T6: Background on vector calculus and Sylvester's criterion

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Optimization and inference techniques for Computer Vision

Material on multivariate calculus

If you have gaps in you background about multivariate calculus, Khan academy has some nice online course. These are the links to the course and the most relevant sections for our class.

`Multivariate calculus at Khan Academy`

`Section on multivariable derivatives`

`Section on applications of multivariable derivatives`

In particular, the section on applications of derivatives contains:

- The Hessian matrix
- First and second order Taylor approximations
- Optimization without constraints
- Optimization with equality constraints using the method of Lagrange multipliers

Remember James Joseph Sylvester, one of the inventors of the SVD? It turns out that he also came up with a criterion to check if a symmetric $n \times n$ matrix $\mathbf{A}$ is positive definite using the principal minors.

$$\mathbf{A} \text{ is positive definite} \iff \text{its } n \text{ leading principal minors are positve.}$$

The **leading** principal minors are the determinants of all the top-left (square) sub-matrices of $\mathbf{A}$. Thus, if we compute the Hessian matrix of a function at a point $\mathbf{x}$ and we find that its n leading principal minors are positive for any $\mathbf{x}$, then the function is **strictly convex** (because the eigenvalues of the Hessian are always strictly positive).

There is also a condition for checking if $\mathbf{A}$ is positive semi-definite:

$$\mathbf{A} \text{ is positive semi-definite} \iff \textbf{all} \text{ principal minors are non-negative.}$$

This requires much more computations, because it the total number of principal minors for an $n \times n$ matrix is $2^n - 1$. Whereas the number of **leading** principal minors is n. Therefore Sylvester's criterion for checking semi-definitness is mainly useful for small matrices.