Master in Computer Vision Barcelona

Module: 3D Vision
Lecture 2: Planar transformations
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Planar transformations

Planar transformations

The relation between an image and the real-World can contain different types of distortion.

In this class we will cover linear distortions of a planar object.

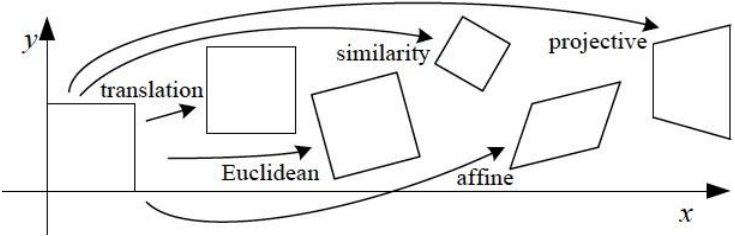
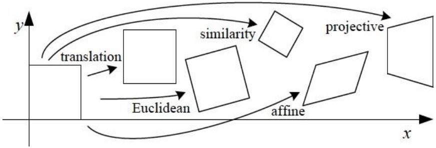
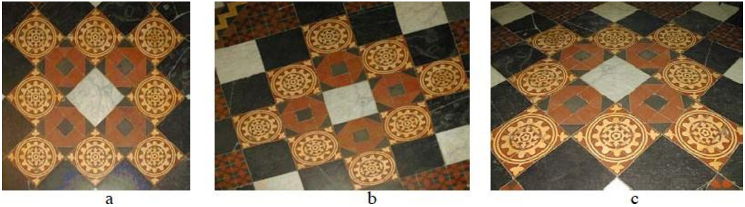


Image source: [Szeliski 2010]

Planar transformations

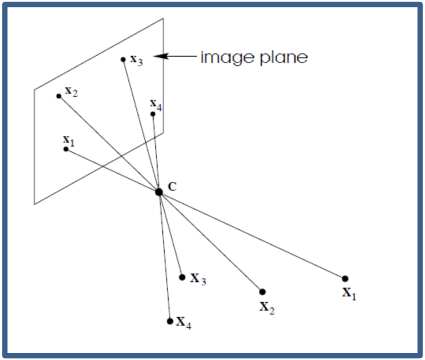


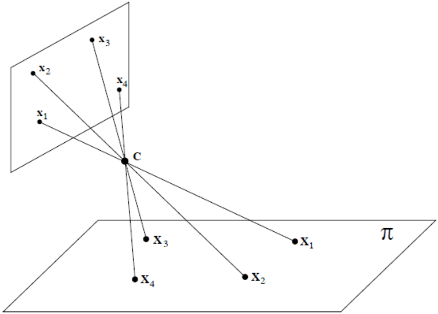


Similarity Affine transformation Projective transformation

Image source: [Hartley Zisserman 2004]

Projective transformations



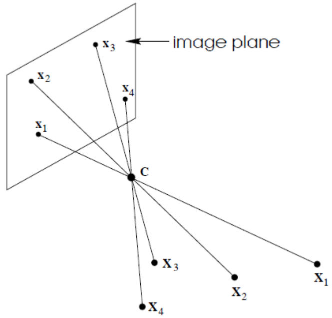


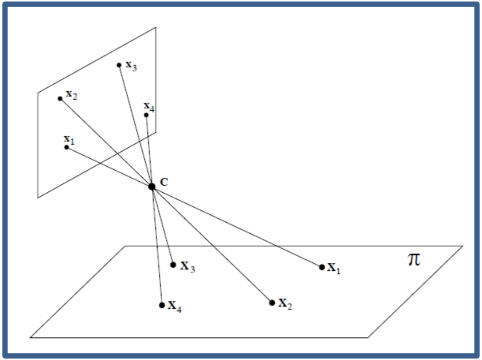
The action of a projective camera on a point in space may be expressed in terms of a linear mapping of homogeneous coordinates.

$$\begin{pmatrix} x \\ y \\ w \end{pmatrix} = \mathrm{P}_{3\times 4} \begin{pmatrix} \mathrm{X} \\ \mathrm{Y} \\ \mathrm{Z} \\ \mathrm{T} \end{pmatrix}$$

Image source: [Hartley Zisserman 2004]

Planar transformations





If all the points lie on a plane, then the linear mapping reduces to:

$$\begin{pmatrix} x \\ y \\ w \end{pmatrix} = \mathrm{H}_{3\times 3} \begin{pmatrix} \mathrm{X} \\ \mathrm{Y} \\ \mathrm{T} \end{pmatrix}$$

Image source: [Hartley Zisserman 2004]

Projective Transformations in P^2

A projectivity is an invertible mappings

- from points in the P^2 (the projective plane)
- to points in P^2
- that maps lines to lines

Definition 2.9. A *projectivity* is an invertible mapping h from $\mathbb{P}^2$ to itself such that three points $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ lie on the same line if and only if $h(\mathbf{x}_1)$, $h(\mathbf{x}_2)$ and $h(\mathbf{x}_3)$ do.

- Projectivities form a group
  - The inverse of a projectivity is a projectivity
  - The composition of two projectivities is a projectivity

- A projectivity is also called
  - A collineation
  - A projective transformation
  - A homography

Definition from [Hartley Zisserman 2004]

Projective Transformations

A projectivity is an invertible mappings from points in the P^2 (the projective plane) to points in P^2 that maps lines to lines

An equivalent algebraic definition is:

Theorem 2.10. *A mapping $h : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ is a projectivity if and only if there exists a non-singular 3×3 matrix $\mathtt{H}$ such that for any point in $\mathbb{P}^2$ represented by a vector $\mathbf{x}$ it is true that $h(\mathbf{x}) = \mathtt{H}\mathbf{x}$.*

- From this theorem we see that
  - Any projectivity arises as a linear transformation in homogeneous coordinates
  - Any linear mapping of homogeneous coordinates is a projectivity

Projective Transformations

Theorem 2.10. *A mapping $h : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ is a projectivity if and only if there exists a non-singular 3×3 matrix $\mathtt{H}$ such that for any point in $\mathbb{P}^2$ represented by a vector $\mathbf{x}$ it is true that $h(\mathbf{x}) = \mathtt{H}\mathbf{x}$.*

- From this theorem we see that
  - Any projectivity arises as a linear transformation in homogeneous coordinates
  - Any linear mapping of homogeneous coordinates is a projectivity

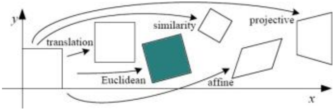
$$\mathbf{x}' = H\,\mathbf{x}$$

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Master in Computer Vision Barcelona

PART I – TYPES OF PLANAR TRANSFORMATIONS

Planar Euclidean Transformation (simplified isometry)



Let **x** and **x**$'$ be homogeneous coordinates.

$$\mathrm{x}' = \mathrm{H}_E \mathrm{x} = \begin{bmatrix} \mathrm{R} & \mathbf{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \mathrm{x}$$

where *R* is a rotation matrix (orthogonal matrix) and
t is a translation vector.

Degrees of freedom: 3
1 for the rotation angle + 2 for the translation coefficients

Invariants: lengths, angles, areas

Invariants

We can describe transformations algebraically as matrices acting on coordinates (e.g. points, lines).

Alternatively, we can also describe a transformation based on the elements or **quantities that are preserved**, or invariant.

A scalar invariant of a geometric configuration is a function of the configuration whose value is unchanged by a particular transformation.

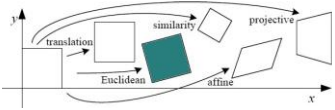
Example:

- The length between two points,
- The angle between two lines, and
- The areas of polygons

Do not change under isometries, and thus are **isometric invariants.**

Isometry

Are transformations of the plane that preserve Euclidean distances



Let **x** and **x**$'$ be homogeneous coordinates.

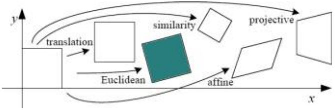
$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \epsilon\cos\theta & -\sin\theta & t_x \\ \epsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

where R is a rotation matrix (orthogonal matrix) and
t is a translation vector
ε is either +1 or -1.

Degrees of freedom: 3
1 for the rotation angle + 2 for the translation coefficients

Invariants: lengths, angles, areas

Isometry



Are transformations of the plane that preserve Euclidean distances

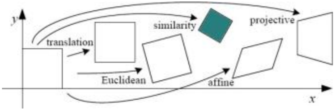
Let $\mathbf{x}$ and $\mathbf{x}'$ be homogeneous coordinates.

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \epsilon\cos\theta & -\sin\theta & t_x \\ \epsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

An isometry is **orientation preserving** if the upper-left 2x2 matrix has determinant +1. Orientation preserving isometries form a group.

An isometry is **orientation reversing** if the upper-left 2x2 matrix has determinant -1. Orientation reversing isometries do NOT form a group.

II. Similarity



Let **x** and **x**$'$ be homogeneous coordinates.

$$\mathrm{x}' = \mathrm{H}_S \mathrm{x} = \begin{bmatrix} s\mathrm{R} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \mathrm{x}$$

where *s* is an isotropic scaling factor,
R is a rotation matrix (orthogonal matrix) and
t is a translation vector.

Degrees of freedom: 4
→ 1 for the rotation angle + 2 for the translation coefficients
+ 1 for scaling factor

Invariants: ratio of lenghts, ratio of two areas,
angles (therefore parallel lines keep parallel)
the circular points **I, J.**

Circular Points

Consider the equation of a conic in homogeneous coordinates:

$$a{x_1}^2 + bx_1x_2 + c{x_2}^2 + dx_1x_3 + ex_2x_3 + f{x_3}^2 = 0$$

A circle is a spacial case of a conic in which a = c and b = 0

$$x_1^2 + x_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

Which intersects the line at infinity at “ideal” points (i.e. $x_3 = 0$):

$$x_1^2 + x_2^2 = 0$$

This equation has 2 solutions:

$$\mathbf{I} = (1, i, 0)^\mathsf{T},\ \mathbf{J} = (1, -i, 0)^\mathsf{T}$$

Each and every circle will intersect the line at infinite at the above points. Thus, they are known as the **circular points**.

Circular Points and Similarities

The circular points are fixed under an orientation-preserving similarity:

$$
\begin{aligned}
\mathbf{I}' &= \mathrm{H}_S \mathbf{I} \\
&= \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = se^{-i\theta} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \mathbf{I}
\end{aligned}
$$

The same can be shown to hold for J.

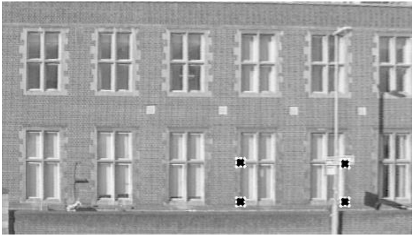
If the similarity is not orientation-preserving, then I and J are swapped

The converse is also true: if the circular points are fixed then H is a similarity

Result 2.21. *The circular points,* $\mathbf{I}, \mathbf{J}$, *are fixed points under the projective transformation* H *if and only if* H *is a similarity.*

Matric Structure

When we talk about metric structure, it implies that the structure is defined up to a similarity.



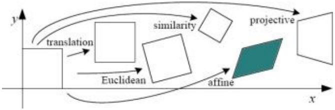
Angles are preserved: squares are squares; parallel lines are parallel; right angles are right angles, etc.

Ratios of lengths and areas are preserved: we can “measure” things on the image in relative terms, not absolute.

And the circular points are preserved as well.

Image source: [Hartley Zisserman 2004]

III. Affine Transformations



Let **x** and **x**$'$ be homogeneous coordinates.

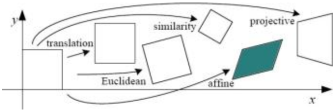
$$\mathbf{x}' = \mathrm{H}_A \mathbf{x} = \begin{bmatrix} \mathrm{A} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \mathbf{x}$$

where A is a non-singular 2×2 matrix
(orientation preserving for $\det(A) > 0$)
t is a translation vector.

Degrees of freedom: 6

Invariants: parallel lines, ratios of parallel lenghts, ratio of two areas, line at infinity l_∞.

III. Affine Transformations



Let **x** and **x**$'$ be homogeneous coordinates.

$$\mathbf{x}' = \mathtt{H}_{\mathtt{A}}\mathbf{x} = \begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix} \mathbf{x}$$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \boxed{\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}} & \begin{matrix} t_x \\ t_y \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

The Singular Value Decomposition of A allows a helpful interpretation of the transformation:

$$\begin{aligned} \mathtt{A} &= \mathtt{UDV}^{\mathsf{T}} = (\underline{\mathtt{UV}^{\mathsf{T}}})(\underline{\mathtt{V}}\mathtt{D}\underline{\mathtt{V}^{\mathsf{T}}}) \\ &= \underline{\mathtt{R}(\theta)}\,(\underline{\mathtt{R}(-\phi)}\,\mathtt{D}\,\underline{\mathtt{R}(\phi)}) \end{aligned}$$

III. Affine Transformations

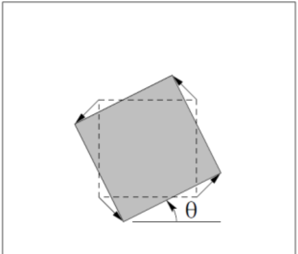
$$\mathbf{x}' = \mathtt{H}_A \mathbf{x} = \begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \mathbf{x}$$

$$\mathtt{A} = \mathtt{R}(\theta)\,\mathtt{R}(-\phi)\,\mathtt{D}\,\mathtt{R}(\phi)$$

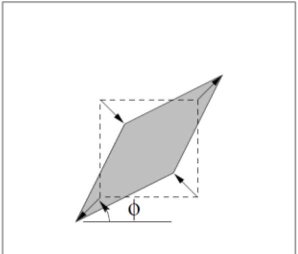
$$\mathtt{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Therefore, an affinity can be though of as

- Rotation (by φ) + Scaling (by matrix **D**) + Rotation back (by -φ)
- Rotation by another angle (θ)



rotation



deformation

Image source: [Hartley Zisserman 2004]

III. Affine Transformations

$$\mathbf{x}' = \mathrm{H}_\mathrm{A}\mathbf{x} = \begin{bmatrix} \mathrm{A} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \mathbf{x}$$

$$\mathrm{A} = \mathrm{R}(\theta)\,\mathrm{R}(-\phi)\,\mathrm{D}\,\mathrm{R}(\phi)$$

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Therefore, an affinity can be though of as

- Rotation (by φ) + Scaling (by matrix **D**) + Rotation back (by -φ)
- Rotation by another angle (θ)

Degrees of freedom: 6

→ *2* for the rotation angles
+2 for the non-isotropic scaling
+2 for the translation

Affine Invariants

$$\mathbf{x}' = \mathrm{H}_A \mathbf{x} = \begin{bmatrix} \mathrm{A} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \mathbf{x}$$

$$\mathrm{A} = \mathrm{R}(\theta)\,\mathrm{R}(-\phi)\,\mathrm{D}\,\mathrm{R}(\phi)$$

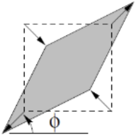
$$\mathrm{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Parallel lines

Parallel lines intersect at ideal points, which under an affinity are also mapped to (other) points at infinity. Therefore lines remain parallel.

Ratio of lengths of parallel lines segments

*T*he amount of scaling of a line segment depends only on its angle (relative to the angles of anisotropic scaling). Thus, it cancels out for parallel segments

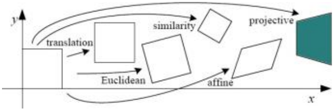


Ratio of areas

Rotations do not affect areas, thus only scaling matters here. It can be shown that the area of any shape is scaled by det(A) and thus cancels out in case of ratios.

The line at infinity

IV. Projective Transformations



Let **x** and **x**′ be homogeneous coordinates.

$$\mathbf{x}' = \mathrm{H_P}\mathbf{x} = \begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^{\mathsf{T}} & v \end{bmatrix} \mathbf{x}$$

Where H_p is a non singular matrix (i.e. it represents an invertible mapping) and it is called a 2D homography

$$H_p = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix}$$

Degrees of freedom: 8

Invariants: concurrency, collinearity, order of contact, cross ratio

IV. Projective Transformations

$$= \mathrm{H_P}\mathbf{x} = \begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^\mathsf{T} & v \end{bmatrix} \mathbf{x}$$

There are 9 elements but they are only defined up to scale, hence we have 8 d.o.f.
Note that we can NOT always set $h_{33} = 1$.

$$H_p = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix}$$

Given 2 planes in general position and orientation (in 3D), the mapping between them is a 2D Homography

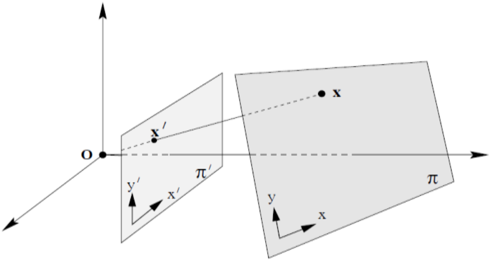
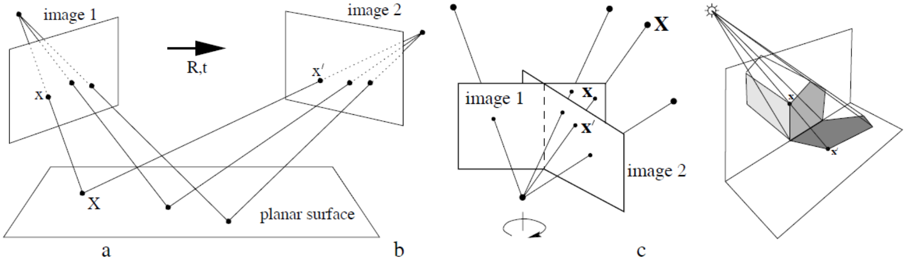


Image source: [Hartley Zisserman 2004]

Examples of Projective Transformations

(a) The projective transformation between 2 images induced by a world plane
(b) The projective transformation between 2 images with the same camera centre
(c) The projective transformation between the image of a plane and the image of its shadow.



2D homographies also provide an approximation with the whole scene is sufficiently far from the camera.

Image source: [Hartley Zisserman 2004]

Planar transformations

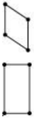
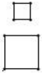
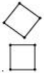
Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, **order of contact**: intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $\mathbf{l}_\infty$.
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, **I**, **J** (see section 2.7.3).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area

Image source: [Hartley Zisserman 2004]

Projective invariants: the cross ratio

The cross ratio is a ratio of ratios:

- It is computed from 4 collinear points A, B, C, D.
- Because the points are collinear, the cross ratio can be explained in a simplified manner using the 1D projective plane P^1
- But it can be generalized to the 2D projective plane since every 2D homography induces a 1D projective transformation of a line
- The cross ratio is valid for any 4 collinear points, i**ncluding ideal points**

$$(A, B; C, D) = \frac{AC \cdot BD}{BC \cdot AD},$$

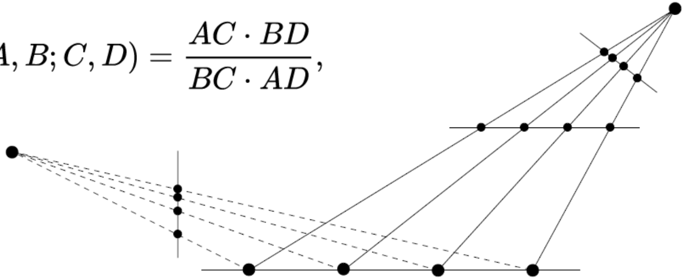


Image source: [Hartley Zisserman 2004]

Projective invariants: cross ratio example

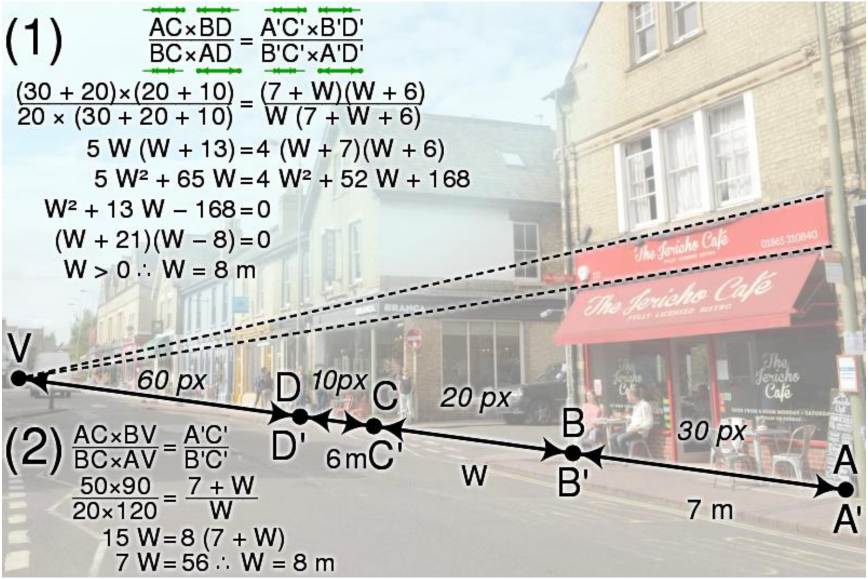


Image source: [Wikipedia]

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PART II – RECOVERY OF AFFINE AND METRIC PROPERTIES FROM IMAGES

Affine and Metric Recovery

- Projective transformations have 8 d.o.f
  - They can be fully specified (recovered) by 4 points
  - This means having the coordinates of 4 points in both the image and the real World. Points must be in correspondence.

- However, we do not need to recover the full projectivity to make useful measurements from images.
  - We can also extract information from affinities and similarities

- An affine transformation has 6 d.o.f
  - We only need -2 d.o.f from a projective image to recover it
  - This can be done by using the line at infinity

- A similarity transformation has 4 d.o.f
  - We only need -4 d.o.f. from a projective image to recover it
  - Or -2 d.o.f with respect to an affinity
  - This can be done by using the circular points

The line at infinity

Under a projective transformation, ideal points can be mapped to real points:

$$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^\mathsf{T} & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathtt{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ v_1 x_1 + v_2 x_2 \end{pmatrix}$$

but this does not happen under an affine transformation:

$$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathtt{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ 0 \end{pmatrix}$$

Therefore, the line of all ideal points, or line at infinity l_∞ is fixed under an affinity.

- Note that the fixing is not point-wise: a point at infinity is mapped to (another) point at infinity.

The line at infinity

This can also be seen by directly transforming the line at infinity with an affine transformation:

$$\mathbf{l}'_\infty = \mathtt{H}_{\mathtt{A}}^{-\mathsf{T}}\mathbf{l}_\infty = \begin{bmatrix} \mathtt{A}^{-\mathsf{T}} & \mathbf{0} \\ -\mathbf{t}^{\mathsf{T}}\mathtt{A}^{-\mathsf{T}} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{l}_\infty .$$

That is, under an affinity the line l_∞ remains at affinity.

The converse is also true because to guarantee that a point at infinity is mapped to an ideal point, we need h_{31} and h_{32} to be zero.

Result 2.17. *The line at infinity, $\mathbf{l}_\infty$, is a fixed line under the projective transformation $\mathtt{H}$ if and only if $\mathtt{H}$ is an affinity.*

Affine Rectification

- We can use two simple facts that we already know
  - Parallel lines intersect at ideal points (e.g. at the line at infinity)
  - Projective transformations preserve intersections
- Therefore, with 2 pairs of parallel lines whose intersection is imaged as a real point, we can identify the line at infinity.

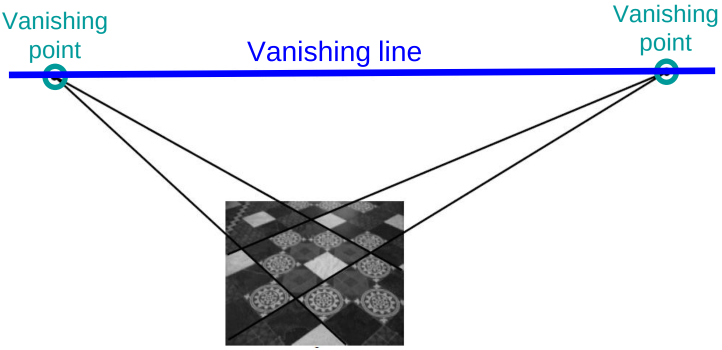
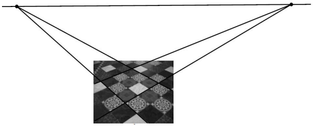


Image source: [Hartley Zisserman 2004]

Affine Rectification from the Vanishing Line



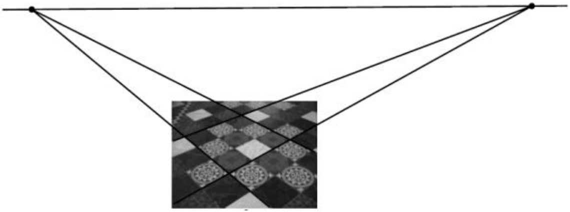
1. Take two sets of two parallel lines in the image of a plane.
2. Each one provides a vanishing point, which can be computed from the cross product.
3. From these two points (which are on the vanishing line), compute the vanishing line $l = (l_1, l_2, l_3)^T$
4. Assuming $l_3 \neq 0$, we can compute $H_{a \leftarrow p}$.

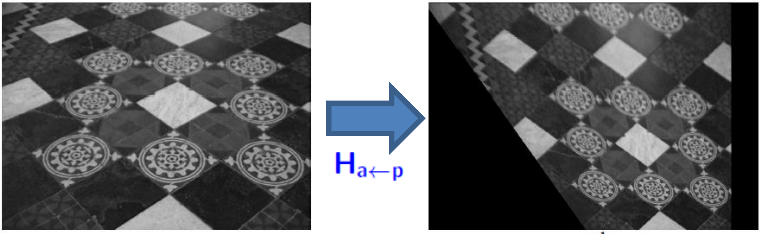
$$\mathbf{H_{a \leftarrow p}} = H_a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{pmatrix}$$

It can be shown that $H_{a \leftarrow p}$ will map $l = (l_1, l_2, l_3)^T$ to $l_\infty = (0, 0, 1)^T$

We apply $H_{a \leftarrow p}$ to the whole image to obtain the affine-rectified image.

Affine Rectification from the Vanishing Line





We apply $H_{a \leftarrow p}$ to the whole image to obtain the affine-rectified image.

Metric Rectification

Metric reconstruction is based on the following result:

Result 2.21. *The circular points,* **I**, **J**, *are fixed points under the projective transformation* H *if and only if* H *is a similarity.*

Notice that, similarly to the line at infinity:

- The imaged circular points, I′ and J′, are the points of intersection of circles and the line at infinity
  - They will be at = $(1, \pm i, 0)^T$ if and only if the image is matric-rectified.

Because of the duality principle:

- If the circular points I and J are fixed under similarity
  - Also their dual is fixed
  - The dual of I and J is known as the dual conic to the circular points

The Dual Conic

The dual conic is: $\mathtt{C}^*_\infty = \mathbf{I}\mathbf{J}^\mathsf{T} + \mathbf{J}\mathbf{I}^\mathsf{T}$

In a Euclidean coordinate frame, we have:

$$\mathtt{C}^*_\infty = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \begin{pmatrix} 1 & -i & 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \begin{pmatrix} 1 & i & 0 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Because it is a "dual" conic, it transforms according to:

$$\mathtt{C}^{*\prime} = \mathtt{H}\mathtt{C}^*\mathtt{H}^\mathsf{T}$$

And it can be shown that:

$$\mathtt{C}^{*\ \prime}_\infty = \mathtt{H}_\mathtt{S}\mathtt{C}^*_\infty\mathtt{H}_\mathtt{S}^\mathsf{T} = \mathtt{C}^*_\infty$$

Result 2.22. *The dual conic* $\mathtt{C}^*_\infty$ *is fixed under the projective transformation* $\mathtt{H}$ *if and only if* $\mathtt{H}$ *is a similarity.*

Angles and the Dual Conic

Given two lines $\boldsymbol{l} = (l_1, l_2, l_3)^T$ and $\mathbf{m} = (m_1, m_2, m_3)^T$

- Their normal would be $(l_1, l_2)^T$ and $(m_1, m_2)^T$
- And (cosine of) the angle between them is:

$$\cos\theta = \frac{l_1 m_1 + l_2 m_2}{\sqrt{(l_1^2 + l_2^2)(m_1^2 + m_2^2)}}.$$

But the above is NOT a projective invariant.
Thus, a more convenient formulation that is projectively invariant, is:

$$\cos\theta = \frac{\mathbf{l}^{\mathsf{T}} \mathtt{C}^*_\infty \mathbf{m}}{\sqrt{(\mathbf{l}^{\mathsf{T}} \mathtt{C}^*_\infty \mathbf{l})(\mathbf{m}^{\mathsf{T}} \mathtt{C}^*_\infty \mathbf{m})}}$$

Result 2.23. *Once the conic $\mathtt{C}^*_\infty$ is identified on the projective plane then Euclidean angles may be measured*

Orthogonality and Length Ratios

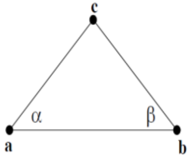
Result 2.24. *Lines* $\mathbf{l}$ *and* $\mathbf{m}$ *are orthogonal if* $\mathbf{l}^{\mathsf{T}} C^{*}_{\infty} \mathbf{m} = 0$.

$$\cos\theta = \frac{\mathbf{l}^{\mathsf{T}} C^{*}_{\infty} \mathbf{m}}{\sqrt{(\mathbf{l}^{\mathsf{T}} C^{*}_{\infty} \mathbf{l})(\mathbf{m}^{\mathsf{T}} C^{*}_{\infty} \mathbf{m})}}$$

Length ratios

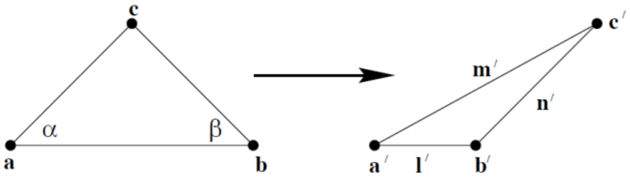
Once the (imaged) dual conic is identified, we can measure length ratios

- Consider a triangle with vertices **a**, **b** and **c**
- By the sine rule we know that
  - dist(**b**, **c**) : dist(**a**, **c**) = sin(α) : sin(β)



- And since we can compute cosines, we may also compute sines between any two lines.

Orthogonality and Length Ratios



Assume the dual conic has been identified in the image plane

- Compute lines

$$\mathbf{l}' = \mathbf{a}' \times \mathbf{b}', \quad \mathbf{m}' = \mathbf{c}' \times \mathbf{a}' \quad \mathbf{n}' = \mathbf{b}' \times \mathbf{c}'$$

- Compute the cosines of the angles from

$$\cos\theta = \frac{\mathbf{l}^{\mathsf{T}} C^{*}_{\infty} \mathbf{m}}{\sqrt{(\mathbf{l}^{\mathsf{T}} C^{*}_{\infty} \mathbf{l})(\mathbf{m}^{\mathsf{T}} C^{*}_{\infty} \mathbf{m})}}$$

- Compute the sines of the angles
- Use the sine rule for the ratios, e.g. dist(**b**, **c**) : dist(**a**, **c**) = sin(α) : sin(β)

Matric Rectification through the Dual Conic

Result 2.25. *Once the conic C^*_∞ is identified on the projective plane then projective distortion may be rectified up to a similarity.*

To see why, we first decompose the projective transformation H into a chain of transformations that separates explicitly the incremental components from a Similarity to a Projectivity, passing through an Affinity:

$$H = H_P\, H_A\, H_S = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{v}^\top & 1 \end{bmatrix} \begin{bmatrix} K & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} sR & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

Replacing the above in $C^{*\prime} = HC^*H^\top$

$$\begin{aligned} C^{*\prime}_\infty &= (H_P\, H_A\, H_S)\, C^*_\infty\, (H_P\, H_A\, H_S)^\top = (H_P\, H_A)\left(H_S\, C^*_\infty H_S^\top\right)\left(H_A^\top\, H_P^\top\right) \\ &= (H_P\, H_A)\, C^*_\infty\left(H_A^\top\, H_P^\top\right) \\ &= \begin{bmatrix} KK^\top & KK^\top\mathbf{v} \\ \mathbf{v}^\top KK^\top & \mathbf{v}^\top KK^\top\mathbf{v} \end{bmatrix}. \end{aligned}$$

Matric Rectification through the Dual Conic Starting from an Affine-rectified image

Consider an affinely-rectified image in which we identify lines **l'** and **m'** that are orthogonal (in the real World, not in the image).

$$\begin{pmatrix} l'_1 & l'_2 & l'_3 \end{pmatrix} \begin{bmatrix} \mathtt{KK}^\mathsf{T} & \mathtt{KK}^\mathsf{T}\mathbf{v} \\ \mathbf{v}^\mathsf{T}\mathtt{KK}^\mathsf{T} & \mathbf{v}^\mathsf{T}\mathtt{KK}^\mathsf{T}\mathbf{v} \end{bmatrix} \begin{pmatrix} m'_1 \\ m'_2 \\ m'_3 \end{pmatrix} = 0$$

But given that the image is affinely-rectified, $\mathbf{v}^\mathsf{T} = 0$

$$\begin{pmatrix} l'_1 & l'_2 & l'_3 \end{pmatrix} \begin{bmatrix} \mathtt{KK}^\mathsf{T} & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 0 \end{bmatrix} \begin{pmatrix} m'_1 \\ m'_2 \\ m'_3 \end{pmatrix} = 0$$

Which is a linear constraint on the 2×2 matrix S = $\mathtt{KK}^\mathsf{T}$

$$(l'_1, l'_2)\mathtt{S}(m'_1, m'_2)^\mathsf{T}$$

Matric Rectification through the Dual Conic Starting from an Affine-rectified image

Matrix S = KK^T is symmetric

- Then it has 3 distinct parameters, and 2 independent ratios
- We need 2 constraints to recover it
  - This matches the concept that we need -2 d.o.f to go from an affinity to a similarity
- Therefore, if we vectorize matrix S and write the constraint from orthogonal lines **l** and **m**:

$$\mathbf{s} = (s_{11}, s_{12}, s_{22})^{\mathsf{T}}$$

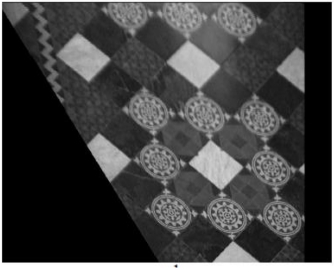
$$(l'_1 m'_1, l'_1 m'_2 + l'_2 m'_1, l'_2 m'_2)\, \mathbf{s} = 0,$$

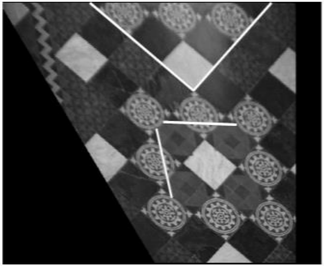
- We see that we need 2 pairs of parallel lines
  - We obtain a 2×3 matrix of constraints
  - Can solve for S (and therefore K) finding the null space.

Affine and Matric Rectification

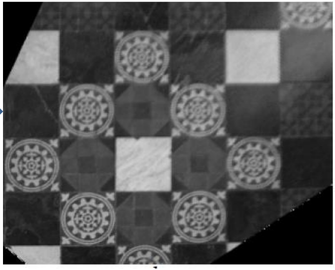
$H_{a \leftarrow p}$

$$\begin{bmatrix} I & 0 \\ v^T & 1 \end{bmatrix}$$





$$\begin{bmatrix} K & 0 \\ 0^T & 1 \end{bmatrix}$$



References

[Hartley and Zisserman 2004] R.I. Hartley and A. Zisserman, Multiple View Geometry in Computer Vision, Cambridge University Press, 2004.

[Szeliski 2010] R. Szeliski, Computer Vision: Algorithms and Applications, Springer, 2010.