of the parameters of a conditional random field

$$\begin{aligned} p(y|x,w) &= \frac{1}{Z(x,w)}\exp[-E(x,y,w)] \\ E(x,y,w) &= \langle\ w, \psi(x,y)\ \rangle \\ Z(x,y) &= \sum_{y\in\mathcal{Y}} \exp[-E(x,y,w)] \\ y^\star &= \operatorname*{arg\,max}_{y\in\mathcal{Y}}\ \langle\ w, \psi(x,y)\ \rangle \end{aligned}$$

was to maximize the conditional likelihood, from which we arrived to

$$w^\star = \operatorname*{arg\,min}_w \sum_{i=1}^N \langle\ w, \psi(x^i, y^i)\ \rangle + \sum_{i=1}^N \log Z(x^i, w)$$

Interpret this formula, that is, tell what is (need to correctly answer all of them):

(a) the number of samples in the training set

(b) the training set itself

(c) the partition function

(d) the model parameters

Problem 8 J.Serrat *0.5 Points*

Which was the solution to the three main problems found when trying to optimize the fomer expression by gradient descent? Answer writing the pairs of problem-solution labels, like 1-a, 2-b, 3-c.
Problems:

(1) $Z(x^i, w)$ or $\mathbb{E}_{y \sim p(y|x^i,w)} \psi(x^i, y)$ impossible to calculate in practice

(2) N large and therefore we have to run belief propagation N times

(3) N small compared to number of parameters, causing overfitting

Solutions:

(a) regularization, assuming w follows a Gaussian distribution

(b) since $\psi(x, y)$ decomposes in factors, we can apply some inference method like belief propagation to compute it

(c) perform stochastic gradient descent

Problem 9 J.Serrat *0.5 Points*

Consider the graphical model of figure 1 where observations are binary images $16 \times 8 = 128$ pixels, that is, $x_i \in \{0, 1\}^{128}$, $y_i \in Y = \{\mathrm{a}, \mathrm{b} \ldots \mathrm{z}\}$ (26 lowercase letters), $x = (x_1 \ldots x_9), y = (y_1 \ldots y_9)$.

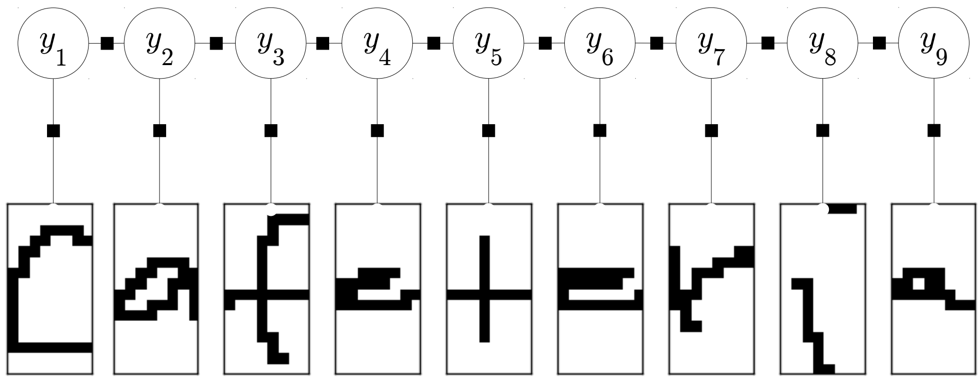


Figura 1

We want to learn w to later infer a word from a series of binary images of letters as

$$\begin{aligned} y^\star &= \arg\max_{y \in \mathcal{Y}} \langle w, \psi(x, y) \rangle \\ &= \arg\max_{y \in Y^9} \sum_{i=1}^{9} \sum_{p=\mathrm{a}}^{\mathrm{z}} \sum_{j=1}^{16} \sum_{k=1}^{8} w_{pjk} \, x_{ijk} \; + \; \sum_{i=1}^{8} \sum_{p=\mathrm{a}}^{\mathrm{z}} \sum_{q=\mathrm{a}}^{\mathrm{z}} w_{pq} \, \mathbf{1}_{y_i=\mathrm{p}, \, y_{i+1}=\mathrm{q}} \end{aligned}$$

where $\mathbf{1}_{y_i=\mathrm{p}, \, y_{i+1}=\mathrm{q}}$ evaluates to 1 if $y_i = \mathrm{p}$ and $y_{i+1} = \mathrm{q}$. In this context,

(a) what's the total number of parameters to learn ? (no need to write the final number, just an expression like $12\times34 + 56^7$ is ok)

(b) what does $w_{p=\mathrm{a},q=\mathrm{b}}$ mean or represent ?

Problem 10 J.Serrat *0.5 Points*

With regard to the problem of question

(a) what do you think the image of figure 2 is, I mean, which specific w_{pq} or w_p ?

(b) what's the total number of unary coefficients to learn if we apply the two-stage training technique? (again, an arithmetic expression is ok)

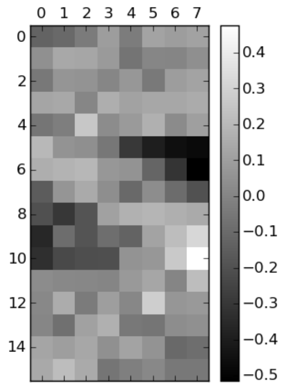


Figura 2

Problem 11 J.Serrat *0.5 Points*

In the exercise of labeling segments of a jacket contour we proposed the model of figure 3 (again a chain). What's the advantage of labeling by inference on this model, that is, to do structured prediction, over the simpler approach of classifying each segment independently with, say, a multiclass SVM ?

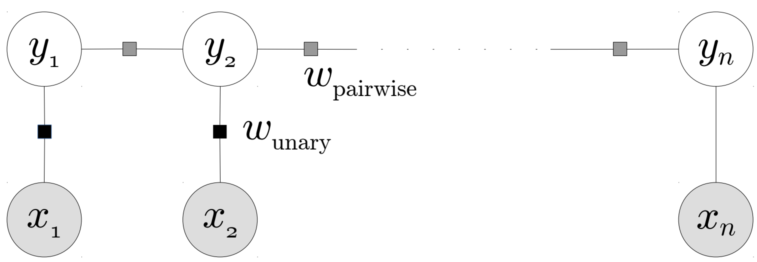
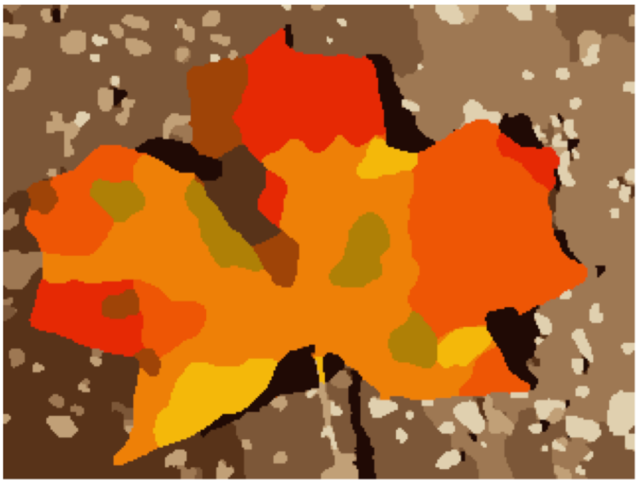


Figura 3

Problem 12 *Oriol Ramos Terrades, 2.5 Points*

Consider the segmentation problem seen in the lectures and illustrated with the following images:

(a) Original Image



(b) Segmented image by color

The goal is, given color image such us for instance the one shown in (a), to segment it into regions of similar colors. Assume that a palette of K colors has already learned and μ_k represents the k-th color in the RGB space.

a) This problem can be modeled as a conditional random field (CRF). Which are the hidden (or latent) variables? and the observed variables? Write the domain of both kind of variables (0.5 point).

b) Draw the factor graph that models this problem and write the associated joint distribution in terms of factor functions (0.5 point).

c) We define the factor function, $\phi(y_i, y_j)$, that models the interactions between hidden variables, by a Potts model of parameter θ. Write the matrix that represents this factor function (0.5 point).

d) If we replace the Potts model by the following feature function:

$$f(y_i, y_j) = \sum_{s,t=1}^{K} \theta_{s,t} 1_{\{y_i=s\}}(y_i) 1_{\{y_j=t\}}(y_j),$$

which is the matrix that represents this new factor function (0.5 point)?

e) Given the kind of problem that we want to solve, which are the most suitable inference algorithms that we can apply to solve it (0.5 point)?